On the forced oscillation of solutions for systems of impulsive parabolic differential equations with several delays

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Abstract

In this paper, we study the forced oscillation of certain systems of impulsive parabolic differential equations with several delays. Some oscillation criteria are established.

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1. Introduction

It is well known that many evolution processes experience changes of state abruptly because of short-term perturbations. We usually regard these perturbations as impulsive type because the duration of these perturbations is negligible in comparison with the duration of the processes considered. In the past few years, the theory of impulsive partial differential equations has been investigated extensively. For instance,
see [1–9] and the references therein. Recently, the oscillations for impulsive delay parabolic differential equations and impulsive delay hyperbolic differential equations were studied by Fu et al. [6] and Cui et al. [3], respectively. But nobody studied the forced oscillation of systems of impulsive delay partial differential equations, as far as we know.

In this paper, we study the forced oscillation of systems of impulsive parabolic differential equations with several delays of the form

$$
\frac{\partial}{\partial t} u_i(x,t) = \sum_{k=1}^{m} a_{ik}(t) \Delta u_k(x,t) + \sum_{k=1}^{m} b_{ik}(t) \Delta u_k(x,t - \tau_{ik}) - c_i(x,t, (u_k(x,t))_{k=1}^{m}, (u_k(x,t - \sigma_{ik}))_{k=1}^{m}) - \sum_{h=1}^{l} q_{ih}(x,t) u_i(x,t - \lambda_{ih}) + f_i(x,t), \quad t \neq j,
$$

$$
u_i(x,t_j^+) - \nu_i(x,t_j^-) = p_i(x,t_j, u_i(x,t_j)), \quad i \in I_m, \quad j \in I_\infty, \quad (x,t) \in \Omega \times R_+ \equiv G, \tag{1}
$$

where $I_m = \{1, 2, \ldots, m\}$, $I_\infty = \{1, 2, \ldots\}$, $R_+ = [0, \infty)$, $\Omega$ is a bounded domain in $R^n$ with a smooth boundary $\partial \Omega$,

$$
\Delta u_i(x,t) = \sum_{r=1}^{n} \frac{\partial^2 u_i(x,t)}{\partial x_r^2}, \quad i \in I_m, \quad 0 < t_1 < t_2 < \cdots < t_j < \cdots
$$

and $\lim_{j \to \infty} t_j = \infty$.

Consider the following boundary condition:

$$
\frac{\partial u_i(x,t)}{\partial N} = \psi_i(x,t), \quad (x,t) \in \partial \Omega \times R_+, \quad t \neq t_j, \quad i \in I_m, \quad j \in I_\infty \tag{2}
$$

and the initial condition

$$
u_i(x,t) = \phi_i(x,t), \quad (x,t) \in \Omega \times [-\delta_i, 0], \tag{3}
$$

where $N$ is the unit exterior normal vector to $\partial \Omega$ and $\psi_i \in PC[\partial \Omega \times R_+, R]$, $i \in I_m$, $PC$ denotes the class of functions, which are piecewise continuous in $t$ with discontinuities of first kind only at $t = t_j$ and left continuous at $t = t_j$, $j \in I_\infty$,

$$
\delta_i = \max\{\tau_{ik}, \sigma_{ik}, \lambda_{ih}; k \in I_m, h \in I_1\}, \quad \phi_i \in C^2(\Omega \times [-\delta_i, 0], R), \quad i \in I_m, \quad I_1 = \{1, 2, \ldots, l\}.
$$

Throughout this paper, we assume that the following conditions hold:

(C1) $a_{ik}, b_{ik} \in PC[R_+, R_+], \quad i, k \in I_m$;

(C2) $\tau_{ik}, \sigma_{ik}, \lambda_{ih} \geq 0$ are constants, $i, k \in I_m, \quad h \in I_1$;

(C3) $c_i \in PC[\overline{G} \times R^{2m}, R]$ and

$$
c_i(x,t, \xi_1, \ldots, \xi_i, \ldots, \xi_m, \eta_1, \ldots, \eta_i, \ldots, \eta_m) \begin{cases} \geq 0 & \text{if } \xi_i \text{ and } \eta_i \in (0, \infty), \\ \leq 0 & \text{if } \xi_i \text{ and } \eta_i \in (-\infty, 0), \end{cases} \quad i \in I_m;
$$
\( q_{ih} \in \text{PC}[\overline{G}, R_+] \), \( q_{ih}(t) = \min_{x \in \Omega} q_{ih}(x, t) \), \( i \in I_m, \ h \in I_l \); 
\( f_i \in \text{PC}[\overline{G}, R], \ p_i \in \overline{G} \times R \rightarrow R, \ i \in I_m, \ j \in I_\infty \); 
\( \text{for any function } \mu_i \in \text{PC}[\overline{G}, R_+] \), the following conditions are satisfied:

\[
p_i(x, t_j, -u_i(x, t_j)) = -p_i(x, t_j, u_i(x, t_j))
\]
and

\[
\int_{\Omega} p_i(x, t_j, u_i(x, t_j)) \, dx \leq \alpha_{ij} \int_{\Omega} u_i(x, t_j) \, dx,
\]
where \( \alpha_{ij} > 0 \) is a constant, \( i \in I_m, \ j \in I_\infty \).

**Definition 1.1.** The vector function \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) is said to be a solution of problem (1) and (2) if the following conditions are satisfied:

(i) \( u_i(x, t) \) is a first differentiable function for \( t, t \neq t_j, j \in I_\infty, \ i \in I_m \);

(ii) \( u_i(x, t) \) is a piecewise continuous function with points of discontinuity of the first kind at \( t = t_j, j \in I_\infty \), and at the moments of impulse the following relations are satisfied:

\[
u_i(x, t_j^-) = u_i(x, t_j), \quad u_i(x, t_j^+) = u_i(x, t_j) + p_i(x, t_j, u_i(x, t_j)), \quad i \in I_m, \ j \in I_\infty;
\]

(iii) \( u_i(x, t) \) is a second-order differentiable function for \( x, i \in I_m \);

(iv) \( u_i(x, t) \) satisfies (1) in the domain \( G \) and boundary condition (2), \( i \in I_m \).

**Definition 1.2.** A nontrivial component \( u_i(x, t) \) of the vector function \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) is said to oscillate in \( \Omega \times [\alpha_0, \infty) \) if for each \( \alpha \neq \alpha_0 \) there is a point \( (x_0, t_0) \in \Omega \times [\alpha, \infty) \) such that \( u_i(x_0, t_0) = 0 \).

**Definition 1.3.** The vector solution \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) of problem (1) and (2) is said to oscillate in the domain \( G = \Omega \times R_+ \), if at least one of its nontrivial component oscillates in \( G \). Otherwise, the vector solution \( u(x, t) \) is said to be nonoscillatory in \( G \).

**Definition 1.4.** The vector solution \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) of the problem (1) and (2) is said to strongly oscillate in the domain \( G = \Omega \times R_+ \) if each of its nontrivial component oscillates in \( G \).

In this paper, we shall use the following notations:

\[
U_i(t) = \int_{\Omega} u_i(x, t) \, dx, \quad \Psi_i(t) = \int_{\partial \Omega} \psi_i(x, t) \, dS, \quad F_i(t) = \int_{\Omega} f_i(x, t) \, dx,
\]

\[
H_i(t) = F_i(t) + \sum_{k=1}^{m} a_{ik}(t) \Psi_k(t) + \sum_{k=1}^{m} b_{ik}(t) \Psi_k(t - \tau_{ik}), \quad t \in R_+, \ i \in I_m,
\]
where \( dS \) is the surface element on \( \partial \Omega \), and

\[
\prod_{t_0 < t_j < t} (1 + \beta_j)w(t_0) = [(1 + \beta_1)(1 + \beta_2) \cdots (1 + \beta_j) \cdots]w(t_0), \quad t_0 < t_j < t, \ j \in I_\infty, \ t \in R_+.
\]
In Section 2, we shall offer some lemmas, which will be used in Section 3 to establish our oscillation criteria of problem (1) and (2).

2. Some lemmas

Lemma 2.1. Suppose that \( u(x, t) = [u_1(x, t), u_2(x, t), \ldots, u_m(x, t)]^T \) is a solution of problem (1) and (2) in \( G \). If there exists some \( i_0 \in I_m \) such that \( u_{i_0}(x, t) > 0, t \geq T \), then \( U_{i_0}(t) \) satisfies the impulsive differential inequality

\[
V'(t) + \sum_{h=1}^{l} q_{i_0 h}(t) V(t - \lambda_{i_0} h) \leq H_{i_0}(t), \quad t \geq T, \quad t \neq t_j,
\]

\[
V(t_{j+}) \leq (1 + z_{i_0 j}) V(t_j), \quad j \in I_\infty,
\]

where \( T \) is a sufficiently large positive number.

Proof. Consider the following equation:

\[
\frac{\partial}{\partial t} u_{i_0}(x, t) = \sum_{k=1}^{m} a_{i_0 k}(t) \Delta u_k(x, t) + \sum_{k=1}^{m} b_{i_0 k}(t) \Delta u_k(x, t - \tau_{i_0 k})
\]

\[
- c_{i_0}(x, t, (u_k(x, t))_{k=1}^{m}, (u_k(x, t - \sigma_{i_0 k}))_{k=1}^{m})
\]

\[
- \sum_{h=1}^{l} q_{i_0 h}(x, t) u_{i_0}(x, t - \lambda_{i_0 h}) + f_{i_0}(x, t), \quad t \neq t_j,
\]

\[
u_{i_0}(x, t_{j+}) - v_{i_0}(x, t_{j-}) = p_{i_0}(x, t_j, u_{i_0}(x, t_j)),
\]

\[
 j \in I_\infty, \quad (x, t) \in \Omega \times R_+ \equiv G.
\]

(5)

From the assumptions of Lemma 2.1, we have \( u_{i_0}(x, t) > 0, (x, t) \in \Omega \times [T, \infty) \). Then there exists a number \( T_1 \geq T \) such that \( u_{i_0}(x, t) > 0, u_{i_0}(x, t - \tau_{i_0 k}) > 0, u_{i_0}(x, t - \sigma_{i_0 k}) > 0 \) and \( u_{i_0}(x, t - \lambda_{i_0 h}) > 0 \) in \( \Omega \times [T_1, \infty), k \in I_m, h \in I_1 \).

Case 1: \( t \neq t_j \). Integrating the first equation in (5) with respect to \( x \) over the domain \( \Omega \), we have

\[
\frac{d}{dt} \int_{\Omega} u_{i_0}(x, t) \, dx = \sum_{k=1}^{m} a_{i_0 k}(t) \int_{\Omega} \Delta u_k(x, t) \, dx + \sum_{k=1}^{m} b_{i_0 k}(t) \int_{\Omega} \Delta u_k(x, t - \tau_{i_0 k}) \, dx
\]

\[
- \int_{\Omega} c_{i_0}(x, t, (u_k(x, t))_{k=1}^{m}, (u_k(x, t - \sigma_{i_0 k}))_{k=1}^{m}) \, dx
\]

\[
- \sum_{h=1}^{l} \int_{\Omega} q_{i_0 h}(x, t) u_{i_0}(x, t - \lambda_{i_0 h}) \, dx
\]

\[
+ \int_{\Omega} f_{i_0}(x, t) \, dx, \quad t \geq T_1, \quad t \neq t_j, \quad j \in I_\infty.
\]

(6)
Using Green’s formula and (2), we have
\[ \int_{\Omega} \Delta u_k(x,t) \, dx = \int_{\partial \Omega} \frac{\partial u_k(x,t)}{\partial N} \, dS = \int_{\partial \Omega} \psi_k(x,t) \, dS = \Psi_k(t) \] (7)
and
\[ \int_{\Omega} \Delta u_k(x,t - \tau_{io_k}) \, dx = \int_{\partial \Omega} \frac{\partial u_k(x,t - \tau_{io_k})}{\partial N} \, dS \]
\[ = \int_{\partial \Omega} \psi_k(x,t - \tau_{io_k}) \, dS \]
\[ = \Psi_k(t - \tau_{io_k}), \quad t \geq T_1, \quad t \neq t_j, \quad j \in I_\infty, \quad k \in I_m. \] (8)
Noting that \( u_{i_0}(x,t) > 0, u_{i_0}(x,t - \sigma_{i_0}) > 0 \), from assumption (C3), we easily obtain
\[ c_{i_0}(x,t, (u_k(x,t))_{k=1}^m, (u_k(x,t - \sigma_{i_0}))_{k=1}^m) \geq 0, \quad t \geq T_1, \quad t \neq t_j, \quad j \in I_\infty. \] (9)
Therefore, combining (6)–(9) and using assumption (C4), we have
\[ U_{i_0}'(t) + \sum_{h=1}^l q_{i_0h}(t) U_{i_0}(t - \lambda_{i_0h}) \leq F_{i_0}(t) + \sum_{k=1}^m a_{i_0k}(t) \Psi_k(t) \]
\[ + \sum_{k=1}^m b_{i_0k}(t) \Psi_k(t - \tau_{i_0k}), \quad t \geq T_1, \quad t \neq t_j, \quad j \in I_\infty. \] (10)

**Case 2: \( t = t_j \).** It follows from the second equation in (5) and assumption (C6) that
\[ U_{i_0}(t_j^+) = \int_{\Omega} u_{i_0}(x,t_j^+) \, dx = \int_{\Omega} u_{i_0}(x,t_j) \, dx + \int_{\Omega} p_{i_0}(x,t_j,u_{i_0}(x,t_j)) \, dx \]
\[ \leq \int_{\Omega} u_{i_0}(x,t_j) \, dx + \alpha_{i_0j} \int_{\Omega} u_{i_0}(x,t_j) \, dx = (1 + \alpha_{i_0j}) U_{i_0}(t_j), \quad t \geq T_1, \quad j \in I_\infty. \] (11)
Therefore, (10) and (11) show that \( U_{i_0}(t) > 0 \) satisfies the impulsive differential inequality (4). The proof is complete. \( \square \)

**Lemma 2.2.** Suppose that \( u(x,t) = \{u_1(x,t), u_2(x,t), \ldots, u_m(x,t)\}^T \) is a solution of problem (1) and (2) in \( G \). If \( u_i(x,t) > 0, t \geq T, i \in I_m \), then \( U_i(t) \) satisfies the impulsive differential inequality
\[ V'(t) + \sum_{h=1}^l q_{ih}(t) V(t - \lambda_{ih}) \leq H_i(t), \quad t \geq T, \quad t \neq t_j, \]
\[ V(t_j^+) \leq (1 + \alpha_{ij}) V(t_j), \quad j \in I_\infty, \quad i \in I_m, \] (12)
where \( T \) is a sufficiently large positive number.

The proof of Lemma 2.2 is similar to that of Lemma 2.1. So we omit it here.
Lemma 2.3. Suppose that \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) is a solution of problem (1) and (2) in \( G \). If there exists some \( i_0 \in I_m \) such that \( u_{i_0}(x, t) < 0, t \geq T \), then \( U_{i_0}(t) \) satisfies the impulsive differential inequality

\[
V'(t) + \sum_{h=1}^{l} q_{i_0h}(t)V(t - \lambda_{i_0h}) \geq H_{i_0}(t), \quad t \geq T, \quad t \neq t_j,
\]

\[
V(t^+_j) \geq (1 + z_{i_0j})V(t_j), \quad j \in I_\infty, \quad (13)
\]

where \( T \) is a sufficiently large positive number.

Proof. Consider Eq. (5) and let \( u_{i_0}(x, t) < 0 \) be a solution of (5). Then there exists a number \( T_1 \geq T \) such that \( u_{i_0}(x, t) < 0, t \geq T \), \( u_{i_0}(x, t - \tau_{i_0k}) < 0 \) and \( u_{i_0}(x, t - \sigma_{i_0k}) < 0 \) in \( \Omega \times [T_1, \infty), k \in I_m, h \in I_l \).

Case 1: \( t \neq t_j \). As in the proof of Lemma 2.1, we obtain (6)–(8). Combining (6)–(8) and using assumptions (C3) and (C4), we have

\[
U_{i_0}'(t) + \sum_{h=1}^{l} q_{i_0h}(t)U_{i_0}(t - \lambda_{i_0h}) \geq F_{i_0}(t) + \sum_{k=1}^{m} a_{i_0k}(t)\Psi_k(t)
\]

\[
+ \sum_{k=1}^{m} b_{i_0k}(t)\Psi_k(t - \tau_{i_0k}), \quad t \geq T_1, \quad t \neq t_j, \quad j \in I_\infty. \quad (14)
\]

Case 2: \( t = t_j \). It follows from the second equation in (5) and assumption (C6) that

\[
U_{i_0}(t^+_j) = \int_{\Omega} u_{i_0}(x, t^+_j)\, dx = \int_{\Omega} u_{i_0}(x, t_j)\, dx + \int_{\Omega} p_{i_0}(x, t_j, u_{i_0}(x, t_j))\, dx
\]

\[
= \int_{\Omega} u_{i_0}(x, t_j)\, dx - \int_{\Omega} p_{i_0}(x, t_j, -u_{i_0}(x, t_j))\, dx
\]

\[
\geq \int_{\Omega} u_{i_0}(x, t_j)\, dx + z_{i_0j}\int_{\Omega} u_{i_0}(x, t_j)\, dx
\]

\[
= (1 + z_{i_0j})U_{i_0}(t_j), \quad t \geq T_1, \quad t = t_j, \quad j \in I_\infty. \quad (15)
\]

Obviously, (14) and (15) show that \( U_{i_0}(t) < 0 \) satisfies the impulsive differential inequality (13). The proof is complete. \( \square \)

Similarly, we can establish the following result.

Lemma 2.4. Suppose that \( u(x, t) = \{u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}^T \) is a solution of problem (1) and (2) in \( G \). If \( u_i(x, t) < 0, t \geq T, i \in I_m \), then \( U_i(t) \) satisfies the impulsive differential inequality

\[
V'(t) + \sum_{h=1}^{l} q_{ih}(t)V(t - \lambda_{ih}) \geq H_i(t), \quad t \geq T, \quad t \neq t_j,
\]

\[
V(t^+_j) \geq (1 + z_{ij})V(t_j), \quad j \in I_\infty, \quad i \in I_m, \quad (16)
\]

where \( T \) is a sufficiently large positive number.
Lemma 2.5 (Zhang [9]). Assume that $0 \leq t_0 < t_1 < t_2 < \cdots < t_j < \cdots$, $\lim_{j \to \infty} t_j = \infty$, $w \in PC^1[R_+, R]$, $h \in PC[R_+, R]$, and $\beta_j > 0$ is a constant, $j \in I_\infty$. If

\[
\begin{align*}
    w'(t) &\leq h(t), \quad t \geq t_0, \quad t \neq t_j, \\
    w(t_j^+) &\leq (1 + \beta_j)w(t_j), \quad j \in I_\infty.
\end{align*}
\]

then

\[
w(t) \leq \prod_{t_0 < t_j < t} (1 + \beta_j)w(t_0) + \int_{t_0}^{t} \prod_{s < t_j < t} (1 + \beta_j)h(s) \, ds, \quad t \geq t_0.
\]

3. Main results

We firstly introduce the following useful definition.

**Definition 3.1.** The solution $V(t)$ of the impulsive differential inequality (4) ((13)) is called eventually positive (negative), if there exists a number $\mu \geq 0$ such that $V(t) > 0$ ($V(t) < 0$) for $t \geq \mu$.

Using Lemmas 2.1 and 2.3, we immediately obtain the following theorem.

**Theorem 3.1.** If there exists some $i_0 \in I_m$ such that the impulsive differential inequality (4) has no eventually positive solutions and the impulsive differential inequality (13) has no eventually negative solutions, then every solution of problem (1) and (2) is oscillatory in $G$.

**Theorem 3.2.** If there exists some $i_0 \in I_m$ such that

\[
\sum_{j=1}^{\infty} x_{i_0j} < \infty,
\]

\[
\liminf_{t \to \infty} \frac{\int_T^t \prod_{s < t_j < t} (1 + x_{i_0j})H_{i_0}(s) \, ds}{\prod_{T < t_j < t} (1 + x_{i_0j})} = -\infty
\]

and

\[
\limsup_{t \to \infty} \frac{\int_T^t \prod_{s < t_j < t} (1 + x_{i_0j})H_{i_0}(s) \, ds}{\prod_{T < t_j < t} (1 + x_{i_0j})} = \infty,
\]

where $j \in I_\infty$, $T$ is a sufficiently large positive number.

Then every solution of problem (1) and (2) is oscillatory in $G$.

**Proof.** We prove that the impulsive differential inequality (4) has no eventually positive solutions and the impulsive differential inequality (13) has no eventually negative solutions.
Firstly, assume to the contrary that (4) has an eventually positive solution \( U_{i_0}(t) \), then there exists \( T_1 \geq T \) such that \( U_{i_0}(t) > 0, U_{i_0}(t - \lambda_{i_0 h}) > 0, t \geq T_1, h \in I_i \). Thus from (4) we have

\[
U_{i_0}'(t) \leq H_{i_0}(t), \quad t \geq T_1, \quad t \neq t_j, \\
U_{i_0}(t_j^+) \leq (1 + \alpha_{i_0 j}) U_{i_0}(t_j), \quad j \in I_i. 
\] (22)

Using Lemma 2.5, we have

\[
U_{i_0}(t) \leq \prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j}) U_{i_0}(T_1) + \int_{T_1}^{t} \prod_{s < t_j < t} (1 + \alpha_{i_0 j}) H_{i_0}(s) \, ds, \quad t \geq T_1. 
\] (23)

From (23), we have

\[
\frac{U_{i_0}(t)}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})} \leq U_{i_0}(T_1) + \frac{\int_{T_1}^{t} \prod_{s < t_j < t} (1 + \alpha_{i_0 j}) H_{i_0}(s) \, ds}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})}, \quad t \geq T_1. 
\] (24)

Noting conditions (19) and (20) and taking \( t \to \infty \), from (24) we obtain

\[
\lim_{t \to \infty} \frac{U_{i_0}(t)}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})} = -\infty,
\]

which contradicts with the assumption that \( U_{i_0}(t) > 0 \).

Nextly, assume to the contrary that the impulsive differential inequality (13) has an eventually negative solution \( \overline{U}_{i_0}(t) \). Then there exists \( T_1 \geq T \) such that \( \overline{U}_{i_0}(t) < 0, \overline{U}_{i_0}(t - \lambda_{i_0 h}) < 0, t \geq T_1, h \in I_i \). Thus from (13) we have

\[
\overline{U}_{i_0}'(t) \geq H_{i_0}(t), \quad t \geq T_1, \quad t \neq t_j, \\
\overline{U}_{i_0}(t_j^+) \geq (1 + \alpha_{i_0 j}) \overline{U}_{i_0}(t_j), \quad j \in I_i. 
\] (25)

Let \( W(t) = -\overline{U}_{i_0}(t) \), then \( W(t) > 0 \). Obviously, it follows from (25) that

\[
W(t) \leq -H_{i_0}(t), \quad t \geq T_1, \quad t \neq t_j, \\
W(t_j^+) \leq (1 + \alpha_{i_0 j}) W(t_j), \quad j \in I_i. 
\] (26)

Using Lemma 2.5, we have

\[
W(t) \leq \prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j}) W(T_1) - \int_{T_1}^{t} \prod_{s < t_j < t} (1 + \alpha_{i_0 j}) H_{i_0}(s) \, ds, \quad t \geq T_1. 
\] (27)

From (27), we obtain

\[
\frac{W(t)}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})} \leq W(T_1) - \frac{\int_{T_1}^{t} \prod_{s < t_j < t} (1 + \alpha_{i_0 j}) H_{i_0}(s) \, ds}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})}, \quad t \geq T_1,
\]

i.e.

\[
\frac{\overline{U}_{i_0}(t)}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})} \geq \overline{U}_{i_0}(T_1) + \frac{\int_{T_1}^{t} \prod_{s < t_j < t} (1 + \alpha_{i_0 j}) H_{i_0}(s) \, ds}{\prod_{T_1 < t_j < t} (1 + \alpha_{i_0 j})}, \quad t \geq T_1. 
\] (28)
Noting condition (19) and (21) and taking $t \to \infty$, from (28) we have
\[
\limsup_{t \to \infty} \frac{U_{i_0}(t)}{\prod_{T_1 < t_j < t} (1 + x_{i_0} j)} = \infty,
\]
which contradicts with the assumption that $U_{i_0}(t) < 0$. This completes the proof. \(\square\)

Using Lemmas 2.2, 2.4 and 2.5, it is easy to see that the following strong oscillation results of problem (1) and (2) are parallel to the above oscillation conclusions.

**Theorem 3.3.** Suppose that for all $i \in I_m$, the impulsive differential inequality
\[
V'(t) + \sum_{h=1}^{l} q_{ih}(t)V(t - \lambda_{ih}) \leq H_i(t), \quad t \geq T, \quad t \neq t_j,
\]
\[
V(t_j^+) \leq (1 + x_{ij})V(t_j), \quad j \in I_\infty,
\]
has no eventually positive solutions and the impulsive differential inequality
\[
V'(t) + \sum_{h=1}^{l} q_{ih}(t)V(t - \lambda_{ih}) \geq H_i(t), \quad t \geq T, \quad t \neq t_j,
\]
\[
V(t_j^+) \geq (1 + x_{ij})V(t_j), \quad j \in I_\infty
\]
has no eventually negative solutions, where $T$ is a sufficiently large positive number. Then every solution of problem (1) and (2) strongly oscillates in $G$.

**Theorem 3.4.** Suppose that for all $i \in I_m$, the following conditions are satisfied:
\[
\sum_{j=1}^{\infty} x_{ij} < \infty,
\]
and
\[
\liminf_{t \to \infty} \frac{\int_{T}^{t} \prod_{s < t_j < t} (1 + x_{ij}) H_i(s) \, ds}{\prod_{T < t_j < t} (1 + x_{ij})} = -\infty
\]
and
\[
\limsup_{t \to \infty} \frac{\int_{T}^{t} \prod_{s < t_j < t} (1 + x_{ij}) H_i(s) \, ds}{\prod_{T < t_j < t} (1 + x_{ij})} = \infty,
\]
where $j \in I_\infty$, $T$ is a sufficiently large positive number. Then every solution of problem (1) and (2) strongly oscillates in $G$. 

Example 3.1. Consider the following system:

\[
\frac{\partial u_1(x, t)}{\partial t} = 2\Delta u_1(x, t) + a_{12}(t)\Delta u_2(x, t) + b_{11}(t)\Delta u_1(x, t - \pi) + 6t\Delta u_2\left(x, t - \frac{3\pi}{2}\right) - \frac{1}{2}u_1(x, t) - \frac{1}{2}u_1(x, t - 2\pi) - e^t u_1 \left(x, t - \frac{\pi}{5}\right) - e^t (\sin t + \cos t)(1 + \cos x),
\]

\[ t \neq \frac{j\pi}{3}, \quad j \in I_\infty, \]

\[ u_1(x, t^+_{j}) - u_1(x, t^-_{j}) = t^{-3}_j u_1(x, t_j) \cos \frac{x}{3}, \quad t_j = \frac{j\pi}{3}, \quad j \in I_\infty, \]

\[
\frac{\partial u_2(x, t)}{\partial t} = a_{21}(t)\Delta u_1(x, t) + 2e^t \Delta u_2(x, t) + \frac{1}{3}\Delta u_1 \left(x, t - \frac{3\pi}{4}\right) + b_{22}(t)\Delta u_2 \left(x, t - \frac{\pi}{4}\right) - \frac{1}{2}u_2(x, t - \pi) - \frac{1}{2}u_2(x, t - 2\pi) - 3e^t u_2 \left(x, t - \frac{\pi}{3}\right) + e^t \sin t \cos x, \quad t \neq \frac{j\pi}{3}, \quad j \in I_\infty, \]

\[ u_2(x, t^+_{j}) - u_2(x, t^-_{j}) = t^{-7}_j u_2(x, t_j) \cos \frac{x}{2}, \quad t_j = \frac{j\pi}{3}, \quad j \in I_\infty, \]

\[ (x, t) \in (0, \pi) \times [0, \infty), \quad (34) \]

with boundary condition

\[
\frac{\partial u_i(0, t)}{\partial x} = \frac{\partial u_i(\pi, t)}{\partial x} = 0, \quad t \geq 0, \quad t \neq \frac{j\pi}{3}, \quad j \in I_\infty, \quad i = 1, 2, \quad (35)
\]

where

\[
a_{12}(t) = \begin{cases} 3, & t = 0, \\ 3 + \frac{1}{1 - \cos 6t}, & t \neq \frac{j\pi}{3}, \quad j \in I_\infty, \end{cases}
\]

\[
b_{11}(t) = \begin{cases} 4, & t = 0, \\ 4 + \frac{3}{1 - \cos 12t}, & t \neq \frac{j\pi}{3}, \quad j \in I_\infty, \end{cases}
\]

\[
a_{21}(t) = \begin{cases} 1, & t = 0, \\ 1 + \frac{2}{1 - \cos^2 3t}, & t \neq \frac{j\pi}{3}, \quad j \in I_\infty \end{cases}
\]

and

\[
b_{22}(t) = \begin{cases} 2, & t = 0, \\ 2 + \frac{1}{1 - \cos^2 9t}, & t \neq \frac{j\pi}{3}, \quad j \in I_\infty. \end{cases}
\]

Here \( \Omega = (0, \pi), n = 1, m = 2, l = 1, c_i(x, t, (u_k(x, t)))_{k=1}^2, (u_k(x, \sigma_{im}(t)))_{k=1}^2 = \frac{1}{2}(u_i(x, t) + u_i(x, \sigma_{i}(t)), (i = 1, 2), f_1(x, t) = -e^t (\sin t + \cos t)(1 + \cos x), f_2(x, t) = e^t \sin t \cos x, p_1(x, t, u_1) = t^{-3}u \cos(x/3), \text{and} p_2(x, t, u_2) = t^{-7}u \cos(x/2). \)
It is easy to see that $\Psi_1(t) = \Psi_2(t) = 0$, then
\[
H_1(t) = F_1(t) = \int_{\Omega} f_1(x, t) \, dx = \int_0^\pi f_1(x, t) \, dx = -\pi e^t (\sin t + \cos t),
\]
\[
H_2(t) = F_2(t) = \int_{\Omega} f_2(x, t) \, dx = \int_0^\pi f_2(x, t) \, dx = 0.
\]
On the other hand, noting that, for $u_1\left(x, \frac{j\pi}{3}\right) \geq 0$ and $u_2\left(x, \frac{j\pi}{2}\right) \geq 0$,
\[
\int_{\Omega} p_1\left(x, \frac{j\pi}{3}, u_1\left(x, \frac{j\pi}{3}\right)\right) \, dx = \int_0^\pi \left(\frac{j\pi}{3}\right)^{-3} u_1\left(x, \frac{j\pi}{3}\right) \cos \frac{x}{3} \, dx
\]
and
\[
\int_{\Omega} p_2\left(x, \frac{j\pi}{3}, u_2\left(x, \frac{j\pi}{2}\right)\right) \, dx = \int_0^\pi \left(\frac{j\pi}{3}\right)^{-7} u_2\left(x, \frac{j\pi}{3}\right) \cos \frac{x}{2} \, dx
\]
we obtain
\[
\alpha_1j = \left(\frac{j\pi}{3}\right)^{-3}, \quad \alpha_2j = \left(\frac{j\pi}{3}\right)^{-7}, \quad j \in I_\infty.
\]
Hence
\[
\sum_{j=1}^{\infty} \alpha_1j = \sum_{j=1}^{\infty} \left(\frac{j\pi}{3}\right)^{-3} < \infty,
\]
\[
\liminf_{t \to \infty} \frac{\int_T^t \prod_{s < t_j < t} (1 + \alpha_{1j}) H_1(s) \, ds}{\prod_{T < t_j < t} (1 + \alpha_{1j})} = -\infty
\]
and
\[
\limsup_{t \to \infty} \frac{\int_T^t \prod_{s < t_j < t} (1 + \alpha_{1j}) H_1(s) \, ds}{\prod_{T < t_j < t} (1 + \alpha_{1j})} = \infty,
\]
where $j \in I_\infty$, $T$ is a sufficiently large positive number.

Therefore, all the conditions of Theorem 3.2 are fulfilled. Then every solution of problem (34), (35) oscillates in $(0, \pi) \times [0, \infty)$. 
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References