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Large deviations of infinite intersections of events in Gaussian processes

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Abstract

Consider events of the form $\{Z_s \geq \zeta(s), s \in S\}$, where Z is a continuous Gaussian process with stationary increments, ζ is a function that belongs to the reproducing kernel Hilbert space R of process Z, and $S \subset \mathbb{R}$ is compact. The main problem considered in this paper is identifying the function $\beta^* \in R$ satisfying $\beta^*(s) \geq \zeta(s)$ on S and having minimal R-norm. The smoothness (mean square differentiability) of Z turns out to have a crucial impact on the structure of the solution. As examples, we obtain the explicit solutions when $\zeta(s) = s$ for $s \in [0, 1]$ and Z is either a fractional Brownian motion or an integrated Ornstein–Uhlenbeck process.

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1. Introduction

The large deviation principle (LDP) for Gaussian measures in Banach space, usually known as the (generalized) Schilder theorem, was established more than two decades ago by [3], see also

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[2,4]. In this LDP, a central role is played by the norm $\|f\|$ of paths f in the reproducing kernel Hilbert space of the underlying Gaussian process. More precisely, the probability of the Gaussian process being in some closed convex set A has exponential decay rate $-\frac{1}{2}\|f^*\|^2$, where f^* is the path in A with minimum norm, i.e., $\operatorname{argmin}_{f\in A}\|f\|$. The path f^* is called the "dominating point" in large deviations literature, and it has the interpretation of the *most probable path* (MPP) in A: if the Gaussian process happens to fall in A, with overwhelming probability it will be close to f^*.

Our interest in this topic stems from large deviation analyses of Gaussian queues, where various MPPs have been found explicitly. Addie et al. [1] consider a queueing system fed by a Gaussian process with stationary increments, and succeed in finding the MPP leading to overflow. This problem is relatively easy as the overflow event can be written as an infinite *union* of events $A = \cup_{t>0} A_t$, and the decomposition $\inf_{f\in A}\|f\| = \inf_{t>0}\inf_{f\in A_t}\|f\|$ applies. Here A_t corresponds to the event of overflow at time t, and due to the fact that finding the infimum over A_t turns out to be just a one-dimensional problem, the problem can be solved. In this paper we look at the intrinsically more involved situation where A is an *intersection*, rather than a union, of events: $A = \cap_t A_t$; decay rates, and the corresponding MPPs, of these intersections are then usually considerably harder to determine. In our setting the norm has to be minimized over a truly infinite-dimensional convex set in a Hilbert space.

Few results are known on MPPs of these infinite intersections of events. Norros [11] showed that the event of a queue with fractional Brownian motion (fBm) input having a busy period longer than, say, 1, corresponds to an infinite intersection of events; the set A consists of all f such that $f(t) \geq t$ for all $t \in [0, 1]$. However, the shape of the MPP in A remained an open problem. Interestingly, it was proven that the straight line, i.e., the path $f(t) = t$, is *not* optimal except for the Brownian case. In the case of Markovian input, the straight line is the asymptotically typical form for long busy periods [15, Thm. 11.24]. Mandjes and van Uitert [9, 10] analyzed buffer overflow in tandem, priority, and generalized processor sharing queues: it was shown that in these queues overflow relates to an infinite intersection of events, and explicit lower bounds on the minimizing norm (corresponding to upper bounds on the overflow probability) were obtained. Conditions were given under which this lower bound is tight — in that case obviously the path corresponding to the lower bound is also the MPP.

An important element in our analysis is the mean square smoothness of the Gaussian process involved. Most of our results assume that the process has the property that its local behaviors around any two distinct points are asymptotically independent, and this is shown to exclude smoothness. On the other hand, we show that the solution can have a simple finite-dimensional structure when the process is smooth.

This paper is organized as follows. Section 2 presents preliminaries on Gaussian processes and a number of other prerequisites. In Section 3 we focus on the most probable path in the set of paths f such that $f(t) \geq \zeta(t)$, for a function ζ and t in some compact set $S \subset \mathbb{R}$. Our general result characterizes the MPP in this infinite intersection of events. In the case where the Gaussian process does not have derivatives, the MPP can be expressed as a conditional mean given the points s where it equals $\zeta(s)$. Section 4 gives explicit results for the case $\zeta(t) = t$ and $S = [0, 1]$, the so-called "busy period problem". We illustrate the impact of the smoothness with examples of both a process without derivatives (fBm) and a process with one derivative (integrated Ornstein–Uhlenbeck process). In the case of fBm, we prove that for $H > \frac{1}{2}$ the MPP is *at* the diagonal in some interval $[0, s^*]$, and also at time 1, but *strictly above* the diagonal in between; for $H < \frac{1}{2}$, the corresponding path departs immediately after time 0 from the diagonal, but returns to it strictly before time 1 and continues along it for the rest of the interval. In the case of the integrated Ornstein–Uhlenbeck process, we show how the MPP is derived by imposing conditions at two points, namely on the derivative at $t = 0$ and on the value of the function at $t = 1$. It this case, the MPP touches the diagonal only at times 0 and 1.

2. Preliminaries

This section presents some prerequisites, including relevant results on Gaussian processes.

2.1. Gaussian process, path space, and reproducing kernel Hilbert space

The following framework will be used throughout the paper. Let $Z = (Z_t)_{t\in\mathbb{R}}$ be a centered Gaussian process with stationary increments and $Z_0 = 0$ a.s., completely characterized by its variance function $v(t) \doteq \mathrm{Var}\,(Z_t)$. The covariance function of Z can be written as $\Gamma(t, s) \doteq \mathrm{Cov}\,(Z_t, Z_s) = \frac{1}{2}(v(s) + v(t) - v(s - t))$. For a finite subset S of $\mathbb{R}$, we denote by $\Gamma(S)$ the matrix $\{\Gamma(s, t) : s \in S, t \in S\}$, by $\Gamma(S, t)$ the column vector $\{\Gamma(s, t) : s \in S\}$, and by $\Gamma(t, S)$ the corresponding row vector.

In addition to the basic requirement that $v(t)$ results in a positive semi-definite covariance function, we impose the following assumptions on $v(t)$:

(i) $v(t)$ is continuous, and $\Gamma(S)$ is non-singular for any finite subset S of $\mathbb{R}$ that does not contain 0;
(ii) there is a number $\alpha_0 \in (0, 2]$ such that $v(h)/h^{\alpha_0}$ is bounded for $h \in (0, 1)$;
(iii) $\lim_{t\to\infty} v(t) = \infty$, and $\lim_{t\to\infty} v(t)/t^{\alpha_\infty} = 0$ for some $\alpha_\infty \in (0, 2)$.

The (ii) guarantees the existence of a version with continuous paths, by virtue of Kolmogorov's lemma. Denote by Ω the function space

$$\Omega \doteq \left\{\omega : \omega \text{ continuous } \mathbb{R} \to \mathbb{R},\ \omega(0) = 0,\ \lim_{t\to\infty} \frac{\omega(t)}{1 + |t|} = \lim_{t\to-\infty} \frac{\omega(t)}{1 + |t|} = 0\right\}.$$

Equipped with the norm $\|\omega\|_\Omega \doteq \sup\{|\omega(t)|/(1 + |t|) : t \in \mathbb{R}\}$, Ω is a separable Banach space. We choose Ω as our basic probability space by letting $\mathbb{P}$ be the unique probability measure on the Borel sets of Ω such that the random variables $Z_t(\omega) = \omega(t)$ form a realization of Z.

The *reproducing kernel Hilbert space R* related to Z is defined by starting from the functions $\Gamma(t, \cdot)$ and defining an inner product by $\langle\Gamma(s, \cdot), \Gamma(t, \cdot)\rangle = \Gamma(s, t)$. The space is then closed with linear combinations, and completed with respect to the norm $\|\cdot\|^2 = \langle\cdot, \cdot\rangle$. Thus, the mapping

$$Z_t \mapsto \Gamma(t, \cdot) \tag{1}$$

is extended to an isometry between the Gaussian space G of Z, i.e., the smallest closed linear subspace of L^2 containing the random variables Z_t, and the function space R. The continuity of Γ entails that G and R are separable. The inner product definition generalizes to the *reproducing kernel property*:

$$\langle f, \Gamma(t, \cdot)\rangle = f(t), \quad f \in R. \tag{2}$$

The topology of R is finer than that corresponding to a weighted supremum distance between the paths: by Cauchy–Schwarz and (2),

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1 + |t|} \le \|f\| \cdot \sup_{t\in\mathbb{R}} \frac{\|\Gamma(t, \cdot)\|}{1 + |t|}, \tag{3}$$

where the supremum on the right hand side is finite by (iii). We see that all elements of R are continuous functions, R is a subset of Ω, and the topology of R is finer than that of Ω.

2.2. Large deviations: generalized Schilder theorem

The generalization of Schilder's theorem on large deviations of Brownian motion to Gaussian measures in a Banach space is originally due to [3]; see also [2,4]. Here is a formulation appropriate to our case.

Theorem 1. *The function* $I : \Omega \to [0, \infty]$,

$$I(\omega) \doteq \begin{cases} \frac{1}{2}\|\omega\|_R^2, & \text{if } \omega \in R, \\ \infty, & \text{otherwise}, \end{cases} \tag{4}$$

is a good rate function for the centered Gaussian measure $\mathbb{P}$*, and* $\mathbb{P}$ *satisfies the large deviations principle:*

$$\text{for } F \text{ closed in } \Omega: \quad \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{Z}{\sqrt{n}} \in F\right) \le -\inf_{\omega\in F} I(\omega);$$

$$\text{for } G \text{ open in } \Omega: \quad \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left(\frac{Z}{\sqrt{n}} \in G\right) \ge -\inf_{\omega\in G} I(\omega).$$

For the definition of *good* rate function, see e.g. [4, Section 2.1].

We call a function $f \in A$ such that $I(f) = \inf_{\omega\in A} I(\omega) < \infty$ a *most probable path* of A. A most probable path can be intuitively understood as a point of maximum likelihood, although there is no counterpart to the Lebesgue measure on Ω. If A is convex and closed and has a non-empty intersection with R, then the most probable path exists and is unique.

2.3. Notes on optimization

For a subset A of a Banach space, we denote its linear span by $\operatorname{sp} A$ and its closed linear span by $\overline{\operatorname{sp}}\, A$. The following standard fact from optimization theory is crucial in our analysis, see, e.g., [8, Ex. 3.13.23].

Proposition 2. *Let* H *be a Hilbert space. Define* $A \doteq \{x \in H : \langle x, y_i\rangle \ge a_i,\ i \in I\}$, *where* I *is a finite index set and* $y_i \in H$. *Assume* $x^* = \operatorname{argmin}\{\|x\| : x \in A\}$ *and* $I^* = \{i \in I : \langle x^*, y_i\rangle = a_i\}$. *Then* $x^* \in \operatorname{sp}\{y_i : i \in I^*\}$.

The intuitive content of Proposition 2 is that conditions which are not tightly met (i.e., satisfied with equality) at the optimal point do not appear in the solution. If the finite set of linear conditions is replaced by an infinite one, the result does not hold without further assumptions. One particular generalization will be considered in Section 3.

We also need the following basic infinite-dimensional result.

Proposition 3. *Let* H *be a Hilbert space and* $y_i \in H$, $a_i \in \mathbb{R}$, $i = 1, 2, \ldots$. *Define* $A_n = \{x \in H : \langle x, y_i\rangle \ge a_i,\ i = 1, \ldots, n\}$; $A_\infty = \{x \in H : \langle x, y_i\rangle \ge a_i, i = 1, 2, \ldots\}$. *Assume that the convex set* A_∞ *is non-empty and let* $\alpha_n = \operatorname{argmin}_{x\in A_n}\|x\|, n = 1, 2, \ldots, \infty$. *Then* $\lim_{n\to\infty} \alpha_n = \alpha_\infty$.

Proof. We show first that $\|\alpha_n\| \to \|\alpha_\infty\|$. Obviously the sequence $\|\alpha_n\|$ is non-decreasing, and $\|\alpha_n\| \le \|\alpha_\infty\|$. The closed ball $B(0, \lim_n \|\alpha_n\|)$ is weakly compact. Let α_0 be a weak accumulation point of the sequence α_n. Then, for each n, there is a subsequence m_j such that $\langle \alpha_0, y_n \rangle = \lim_{j\to\infty} \langle \alpha_{m_j}, y_n \rangle \ge a_n$. Thus, $\alpha_0 \in A_n$ for every n. It follows that $\alpha_0 \in A_\infty$ and hence $\|\alpha_\infty\| \le \|\alpha_0\| \le \lim_n \|\alpha_n\|$.

Now, by a basic characterization of minimum norm elements in closed convex sets, we have $\langle \alpha_n, \alpha_\infty - \alpha_n \rangle \ge 0$, since $\alpha_\infty \in A_\infty \subseteq A_n$ and α_n is the minimum norm element of A_n. But then

$$\|\alpha_n - \alpha_\infty\|^2 = \|\alpha_\infty\|^2 - \|\alpha_n\|^2 - 2\langle \alpha_n, \alpha_\infty - \alpha_n \rangle \le \|\alpha_\infty\|^2 - \|\alpha_n\|^2 \to 0. \quad \square$$

2.4. Derivatives and the infinitesimal space

The character of the "most probable paths" considered in this paper turns out to depend strongly on the smoothness and certain other, rather subtle properties of the Gaussian process Z. First we have to note some general facts about shrinking sequences of subspaces.

Consider one fixed separable Hilbert space and denote its closed subspaces by X, Y, etc. Write $X_n \searrow Y$ if $X_1 \supseteq X_2 \supseteq \cdots$ and

$$\bigcap_{n=1}^{\infty} X_n = Y.$$

This kind of decreasing sequence of subspaces is rather deceptive for intuitive perception. For example, $X_n \searrow \{0\}$ does not imply $X_n + Y \searrow Y$ in general. The next lemma provides some rules for such situations.

For two subspaces X and Y, define the cosine of the angle between X and Y as

$$\alpha(X, Y) = \sup\{\langle x, y \rangle : x \in X,\ y \in Y, \|x\| = \|y\| = 1\}.$$

Two decreasing sequences of subspaces X_n and Y_n are called asymptotically orthogonal if $\lim_{n\to\infty} \alpha(X_n, Y_n) = 0$.

Lemma 4. (i) *If $X_n \searrow \{0\}$, $Y_n \searrow \{0\}$, and $\alpha(X_n, Y_n) < 1$ for some n, then*

$$X_n + Y_n \searrow \{0\}.$$

(ii) *Assume that the k sequences of subspaces $X_n^1, \dots, X_n^k$ are pairwise asymptotically orthogonal and satisfy, for every $i = 1, \dots, k$, $X_n^i \searrow \{0\}$ as $n \to \infty$. Then*

$$X_n^1 + \cdots + X_n^k \searrow \{0\} \quad \text{as } n \to \infty.$$

(iii) *If $Y_n \searrow \{0\}$ and $\alpha(X, Y_n) < 1$ for some n, then*

$$X + Y_n \searrow X.$$

If X is finite-dimensional, then the additional condition $\alpha(X, Y_n) < 1$ holds automatically for some n.

Proof. To prove (i), assume the contrary: there exists some $z \in \bigcap(X_n + Y_n)$ with $\|z\| = 1$. Then for each n there are $x_n \in X_n$ and $y_n \in Y_n$ such that $z = x_n + y_n$. Since

$$\|z\|^2 \ge (\|x_n\| - \|y_n\|)^2 + 2(1 - \alpha(X_n, Y_n))\|x_n\|\|y_n\|, \tag{5}$$

and $\alpha(X_n, Y_n)$ is decreasing, the sequences x_n and y_n are bounded. Since the unit ball of Hilbert space is weakly compact, the sequence of pairs (x_n, y_n) has an accumulation point (x^*, y^*) in the product space (with weak topologies). In particular, there is a subsequence n_i such that

$$1 = \|z\|^2 = \lim_{i\to\infty} (\langle z, x_{n_i}\rangle + \langle z, y_{n_i}\rangle) = \langle z, x^*\rangle + \langle z, y^*\rangle.$$

However, necessarily $x^* \in \bigcap X_n$ and $y^* \in \bigcap Y_n$, so the right hand side must be zero — a contradiction.

To prove (ii), it is sufficient to show that $\lim_{n\to\infty} \alpha(X_n^1 + X_n^2, X_n^3) = 0$. The rest follows by induction from claim (i). Define for brevity $X_n^1 = X$, $X_n^2 = Y$, and $X_n^3 = Z$, and let $x \in X$, $y \in Y$, and $z \in Z$ be arbitrary.

Note first that the inequality (5), adapted to the present notation, implies the two inequalities:

$$\text{if} \quad \|x\| \wedge \|y\| \le \frac{1}{2}(\|x\| \vee \|y\|) \Rightarrow \|x+y\|^2 \ge \frac{1}{4}(\|x\| \vee \|y\|)^2,$$
$$\text{if} \quad \|x\| \wedge \|y\| > \frac{1}{2}(\|x\| \vee \|y\|) \Rightarrow \|x+y\|^2 \ge (1-\alpha(X,Y))(\|x\| \vee \|y\|)^2.$$

Now,

$$\begin{aligned}\langle z, x+y\rangle &\le (\alpha(X,Z) \vee \alpha(Y,Z)) \cdot 2 \cdot (\|x\| \vee \|y\|)\|z\| \\ &\le 2(\alpha(X,Z) \vee \alpha(Y,Z))(2 \vee (1-\alpha(X,Y))^{-1/2}\|x+y\|\|z\|.\end{aligned}$$

Thus, we have

$$\alpha(X_n^1 + X_n^2, X_n^3) \le 2(\alpha(X_n^1, X_n^3) \vee \alpha(X_n^2, X_n^3))(2 \vee (1-\alpha(X_n^1, X_n^2))^{-1/2} \to 0$$

as $n \to \infty$.

The proof of (iii) resembles that of claim (i). Assume that $z \in \bigcap_n (X + Y_n)$, $\|z\| = 1$. Like in the proof of (i), write $z = x_n + y_n$ and find an accumulation point pair (x^*, y^*) and a weakly converging subsequence. For any vector u,

$$\langle u, z\rangle = \langle u, x_{n_k}\rangle + \langle u, y_{n_k}\rangle \to \langle u, x^*\rangle + \langle u, y^*\rangle.$$

Now, $x^* \in X$ and $y^* \in \bigcap Y_n = \{0\}$. Thus, we must have $z = x^*$.

Finally, if X is finite-dimensional and $\alpha(X, Y_n) = 1$ for all n, there exist sequences $x_n \in X$ and $y_n \in Y_n$ such that $\|x_n\| = \|y_n\| = 1$ and $\langle x_n, y_n\rangle > 1 - 1/n$. Take again a weak accumulation point pair (x^*, y^*) and a weakly converging subsequence (x_{n_k}, y_{n_k}). Then $\langle x^*, y_{n_k}\rangle \to \langle x^*, y^*\rangle$. Now, $x_{n_k} \to x^*$ also in the norm, and it follows that

$$\langle x^*, y^*\rangle = \lim \langle x^*, y_{n_k}\rangle = \lim \langle x_{n_k}, y_{n_k}\rangle = 1.$$

Thus, $\bigcap Y_n$ cannot be trivial. □

Let us now turn to our Gaussian process Z. For any set $V \subseteq \mathbb{R}$, define

$$G_V \doteq \overline{\text{sp}}\{Z_t : t \in V\}.$$

For $t \in \mathbb{R}$, denote by

$$U_t^\varepsilon = \overline{\text{sp}}\{Z_u - Z_t : |t-u| \le \varepsilon\}, \tag{6}$$

the space generated by increments of Z_t in an ε-environment of t (note that Z_t does generally not belong to this space). The *infinitesimal space* of Z at time point t is defined as

$$\partial G_t \doteq \bigcap_{\varepsilon>0} U_t^\varepsilon.$$

By the stationarity of increments, the structure of ∂G_t is the same for all t. The R-counterparts of the subspaces G_V and ∂G_t in the isometry $Z_t \mapsto \Gamma(t,\cdot)$ are denoted as R_V and ∂R_t, respectively.

We say that Z possesses *local independence* if for any distinct s and t the subspace sequences $U_s^{1/n}$ and $U_t^{1/n}$ are asymptotically orthogonal. This means that the local behavior of Z around one time point is asymptotically independent of that around another time point.

Proposition 5. *If Z possesses local independence, then the infinitesimal spaces ∂G_t are trivial.*

Proof. Assume that ∂G_0 contains a Gaussian random variable Y. Define the time-shift operators

$$\theta_t \omega(s) = \omega(s+t) - \omega(s).$$

By the assumption on stationary increments, the process $Y_t(\omega) = Y(\theta_t \omega)$ is stationary. Moreover, $Y_t \in U_t^{1/n}$ for each t and n. The assumption on local independence then implies that Y_s and Y_t are orthogonal for any distinct s and t. Now, however, G is a separable Hilbert space, and it cannot contain an uncountable number of pairwise orthogonal distinct elements. Therefore, we must have $Y = 0$ almost surely. □

The converse of Proposition 5 is not true. The simplest counterexample is the periodic Brownian bridge, defined by variance function $v = [t]_1(1 - [t]_1)$, where $[t]_d = t \bmod d$ [1]. To obtain a counterexample that satisfies also assumption (iii) of Section 2.1, add the periodic Brownian bridge to an independent smooth process.

We call Z *smooth* at t, if it has a mean square derivative at t, that is, there exists a random variable $Z'_t \in G$ such that

$$\lim_{h\to 0} \mathbb{E}\left\{\left(\frac{Z_{t+h} - Z_t}{h} - Z'_t\right)^2\right\} = 0.$$

It follows from the stationarity of increments that if Z is smooth at 0, then it is smooth at all $t \in \mathbb{R}$. On the other hand, applying the above definition at $t = 0$, we see that process Z is non-differentiable if $\lim_{h\to 0} v(h)/h^2 = \infty$.

Here are some more properties of a smooth Gaussian process with stationary increments. The proofs are straightforward.

Proposition 6. *Assume that Z is smooth. Then*

(i) *$\Gamma(s,t)$ has partial derivatives, and the isometry counterpart of Z'_t in R is the function $\Gamma'(t,s) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t,s)$;*
(ii) *all functions $f \in R$ are differentiable at every point, and $f'(t) = \langle f, \Gamma'(t,\cdot)\rangle$ for $f \in R$ and $t \in \mathbb{R}$;*
(iii) *v is twice differentiable everywhere, and* $\mathrm{Var}\left(Z'_0\right) = \frac{1}{2}v''(0)$;
(iv) *for any $s, t \in \mathbb{R}$, $\langle \Gamma'(s,\cdot), \Gamma'(t,\cdot)\rangle = \frac{v''(t-s)}{2}$.*

If Z is smooth, then obviously $Z'_t \in \partial G_t$. The higher derivatives also belong to the infinitesimal space whenever they exist.

The infinitesimal space of a stationary Gaussian process X was characterized by Tutubalin and Freidlin [16] in the case where its spectral measure has an absolutely continuous part, whose density $f(\lambda)$ satisfies $f(\lambda) \geq 1/\lambda^p$ for some $p > 0$ and large λ. Then, the number k of derivatives of the process is finite, and the infinitesimal space ∂G_0 is generated by the existing derivatives $X_0^{(1)}, \ldots, X_0^{(k)}$. Moreover, the corresponding infinitesimal σ-algebra is, up to sets of measure zero, also generated by these random variables. (For a generalization for non-stationary processes, see [14].) The proof in [16] can be transferred in a straightforward way to the case of a continuous process with stationary increments, where the spectral measure appears in the representation (cf. [6])

$$\mathbb{E}\{Z_s Z_t\} = \int_{-\infty}^{\infty} (\mathrm{e}^{\mathrm{i}2\pi s\lambda} - 1)\overline{(\mathrm{e}^{\mathrm{i}2\pi t\lambda} - 1)} \frac{1+\lambda^2}{4\pi^2\lambda^2} \mu(\mathrm{d}\lambda). \tag{7}$$

Theorem 7. *Assume that the density $g(\lambda)$ of the absolutely continuous part of the spectral measure μ in representation (7) satisfies $g(\lambda) \geq 1/\lambda^p$ for some $p > 0$ and large λ. Then, Z has at most a finite number of L^2-derivatives, and*

$$\partial G_0 = \overline{\mathrm{sp}}\{Z_0^{(i)} : \text{ the } L^2\text{-derivative } Z_0^{(i)} \text{ exists}, \ i = 1, 2, \ldots\}.$$

2.5. A note on conditional expectations

For a finite-dimensional Gaussian vector X, the conditional distribution with respect to any linear condition $AX = a$ is again Gaussian. Moreover, the mean of this distribution is linear in a, whereas its covariance is independent of a. It is less obvious how conditional distributions and expectations with respect to linear conditions should be defined in the infinite-dimensional case. In this subsection we show how certain conditional expectations with respect to infinite-dimensional linear conditions can be defined in an elementary way.

Let $S \subset \mathbb{R}$ be a non-empty finite set of time points. For any $u \in \mathbb{R}$, the conditional expectation of Z_u given the vector Z_S has the expression $\mathbb{E}[Z_u|Z_S] = \Gamma(u, S)\Gamma(S)^{-1}Z_S$. Thus, we have for any particular vector $\boldsymbol{x}$ a natural expression for a particular condition (although evidently the probability of the condition is zero):

$$\mathbb{E}[Z_u|Z_S = \boldsymbol{x}] = \Gamma(u, S)\Gamma(S)^{-1}\boldsymbol{x}.$$

Note that the expression is linear in $\boldsymbol{x}$. We give another point of view on the above formula by defining for each $\boldsymbol{x}$ a random variable

$$Y_{\boldsymbol{x}} = \boldsymbol{x}^T \Gamma(S)^{-1} Z_S. \tag{8}$$

We obtain, for the one particular condition $\{Z_S = \boldsymbol{x}\}$, the conditional expectations of all Z_u's as covariances with one and the same random variable $Y_{\boldsymbol{x}}$:

$$\mathbb{E}\{Y_{\boldsymbol{x}} Z_u\} = \mathbb{E}[Z_u|Z_S = \boldsymbol{x}] \quad \text{for all } u \in \mathbb{R}. \tag{9}$$

Further, the isometry counterpart of $Y_{\boldsymbol{x}}$ in R is the element f that satisfies

$$\langle f, \Gamma(u, \cdot)\rangle = \mathbb{E}\{Y_{\boldsymbol{x}} Z_u\} \quad \text{for all } u \in \mathbb{R}.$$

By the reproducing kernel property, this element is the function $u \mapsto \mathbb{E}[Z_u|Z_S = \boldsymbol{x}]$. From this, we deduce the following characterization of the most probable path going through a finite number of specified points.

Proposition 8. *For any finite $S \subset \mathbb{R}$ and any $\boldsymbol{x} \in \mathbb{R}^{|S|}$, the conditional expectation given the values on S and the most probable path satisfying $f(S) = \boldsymbol{x}$ are equal, i.e., $f^*(u) = \mathbb{E}[Z_u|Z_S = \boldsymbol{x}]$ for all $u \in \mathbb{R}$.*

Proof. As shown above, the random variable $Y_{\boldsymbol{x}}$ defined in (8) is the random variable with smallest variance that satisfies $\mathbb{E}\{Y_{\boldsymbol{x}} Z_s\} = \mathbb{E}[Z_s | Z_S = \boldsymbol{x}]$ for all $s \in S$. By this minimum variance characterization, its isometry counterpart in R is the most probable path f^*. The claim follows now from (9). $\square$

Remark 9. In the case where Z is smooth, we could also condition on values of the existing derivatives of Z at a finite number of points. The generalization of Proposition 8 to those cases is straightforward and we skip the details.

It is not clear to us how far Proposition 8 can be generalized to an infinite-dimensional setting. We now show how this can be done when the conditions are in R.

Proposition 10. *Let S be a closed subset of $\mathbb{R}$ and let $\zeta \in R$. Let f^* be the most probable path satisfying $f(s) = \zeta(s)$ for every $s \in S$. Then, for every increasing sequence of finite subsets of S such that $\overline{\bigcup_n S_n} = S$, and for every $u \in \mathbb{R}$,*

$$f^*(u) = \lim_{n\to\infty} \mathbb{E}[Z_u | Z_s = \zeta(s)\ \forall s \in S_n].$$

Proof. Take any sequence S_n, and define $A_n = \{f \in R : f(s) = \zeta(s)\ \forall s \in S_n\}$, $f_n = \operatorname{argmin}_{f\in A_n} \|f\|$, $n = 1, 2, \ldots$ and $A = \bigcap_n A_n$. Since an equality can be obtained as a pair of non-strict inequalities in opposite directions, and since $f^* \in A_\infty$, we can apply Proposition 3 to see that $f_n \to f^*$ in R as $n \to \infty$. The expression of $f_n(u)$ is obtained from Proposition 8. $\square$

Consequently, it is unambiguous to define, for any closed set $S \subset \mathbb{R}$ and any $\zeta \in R$,

$$\mathbb{E}[Z_u | Z_s = \zeta(s)\ \forall s \in S] \doteq \lim_{n\to\infty} \mathbb{E}[Z_u | Z_s = \zeta(s)\ \forall s \in S_n], \tag{10}$$

where S_n is any sequence of finite sets that approaches S from within.

3. The most probable path in $\{Z \geq \zeta$ on $S\}$

The central problem in this paper is of the following form: given a function ζ and a set of time points S, what is the most probable path in the event $\{Z \geq \zeta$ on $S\}$? In the rest of the paper, we assume that Gaussian process Z satisfies the conditions of Theorem 7 so that the infinitesimal spaces are generated simply by $Z_t, \ldots, Z_t^{(k)}$, where k is the number of derivatives.

In order to keep the presentation simpler, we only consider sets $\{Z \geq \zeta$ on $S\}$, with $\zeta \in R$. We mention two immediate generalizations that require little additional effort. First, sets like $\{Z_t \geq t - 1\ \forall t \in [0, 2]\}$, where the function $t \mapsto t - 1$ does not belong to R, but the problem of most probable path can be reduced to the previous case. Second, our analysis also goes through with sets $\{Z \operatorname{sign}(\zeta) \geq \zeta \operatorname{sign}(\zeta)$ on $S\}$.

3.1. General results

Our first general result is a generalization of Proposition 2.

Theorem 11. *Let $\zeta \in R$, and let $S \subseteq \mathbb{R}$ be compact. Consider the set*

$$B_S \doteq \{f \in R : f(s) \geq \zeta(s)\ \forall s \in S\}.$$

There exists a function $\beta^* \in B_S$ with minimal norm, i.e., $\beta^* \doteq \operatorname{argmin}_{f \in B_S} \|f\|$. Moreover, $\beta^* \in R^o_{S^*} \cap R_S$, where

$$S^* = \left\{t \in S : \beta^*(t) = \zeta(t)\right\},$$

and the notation R^o_V, $V \subseteq \mathbb{R}$, means

$$R^o_V \doteq \bigcap_{\varepsilon>0} R_{V+[-\varepsilon,\varepsilon]}.$$

If $R^o_{S^*} \cap R_S = R_{S^*}$, then $\beta^*(t) = \mathbb{E}[Z_t | Z_s = \zeta(s) \forall s \in S^*]$.

Proof. Since B_S contains ζ and it is convex and closed, it has a unique element with minimum norm. Let S_n be a non-decreasing sequence of finite subsets of S such that $S_\infty \doteq \bigcup S_n$ is dense in S. Define $B_n = \{f \in R : f(s) \geq \zeta(s)\ \forall s \in S_n\}$, for $n = 1, 2, \ldots$, and let β_n be the element in B_n with smallest norm. By Proposition 3, the sequence β_n converges, and since the functions in R are continuous, the limit is β^*.

Let U be a bounded open interval such that $S \subset U$. For $m = 1, 2, \ldots$, define $U_m = \left\{t \in U : \beta^*(t) > \zeta(t) + \frac{1}{m}\right\}$. Since

$$|\beta_n(t) - \beta^*(t)| = |\langle \beta_n - \beta^*, \Gamma(t, \cdot)\rangle| \leq \|\beta_n - \beta^*\| \sup_{u \in U} \sqrt{\Gamma(u, u)}$$

for all $t \in U$, there is a number n_m such that $\beta_{n_m}(t) > \zeta(t) + 1/(2m)$ for all $t \in U_m$.

By Proposition 2,

$$\beta_{n_m} \in \overline{\mathrm{sp}} \left\{\Gamma(s, \cdot) : s \in S_{n_m} \cap U^c_m\right\} \subseteq R_{S \setminus U_m}.$$

Since the sequence of closed subspaces $R_{S \setminus U_m}$ is decreasing in m and $\beta_{n_m} \to \beta^*$, it follows that

$$\beta^* \in \bigcap_{m=1}^{\infty} R_{S \setminus U_m} = R^o_{S^*} \cap R_S.$$

The last assertion follows directly from Proposition 10. □

Remark 12. The condition $R^o_{S^*} \cap R_S = R_{S^*}$ is trivially satisfied when S is finite. When $\partial R_0 = \{0\}$ and the boundary of S^* is finite, Lemma 4(iii) can be used to deduce $R^o_{S^*} \cap R_S = R_{S^*}$ in most cases of interest.

Remark 13. The set S^* in Theorem 11 need not be the smallest set fulfilling the assertions. For example, let $\zeta(t) = \Gamma(1, .)$, which is the minimum norm function satisfying the one-dimensional condition $\zeta(1) = v(1)$, and let $S = [0, 1]$. Now, ζ is the minimum norm solution of our problem, and our definition of S^* gives $S^* = [0, 1]$. However, the singleton $\{1\}$ would in fact suffice for the role of S^* in this case.

Remark 14. In the case $R^o_{S^*} \cap R_S = R_{S^*}$, Theorem 11 has a clear intuitive content: the 'cheapest' way to push the process above ζ is to push it exactly to the curve $t \mapsto \zeta(t)$ in the subset S^*; the points in $S \setminus S^*$ then come 'for free'.

The information provided by Theorem 11 is still insufficient for characterizing the MPP in any concrete case. Such a characterization can often be obtained by studying 'least likely' finite-dimensional approximations of β^*, defined in such a way that their norm is always less than or equal to $\|\beta^*\|$. This idea is borrowed from Mandjes and van Uitert [9,10].

For any set $V \subseteq S$, define

$$B_V \doteq \{f \in R : f(t) \geq \zeta(t)\ \forall t \in V\}, \qquad L_V \doteq \{f \in R : f(t) = \zeta(t)\ \forall t \in V\}.$$

Let the unique element with smallest norm be in B_V and L_V, respectively,

$$\varphi^V \doteq \operatorname{argmin}_{\varphi \in B_V} \|\varphi\|, \qquad \overline{\varphi}^V \doteq \operatorname{argmin}_{\varphi \in L_V} \|\varphi\|.$$

In this context we identify a vector $\boldsymbol{t} \in \mathbb{R}^n$ with the set of its distinct components. Note that for any $V \subseteq S$, $\|\varphi^V\|$ is a lower bound of $\|\beta^*\|$, but it is possible that $\|\overline{\varphi}^V\| > \|\beta^*\|$.

Next, we state a proposition showing that the coefficients of the $\Gamma(v, \cdot)$, $v \in V$, in the representation of $\overline{\varphi}^V$ are strictly positive if every v is needed to make function φ^V feasible.

Proposition 15. *Let V be finite. If for each $v \in V$ it holds that $\overline{\varphi}^{V\setminus\{v\}}(v) < \zeta(v)$, then the coefficients θ_v in the representation $\overline{\varphi}^V = \sum_{v \in V} \theta_v \Gamma(v, \cdot)$ are all strictly positive.*

Proof. Take $v \in V$ and define $\overline{\varphi}^{V\setminus\{v\}} = \sum_{t \in V\setminus\{v\}} \tilde{\theta}_t \Gamma(t, \cdot)$. The assumption that $\overline{\varphi}^{V\setminus\{v\}}(v) < \zeta(v)$ implies that $\|\overline{\varphi}^V\| > \|\overline{\varphi}^{V\setminus\{v\}}\|$. Thus

$$0 < \|\overline{\varphi}^V - \overline{\varphi}^{V\setminus\{v\}}\|^2 = \left\langle \overline{\varphi}^V - \overline{\varphi}^{V\setminus\{v\}}, \sum_{t \in V\setminus\{v\}} (\theta_t - \tilde{\theta}_t)\Gamma(t, \cdot) + \theta_v \Gamma(v, \cdot) \right\rangle$$
$$= \theta_v(\zeta(v) - \overline{\varphi}^{V\setminus\{v\}}(v)). \quad \square$$

The nature of the MPP depends crucially on the smoothness of Z. The analysis must be divided into the non-smooth and the smooth cases.

3.2. The case of non-smooth Z

In the non-smooth case we make the additional assumption that Z possesses local independence, which implies that the infinitesimal space is trivial (Proposition 5) and thus (assuming that Z is not identically zero) that Z is non-smooth. The main result will be Theorem 18, which requires a continuity result provided by the following proposition.

Proposition 16. *If our Gaussian process Z possesses local independence, then the mappings $T \mapsto \overline{\varphi}^T$ and $T \mapsto \varphi^T$ from $\{T \subset \mathbb{R} : |T| < \infty\}$ to R are for every fixed $\zeta \in R$ continuous with respect to the Hausdorff metric. On the other hand, if Z is mean square differentiable, then there exist many ζ's such that neither of the above mappings is continuous.*

Proof. First we show the continuity of $\overline{\varphi}^T$ and φ^T under the triviality assumption, i.e., $\partial G_0 = \{0\}$. Consider the map $T \mapsto \overline{\varphi}^T$.

1. Let T_n and T be finite subsets of $\mathbb{R}$ such that $T_n \to T$. (Notice that in principle T can have a lower cardinality than the T_n.) For every $\varepsilon > 0$, let n_ε be the smallest number such that $T_n \subset T + [-\varepsilon, \varepsilon]$ for all $n \geq n_\varepsilon$.
2. For a closed subspace Y of R, denote by P_Y the orthogonal projection on Y. For closed sets $V \subset \mathbb{R}$ we also use the shorthand notation $P_V \doteq P_{R_V}$. Note that evidently $\overline{\varphi}^{T_n} = P_{T_n}\zeta$, and $\overline{\varphi}^T = P_T\zeta$.
3. Define $Y^\varepsilon = \overline{\text{sp}} \bigcup_{t \in T} U_t^\varepsilon$, where U_t^ε is as in (6). By the assumption on local independence and claim (ii) of Lemma 4, $Y^\varepsilon \searrow \{0\}$ as $\varepsilon \to 0$. Since R_T is finite-dimensional, claim (iii) of Lemma 4 yields that $R_T + Y^\varepsilon \searrow R_T$, and consequently $(R_T + Y^\varepsilon) \ominus R_T \searrow \{0\}$. ($A \ominus B$

denotes the orthogonal complement of a closed subspace B with respect to a larger closed subspace A.)

Now, for $n \geq n_\varepsilon$,

$$\overline{\varphi}^{T_n} = P_{T_n}\zeta = P_{T_n}P_{R_T+Y^\varepsilon}\zeta = P_{T_n}P_T\zeta + P_{T_n}P_{(R_T+Y^\varepsilon)\ominus R_T}\zeta.$$

As $n \to \infty$, the first term converges to $P_T\zeta$, due to the assumed convergence $T_n \to T$ (note that $P_T\zeta$ is a *finite* combination of $\Gamma(t,\cdot)$'s, $t \in T$). On the other hand,

$$\|P_{T_n}P_{(R_T+Y^\varepsilon)\ominus R_T}\zeta\| \leq \|P_{(R_T+Y^\varepsilon)\ominus R_T}\zeta\| \to 0 \quad \text{as } \varepsilon \to 0.$$

Then consider the map $T \mapsto \varphi^T$.

1. For any finite T, define

$$\overline{T} \doteq \{t \in T : \varphi^T(t) = \zeta(t)\},$$

and note that $\varphi^T = \overline{\varphi}^{\overline{T}}$. Choose $\varepsilon > 0$ such that for all $t_i, t_j \in T$, it holds that $|t_i - t_j| > 2\varepsilon$. Define also $\tilde{T}_n \doteq T_n \cap (\overline{T} + [-\varepsilon, \varepsilon])$. Then $\tilde{T}_n \to \overline{T}$ as $n \to \infty$, and by the first part of the proposition we have

$$\overline{\varphi}^{\tilde{T}_n} \to \overline{\varphi}^{\overline{T}} = \varphi^T. \tag{11}$$

2. Let then T' be any accumulation point of the sequence $\overline{T_n}$, and let (n_k) be a subsequence such that $\overline{T_{n_k}} \to T'$. By the continuity of $\overline{\varphi}^T$,

$$\varphi^{T_{n_k}} = \overline{\varphi}^{\overline{T_{n_k}}} \to \overline{\varphi}^{T'}. \tag{12}$$

3. For any $t \in T$, take $t_k \in T_{n_k}$ such that $t_k \to t$. Because convergence in R implies uniform convergence on compacts by (3),

$$\overline{\varphi}^{T'}(t) = \lim_{k\to\infty} \overline{\varphi}^{T'}(t_k) = \lim_{k\to\infty} \varphi^{T_{n_k}}(t_k) \geq \lim_{k\to\infty} \zeta(t_k) = \zeta(t),$$

where the first equality is due to $\overline{\varphi}^{T'}$ being continuous, the second by virtue of (12), the inequality because $t_k \in T_{n_k}$, and the last equality due to ζ being continuous. Thus, $\overline{\varphi}^{T'} \in B_T$. As φ^T is the element of B_T with minimal norm, we conclude that $\|\overline{\varphi}^{T'}\| \geq \|\varphi^T\|$.

4. Now we prove that $\overline{\varphi}^{\tilde{T}_{n_k}} \in B_{T_{n_k}}$ for large k. For any $t \in \tilde{T}_{n_k}$ evidently $\overline{\varphi}^{\tilde{T}_{n_k}}(t) = \zeta(t)$. Now pick $t \in T_{n_k}\backslash\tilde{T}_{n_k}$. By (11) and the continuity of $\overline{\varphi}^{\tilde{T}_{n_k}}$ and ζ, we see that $\overline{\varphi}^{\tilde{T}_{n_k}}(t) > \zeta(t)$ for k large enough.

5. The fact that $\overline{\varphi}^{\tilde{T}_{n_k}} \in B_{T_{n_k}}$ for large k, in conjunction with the property that $\varphi^{T_{n_k}}$ is the element of $B_{T_{n_k}}$ with minimum norm, implies the inequality $\|\varphi^{T_{n_k}}\| \leq \|\overline{\varphi}^{\tilde{T}_{n_k}}\|$ for large k. Thus, we have obtained the chain

$$\|\varphi^T\| \leq \|\overline{\varphi}^{T'}\| = \lim_{k\to\infty}\|\varphi^{T_{n_k}}\| \leq \lim_{k\to\infty}\|\overline{\varphi}^{\tilde{T}_{n_k}}\| = \|\varphi^T\|$$

and see that equality must hold everywhere. By the uniqueness of the minimum norm element, we deduce that $\overline{\varphi}^{T'} = \varphi^T$. Finally, because the limit is independent of the accumulation point T', we get the desired convergence $\varphi^{T_n} \to \varphi^T$.

Finally, let us show that the existence of Z_0' implies that the mappings φ^T and $\overline{\varphi}^T$ cannot be continuous. We first verify this statement for $\overline{\varphi}^T$. Suppose the mean square derivative Z_0' exists.

Take $T_n = \{1/n\}$ and let ζ be any element in R such that $\zeta'(0) > 0$. Then $\lim T_n = \{0\}$ and $\overline{\varphi}^{\{0\}} = 0$, but

$$\overline{\varphi}^{T_n} = \frac{\zeta\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}, \frac{1}{n}\right)} \Gamma\left(\frac{1}{n}, \cdot\right) \to \frac{\zeta'(0)}{2v''(0)} \Gamma'(0, \cdot).$$

Since $\varphi^{\{s\}} = \overline{\varphi}^{\{s\}}$ whenever $\zeta(s) \geq 0$, we obtain a counterexample for φ^T as well. □

We now consider sets V of at most n time points such that the norm of φ^V is as *large* as possible: let

$$b_n \doteq \sup\{\|\varphi^V\| : V \subseteq S, |V| \leq n\}.$$

Lemma 17. $b_n \uparrow \|\beta^*\|$.

Proof. The sequence b_n is clearly non-decreasing. Let S_n and β_n be as in the proof of Theorem 11. Recall that $\beta_n \to \beta^*$ by Proposition 3. But $\|\beta_n\| \leq b_{|S_n|} \leq \|\beta^*\|$, and the claim follows. □

The following theorem shows that for each n, the value b_n is attained at some set S_n, and provides detailed information on this set. This theorem is the key element in our method for identifying most probable paths satisfying an infinite number of conditions. We shall see later that the theorem does not hold in the smooth case.

Theorem 18. *Assume that our Gaussian process Z possesses local independence. Let b_n be as above, and denote by n^* the possibly infinite number*

$$n^* = \inf\{n \in \mathbb{N} : b_n = b_{n+1}\}.$$

Then

(i) *for each n, there exists a (generally non-unique) set $S_n \subseteq S$ with at most n elements such that $\|\varphi^{S_n}\| = b_n$;*
(ii) *if $\|\varphi^{S_n}\| = \|\varphi^{S_{n+1}}\|$ for some n, then $\beta^* = \varphi^{S_{n^*}}$;*
(iii) *if $n \leq n^*$, then $\varphi^{S_n} = \overline{\varphi}^{S_n}$;*
(iv) $\lim_{n\to\infty} \varphi^{S_n} = \beta^*$*;*
(v) *assume that $n^* = \infty$; then*

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} S_n} \subseteq S^*,$$

where S^* is the set defined in Theorem 11.

Proof. (i) Take any n if $n^* = \infty$, otherwise any $n \leq n^*$. For $m = 1, 2, \ldots$, choose an n-element set $T_m \subseteq S$ such that

$$\|\varphi^{T_m}\| > b_{n-1} + \left(1 - \frac{1}{m}\right)(b_n - b_{n-1}).$$

If there were a point $t \in T_m$ such that $\varphi^{T_m}(t) > \zeta(t)$, we could, by Proposition 2, remove it from the optimization without changing the optimal point, i.e., we would have $\varphi^{T_m\setminus\{t\}} = \varphi^{T_m}$. This is not possible however, because we required $\|\varphi^{T_m}\| > b_{n-1}$. Thus we have $\varphi^{T_m} = \overline{\varphi}^{T_m}$.

Let us identify the sets T_m with elements in

$$D_S^n \doteq \left\{ \boldsymbol{t} \in \mathbb{R}^n : t_1 \leq \cdots \leq t_n,\ t_i \in S\ \forall i \right\}.$$

Since D_S^n is compact, the sequence T_m has a subsequence T_{m_k} converging to some element $S_n \in D_S^n$, that might have less than n distinct elements. In any case, Proposition 16 yields that

$$\|\varphi^{S_n}\| = \lim_{k\to\infty} \|\varphi^{T_{m_k}}\| = b_n. \tag{13}$$

Finally, the proof of the next claim shows that in the case $n^* < \infty$ we can just take $S_n = S_{n^*}$ for $n > n^*$.

(ii) If $\|\varphi^{S_n}\| = \|\varphi^{S_{n+1}}\|$ but $\varphi^{S_n} \neq \beta^*$, then $\varphi^{S_n} \notin B_S$. Then some of the hyperplanes $L_{\{t\}}$ strictly separate φ^{S_n} from B_S, that is, $\varphi^{S_n}(t) < \zeta(t)$. Thus, $\varphi^{S_n \cup \{t\}} \neq \varphi^{S_n}$, which by the uniqueness of minimum norm elements implies that $\|\varphi^{S_n\cup\{t\}}\| > \|\varphi^{S_n}\|$.

(iii) This was shown already in the proof of claim (i).

(iv) By Lemma 17, $\|\varphi^{S_n}\| \to \|\beta^*\|$. It suffices to show that $\|\varphi^{S_n} - \beta^*\|^2 \leq \|\beta^*\|^2 - \|\varphi^{S_n}\|^2$. But this is easily seen to be equivalent to the condition $\langle \varphi^{S_n}, \beta^* - \varphi^{S_n}\rangle \geq 0$, which is true since β^* is on the same side of the hyperplane $\left\{ f : \langle \varphi^{S_n}, f\rangle = \|\varphi^{S_n}\|^2 \right\}$ as the set B_S.

(v) By Cauchy–Schwarz,

$$\|\beta^* - \varphi^{S_n}\| \geq \frac{\beta^*(s) - \varphi^{S_n}(s)}{\|\Gamma(s,\cdot)\|} = \frac{\beta^*(s) - \zeta(s)}{\|\Gamma(s,\cdot)\|}$$

for any n and any $s \in S_n$. Define

$$\tilde{U}_\varepsilon = \left\{ t \in S : \frac{\beta^*(t) - \zeta(t)}{\|\Gamma(t,\cdot)\|} > \varepsilon \right\}.$$

If $\|\beta^* - \varphi^{S_n}\| \leq \varepsilon$, then $S_n \subseteq \tilde{U}_\varepsilon^c$. On the other hand, $\bigcap_{\varepsilon>0} \tilde{U}_\varepsilon^c = S^*$. □

The claim (iii) of the previous proposition is crucial, because it makes it possible to compute the paths φ^{S_n} when the set S_n is known. Our example with fractional Brownian motion in Section 4.2 indicates that the explicit identification of the S_n's is usually impossible in practice, but general properties can often be deduced.

Here are some other useful properties of the paths φ^{S_n}:

Proposition 19. *Assume that Z possesses local independence, and let $n \leq n^*$.*

(i) *For each $s \in S_n$, $\overline{\varphi}^{S_n\setminus\{s\}}(s) < \zeta(s)$.*

(ii) *All coefficients θ_s in the unique representation $\varphi^{S_n} = \sum_{s\in S_n} \theta_s \Gamma(s,\cdot)$ are strictly positive.*

Proof. (i) By claim (ii) of Theorem 18, all points in S_n are relevant. It follows that we cannot have $\overline{\varphi}^{S_n\setminus\{s\}}(s) = \zeta(s)$, because otherwise we would have $\overline{\varphi}^{S_n\setminus\{s\}} = \overline{\varphi}^{S_n} = \varphi^{S_n}$. Assume that $\overline{\varphi}^{S_n\setminus\{s\}}(s) > \zeta(s)$. Then $\overline{\varphi}^{S_n\setminus\{s\}} \in B_{S_n}$. Since $\overline{\varphi}^{S_n\setminus\{s\}} \neq \overline{\varphi}^{S_n}$ and $\overline{\varphi}^{S_n} \in L_{S_n\setminus\{s\}}$, we obtain the contradictory chain of inequalities

$$\|\overline{\varphi}^{S_n\setminus\{s\}}\| < \|\overline{\varphi}^{S_n}\| = \|\varphi^{S_n}\| < \|\overline{\varphi}^{S_n\setminus\{s\}}\|.$$

Thus, $\overline{\varphi}^{S_n\setminus\{s\}}(s) < \zeta(s)$.

(ii) Follows from Proposition 15. □

So far we have made rather few assumptions on the variance function. In the last general proposition in the non-smooth case, we make the additional assumption that $v(t) = \Gamma(t,t)$ be everywhere differentiable, including at the origin (necessarily then $v'(0) = 0$). We show that φ^{S_n} then touches ζ smoothly at the points of S_n that are interior points of S.

Proposition 20. *Assume that Z possesses local independence. Let S be an interval. Assume that v is differentiable on the whole $\mathbb{R}$. Let $n \le n^*$, and let S_n be the extreme set as above. Define the points (s_i) by $S_n = \{s_i\}_{i=1}^n$, where $\min\{s \in S\} \le s_1 < s_2 < \cdots < s_n \le \max\{s \in S\}$. Then we have:*

(i)

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)|_{t=s_i} = \zeta'(s_i), \quad i = 2, \ldots, n-1,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)|_{t=s_1} \ge \zeta'(s_1), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)|_{t=s_n} \le \zeta'(s_n),$$

where an inequality can be replaced by an equality, if point s_1 or s_n is an inner point of S.

(ii) *Assume additionally that $v(t)$ is twice differentiable outside the origin, and $v''(0) = \infty$. Then the curve $\varphi^{S_n}(t)$ touches the line $\zeta(t)$ from below at the points $s_i \in S_n$ which are inner points of S.*

Proof. (i) Define $\mathbf{t} = (t_1, \ldots, t_n)$, $\zeta(\mathbf{t}) = (\zeta(t_1), \ldots, \zeta(t_n))^{\mathrm{T}}$ and

$$f(\cdot) = \zeta(\mathbf{t})^{\mathrm{T}}\Gamma(\mathbf{t})^{-1}\Gamma(\mathbf{t}, \cdot) = \theta(\mathbf{t})\Gamma(\mathbf{t}, \cdot),$$

where $\theta(\mathbf{t}) = \zeta(\mathbf{t})^{\mathrm{T}}\Gamma(\mathbf{t})^{-1}$. Thus $f(t_i) = \zeta(t_i)$ for $i = 1, \ldots, n$. Taking the derivative of f at points t_k, $k = 1, \ldots, n$, gives

$$f'(t_k) = \sum_{i \ne k} \theta_i(\mathbf{t}) \frac{\partial}{\partial t_k}\Gamma(t_i, t_k) + \frac{1}{2}\theta_k(\mathbf{t})\, v'(t_k) \tag{14}$$

(note that here we need that $v'(0) = 0$).

Since the s_i maximize the norm,

$$\left.\frac{\partial}{\partial t_k}\|f\|^2\right|_{\mathbf{t}=\mathbf{s}} = 0 \quad \text{for } k = 2, \ldots, n-1. \tag{15}$$

Observing that $\|f\|^2 = \langle f, \boldsymbol{\theta}(\mathbf{t})\Gamma(\mathbf{t}, \cdot)\rangle = \boldsymbol{\theta}(\mathbf{t})\zeta(\mathbf{t})$, this condition can be written as

$$(\partial_k \boldsymbol{\theta}(\mathbf{s}))\, \zeta(\mathbf{s}) = -\zeta'(s_k)\theta_k(\mathbf{s}), \quad k = 2, \ldots, n-1, \tag{16}$$

where $\partial_k \boldsymbol{\theta}(\mathbf{t}) = \frac{\partial}{\partial t_k}\boldsymbol{\theta}(\mathbf{t})$.

On the other hand, we can write $\|f\|^2 = \boldsymbol{\theta}(\mathbf{t})\Gamma(\mathbf{s})\boldsymbol{\theta}(\mathbf{t})^{\mathrm{T}}$ and obtain the expressions

$$\begin{aligned}
\frac{\partial}{\partial t_k}\|f\|^2 &= \frac{\partial}{\partial t_k}\sum\sum \theta_i(\mathbf{t})\Gamma(t_i, t_j)\theta_j(\mathbf{t}) \\
&= \sum\sum 2\theta_i(\mathbf{t})\Gamma(t_i, t_j)\left(\partial_k\theta_j(\mathbf{t})\right) \\
&\quad + \sum_{i \ne k} 2\theta_k(\mathbf{t})\theta_i(\mathbf{t})\left(\frac{\partial}{\partial t_k}\Gamma(t_i, t_k)\right) + \theta_k(\mathbf{t})^2 v'(t_k) \\
&= 2\,(\partial_k \boldsymbol{\theta}(\mathbf{t}))\,\Gamma(\mathbf{t})\boldsymbol{\theta}(\mathbf{t})^{\mathrm{T}} + 2\theta_k(\mathbf{t}) f'(t_k), \quad k = 1, \ldots, n-1,
\end{aligned}$$

where the last line follows from (14). Finally, notice that $\Gamma(\mathbf{t})\boldsymbol{\theta}(\mathbf{t})^{\mathrm{T}} = \zeta(\mathbf{t})$, replace $\mathbf{t}$ by $\mathbf{s}$, and use (16) to get

$$f'(s_k) = -\frac{1}{\theta_k(\mathbf{s})}\,(\partial_k \boldsymbol{\theta}(\mathbf{s}))\,\zeta(\mathbf{s}) = \zeta'(s_k).$$

For points s_1 and s_n, the equality in (15) is replaced by an inequality. Otherwise, the proof is similar.

(ii) By claim (i), it is enough to show that

$$\frac{d^2}{d^2 t}\varphi^{S_n}(t) < 0$$

at the points $s_1, \ldots, s_{n-1}$. A direct computation yields

$$\frac{d^2}{d^2 t}\varphi^{S_n}(t) = \frac{1}{2}\sum_i \theta_i(\mathbf{s})(v''(t) - v''(t - s_i)).$$

By claim (ii) of Proposition 19 and the assumption $v''(0) = \infty$, this expression equals $-\infty$ at all the points s_i. □

3.3. The case of smooth Z

When process Z has a mean square derivative, the analysis gets more involved since the mappings $T \mapsto \varphi^T$ and $T \mapsto \overline{\varphi}^T$ are not continuous any longer. The example presented in Section 4.3 indicates that when it is possible to impose conditions on the derivatives of Z, a small finite number of points is often enough to determine the most probable path. The general approach is left for future studies.

4. Example: Busy periods of Gaussian queues

4.1. The Gaussian queue

Our motivation for doing this study came from queues with Gaussian input, where we encountered the problem of identifying the most probable paths in sets of the type $\{Z_t \geq \zeta(t), \forall t \in S\}$. We here present two prominent examples of this.

Busy period. The first example relates to the busy period in a queue fed by Gaussian input. The queue length process with input Z and service rate 1 is commonly defined as $Q_t = \sup_{s \leq t}(Z_t - Z_s - (t - s))$. For each $T > 0$, denote by K_T the event that the ongoing busy period at time 0 is longer than T: $K_T \doteq \{\tau_+ - \tau_- > T\}$, with $\tau_- \doteq \sup\{t \leq 0 : Q_t = 0\}$ and $\tau_+ \doteq \inf\{t \geq 0 : Q_t = 0\}$. When one is interested in the decay rate of the probability of a *long* busy period, [11] showed that for fBm, with $v(t) = t^{2H}$, without losing generality, attention can be restricted to the set $B = \{f \in R : f(s) \geq s, \forall s \in [0, 1]\}$ of paths in R that create non-proper busy periods starting at 0 and straddling the interval [0, 1]; this is due to

$$\lim_{T\to\infty} \frac{1}{T^{2-2H}} \log \mathbb{P}(Z \in K_T) = -\inf_{f\in B} \frac{1}{2}\|f\|^2.$$

The problem is determining the MPP in B, i.e., $\beta^* \doteq \operatorname{argmin}_{f\in B}\|f\|$. Since B is convex and closed, β^* is uniquely determined, but [11] does not succeed in finding an explicit characterization. Both Kozachenko et al. [7] and Dieker [5] consider the extension of this set-up to a regularly varying (rather than purely polynomial) variance function: $v(t) = L(t)t^{2H}$ for a slowly varying $L(\cdot)$, and show that, under specific conditions,

$$\lim_{T\to\infty} \frac{L(T)}{T^{2-2H}} \log \mathbb{P}(Z \in K_T) = -\inf_{f\in B} \frac{1}{2}\|f\|^2.$$

Hence, the same minimization problem appears again.

Tandem queue. The second example involves a tandem queueing network, i.e., a two-node network in which the output of a first queue feeds into a second queue. Assume that the first queue is emptied at a constant service rate c_1, whereas the second has a constant service rate c_2 (with $c_1 > c_2$, to avoid the trivial situation that the second queue remains empty all the time). The steady-state queue length of the first queue of the tandem system can be represented as $Q_1 = \sup_{t\geq 0}(Z_{-t} - c_1 t)$, see [1], and hence $\{Q_1 > b\} = \{\exists t \geq 0 : Z_{-t} - c_1 t \geq b\}$. Using the generalized Schilder theorem, the decay rate for exceeding b in the first queue equals $-\inf_{f\in F} \frac{1}{2}\|f\|^2$, where

$$F \doteq \bigcup_{t\geq 0} F_t, \quad \text{with } F_t \doteq \{f \in R : f(t) \geq b + c_1 t\}.$$

The decay rate of $\{Q_1 \geq b\}$ now follows from Addie et al. [1]:

$$-\inf_{f\in F} \frac{1}{2}\|f\|^2 = -\inf_{t\geq 0} \inf_{f\in F_t} \frac{1}{2}\|f\|^2 = -\inf_{t\geq 0} \frac{1}{2} \frac{(b + c_1 t)^2}{v(t)}.$$

The large deviations of the first queue having been analyzed in detail in [1], let us now consider the second queue, i.e., the decay rate of $\{Q_2 \geq b\}$. We will show that the event of interest is very similar to that of a busy period in a single queue, in that we can rewrite it in terms of an infinite intersection. To this end, notice that the *total* queue length behaves as a queue with constant service rate c_2:

$$Q_1 + Q_2 = \sup_{t\geq 0}(Z_{-t} - c_2 t),$$

see e.g. [10]. Therefore, we express the occupancy of the second queue as the difference of the total buffer content and the content of the first queue,

$$\{Q_2 \geq b\} = \{\exists t \geq 0 : \forall s \geq 0 : Z_{-t} - Z_{-s} - c_2 t + c_1 s \geq b\};$$

it is easily seen that we can restrict ourselves to $s \in [0, t]$, and $t \geq t_b \doteq b/(c_1 - c_2)$. By a straightforward time-shift, we conclude that the decay rate of interest to us equals $-\inf_{f\in U} \frac{1}{2}\|f\|^2$, where

$$U \doteq \bigcup_{t\geq t_b} U_t, \quad \text{with } U_t \doteq \{f \in R : \forall s \in [0, t] : f(s) \geq b + c_2 t - c_1(t - s)\}.$$

This decay rate obviously reads $-\inf_{t\geq t_b} \inf_{f\in U_t} \frac{1}{2}\|f\|^2$. Mandjes and van Uitert [10] partly solve the problem of finding the MPP in U_t: for large values of c_1 (above some explicit threshold value c_1^F) the MPP is known, and for small c_1 the MPP is known under some additional condition (that is *not* fulfilled in the case of fBm).

As an application of the results derived in Section 3, we now consider the “busy period problem”, with examples of both non-smooth (fBm, Section 4.2) and smooth (integrated Ornstein–Uhlenbeck, Section 4.3) input process Z.

4.2. Fractional Brownian motion

Our results enable an explicit characterization of β^* in the case where Z is a fractional Brownian motion (fBm), $S = [0, 1]$, and $\zeta(t) = t$ for $t \in S$. As discussed in Section 4.1, this gives the logarithmic asymptotics of the probability of long busy periods in a queue with fBm input.

Assume that the fBm Z has self-similarity parameter $H \in (0, 1)$, such that

$$\Gamma(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

It is "well known" that fBm possesses local independence, and this is indeed true [12]. We have, however, not found a proof of this in the literature.

Let us first state some properties of the derivative of φ^{S_n} for fixed $n \le n^*$.

Proposition 21. *Let $H > 1/2$, and let $n \le n^*$. Define $\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)$ and $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n \le 1$. Then*

(i) $s_n = 1$;
(ii) $\psi(s_i) = 1$ *and* $\psi'(s_i) = -\infty$ *for* $i = 1, \ldots, n-1$;
(iii) $\psi(0) < 1$, *and* $\psi(t) = 1$ *for only one point in* $(0, s_1)$;
(iv) *for* $i = 1, \ldots, n-1$, *there are two points* $\tau_i', \tau_i'' \in (s_i, s_{i+1})$ *such that* $\psi(t)$ *is strictly decreasing on* (s_i, τ_i') *and* (τ_i'', s_{i+1}), *and strictly increasing on* (τ_i', τ_i'');
(v) *for* $i = 1, \ldots, n-2$, $\psi(t) = 1$ *for exactly one point in* (s_i, s_{i+1});
(vi) $\psi(1) < 1$, *and* $\psi(t) = 1$ *for two points in* $(s_{n-1}, 1)$.

Proof. (i) Define $\mathbf{s} = (s_1, \ldots, s_n)$. The self-similarity of fBm gives $\Gamma(s_i, s_j) = s_n^{2H}\Gamma(s_i/s_n, s_j/s_n)$. Thus,

$$\|\overline{\varphi}^{S_n}\|^2 = \mathbf{s}\,\Gamma(\mathbf{s})^{-1}\mathbf{s}^T = s_n^{2-2H}\tilde{\mathbf{s}}\,\Gamma(\tilde{\mathbf{s}})^{-1}\tilde{\mathbf{s}}^T,$$

where $\tilde{\mathbf{s}} = (s_1/s_n, \ldots, s_{n-1}/s_n, 1) = (\tilde{s}_1, \ldots, \tilde{s}_{n-1}, 1)$. Since $\varphi^{S_n} = \overline{\varphi}^{S_n}$ for $n \le n^*$, and recalling that S_n maximizes the norm, we conclude $s_n = 1$.

(ii) This follows from Proposition 20; note that $v''(0) = \infty$.

(iii) Write $\psi(t)$ in the form

$$\psi(t) = C\left[t^\alpha + \sum_{s\in S_n,\ s>t}\rho_s(s-t)^\alpha - \sum_{s\in S_n,\ s<t}\rho_s(t-s)^\alpha\right], \tag{17}$$

where

$$\alpha \doteq 2H-1 \in (0,1), \quad C \doteq H\sum_{s\in S_n}\theta_s, \quad \rho_s \doteq \frac{\theta_s}{\sum_{r\in S_n}\theta_r} \in (0,1).$$

Note that in the right hand side of (17), the first term is increasing and concave, the second is decreasing and concave, and the third (negative) is decreasing and convex. Hence ψ is strictly concave between 0 and s_1. Due to this property, in conjunction with $\psi(s_1) = 1$, ψ can attain the value 1 at most once in $(0, s_1)$. On the other hand, this does happen at least once by the mean value theorem, since $\varphi^{S_n}(s_1) = \int_0^{s_1}\psi(\tau)\mathrm{d}\tau = s_1$.

(iv)–(vi) Since $\psi'(s_i) < 0$, $i = 1, \ldots, n-1$, we have to show that ψ' changes its sign exactly twice within (s_i, s_{i+1}). Write

$$\psi'(t) = C\alpha\left[t^\beta - \sum_{s\in S_n, s>t}\rho_s(s-t)^\beta - \sum_{s\in S_n, s<t}\rho_s(t-s)^\beta\right],$$

where $\beta \doteq \alpha - 1 \in (-1, 0)$. Consider $t \in (s_i, s_{i+1})$ and make the change of variable $x = s_i^\beta - t^\beta$, i.e., $t = (s_i^\beta - x)^{1/\beta}$, where $x \in (0, s_i^\beta - s_{i+1}^\beta)$. This transforms the first term t^β into a linear function. The powers in the first sum read, in terms of x:

$$g_j(x) \doteq \left(s_j - \left(s_i^\beta - x\right)^{1/\beta}\right)^\beta, \quad j > i.$$

A straightforward calculation shows that $g_j''(x) > 0$, thus g_j is convex. An essentially identical calculation shows the convexity of the functions

$$h_j(x) \doteq \left(\left(s_i^\beta - x\right)^{1/\beta} - s_j\right)^\beta, \quad j \le i,$$

appearing in the second sum. Now, the convex function

$$\sum_{j=i+1}^{n} \rho_{s_j} g_j(x) + \sum_{j=1}^{i} \rho_{s_j} h_j(x)$$

can cross the linear function $s_i^\beta - x$ at most twice. It follows that ψ' changes its sign at most twice in (s_i, s_{i+1}). For $i \le n-2$, claim (ii) entails that the sign change must happen exactly twice, and (v) follows.

It remains to consider the interval $(s_{n-1}, 1)$. The mean value theorem implies that ψ obtains the value 1 somewhere in this interval. Moreover, it has to exceed this value. On the other hand,

$$\psi(1) = \sum_{s \in S_n} \theta_s \frac{\mathrm{d}}{\mathrm{d}t} \Gamma(s, t)\Bigg|_{t=1} < \sum_{s \in S_n} \theta_s \Gamma(s, 1) = 1,$$

as a consequence of the fact

$$\frac{\mathrm{d}}{\mathrm{d}t} \Gamma(s, t) < \frac{\Gamma(s, t)}{t}, \quad 0 < s \le t.$$

Thus, there are exactly two points in $(s_{n-1}, 1)$ such that $\psi(t) = 1$. This completes the proof of (vi), and also that of (iv). □

Applying the previous proposition together with results of Section 3, we get the following qualitative characterizations of the paths φ^{S_n}.

Proposition 22. *Let $H > 1/2$ and $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n = 1$. Then*

(i) *the function $\varphi^{S_n}(t)$ is concave for $t \ge 1/2$;*
(ii) *for $n \ge 2$, $s_{n-1} \le 1/2$;*
(iii) *there exists a time point $u_n \in (s_{n-1}, 1)$ such that*

$$\varphi^{S_n}(t) \le t, \quad t \in [0, u_n],$$
$$\varphi^{S_n}(t) \ge t, \quad t \in [u_n, 1];$$

(iv) *$\varphi^{S_n}(t) < t$ on $[0, u_n]$ unless $t \in S_n \cup \{0, u_n\}$, and $\varphi^{S_n}(t) > t$ on $(u_n, 1)$;*
(v) *the number n^* is infinite.*

Proof. (i) Since for any $t > 0$ the second derivative of $\Gamma(t, \cdot)$ is negative after the point $t/2$ (i.e., $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Gamma(t, s) \le 0$ for all $s \ge t/2$), and the coefficients θ_s in the representation

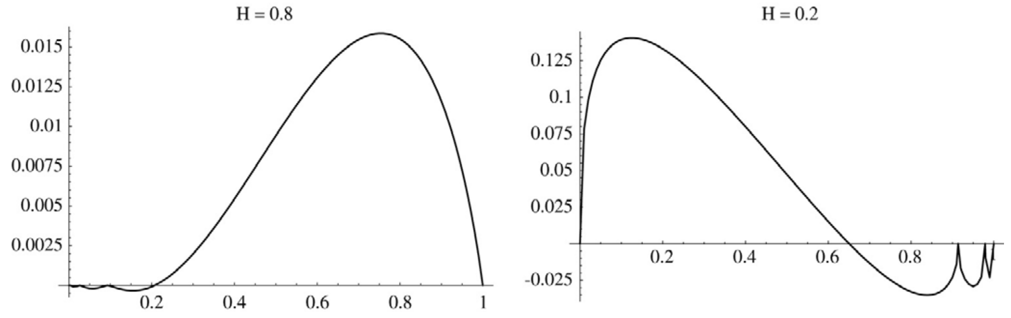

Fig. 1. The shapes of $\varphi^{S_3}(t) - t$ for fBm with $H = 0.8$ (left; in this case s_1 is too close to 0 to be seen in the figure) and $H = 0.2$ (right).

$\varphi^{S_n} = \sum_{s \in S_n} \theta_s \Gamma(s, \cdot)$ are positive by claim (ii) of Proposition 19, the second derivative of φ^{S_n} is negative after the time point $1/2$. This proves the claim of concavity for $t \geq 1/2$.

(ii) By Propositions 20 and 21, $\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)$ must be increasing somewhere after s_{n-1}, i.e., there is a subinterval of $(s_{n-1}, 1)$ where $\varphi^{S_n}(t)$ is convex. However by (i), $\varphi^{S_n}(t)$ is concave in $[1/2, 1]$.

(iii) and (iv) Follow directly from Proposition 21.

(v) The infiniteness of n^* follows from the fact that the above characterization of the S_n's was shown to hold for any n. (If n^* were finite, we would have $\varphi^{S_{n^*}}(t) \geq t$ for all $t \in [0, 1]$.) □

Proposition 23. *Let $H < 1/2$. The number n^* is infinite. Let $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n \leq 1$. The number s_n is 1 for all n. The function $\varphi^{S_n}(t)$ is concave for $t \leq 1/2$. There exists a time point $u_n \in (0, s_1)$ such that*

$$\varphi^{S_n}(t) \geq t, \quad t \in [0, u_n],$$
$$\varphi^{S_n}(t) \leq t, \quad t \in [u_n, 1].$$

Moreover, $\varphi^{S_n}(t) < t$ on $[u_n, 1]$ unless $t \in S_n \cup \{u_n\}$, and $\varphi^{S_n}(t) > t$ on $(0, u_n)$.

Proof. The proof is a simpler variant of the case $H > 1/2$, since φ^{S_n} turns out to be convex inside each interval (s_j, s_{j+1}). This is seen by applying the change of variable used in item (v) in the proof of Proposition 21, applied directly to the path itself instead of the second derivative. As regards the form of φ^{S_n} in $(0, s_1)$, we only need to note that the derivative of φ^{S_n} is convex in this interval. □

Examples of the shapes of the paths φ^{S_n} are shown in Fig. 1. We can now prove our main result on fBm:

Theorem 24. *For an fBm with $H > 1/2$, the set S^* has the form $S^* = [0, s^*] \cup \{1\}$, where $s^* \in (0, 1)$. The function β^* has the expression*

$$\begin{aligned}\beta^*(t) &= \mathbb{E}[Z_t | Z_s = s, \forall s \in [0, s^*], Z_1 = 1] \\ &= \chi_{[0,s^*]}(t) + \frac{\mathrm{Cov}\,[Z_t, Z_1 | \mathscr{F}]}{\mathrm{Var}\,[Z_t | \mathscr{F}]}(1 - \chi_{[0,s^*]}(1)),\end{aligned}$$

where $\mathscr{F} = \sigma(Z_s : s \in [0, s^*])$, and

$$\|\beta^*\|^2 = \|\chi_{[0,s^*]}\|^2 + \frac{(1 - \chi_{[0,s^*]}(1))^2}{\mathrm{Var}\,(Z_1 - \mathbb{E}[Z_1 | \mathscr{F}])},$$

where $\chi_{[0,t]}$ *is the most probable path in* R *satisfying* $\chi_{[0,t]}(s) = s$ *for all* $s \in [0, t]$.

For an fBm with $H = 1/2$ *(i.e., the Brownian motion), we have* $S^* = [0, 1]$.

For an fBm with $H < 1/2$*, we have* $S^* = [s^*, 1]$*, where* $s^* \in (0, 1)$*,*

$$\beta^*(t) = \mathbb{E}[Z_t | Z_s = s, \forall s \in [s^*, 1]] = \chi_{[s^*,1]} \qquad \textit{and} \qquad \|\beta^*\|^2 = \|\chi_{[s^*,1]}\|^2,$$

where $\chi_{[t,1]}$ *is the most probable path in* R *satisfying* $\chi_{[t,1]}(s) = s$ *for all* $s \in [t, 1]$.

Remark 25. For the case $H = 1/2$, S^* is not the minimal set for characterizing β^*, the singleton $\{1\}$ would suffice — cf. Remark 13.

Proof. $H > 1/2$:

1° Set S^* cannot be the whole interval since the case $\beta^*(t) = t$ for all $t \in [0, 1]$ is ruled out because we know from Norros [11] that $\chi_{[0,1]}$ is *not* the optimal busy period path. On the other hand, $S^* \neq \{0, 1\}$, since $R^o_{\{0,1\}} = R_1$ and $\Gamma(1, \cdot)$ does not belong to the set B.

By claim (iv) of Theorem 18, β^* is a limit of the functions φ^{S_n}. By Proposition 22, $\varphi^{S_n}(t)$ is at or below the diagonal on $[0, u_n]$ and strictly above it on $(u_n, 1)$. On the other hand, Proposition 21 (iv) tells us that on $(s_{n-1}, 1)$, φ^{S_n} is first concave, then convex, and finally concave again. Thus, on interval $[u_n, 1]$, φ^{S_n} is either concave or first convex and then concave; this behavior is qualitatively illustrated by the function φ^{S_3} shown in Fig. 1. Combine this with the properties mentioned in the first paragraph to deduce the existence of $s^* \in (0, 1)$ such that

$$\lim_{n\to\infty} \varphi^{S_n}(t) = t, \forall t \in [0, s^*] \cup \{1\} \qquad \text{and} \qquad \lim_{n\to\infty} \varphi^{S_n}(t) > t, \forall t \in (s^*, 1).$$

2° Since it holds for the fBm that $R^o_{[0,s^*]\cup\{1\}} = R_{[0,s^*]\cup\{1\}}$, Theorem 11 yields that $\beta^* = \mathbb{E}[Z_t | Z_s = s \; \forall s \in [0, s^*] \cup \{1\}]$. It remains to compute the claimed expressions.

For any function $f \in R$, define

$$\varphi_f(t) = \mathbb{E}[Z_t | Z_s = f(s) \; \forall s \in [0, s^*]],$$
$$\psi_f(t) = \mathbb{E}[Z_t | Z_s = f(s) \; \forall s \in [0, s^*]; \; Z_1 = 1].$$

The conditional distribution of the pair (Z_t, Z_1) w.r.t. $\mathscr{F}$ is a two-dimensional Gaussian distribution with (random) mean $\mathbb{E}[(Z_t, Z_1)|\mathscr{F}]$. Thus, the further conditioning on $\{Z_1 = 1\}$ can be computed according to the formula of conditional expectation in a bivariate Gaussian distribution:

$$\psi_f(t) = \varphi_f(t) + \frac{\mathrm{Cov}\,[Z_t, Z_1|\mathscr{F}]}{\mathrm{Var}\,[Z_1|\mathscr{F}]}(1 - \varphi_f(1)) = \varphi_f(t) + c(t)((1 - \varphi_f(1)),$$

where $c(t) = \mathrm{Cov}\,[Z_t, Z_1|\mathscr{F}]\,/\mathrm{Var}\,[Z_1|\mathscr{F}]$ does not depend on f. Applying this to the function $f(t) \equiv 0$ yields $c(t) = \psi_0(t)$. Since $\langle \psi_0, \Gamma(u, \cdot)\rangle = 0$ for $u \in [0, s^*]$, ψ_0 minimizes the R-norm in the set

$$R^{\perp}_{[0,s^*]} \cap \{f : f(1) = 1\}.$$

Denote by P the orthogonal projection on the subspace $R_{[0,s^*]}$. For $g \in R^{\perp}_{[0,s^*]}$, we have

$$g(1) = \langle g, \Gamma(1, \cdot)\rangle = \langle g, (I - P)\Gamma(1, \cdot)\rangle,$$

and it follows that the element g in $R^{\perp}_{[0,s^*]} \cap \{f : f(1) = 1\}$ with minimal norm must be a multiple of $(I - P)\Gamma(1, \cdot)$. Thus,

$$\psi_0 = \frac{1}{\|(I - P)\Gamma(1, \cdot)\|^2}(I - P)\Gamma(1, \cdot).$$

The counterpart of $P\,\Gamma(1,\cdot)$ in the isometry (1) is $\mathbb{E}[Z_1|\mathscr{F}]$, and it follows that the counterpart of ψ_0 is the random variable

$$\frac{Z_1-\mathbb{E}[Z_1|\mathscr{F}]}{\mathrm{Var}\,(Z_1-\mathbb{E}[Z_1|\mathscr{F}])}.$$

Thus,

$$\|\psi_0\|^2=\mathrm{Var}\,(Z_1-\mathbb{E}[Z_1|\mathscr{F}])^{-1}.$$

Now, note that

$$\beta^*(t)=\mathbb{E}[Z_t|Z_s=s,\forall s\in[0,s^*],Z_1=1]=\psi_{\chi_{[0,s^*]}},$$

$\varphi_{\chi_{[0,s^*]}}=\chi_{[0,s^*]}$, and ψ_0 is orthogonal to $\chi_{[0,s^*]}$. Thus,

$$\|\beta^*\|^2=\|\chi_{[0,s^*]}\|^2+\frac{(1-\chi_{[0,s^*]}(1))^2}{\mathrm{Var}\,(Z_1-\mathbb{E}[Z_1|\mathscr{F}])}.$$

$H=1/2$: A well known result.

$H<1/2$: Using a type of argument similar to that for $H>1/2$, it is seen that the shapes of the φ^{S_n} (see Fig. 1) are such that the limiting path must be of the form $\beta^*(t)>t$ if $t\in(0,s^*)$ and $\beta^*(t)=t$ if $t\in\{0\}\cup[s^*,1]$ for some $s^*\in(0,1)$. □

The quantities in the expression of β^* can be computed. The function $\chi_{[0,s^*]}$ is the counterpart of the random variable M_{s^*} in [13] in the isometry (1); see also [11]. Let us focus on the case $H>1/2$. Note first that for a multivariate Gaussian distribution the conditional variances and covariances, given a subset of the variables, are constants, and this carries over to Gaussian processes as well. Then apply the general relation

$$\mathrm{Cov}\,[Z_s,Z_t|\mathscr{F}]=\mathbb{E}\,\{Z_sZ_t\}-\mathrm{Cov}\,(\mathbb{E}[Z_s|\mathscr{F}],\mathbb{E}[Z_t|\mathscr{F}])\,,$$

recall the prediction formula [13, Thm. 5.3]

$$\mathbb{E}[Z_t|Z_u,u\in[0,s^*]]=\int_0^{s^*}\Psi_t(s^*,u)\,\mathrm{d}Z_u,$$

and use the covariance formula

$$\mathrm{Cov}\left(\int_0^{s^*}\Psi_s(s^*,u)\,\mathrm{d}Z_u,\int_0^{s^*}\Psi_t(s^*,v)\,\mathrm{d}Z_v\right)$$
$$=H(2H-1)\int_0^{s^*}\int_0^{s^*}\Psi_s(s^*,u)\,\Psi_t(s^*,v)|u-v|^{2H-2}\,\mathrm{d}u\mathrm{d}v.$$

The expression for $\Psi_s(s^*,u)$ contains an integral, and numerical computation of β^* from an expression containing multiple integrals may be hard. As regards the number s^*, we have not found how to obtain any explicit expression for it.

However, knowing the structure of S^*, or even just knowing from Theorem 11 that the MPP is determined by a set where it touches the diagonal, it is easy to obtain discrete approximations of the MPPs using some graphical mathematical tool. Figs. 2 and 3 show the shapes of the paths β^* in two fBm cases.

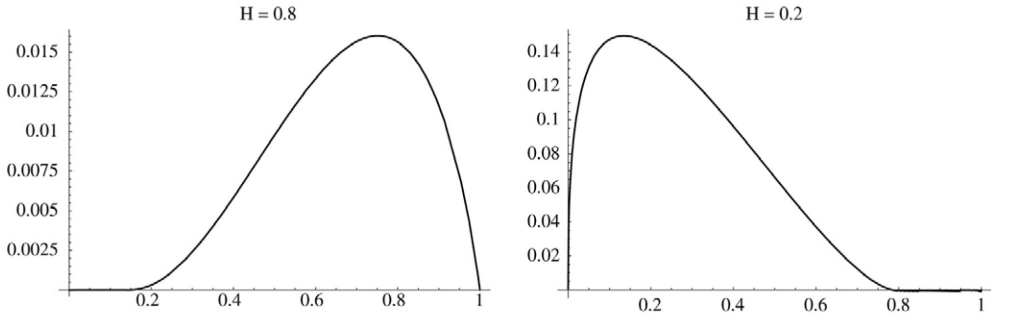

Fig. 2. The difference $\beta^*(t) - t$ for fBm with $H = 0.8$ (left) and $H = 0.2$ (right).

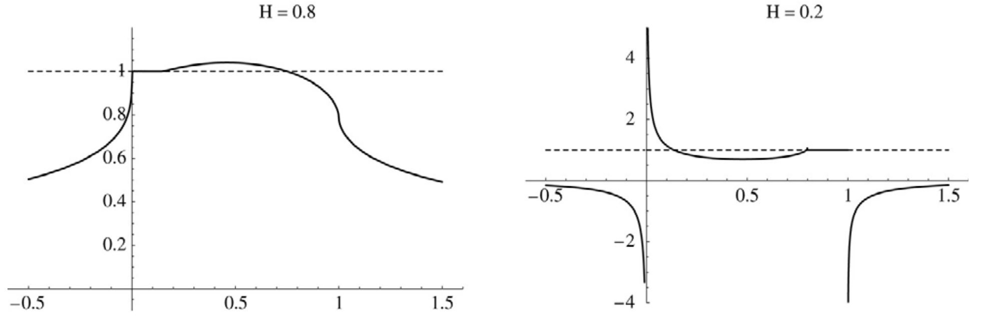

Fig. 3. The derivative of $\beta^*(t)$ for fBm with $H = 0.8$ and $H = 0.2$. The dashed lines correspond to the server rate, 1.

4.3. Integrated Ornstein–Uhlenbeck process

Consider a Gaussian process Z_t with stationary increments and variance $v(t) = t - 1 + \mathrm{e}^{-t}$. This is an integrated Ornstein–Uhlenbeck model, which can be interpreted as the Gaussian counterpart of the Anick–Mitra–Sondhi model; see e.g. [1]. Since the rate process is defined by the stochastic differential equation

$$\mathrm{d}X_t = -X_t \mathrm{d}t + \mathrm{d}W_t,$$

where W denotes the standard Brownian motion, Z is exactly once differentiable and the space $G_{\{t\}} + \partial G_t$ is generated by Z_t and Z_t'. The differentiability property can also be deduced by observing the spectral density of Z_t', which is $1/(4\pi(1+\lambda^2))$.

Let $B = \{f \in R : f(s) \geq s \ \forall s \in [0, 1]\}$ as in the previous subsection. Now, paths in B are differentiable and thus necessarily belong to the set

$$F = \left\{ f \in R : f'(0) \geq 1, f(1) \geq 1 \right\}.$$

The next theorem shows that the most probable path in F is also the most probable path in B, despite $B \subseteq F$. The resulting path is shown in Fig. 4.

Theorem 26. *Assume that* $v(t) = t - 1 + \mathrm{e}^{-t}$. *Then the most probable path in* $B = \{f \in R : f(s) \geq s, \forall s \in [0, 1]\}$ *is given by*

$$\beta^*(t) = t + \frac{(e-1)^2(t - 1 + \mathrm{e}^{-t}) - (\mathrm{e}^t - 1)^2 \mathrm{e}^{-t}}{4e - 1 - e^2}. \tag{18}$$

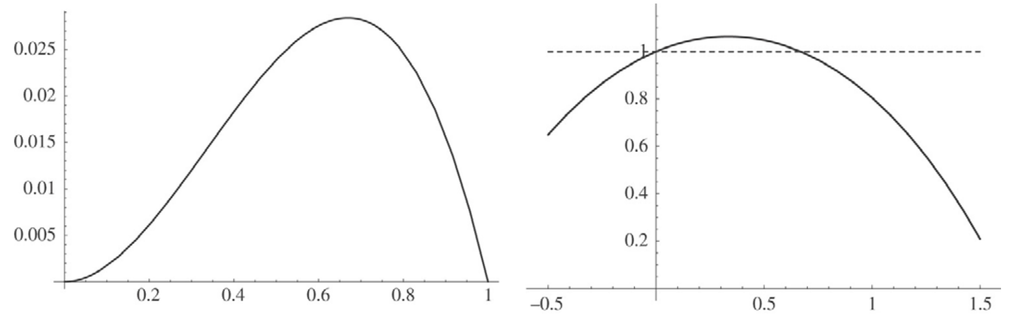

Fig. 4. Integrated Ornstein–Uhlenbeck model with $v(t) = t - 1 + \mathrm{e}^{-t}$. On the left, the difference $\beta^*(t) - t$. On the right, the derivative of $\beta^*(t)$ (solid line) and the server rate (dashed line).

Proof. Application of Proposition 6 gives that the minimizing path in F is

$$f^* = \operatorname{argmin}\{\|f\| : f \in R, \langle f, \Gamma'(0,\cdot)\rangle \geq 1, \langle f, \Gamma(1,\cdot)\rangle \geq 1\}.$$

It is easy to see that both conditions $\langle f, \Gamma'(0,\cdot)\rangle \geq 1$ and $\langle f, \Gamma(1,\cdot)\rangle \geq 1$ are needed, and by Proposition 2, $f^* \in \overline{\mathrm{sp}}\left\{\Gamma'(0,\cdot), \Gamma(1,\cdot)\right\}$. Thus,

$$f^* = (1, 1) \begin{pmatrix} \frac{1}{2}v''(0) & \frac{1}{2}v'(1) \\ \frac{1}{2}v'(1) & v(1) \end{pmatrix}^{-1} \begin{pmatrix} \Gamma'(0,\cdot) \\ \Gamma(1,\cdot) \end{pmatrix}$$

Inserting $v(t) = t - 1 + \mathrm{e}^{-t}$ and doing some simple manipulations gives that $f^*(t)$ equals the formula in the right hand side of (18). One can show that $f^*(t) \geq t$ for all $t \in [0, 1]$, for example, using the Taylor series representation. Thus the optimum path f^* in the 'larger set' F is also in the 'smaller set' B. We conclude that $\beta^* = f^*$. □

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References

[1] R. Addie, P. Mannersalo, I. Norros, Most probable paths and performance formulae for buffers with Gaussian input traffic, Eur. Trans. Telecommun. 13 (3) (2002) 183–196.

[2] R. Azencott, Ecole d'Eté de Probabilités de Saint-Flour VIII-1978, in: Lecture Notes in Mathematics, vol. 774, Springer, Berlin, 1980, pp. 1–176 (Ch. Grandes deviations et applications).

[3] R. Bahadur, S. Zabell, Large deviations of the sample mean in general vector spaces, Ann. Probab. 7 (4) (1979) 587–621.

[4] J.-D. Deuschel, D. Stroock, Large Deviations, Academic Press, Boston, 1989.

[5] A. Dieker, Conditional limit theorems for queues with Gaussian input, a weak convergence approach, Stochastic Process Appl. 115 (2005) 849–873.

[6] J.L. Doob, Stochastic Processes, John Wiley & Sons, 1953.

[7] Y. Kozachenko, O. Vasylyk, T. Sottinen, Path space large deviations of a large buffer with Gaussian input traffic, Queueing Syst. 42 (2) (2002) 113–129.
[8] D. Luenberger, Optimization by Vector Space Methods, John Wiley & Sons, 1969.
[9] M. Mandjes, M. van Uitert, Sample-path large deviations for generalized processor sharing queues with Gaussian inputs, Perform. Evaluation 61 (2005) 225–256.
[10] M. Mandjes, M. van Uitert, Sample-path large deviations for tandem and priority queues with Gaussian inputs, Ann. Appl. Probab. 15 (2005) 1193–1226.
[11] I. Norros, Busy periods of fractional Brownian storage: a large deviations approach, Adv. Performance Anal. 2 (1) (1999) 1–20.
[12] I. Norros, E. Saksman, 2005. Local independence of fractional Brownian motion (Manuscript in preparation).
[13] I. Norros, E. Valkeila, J. Virtamo, An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions, Bernoulli 5 (1999) 571–587.
[14] L. Piterbarg, On the structure of the infinitesimal σ-algebra of Gaussian processes and fields, Theory Probab. Appl. 31 (1987) 484–492.
[15] A. Shwartz, A. Weiss, Large Deviations for Performance Analysis, Chapman & Hall, 1995.
[16] V. Tutubalin, M. Freidlin, On the structure of the infinitesimal σ-algebra of a Gaussian process, Theory Probab. Appl. 7 (1962) 196–199.