Exactness of $G$-sequences and monomorphisms

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Abstract

In this paper, we first show monomorphicity of the inclusion map of a CW-pair implies exactness of the $G$-sequence of the pair. Next we apply exactness of $G$-sequences to solve monomorphicity of maps, that is, if $n \neq 7$, any map from the $n$-sphere to $S^7$ is not monomorphic. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Gottlieb [3] introduced the Gottlieb group, $G_n(X)$, of a space $X$ which consists of all $\alpha \in \pi_n(X, x_0)$ such that there exists an affiliated map $A: S^n \times X \to X$ such that $[A|_{S^n \times x_0}] = [\alpha]$ and $A|_{x_0 \times X} = id_X$, where $s_0$ and $x_0$ are base points of $S^n$ and $X$, respectively. This group, $G_n(X)$, is also characterized by

$G_n(X) = w_\omega(\pi_n(X^X, id_X)) \subset \pi_n(X, x_0),$

where $\omega: X^X \to X$ is an evaluation map at $x_0 \in X$. Thus $G_n(X)$ is also called an evaluation subgroup of $\pi_n(X, x_0)$. Gottlieb extensively studied $G_1(X)$ in [2] and $G_n(X)$ for $n \geq 2$ in

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Among other things he has shown that if $X$ is an $H$-space, then $G_n(X) = \pi_n(X)$ for all $n$. He also had computed

$$G_n(S^n) = \begin{cases} 0 & \text{for } n \text{ even,} \\ Z & \text{for } n = 1, 3, 7, \\ 2Z & \text{for other odd } n. \end{cases}$$

In [5–7], the Gottlieb groups were generalized by Lee, Kim and Woo as the notions of generalized evaluation subgroups and relative evaluation subgroups. As the homotopy sequence of a topological pair plays an important role in computing homotopy groups, Lee and Woo introduced the $G$-sequence of a pair which is consisted by subgroups of homotopy groups, that is, Gottlieb groups, generalized evaluation subgroups and relative evaluation subgroups.

Here we introduce the $G$-sequence of a $CW$-pair from [6,7]. For convenience, from here on we assume a space is a homotopy type of a $CW$-complex and a topological pair is a pointed $CW$-pair.

Let $(X, A)$ be a $CW$-pair and $X^A$ (or $A^X$) be the space of all maps from $A$ into $X$ (or from $X$ into $A$). Then the inclusion map $i: A \to X$ induces the inclusion map $\tilde{i}: A^X \to X^A$ given by $\tilde{i}(f) = if$. Let $\omega: (X^A, i) \to (X, x_0)$ and $\omega: (X^A, A^X, id) \to (X, A, x_0)$ be the corresponding evaluation maps at the base point $x_0 \in A \subset X$. Then these induce homomorphisms

$$\omega_{\ast}: \pi_n(X^A, i) \to \pi_n(X, x_0) \quad \text{and} \quad \omega_{\ast}: \pi_n(X^A, A^X, id) \to \pi_n(X, A, x_0).$$

The generalized evaluation subgroups $G_n(X, A)$ are defined by $[\omega_{\ast}(\pi_n(X^A, i))] = \{[f] \in \pi_n(X) \mid \exists \text{ map } H: A \times I^n \to X \text{ such that } [H|_{\partial I^n}] = [f] \}$ and the relative evaluation subgroups $G_n^{rel}(X, A)$ are defined by $[\omega_{\ast}(\pi_n(X^A, A^X, i))] = \{[f] \in \pi_n(X, A) \mid \exists \text{ map } H: (X \times I^n, A \times \partial I^n) \to (X, A) \text{ such that } [H|_{\partial I^n}] = [f] \}$. The inclusion maps and the evaluation maps induce the following commutative diagram

$$\cdots \to \pi_n(A^X) \overset{i_\ast}{\to} \pi_n(X^A) \overset{j_\ast}{\to} \pi_n(X^A, A^X) \overset{\partial}{\to} \pi_{n-1}(A^X) \to \cdots$$

$$\cdots \to G_n(A) \overset{i_\ast}{\to} G_n(X, A) \overset{j_\ast}{\to} G_n^{rel}(X, A) \overset{\partial}{\to} G_{n-1}(A) \to \cdots$$

$$\cdots \to \pi_n(A) \overset{i_\ast}{\to} \pi_n(X) \overset{j_\ast}{\to} \pi_n(X, A) \overset{\partial}{\to} \pi_{n-1}(A) \to \cdots$$

Since the top and the bottom rows are exact, the middle part make sequence. We call this middle sequence the $G$-sequence of a $CW$-pair $(X, A)$. This sequence is not necessarily exact [7,8]. In [6,7], it was known that if the inclusion map $i: A \to X$ has a left homotopy inverse or is null homotopic, then the $G$-sequence of $(X, A)$ is exact.

A map $f: X \to Y$ is a monomorphism [4] in the category of based topological spaces and based homotopy classes of maps if, for any space $Z$ and any two maps $u, v: Z \to X$, $f \circ u \simeq f \circ v$ implies $u \simeq v$. The monomorphicity of a map is a weaker condition than the existence of a left homotopy inverse of the map. For example, the Hopf map
$h : S^3 \to S^2$ is monomorphic but it does not have a left homotopy inverse. Ganea [1] studied monomorphism of the Hopf fibrations, especially, he showed that $h : S^7 \to S^4$ and $h : S^5 \to S^8$ are not monomorphic.

In this paper, we show that if the inclusion map of a CW-pair is monomorphic in the category of based CW-complexes, then the pair has an exact $G$-sequence. As applications of exact $G$-sequences, we can show that if a map from a space $X$ to a $G$-space is monomorphic, then $X$ is also a $G$-space. It is also shown that the Hopf map $h : S^7 \to S^4$ is not monomorphic (which was erroneously listed as monomorphic by Hilton) by a different method from Ganea.

2. $G$-sequences and monomorphisms

Let $p : (X, x_0) \to (Y, y_0)$ be a map in the category of based CW-complexes and let $\hat{p} : (X^X, id) \to (Y^X, p)$ and $\tilde{p} : (X^X, id) \to (Y^X, p)$ be the induced maps given by $\hat{p}(f) = pf$ and $\tilde{p}(f) = pf$, respectively, where $Y^X = (Y, y)^{X, x}$ is a subspace of the function space $Y^X$ consisting of based maps and $pf$ denotes composition of $p$ and $f$.

In order to prove the main theorem, we need to introduce a lemma.

**Lemma 2.1.** If a map $p : (X, x_0) \to (Y, y_0)$ is monomorphic in the category of based CW-complexes and based homotopy classes of maps, then the induced maps $\hat{p}$ and $\tilde{p}$ are monomorphic.

**Proof.** We prove only the case for $\hat{p}$. Let $(Z, z_0)$ be a based CW-complex and $u, v : (Z, z_0) \to (X^X, id)$ be pointed maps such that $\hat{p}u$ is homotopic to $\hat{p}v$ relative to a base point $z_0$. Since the conjugate map

$$\mu : (Y^X, p)^{(Z, z_0)} \to (Y, y_0)^{(Z \times X)/(Z \times z_0, q_0)}$$

is the natural homeomorphism, $\mu(\hat{p}u)$ is homotopic to $\mu(\hat{p}v)$ as a map from $((Z \times X)/(Z \times z_0), q_0)$ to $(Y, y_0)$, where $q_0$ is the base point of $(Z \times X)/(Z \times z_0)$. Since $\mu(\hat{p}u) = p\mu(u)$ and $\mu(\hat{p}v) = p\mu(v)$, we have $\mu(u)$ is homotopic to $\mu(v)$ by the fact that $p$ is a monomorphism. This implies that $u$ is homotopic to $v$ (not necessarily relative to base point). Now we need to show these two maps to be homotopic relative to base point.

Let $H : Z \times I \to X^X$ be a homotopy from $u$ to $v$ and $\sigma(t) = H(z_0, t)$ be the loop at $id_X$ in $X^X$. We define a map

$$K : (Z \times I \times 0) \cup (z_0 \times I \times I) \cup (Z \times 0 \times I) \cup (Z \times I \times I) \to X^X$$

by $K(z, s, 0) = H(z, s) \circ \sigma(1 - s)$, $K(z_0, s, t) = \sigma(s(1 - t)) \circ \sigma(1 - s(1 - t))$, $K(z, 0, t) = u(z)$ and $K(z, 1, t) = H(z, 1) \circ \sigma(1 - t) \circ \sigma(t)$. By the homotopy extension property, there is an extension $\overline{K} : (Z \times I \times I) \to X^X$ of $K$. If we take a homotopy $G : Z \times I \to X^X$ by $G(z, s) = \overline{K}(z, s, 1)$, then $G$ is a homotopy from $u$ to $v$ relative to $z_0$. Therefore $\hat{p}$ is a monomorphism. □

**Theorem 2.2.** Let $(X, A)$ be a connected CW-pair. If the inclusion map $i : A \to X$ is monomorphic, then the $G$-sequence of $(X, A)$ is exact.
Proof. If we consider the following commutative diagram

\[
\begin{array}{ccc}
(A^A, id) & \xrightarrow{i} & (X^A, i) \\
\downarrow{k_1} & & \downarrow{k_2} \\
(A^A, id) & \xrightarrow{i} & (X^A, i) \\
\downarrow{\omega_1} & & \downarrow{\omega_2} \\
A & \xrightarrow{i} & X
\end{array}
\]

where \(k_i’s\) are inclusions, \(\omega_i’s\) evaluation maps, \(\bar{i}(f) = if\) and \(\bar{i}(f) = if\), then the map \((\omega_1, \omega_2): i \rightarrow i\) is a fibre map in the category of pairs with the induced map, \(\bar{i}\), of the fibers \(A^A, X^A_1\) of \(\omega_1, \omega_2\), as the fiber, because \((A^A, id) \xrightarrow{\bar{k}_1} (A^A, id) \xrightarrow{\omega_1} A\) and \((X^A_1, i) \xrightarrow{\omega_2} (X^A, i) \xrightarrow{\omega_0} X\) are fibration sequences. If we use the fibre map \((\omega_1, \omega_2): \bar{i} \rightarrow i\) in the category of pairs, the sequence

\[
\cdots \rightarrow \pi_n(X^A_1, A^A, id) \rightarrow \pi_n(X^A, A^A, id) \rightarrow \pi_n(X, A) \rightarrow \cdots
\]

is exact (see [4, p. 77]).

Since the inclusion map \(i: A \rightarrow X\) is monomorphic, \(\bar{i}: (A^A, id) \rightarrow (X^A_1, i)\) and \(\bar{i}: (A^A, id) \rightarrow (X^A, i)\) are also monomorphic by Lemma 2.1. Thus we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \pi_n(A^A, id) \\
\downarrow{k_{1z}} & & \downarrow{k_{2z}} \\
0 & \rightarrow & \pi_n(A, id) \\
\downarrow{\omega_{1z}} & & \downarrow{\omega_{2z}} \\
0 & \rightarrow & \pi_n(A) \\
\downarrow{i_z} & & \downarrow{j_z} \\
G_n(A) & \rightarrow & G_n(X, A) \\
\downarrow{\delta} & & \downarrow{j_z} \\
G^\text{Rel}_n(X, A) & \rightarrow & \cdots
\end{array}
\]

such that each row and each column is exact. Let us consider the \(G\)-sequence of \((X, A)\)

\[
\cdots \rightarrow G_n(A) \rightarrow G_n(X, A) \rightarrow G^\text{Rel}_n(X, A) \rightarrow \cdots
\]

It is easy to prove exactness at \(G_n(A)\). By the surjectivity of \(\bar{j}_z\) and the commutativity of above diagram of homotopy groups, we have

\[
G_n^\text{Rel}(X, A) = (\omega_1, \omega_2)z \bar{j}_z(\pi_n(X^A, i)) = j_z \omega_{2z}^z(\pi_n(X^A, i)) = j_z(G_n(X, A))
\]

and hence the \(G\)-sequence is exact at \(G^\text{Rel}_n(X, A)\) by using the triviality of \(\delta\).

To prove the \(G\)-sequence is exact at \(G_n(X, A)\), we use the above diagram of homotopy groups. Let \(\alpha \in G_n(X, A)\) with \(j_z(\alpha) = 0\). Then there is \(\beta \in \pi_n(X^A, i)\) such that \(\alpha = \omega_{2z}(\beta)\). Let \(\gamma = j_z(\beta)\). By the commutativity of the diagram, we have \((\omega_1, \omega_2)z(\gamma) = 0\) and hence \(\gamma = (k_1, k_2)z(\delta)\) for some \(\delta \in \pi_n(X^A_1, A^A, id)\). By surjectivity of \(j_z\), there exists
\[ \eta \in \pi_n(X^4_A, i) \text{ such that } \delta = \tilde{f}_2(\eta). \] Since \( \tilde{f}_2(\beta - k_{2\xi}(\eta)) = 0 \), there exists \( \xi \in \pi_n(A^4, \text{id}) \) such that \( \beta - k_{2\xi}(\eta) = \tilde{i}_2(\xi) \). Therefore we have
\[
\alpha = \omega_2(\beta) = \omega_2(k_{2\xi}(\eta) + \tilde{i}_2(\xi)) = \omega_2\tilde{i}_2(\xi) = \tilde{i}_2\omega_1(\xi).
\]
Therefore \( \alpha \) belongs to the image of \( \tilde{i}_2 \) and hence the \( G \)-sequence of \((X, A)\) is exact. \( \square \)

**Corollary 2.3** [6]. If the inclusion map \( i : A \to X \) has a left homotopy inverse, then the \( G \)-sequence of \((X, A)\) is exact.

### 3. Applications

It is difficult to determine whether a map is monomorphic or not. However if we use exactness of the \( G \)-sequence, we can obtain a useful result to determine the monomorphicity of a map. A space \( X \) is a \( G \)-space if \( G_n(X) = \pi_n(X) \) for each \( n \geq 1 \) [9]. Every \( H \)-space is a \( G \)-space, but the converse is not true. It is well known that the \( n \)-sphere is a \( G \)-space if and only if \( n \) is 1, 3 or 7. In general, the image of \( G \)-spaces under monomorphisms need not be \( G \)-spaces. For example, let \( h : S^3 \to S^2 \) be the Hopf bundle, then \( h \) is monomorphic [4] and \( S^3 \) is a \( G \)-space but \( S^2 \) is not a \( G \)-space. However we have the following.

**Theorem 3.1.** Let \( p : X \to Y \) be a monomorphism and \( Y \) be a \( G \)-space. Then \( X \) is also a \( G \)-space.

**Proof.** Let \( p : X \to Y \) be a monomorphism and \( M_p \) be the mapping cylinder of \( p \). Then the inclusion \( i : X \to M_p \) is also monomorphic. By Theorem 2.2, the \( G \)-sequence of \((M_p, X)\) is exact. Since \( Y \) is homotopy equivalent to \( M_p \) and \( Y \) is a \( G \)-space, we have \( G_n(M_p, X) = \pi_n(M_p) \). Therefore we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & G_n(X) \to i_2^* \to G_n(M_p, X) \to j_2^* \to G_{n\text{Rel}}(M_p, X) \to 0 \\
& \downarrow & \downarrow & \downarrow & \\
0 & \to & \pi_n(X) \to i_2^* \to \pi_n(M_p) \to j_2^* \to \pi_n(M_p, X) \to 0
\end{array}
\]

It is sufficient to show \( \pi_n(X) \subset G_n(X) \). Let \([\alpha]\) be an element of \( \pi_n(X) \). Then \( i_2^*([\alpha]) \) belongs to \( G_n(M_p, X) \) and \( j_2^*(i_2^*([\alpha])) = 0 \). By exactness of \( G \)-sequence of \((M_p, X)\), there is an element \( \beta \in G_n(X) \) such that \( i_2^*([\alpha]) = i_2^*(\beta) \). Since \( i_2^* \) is monomorphism, \([\alpha]\) belongs to \( G_n(X) \). \( \square \)

**Corollary 3.2.** If a map from the \( n \)-sphere to a \( G \)-space is a monomorphism, then \( n = 1, 3 \) or 7. Especially, if \( n \neq 7 \), then any map from the \( n \)-sphere to \( S^7 \) is not monomorphic.

The Hopf map \( h : S^7 \to S^4 \) had first been listed as monomorphic by Hilton [4, p. 169] but it was shown to be not monomorphic by Ganea [1]. In [8, p. 293], the authors showed
that the $G$-sequence of the $CW$-pair $(M_h, S^7)$ is not exact, where $M_h$ is the mapping cylinder of the Hopf map $h$. Suppose the Hopf map $h: S^7 \to S^4$ is monomorphic. Then the inclusion map $i: S^7 \to M_h$ is also monomorphic. By Theorem 2.2, the $G$-sequence of the pair $(M_h, S^7)$ is exact. Therefore this also yields an alternative proof of the following.

**Remark 3.3.** The Hopf map $h: S^7 \to S^4$ is not a monomorphism.

References