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The Cantor function

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Abstract

This is an attempt to give a systematic survey of properties of the famous Cantor ternary function.

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1. Introduction

The Cantor function G was defined in Cantor's paper [10] dated November 1883, the first known appearance of this function. In [10], Georg Cantor was working on extensions of the Fundamental Theorem of Calculus to the case of discontinuous functions and G serves as a counterexample to some Harnack's affirmation about such extensions [33, p. 60]. The interesting details from the early history of the Cantor set and Cantor function can be found in Fleron's note [28]. This function was also used by H. Lebesgue in his famous "Leçons sur l'intégration et la recherche des fonctions primitives" (Paris, Gauthier-Villars, 1904). For this reason G is sometimes referred to as the Lebesgue function. Some interesting function

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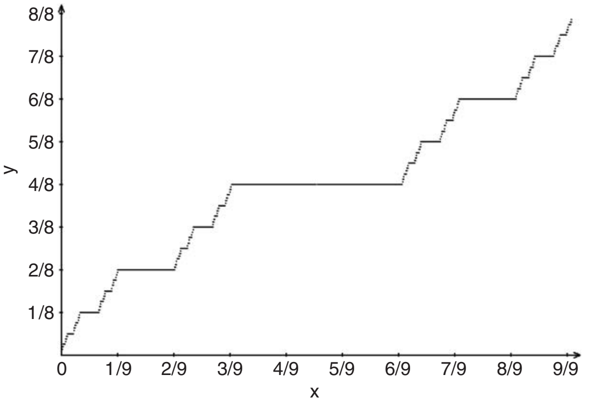

Fig. 1. The graph of the Cantor function G. This graph is sometimes called "Devil's staircase".

classes, motivated by the Cantor function, have been introduced to modern real analysis, see, for example, [8] or [53, Definition 7.31]. There exist numerous generalizations of G which are obtained as variations of Cantor's constructions but we do not consider these in our work. Since G is a distribution function for the simplest nontrivial self-similar measure, fractal geometry has shown new interest in the Cantor function (Fig. 1).

We recall the definitions of the ternary Cantor function G and Cantor set C. Let $x \in [0, 1]$ and expand x as

$$x = \sum_{n=1}^{\infty} \frac{a_{nx}}{3^n}, \quad a_{nx} \in \{0, 1, 2\}. \tag{1.1}$$

Denote by N_x the smallest n with $a_{nx} = 1$ if it exists and put $N_x = \infty$ if there is no such a_{nx}. Then the Cantor function $G : [0, 1] \to \mathbb{R}$ can be defined as

$$G(x) := \frac{1}{2^{N_x}} + \frac{1}{2} \sum_{n=1}^{N_x - 1} \frac{a_{nx}}{2^n}. \tag{1.2}$$

Observe that it is independent of the choice of expansion (1.1) if x has two ternary representations.

The Cantor set C is the set of all points from $[0, 1]$ which have expansion (1.1) using only the digits 0 and 2. In the case $x \in C$ ($a_{nx} \in \{0, 2\}$) the equality (1.2) takes the form

$$G(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_{nx}}{2^n}. \tag{1.3}$$

The following classical generative construction for the triadic Cantor set C is more popular.

Starting with the interval $C_0 := [0, 1]$ define closed subsets $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k \supseteq \cdots$ in C_0 as follows. We obtain the set C_1 by the removing the "middle third" interval open

$(\frac{1}{2}, \frac{2}{3})$ from C_0. Then the set C_2 is obtained by removing from C_1 the open intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, C_k consists of 2^k disjoint closed intervals and, having C_k, C_{k+1} is obtained by removing middle thirds from each of the intervals that make up C_k. Then it is easy to see that

$$C = \bigcap_{k=0}^{\infty} C_k. \tag{1.4}$$

Denote by C^1 the set of all endpoints of complementary intervals of C and set

$$C^{\circ} := C \backslash C^1, \quad I^{\circ} := [0, 1] \backslash C. \tag{1.5}$$

Let also $\mathscr{I}$ be a family of all components of the open set I°.

2. Singularity, measurability and representability by absolutely continuous functions

The well-known properties of the Cantor function are collected in the following.

Proposition 2.1.

2.1.1. *G is continuous and increasing but not absolutely continuous.*
2.1.2. *G is constant on each interval from I°.*
2.1.3. *G is a singular function.*
2.1.4. *G maps the Cantor set C onto* $[0, 1]$.

Proof. It follows directly from (1.2) that G is an increasing function, and moreover (1.2) implies that G is constant on every interval $J \subseteq I^{\circ}$. Observe also that if $x, y \in [0, 1]$, $x \neq y$, and x tends to y, then we can take ternary representations (1.1) so that

$$\min\{n : |a_{nx} - a_{ny}| \neq 0\} \to \infty.$$

Thus the continuity of G follows from (1.2) as well. Since the one-dimensional Lebesgue measure of C is zero,

$$m_1(C) = 0,$$

the monotonicity of G and constancy of G on each interval $J \subseteq I^{\circ}$ imply Statement 2.1.3. One easy way to check the last equality is to use the obvious recurrence relation

$$m_1(C_k) = \tfrac{2}{3} m_1(C_{k-1})$$

for closed subset C_k in (1.4). Really, by (1.4)

$$m_1(C) = \lim_{k\to\infty} m_1(C_k) = \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0.$$

It still remains to note that by (1.3) we have 2.1.4 and that G is not absolutely continuous, since G is singular and nonconstant. $\square$

Remark 2.2. Recall that a monotone or bounded variation function f is called singular if $f' = 0$ a.e.

By the Radon–Nikodym theorem we obtain from 2.1.1 the following.

Proposition 2.3. *The function G cannot be represented as*

$$G(x) = \int_0^x \varphi(t)\,\mathrm{d}t,$$

where φ is a Lebesgue integrable function.

In general, a continuous function need not map a measurable set onto a measurable set. It is a consequence of 2.1.4 that the Cantor function is such a function.

Proposition 2.4. *There is a Lebesgue measurable set $A \subseteq [0, 1]$ such that $G(A)$ is not Lebesgue measurable.*

In fact, a continuous function $g : [a, b] \to \mathbb{R}$ transforms every measurable set onto a measurable set if and only if g satisfies Lusin's condition (N):

$$(m_1(E) = 0) \Rightarrow (m_1(g(E)) = 0)$$

for every $E \subseteq [a, b]$ [48, p. 224].

Let L_f denote the set of points of constancy of a function f, i.e., $x \in L_f$ if f is constant in a neighborhood of x. In the case $f = G$ it is easy to see that $L_G = I^\circ = [0, 1] \backslash C$.

Proposition 2.5. *Let $f : [a, b] \to \mathbb{R}$ be a monotone continuous function. Then the following statements are equivalent*:

2.5.1. *The inverse image $f^{-1}(A)$ is a Lebesgue measurable subset of $[a, b]$ for every $A \subseteq \mathbb{R}$.*

2.5.2. *The Lebesgue measure of the set $[a, b] \backslash L_f$ is zero,*

$$m_1([a, b] \backslash L_f) = 0. \tag{2.6}$$

Proof. 2.5.2 $\Rightarrow$ 2.5.1. Evidently, L_f is open (in the relative topology of $[a, b]$) and f is constant on each component of L_f. Let E be a set of endpoints of components of L_f. Let us denote by f_0 and f_1 the restrictions of f to $L_f \cup E$ and to $[a, b] \backslash (L_f \cup E)$, respectively.

$$f_0 := f|_{L_f \cup E}, \quad f_1 := f|_{[a,b] \backslash (L_f \cup E)}.$$

If A is an arbitrary subset of $\mathbb{R}$, then it is easy to see that $f_0^{-1}(A)$ is a F_σ subset of $[a, b]$ and

$$f_1^{-1}(A) \subseteq [a, b] \backslash L_f.$$

Since $f^{-1}(A) = f_0^{-1}(A) \cup f_1^{-1}(A)$, the equality $m_1([a, b] \backslash L_f) = 0$ implies that $f^{-1}(A)$ is Lebesgue measurable as the union of two measurable sets.

2.5.1 ⇒ 2.5.2. The monotonicity of *f* implies that f_1 is one-to-one and

$$(f_0(L_f \cup E)) \cap (f_1([a, b] \backslash (L_f \cup E))) = \emptyset. \tag{2.7}$$

Suppose that (2.6) does not hold. For every $B \subseteq \mathbb{R}$ with an outer measure $m_1^*(B) > 0$ there exists a nonmeasurable set $A \subseteq B$. See, for instance, [45, Chapter 5, Theorem 5.5]. Thus, there is a nonmeasurable set $A \subseteq [a, b] \backslash (L_f \cup E)$. Since f_1 is one-to-one, equality (2.7) implies that $f^{-1}(f(A)) = A$, contrary to 2.5.1. □

Corollary 2.8. *The inverse image* $G^{-1}(A)$ *is a Lebesgue measurable subset of* [0, 1] *for every* $A \subseteq \mathbb{R}$.

Remark 2.9. It is interesting to observe that $G(A)$ is a Borel set for each Borel set $A \subseteq [0, 1]$. Indeed, if $f : [a, b] \to \mathbb{R}$ is a monotone function with the set of discontinuity *D*, then

$$f(A) = f(A \cap D) \cup f(A \cap L_f) \cup f(A \cap E) \cup f(A \backslash (D \cup L_f \cup E)), \quad A \subseteq [a, b],$$

where E is the set of endpoints of components of L_f. The sets $f(A \cap D)$, $f(A \cap L_f)$ and $f(A \cap E)$ are at most countable for all $A \subseteq [0, 1]$. If A is a Borel subset of $[a, b]$, then $f(A \backslash (D \cup L_f \cup E))$ is Borel, because it is the image of the Borel set under the homeomorphism $f|_{[a,b] \backslash (D \cup L_f \cup E)}$.

A function $f : [a, b] \to \mathbb{R}$ is said to satisfy the Banach condition (T_1) if

$$m_1(\{y \in \mathbb{R} : \mathrm{card}(f^{-1}(y)) = \infty\}) = 0.$$

Since a restriction $G|_{C^\circ}$ is one-to-one and $G(I^\circ)$ is a countable set, 2.1.2 implies

Proposition 2.10. *G satisfies the condition* (T_1).

Bary and Menchoff [3] showed that a continuous function $f : [a, b] \to \mathbb{R}$ is a superposition of two absolutely continuous functions if and only if *f* satisfies both the conditions (T_1) and (N). Moreover, if *f* is differentiable at every point of a set which has positive measure in each interval from $[a, b]$, then *f* is a sum of two superpositions,

$$f = f_1 \circ f_2 + f_3 \circ f_4,$$

where f_i, $i = 1, \ldots, 4$, are absolutely continuous [2]. Thus, we have

Proposition 2.11. *There are absolutely continuous functions* $f_1, \ldots, f_4$ *such that*

$$G = f_1 \circ f_2 + f_3 \circ f_4,$$

but G is not a superposition of any two absolutely continuous functions.

Remark 2.12. A superposition of any finite number of absolutely continuous functions $f_1 \circ f_2 \circ \cdots \circ f_n$ is always representable as $q_1 \circ q_2$ with two absolutely continuous q_1, q_2. Every continuous function is the sum of three superpositions of absolutely continuous functions [2]. An application of Proposition 2.10 to the nondifferentiability set of the Cantor function will be formulated in Proposition 8.1.

3. Subadditivity, the points of local convexity

An extended Cantor's ternary function $\hat{G}$ is defined as follows

$$\hat{G}(x) = \begin{cases} 0 & \text{if } x < 0, \\ G(x) & \text{if } 0 \leqslant x \leqslant 1, \\ 1 & \text{if } x > 1. \end{cases}$$

Proposition 3.1. *The extended Cantor function $\hat{G}$ is subadditive, that is*

$$\hat{G}(x+y) \leqslant \hat{G}(x) + \hat{G}(y)$$

for all $x, y \in \mathbb{R}$.

This proposition implies the following corollary.

Proposition 3.2. *The Cantor function G is a first modulus of continuity of itself, i.e.,*

$$\sup_{\substack{|x-y| \leqslant \delta \\ x,y \in [0,1]}} |G(x) - G(y)| = G(\delta)$$

for every $\delta \in [0, 1]$.

The proof of Propositions 3.1 and 3.2 can be found in Timan's book [54, Section 3.2.4] or in the paper of Doboš [18].

It is well-known that a function $\omega : [0, \infty) \to [0, \infty)$ is the first modulus of continuity for a continuous function $f : [a, b] \to \mathbb{R}$ if and only if ω is increasing, continuous, subadditive and $\omega(0) = 0$ holds.

A particular way to prove the subadditivity for an increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is to show that the function $\delta^{-1}\omega(\delta)$ is decreasing [54, Section 3.2.3]. The last condition holds true in the case of concave functions. The following propositions show that this approach is not applicable for the Cantor function.

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Let us say that f is *locally concave–convex* at point $x \in [a, b]$ if there is a neighborhood U of the point x for which $f|_U$ is either convex or concave. Similarly, a continuous function $f : (a, b) \to \mathbb{R}$ is said to be *locally monotone* at $x \in (a, b)$ if there is an open neighborhood U of x such that $f|_U$ is monotone.

Proposition 3.3. *The function $x^{-1}G(x)$ is locally monotone at point x_0 if and only if $x_0 \in I^\circ$.*

Proposition 3.4. *$G(x)$ is locally concave–convex at x_0 if and only if $x_0 \in I^\circ$.*

Propositions 3.3 and 3.4 are particular cases of some general statements.

Theorem 3.5. *Let $f : [a, b] \to \mathbb{R}$ be a monotone continuous function. Suppose that L_f is an everywhere dense subset of $[a, b]$. Then f is locally concave–convex at $x \in [a, b]$ if and only if $x \in L_f$.*

For the proof we use of the following.

Lemma 3.6. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then $[a, b]\backslash L_f$ is a compact perfect set.*

Proof. By definition L_f is relatively open in $[a, b]$. Hence $[a, b]\backslash L_f$ is a compact subset of $\mathbb{R}$. If p is an isolated point of $[a, b]\backslash L_f$, then either it is a common endpoint of two intervals I, J which are components of L_f or $p \in \{a, b\}$. In the first case it follows by continuity of f that $I \cup \{p\} \cup J \subseteq L_f$. That contradicts to the maximality of the connected components I, J. The second case is similar. □

Proof of Theorem 3.5. It is obvious that f is locally concave–convex at x for $x \in L_f$. Suppose now that $x \in [a, b]\backslash L_f$ and f is concave–convex in a neighborhood U_0 of the point x. By Lemma 3.6 x is not an isolated point of $[a, b]\backslash L_f$. Hence, there exist y, z in $[a, b]\backslash L_f$ and $\delta > 0$ such that

$$z < y, \quad (z - \delta, y + \delta) \subset U_0 \cap [a, b], \quad (z, y) \subset L_f.$$

Since $[a, b]\backslash L_f$ is perfect, we can find z_0, y_0 for which

$$z_0 \in (z - \delta, z) \cap ([a, b]\backslash L_f), \quad y_0 \in (y, y + \delta) \cap ([a, b]\backslash L_f).$$

Denote by l_z and l_y be the straight lines which pass through the points $(z_0, f(z_0))$, $(z + y/2, f(z + y/2))$ and $(y_0, f(y_0))$, $(z + y/2, f(z + y/2))$, respectively. Suppose that f is increasing. Then the point $(z, f(z))$ lies over l_z but $(y, f(y))$ lies under l_y (See Fig. 2). Hence, the restriction $f|_{(z_0, y_0)}$ is not concave–convex. This contradiction proves the theorem since the case of a decreasing function f is similar. □

Theorem 3.7. *Let $f : (a, b) \to [0, \infty)$ be an increasing continuous function and let $\varphi : (a, b) \to [0, \infty)$ be a strictly decreasing function with finite derivative $\varphi'(x)$ at every $x \in (a, b)$. If*

$$m_1(L_f) = |b - a|, \tag{3.8}$$

then the product $f \cdot \varphi$ is locally monotone at a point $x \in (a, b)$ if and only if $x \in L_f$.

For a function $f : (a, b) \to \mathbb{R}$ and $x \in (a, b)$ set

$$\mathrm{SD} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

provided that the limit exists, and write

$$V_f := \{x \in (a, b) : \mathrm{SD}\, f(x) = +\infty\}.$$

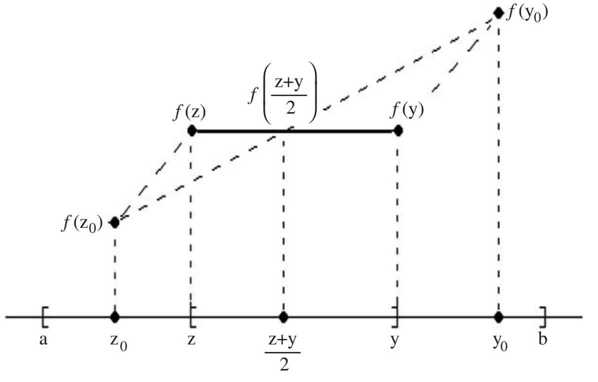

Fig. 2. The set of points of constancy is the same as the set of points of convexity.

Using the techniques of differentiation of Radon measures (see [23, in particular, Section 1.6, Lemma 1]), we can prove the following.

Lemma 3.9. *Let* $f : (a, b) \to \mathbb{R}$ *be continuous increasing function. If equality* (3.8) *holds, then* V_f *is a dense subset of* $(a, b)\backslash L_f$.

Proof of Theorem 3.7. It is obvious that $\varphi \cdot f$ is locally monotone at x_0 for every $x_0 \in L_f$. Suppose that $\varphi \cdot f$ is monotone on an open interval $J \subset (a, b)$ and $x_0 \in ((a, b)\backslash L_f) \cap J$. Since L_f is an everywhere dense subset of (a, b), there exists an interval $J_1 \subseteq J \cap L_f$. Then $\varphi \cdot f|_{J_1}$ is a decreasing function. Consequently, $\varphi \cdot f|_J$ is decreasing too. By Lemma 3.9 we can select a point $t_0 \in V_f \cap J$. Hence, by the definition of V_f

$$\mathrm{SD}\, \varphi(x) f(x)|_{x=t_0} = \varphi'(t_0) f(t_0) + \varphi(t_0) \mathrm{SD}\, f(t_0) = +\infty.$$

This is a contradiction, because $\varphi \cdot f|_J$ is decreasing. □

Remark 3.10. Density of L_f is essential in Theorem 3.5. Indeed, if A is a closed subset of $[a, b]$ with nonempty interior Int A, then it is easy to construct a continuous increasing function f such that $L_f = [a, b]\backslash A$ and f is linear on each interval which belongs to Int A. Theorem 3.7 remains valid if the both functions f and φ are negative, but if f and φ have different signs, then the product $f \cdot \varphi$ is monotone on (a, b). Functions having a dense set of constancy have been investigated by Bruckner and Leonard in [8]. See also Section 7 in the present work.

4. Characterizations by means of functional equations

There are several characterizations of the Cantor function G based on the self-similarity of the Cantor ternary set. We start with an iterative definition for G.

Define a sequence of functions $\psi_n : [0, 1] \to \mathbb{R}$ by the rule

$$\psi_{n+1}(x) = \begin{cases} \frac{1}{2}\,\psi_n(3x) & \text{if } 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}\,\psi_n(3x-2) & \text{if } \frac{2}{3} \leqslant x \leqslant 1, \end{cases} \tag{4.1}$$

where $\psi_0 : [0, 1] \to \mathbb{R}$ is an arbitrary function. Let $\mathscr{M}[0, 1]$ be the Banach space of all uniformly bounded real-valued functions on $[0, 1]$ with the supremum norm.

Proposition 4.2. *The Cantor function G is the unique element of $\mathscr{M}[0, 1]$ for which*

$$G(x) = \begin{cases} \frac{1}{2}\,G(3x) & \text{if } 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{1}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2}\,G(3x-2) & \text{if } \frac{2}{3} \leqslant x \leqslant 1. \end{cases} \tag{4.3}$$

If $\psi_0 \in \mathscr{M}[0, 1]$, then the sequence $\{\psi_n\}_{n=0}^{\infty}$ converges uniformly to G.

Proof. Define a map $F : \mathscr{M}[0, 1] \to \mathscr{M}[0, 1]$ as

$$F(f)(x) = \begin{cases} \frac{1}{2} f(3x), & 0 \leqslant x \leqslant \frac{1}{3}, \\ \frac{1}{2}, & \frac{1}{3} < x < \frac{2}{3}, \\ \frac{1}{2} + \frac{1}{2} f(3x-2), & \frac{2}{3} \leqslant x \leqslant 1. \end{cases}$$

Since

$$\|F(f_1) - F(f_2)\| \leqslant \tfrac{1}{2}\,\|f_1 - f_2\|,$$

where $f_1, f_2 \in \mathscr{M}[0, 1]$ and $\|\cdot\|$ denotes the norm in $\mathscr{M}[0, 1]$, F is a contraction map on complete space $\mathscr{M}[0, 1]$ and, consequently, by the Banach theorem F has an unique fixed point f_0, $f_0 = F(f_0)$, and that $\psi_n \to f_0$ uniformly on $[0, 1]$. It follows from the definition of G, that (4.3) holds. Hence, $F(G) = G$ and by uniqueness $f_0 = G$. □

It should be observed here that there exist several iterative definitions for the Cantor ternary function G. The above method is a simple modification of the corresponding one from Doboš's article [18]. It is interesting to compare Proposition 4.2 with the self-similarity property of the Cantor set C.

Let for $x \in \mathbb{R}$

$$\varphi_0(x) := \tfrac{1}{3}\,x, \quad \varphi_1(x) := \tfrac{1}{3}\,x + \tfrac{2}{3}. \tag{4.4}$$

Proposition 4.5. *The Cantor set C is the unique nonempty compact subset of $\mathbb{R}$ for which*

$$C = \varphi_0(C) \cup \varphi_1(C) \tag{4.6}$$

holds. Further if F is an arbitrary nonempty compact subset of $\mathbb{R}$, *then the iterates*

$$\Phi^{k+1}(F) := \varphi_0(\Phi^k(F)) \cup \varphi_1(\Phi^k(F)), \quad \Phi^0(F) := F,$$

convergence to the Cantor set C in the Hausdorff metric as $k \to \infty$.

Remark 4.7. This theorem is a particular case of a general result by Hutchinson about a compact set which is invariant with respect to some finite family contraction maps on $\mathbb{R}^n$ [36].

In the case of a two-ratio Cantor set a system which is similar to (4.3) was found by Coppel in [12]. The next theorem follows from Coppel's results.

Proposition 4.8. *Every real-valued $F \in \mathscr{M}[0,1]$ that satisfies*

$$F\left(\frac{x}{3}\right) = \frac{F(x)}{2}, \tag{4.9}$$

$$F\left(1 - \frac{x}{3}\right) = 1 - \frac{F(x)}{2}, \tag{4.10}$$

$$F\left(\frac{1}{3} + \frac{x}{3}\right) = \frac{1}{2} \tag{4.11}$$

is the Cantor ternary function.

The following simple characterization of the Cantor function G has been suggested by Chalice in [11].

Proposition 4.12. *Every real-valued increasing function $F : [0,1] \to \mathbb{R}$ that satisfying* (4.9) *and*

$$F(1-x) = 1 - F(x) \tag{4.13}$$

is the Cantor ternary function.

Remark 4.14. Chalice used the additional condition $F(0) = 0$, but it follows from (4.9).

The functional equations, given above, together with Proposition 4.5 are sometimes useful in applications. As examples, see Proposition 5.5 in Section 5 and Proposition 6.1 in Section 6.

The system (4.9) + (4.13) is a particular case of the system that was applied in the earlier paper of Evans [22] to the calculation of the moments of some Cantor functions. In this interesting paper Evans noted that (4.9) and (4.13) together with continuity do not determine G uniquely. However, they imply that

$$F(x) = \tfrac{1}{2} + F\left(x - \tfrac{2}{3}\right), \quad \tfrac{2}{3} \leqslant x \leqslant 1. \tag{4.15}$$

Next we show that a variational condition, together with (4.9) and (4.13), determines the Cantor function G (Fig. 3).

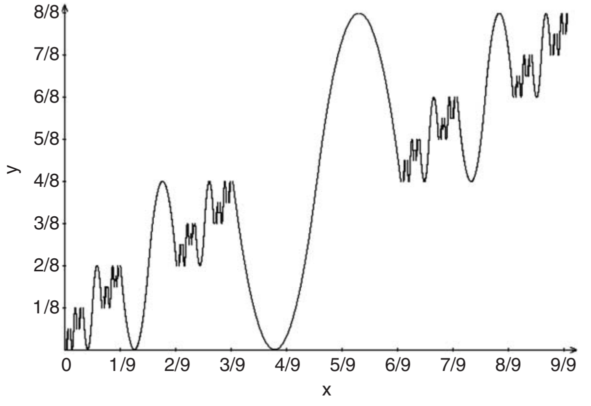

Fig. 3. The continuous function F which satisfies "the two Cantor function equations" (4.9) and (4.13) but $F(x) = \frac{1}{2}(12x-5)^2$ on $\left[\frac{1}{3}, \frac{1}{2}\right]$.

Proposition 4.16. *Let* $F : [0,1] \to \mathbb{R}$ *be a continuous function satisfying* (4.9) *and* (4.13). *Then for every* $p \in (1, +\infty)$ *it holds*

$$\int_0^1 |F(x)|^p \, \mathrm{d}x \geqslant \int_0^1 |G(x)|^p \, \mathrm{d}x \tag{4.17}$$

and if for some $p \in (1, \infty)$ *an equality*

$$\int_0^1 |F(x)|^p \, \mathrm{d}x = \int_0^1 |G(x)|^p \, \mathrm{d}x \tag{4.18}$$

holds, then $G = F$.

Lemma 4.19. *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function and let* $p \in (1, \infty)$. *Suppose that the graph of f is symmetric with respect to the point* $(a+b/2;\ f\ (a+b/2))$. *Then the inequality*

$$\frac{1}{|b-a|} \int_a^b |f(x)|^p \, \mathrm{d}x \geqslant \left| f\left(\frac{a+b}{2}\right) \right|^p \tag{4.20}$$

holds with equality only for

$$f(x) \equiv f\left(\frac{a+b}{2}\right).$$

Proof. We may assume without loss of generality that $a = -1$, $b = 1$. Now for $f(0) = 0$ inequality (4.20) is trivial. Hence replacing f with $-f$, if necessary, we may assume that $f(0) < 0$. Write

$$\Psi(x) = f(x) - f(0).$$

Given $m > 0$, we decompose $[-1, 1]$ as

$$A^+ := \{x \in [-1, 1] : |\Psi(x)| > m\}, \quad A^0 := \{x \in [-1, 1] : |\Psi(x)| = m\},$$
$$A^- := \{x \in [-1, 1] : |\Psi(x)| < m\}.$$

Set $J_m := \int_a^b |\Psi(x) - m|^p \, dx$. Since Ψ is an odd function, we obtain

$$\begin{aligned} J_m = &\int_{A^+\cap[0,1]} [(|\Psi(x)| - m)^p + (|\Psi(x)| + m)^p] \, dx + \int_{A^0\cap[0,1]} (2m)^p \, dx \\ &+ \int_{A^-\cap[0,1]} [(m - |\Psi(x)|)^p + (m + |\Psi(x)|)^p] \, dx := J_m^+ + J_m^0 + J_m^-. \end{aligned}$$

Elementary calculation shows that the function

$$g(y) := (B - y)^p + (B + y)^p$$

is strictly increasing on $(0, B)$ for $p > 1$. Hence,

$$J_m^+ \geqslant 2 \int_{A^+\cap[0,1]} |\Psi(x)|^p \, dx = \int_{A^+} |\Psi(x)|^p \, dx \geqslant \int_{A^+} m^p \, dx. \tag{4.21}$$

If $\Psi(x) \not\equiv 0$, then A^- is nonempty and open, as $\Psi(0) = 0$. Thus,

$$J_m^- > 2 \int_{A^-\cap[0,1]} m^p \, dx = \int_{A^-} m^p \, dx \tag{4.22}$$

for $\Psi(x) \not\equiv 0$. Moreover, it is obvious that

$$J_m^0 = 2^{p-1} \left(2 \int_{A^0\cap[0,1]} m^p \, dx \right) \geqslant \int_{A^0} m^p \, dx. \tag{4.23}$$

Choose $m = -f(0)$. Then, in the case where $f(x) \not\equiv f(0)$, from (4.21) to (4.23) we obtain the required inequality

$$\frac{1}{2} \int_{-1}^{1} |f(x)|^p > |f(0)|^p. \qquad \square$$

Remark 4.24. Inequality (4.20) holds also for $p = 1$, but, as simple examples show, in this case the equality in (4.20) is also attained by functions different from the constant function.

For every interval $J \in \mathscr{I}$ (where $\mathscr{I}$ is a family of components of the open set $I^\circ = [0, 1]\backslash C$) let us denote by x_J the center of J and by G_J the value of the Cantor function G at x_J.

Lemma 4.25. *If* $F : [0, 1] \to \mathbb{R}$ *is a continuous function satisfying* (4.9) *and* (4.13), *then the graph of the restriction* $F|_J$ *is symmetric with respect to the point* (x_J, G_J) *for each* $J \in \mathscr{I}$.

Proof. It follows from (4.13) that

$$F\left(\tfrac{1}{2}\right) = G\left(\tfrac{1}{2}\right) = \tfrac{1}{2},$$

thus we can rewrite equation (4.13) as

$$F\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\left(F\left(\tfrac{1}{2}+x\right) + F\left(\tfrac{1}{2}-x\right)\right).$$

Consequently a graph of $F|_{\left(\frac{1}{3},\frac{2}{3}\right)}$ is symmetric with respect to the point $\left(\frac{1}{2},\frac{1}{2}\right)$. If J is an arbitrary element of $\mathscr{I}$, then there is a finite sequence of contractions $\varphi_{i1},\dots,\varphi_{in}$ such that J is an image of the interval $\left(\frac{1}{3},\frac{2}{3}\right)$ under superposition $\varphi_{i1}\circ\dots\circ\varphi_{in}$ and each $\varphi_{ik}, k=1,\dots,n$ belongs to $\{\varphi_0,\varphi_1\}$ (see formula (4.4)). Now, the desired symmetry follows from (4.9) and (4.15) by induction. □

Proof of Proposition 4.16. Let $J \in \mathscr{I}$. If F satisfies (4.9) and (4.13), then Lemma 4.25 and Lemma 4.19 imply that

$$\int_J |F(x)|^p \,\mathrm{d}x \geqslant \int_J |G(x)|^p \,\mathrm{d}x \tag{4.26}$$

for every $p \in (1,\infty)$. Thus, we obtain (4.17) from the condition $m_1(C)=0$. Suppose now that (4.18) holds. Then we have the equality in (4.26) for every $J \in \mathscr{I}$. Thus, by Lemma 4.19

$$F|_J = G|_J, \quad J \in \mathscr{I}.$$

Since $I^0 = \bigcup_{J\in I} J$ is a dense subset of [0, 1] and $F = G$ as required. □

In the special case $p=2$ we can use an orthogonal projection to prove Proposition 4.16.

Define a subspace $L^2_C[0,1]$ of the Hilbert space $L^2[0,1]$ by the rule: $f \in L^2_C[0,1]$ if $f \in L^2[0,1]$ and for every $J \in \mathscr{I}$ there is a constant C_J such that

$$\int_J |f(x) - C_J|^2 \,\mathrm{d}x = 0.$$

It is obvious that $G \in L^2_C[0,1]$.

Let us denote by P_C the operator of the orthogonal projection from $L^2[0,1]$ to $L^2_C[0,1]$.

Proposition 4.27. *Let f be an arbitrary function in $L^2[0,1]$ and let $J \in \mathscr{I}$ be an interval with endpoints a_J, b_J.*

4.27.1. *Suppose that f is continuous. Then the image $P_C(f)$ is continuous if and only if*

$$f(a_J) = f(b_J) = \frac{1}{|b_J - a_J|}\int_J f(x)\,\mathrm{d}x$$

for each $J \in \mathscr{I}$.

4.27.2. *Function* $f \in L^2[0, 1]$ *is a solution of the equation* $P_C(f) = G$ *if and only if*

$$\int_J f(x)\,\mathrm{d}x = \int_J G(x)\,\mathrm{d}x$$

for each $J \in \mathscr{I}$.

We leave the verification of this simple proposition to the reader.

Now, the conclusion of Proposition 4.16 follows directly from Lemma 4.25 by usual properties of orthogonal projections. See, for example, [5, Chapter VII, 9].

5. The Cantor function as a distribution function

It is well known that there exists an one-to-one correspondence between the set of the Radon measures μ in $\mathbb{R}$ and the set of a finite valued, increasing, right continuous functions F on $\mathbb{R}$ with $\lim_{x\to-\infty} F(x) = 0$. Here we study the corresponding measure for the Cantor function.

Let $\varphi_0, \varphi_1 : \mathbb{R} \to \mathbb{R}$ be similarity contractions defined by formula (4.4). Write

$$s_c := \frac{\lg 2}{\lg 3}. \tag{5.1}$$

We let $\mathscr{H}^{s_c}$ denote the s_c-dimensional Hausdorff measure in $\mathbb{R}$. See [24] for properties of Hausdorff measures.

Proposition 5.2. *There is the unique Borel regular probability measure* μ *such that*

$$\mu(A) = \tfrac{1}{2}\,\mu(\varphi_0^{-1}(A)) + \tfrac{1}{2}\,\mu(\varphi_1^{-1}(A)) \tag{5.3}$$

for every Borel set $A \subseteq \mathbb{R}$. *Furthermore, this measure* μ *coincides with the restriction of the Hausdorff measure* $\mathscr{H}^{s_c}$ *to* C, *i.e.*,

$$\mu(A) = \mathscr{H}^{s_c}(A \cap C) \tag{5.4}$$

for every Borel set $A \subseteq \mathbb{R}$.

For the proof see [24, Theorem 2.8, Lemma 6.4].

Proposition 5.5. *Let* m_1 *be the Lebesgue measure on* $\mathbb{R}$. *Then*

$$m_1(G(A)) = \mathscr{H}^{s_c}(A) \tag{5.6}$$

for every Borel set $A \subseteq C$.

Proof. Write

$$\mu(A) := m_1(G(A \cap C))$$

for every Borel set $A \subseteq \mathbb{R}$. If suffices to show that μ is a Borel regular probability measure which fulfils (5.3). Since C^1 is countable (see formula (1.5)), $m_1(G(C^1)) = 0$. Hence, we have

$$\mu(A) = m_1(G(C^\circ \cap A)), \quad A \subseteq \mathbb{R}.$$

The restriction $G|_{C^\circ} : C^\circ \to G(C^\circ)$ is a homeomorphism and

$$\mu(\mathbb{R}) = m_1(G(C^\circ)) = 1.$$

Hence, μ is a Borel regular probability measure. (Note that $G(A)$ is a Borel set for each Borel set $A \subseteq C$. See Remark 2.9.) To prove (5.3) we will use following functional equations (see (4.9), (4.15)).

$$G\left(\frac{x}{3}\right) = \frac{1}{2}G(x), \quad 0 \leqslant x \leqslant 1, \tag{5.7}$$

$$G(x) = \tfrac{1}{2} + G\left(x - \tfrac{2}{3}\right), \quad \tfrac{2}{3} \leqslant x \leqslant 1. \tag{5.8}$$

Let A be a Borel subset of $\mathbb{R}$. Put

$$A_0 := A \cap \left[0, \tfrac{1}{3}\right], \quad A_1 := A \cap \left[\tfrac{2}{3}, 1\right].$$

Since $\varphi_0^{-1}(x) = 3x$ and $\varphi_1^{-1}(x) = 3\left(x - \frac{2}{3}\right)$, we have

$$\mu(\varphi_i^{-1}(A)) = \mu(\varphi_i^{-1}(A_i))$$

for $i = 1, 2$. It follows from (5.7) that

$$2G(x) = G(\varphi_0^{-1}(x))$$

for $x \in \left[0, \frac{1}{3}\right]$. Hence,

$$\tfrac{1}{2}\mu(\varphi_0^{-1}(A_0)) = \tfrac{1}{2}m_1(G(\varphi_0^{-1}(A_0))) = \tfrac{1}{2}m_1(2G(A_0)) = m_1(G(A_0)).$$

Similarly, (5.7) implies that

$$2G\left(x - \tfrac{2}{3}\right) = G(\varphi_1^{-1}(x))$$

for $x \in \left[\frac{2}{3}, 1\right]$, and we obtain

$$\begin{aligned}\tfrac{1}{2}\mu(\varphi_1^{-1}(A_1)) &= \tfrac{1}{2}m_1(G(\varphi_1^{-1}(A_1)))\\ &= \tfrac{1}{2}m_1\left(2G\left(A_1 - \tfrac{2}{3}\right)\right) = m_1\left(G\left(A_1 - \tfrac{2}{3}\right)\right).\end{aligned}$$

Now using (5.8) we get

$$m_1\left(G\left(A_1-\tfrac{2}{3}\right)\right)=m_1\left(-\tfrac{1}{2}+G(A_1)\right)=m(G(A_1)).$$

The set $G(A_1)\cap G(A_0)$ is empty or contains only the point $\frac{1}{2}$. Hence we obtain

$$m_1(G(A_0))+m_1(G(A_1))=m_1(G(A_0\cup A_1))=\mu(A).$$

Therefore $\mu(A)$ satisfies (5.3). $\square$

Corollary 5.9. *The extended Cantor function $\hat{G}$ is the cumulative distribution function of the restriction of the Hausdorff measure $\mathscr{H}^{s_c}$ to the Cantor set C.*

This description of the Cantor function enables us to suggest a method for the proof of the following proposition. Write

$$\hat{G}_h(x) := \hat{G}(x+h)-\hat{G}(x)$$

for each $h\in\mathbb{R}$. The function $\hat{G}_h$ is of bounded variation, because it equals a difference of two increasing bounded functions.

Proposition 5.10 (*Hille and Tamarkin [34]*). *Let* $\mathrm{Var}(\hat{G}_h)$ *be a total variation of* $\hat{G}_h$. *Then we have*

$$\sup_{0\leqslant h\leqslant\delta}\mathrm{Var}(\hat{G}_h)=2$$

for every $\delta>0$.

This holds because sets $C\cap(C\pm 3^{-n})$ have finite numbers of elements for all positive integer n.

Remark 5.11. If F is an absolutely continuous function of bounded variation and $F_h(x) := F(x+h)-F(x)$, then

$$\lim_{h\to 0}\mathrm{Var}(F_h)=0. \tag{5.12}$$

Really, if F is absolutely continuous on $[a,b]$, then F' exists a.e. in $[a,b]$, $F'\in L^1[a,b]$, and

$$\mathrm{Var}(F)=\int_a^b|F'(x)|\,\mathrm{d}x.$$

Approximating of F' by continuous functions, for which the property is obvious, we obtain (5.12).

In fact, Proposition 5.10 remains valid for an arbitrary singular function of bounded variation.

Theorem 5.13. *Let $F : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation with singular part φ. Then the limit relation*

$$\lim_{h\to 0} \sup(\mathrm{Var}(F_h)) = 2\mathrm{Var}(\varphi)$$

holds.

The theorem is an immediate adaptation of the result that was proved by Wiener and Young [57].

6. Calculation of moments and the length of the graph

In the article [22], Evans proved the recurrence relations for the moments

$$\int_0^1 x^n G_\alpha(x)\,\mathrm{d}x,$$

where G_α is a “Cantor function” for a α-middle Cantor set. In the classic middle-third case Evans’s results can be written in the form.

Proposition 6.1. *Let n be a natural number and let M_n be a moment of the form*

$$M_n = \int_0^1 x^n G(x)\,\mathrm{d}x.$$

Then the following relations hold:

$$2M_n = \frac{1}{n+1} + \frac{1}{3^{n+1}-1}\sum_{k=0}^{n-1}\binom{n}{k}2^{n-k}M_k, \tag{6.2}$$

$$(1+(-1)^n)M_n = \frac{1}{n+1} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n}{k}M_k \tag{6.3}$$

for all positive integers n where $\binom{n}{k}$ are the binomial coefficients.

It follows immediately from $G(x)+G(1-x)=1$ that $M_0=\frac{1}{2}$. Hence, (6.2) can be used to compute all moments M_n. Let μ_C be the restriction of the Hausdorff measure $\mathscr{H}^{s_c}$ to the Cantor set C (see Corollary 5.9). Now set

$$m_n := \int_0^1 x^n\,\mathrm{d}\mu_C(x).$$

Proposition 6.4. *The following equality holds*:

$$m_{n+1} = \frac{\sum_{k=0}^{n}\binom{n+1}{k}2^{n-k}m_k}{3^{n+1}-1} \tag{6.5}$$

for every natural n and we have

$$2m_n = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} m_k, \tag{6.6}$$

$$(n+1)m_n = \sum_{k=0}^{n-1} (-1)^k \binom{n+1}{k} m_k \tag{6.7}$$

for every odd n.

Proof. First we prove (6.5).

The Cantor set C can be defined as the intersection $\bigcap_{k=0}^{\infty} C_k$, see (1.4). Each C_k consists of 2^k disjoint closed intervals $C_k^i = [a_k^i, b_k^i]$, $i = 1, \ldots, 2^k$. The length of C_k^i is 3^{-k} and

$$\mu_C(C_k^i) = G(b_k^i) - G(a_k^i) = 2^{-k}.$$

Let L_k be a set of all left-hand points of intervals $C_k^i \subseteq C_k$, i.e.,

$$L_k = \{a_k^i : i = 1, \ldots, 2^k\}.$$

It is easy to see that $L_0 = \{0\}$ and

$$L_k = \left(\tfrac{1}{3} L_{k-1}\right) \cup \left(\tfrac{2}{3} + \left(\tfrac{1}{3} L_{k-1}\right)\right) \tag{6.8}$$

for $k \geqslant 1$. Observe that

$$\int_0^1 x^n \, \mathrm{d}\mu_C(x) = \int_0^1 x^n \, \mathrm{d}G(x)$$

because the integrand is continuous. Hence we have

$$m_k = \lim_{l \to \infty} \sum_{j=1}^{l} (\xi_i)^n [G(x_{j+1}) - G(x_j)],$$

where $0 = x_1 < \cdots < x_l < x_{l+1} = 1$ is any subdivision of [0, 1] with $\max_j (x_{j+1} - x_j) \to 0$ as $l \to \infty$ and $\xi_j \in [x_j, x_{j+1}]$. Subdivide [0, 1] into $l = 3^k$ equal intervals $[x_j, x_{j+1}]$ and take $\xi_j = x_j$. Then

$$m_n = \lim_{k \to \infty} \frac{1}{2^k} \sum_{x \in L_k} x^n.$$

The last relation and (6.8) imply that

$$\begin{aligned} m_n &= \lim_{k\to\infty} \frac{1}{2^k}\left(\sum_{x\in \frac{1}{3} L_{k-1}} x^k + \sum_{x\in \frac{2}{3}+\frac{1}{3} L_{k-1}} x^n\right) \\ &= \frac{1}{2}\cdot\frac{1}{3^n}\lim_{k\to\infty}\left(\frac{1}{2^{k-1}}\sum_{x\in L_{k-1}} x^n\right) + \frac{1}{2}\cdot\frac{1}{3^n}\lim_{k\to\infty}\left(\frac{1}{2^{k-1}}\sum_{x\in L_{k-1}} (x+2)^n\right) \\ &= \frac{1}{2}\cdot\frac{1}{3^n} m_n + \frac{1}{2}\cdot\frac{1}{3^n}\sum_{p=0}^{n}\binom{n}{p} m_p 2^{n-p} \end{aligned}$$

and we obtain (6.5).

In order to prove (6.6) we can use (6.3), because

$$\begin{aligned} m_{n+1} &= \int_0^1 x^{n+1}\,\mathrm{d}G(x) = x^{n+1}G(x)\Big|_0^1 - (n+1)\int_0^1 x^n G(x)\,\mathrm{d}x \\ &= 1-(n+1)M_n. \end{aligned} \tag{6.9}$$

Suppose that n is even, then it follows from (6.3) and (6.9) that

$$\begin{aligned} 2(1-m_{n+1}) = 2(n+1)M_n &= 1+(n+1)\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n}{k}M_k \\ &= 1+\sum_{k=0}^{n-1}(-1)^{k+1}\frac{n+1}{k+1}\binom{n}{k}(1-m_{k+1}). \end{aligned}$$

Hence from

$$\frac{n+1}{k+1}\binom{n}{k} = \binom{n+1}{k+1} \quad \text{and} \quad (1-m_0)=0$$

we get

$$\begin{aligned} 2(1-m_{n+1}) &= 1+\sum_{k=0}^{n-1}(-1)^{k+1}\binom{n+1}{k+1}(1-m_{k+1}) \\ &= 1+\sum_{k=1}^{n}(-1)^{k}\binom{n+1}{k}(1-m_k) = 1+\sum_{k=0}^{n}(-1)^{k}\binom{n+1}{k} \\ &\quad -\sum_{k=0}^{n}(-1)^{k}\binom{n+1}{k}m_k. \end{aligned}$$

Observe that

$$\sum_{k=0}^{n}(-1)^{k}\binom{n+1}{k} = (1-1)^{n+1} - \binom{n+1}{n+1}(-1)^{n+1} = 1$$

because n is even. Consequently,

$$2(1 - m_{n+1}) = 2 - \sum_{k=0}^{n} (-1)^k m_k.$$

The last formula implies (6.6). The proof of (6.7) is analogous to that of (6.6). $\square$

Remark 6.10. The measure μ_C is frequently referred to as the Cantor measure. In the proof of (6.5) we used an idea from Hille and Tamarkin [34]. By (6.5) with $\tau_n = m_n/n!2^n$ we obtain

$$\tau_{n+1} = \frac{1}{2(3^{n+1}-1)} \left(\frac{\tau_n}{1!} + \frac{\tau_{n-1}}{2!} + \cdots + \frac{\tau_0}{(n+1)!} \right).$$

The asymptotics of the moments m_n was determined in [31]. Write

$$v(m) := 1 - \frac{2\pi \mathrm{i} m}{\lg 3}$$

for $m \in \mathbb{Z}$.

Theorem 6.11. *Let m_n be the nth moment of the Cantor measure μ_C, then*

$$m_n \sim 2^{1/2 - 3s_c/2} n^{-s_c} \exp(-2H(n)), \quad n \to \infty, \tag{6.12}$$

where

$$H(x) := \frac{1}{2\pi \mathrm{i}} \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}}^{+\infty\,\prime} \frac{1}{m} 2^{-v(m)} (1 - 2^{1-v(m)}) \zeta(v(m)) \Gamma((v(m)) x^{1-v(m)}$$

and

$$s_c = \frac{\lg 2}{\lg 3}.$$

Note that $H(x)$ is a real-valued function for $x > 0$ and periodic in the variable $\log x$. See also [27] for an example of an absolutely continuous measure with asymptotics of moments containing oscillatory terms.

The behavior of the integrals

$$I(\lambda) := \int_0^1 (G(x))^\lambda \, \mathrm{d}x, \quad E(\lambda) = \int_0^1 \exp(\lambda G(x)) \, \mathrm{d}x \tag{6.13}$$

was described in [32]. It was noted that I extends to a function which is analytic in the half-plane $\mathrm{Re}(\lambda) > -s_c$ and E extends to an entire function. The following theorem is a particular case of the results from [32].

Theorem 6.14.

6.14.1. *For* $\mathrm{Re}(\lambda) > -s_c$, *the function I obeys the formula*

$$(3 \cdot 2^{\lambda} - 1) I(\lambda) = 1 + \int_0^1 (1 + G(x))^{\lambda} \, \mathrm{d}x$$

and for all $\lambda \in \mathbb{C}$ *the function E obeys the formula*

$$3E(2\lambda) = \mathrm{e}^{\lambda} + (\mathrm{e}^{\lambda} + 1) E(\lambda).$$

6.14.2. *For all natural* $n \geqslant 2$, *we have*

$$I(n) = \frac{1}{n+1} - \sum_{k \leqslant n}{}'' \binom{n}{k} \frac{2^{k-1} - 1}{3 \cdot 2^{k-1} - 1} \frac{B(k)}{n - k + 1},$$

where the primes mean that summation is over even positive k and $B(k)$ *are Bernoulli numbers*

6.14.3. *For* $\lambda \to \infty$, *we have*

$$\lambda^{s_c} I(\lambda) = \Phi(\log_2 \lambda) + O(\lambda^{-s_c - 1}),$$
$$\lambda^{s_c} E(-\lambda) = \Phi(\log_2 \lambda) + O(\lambda^{-s_c - 1}),$$

where Φ *is the function analytic in the strip* $|\mathrm{Im}(z)| < \pi/(2 \lg 2)$,

$$\Phi(z) = \sum_{-\infty}^{+\infty} \frac{2}{3 \lg 2} \Gamma\left(s_c + \frac{2\pi \mathrm{i} n}{\lg 2}\right) \zeta\left(s_c + \frac{2\pi \mathrm{i} n}{\lg 2}\right) \exp(-2\pi \mathrm{i} n z).$$

Proposition 6.15. *The equality*

$$\int_0^1 \mathrm{e}^{ax} \, \mathrm{d}G(x) = \exp\left(\frac{a}{2}\right) \prod_{k=1}^{\infty} \cosh\left(\frac{a}{3^k}\right) \tag{6.16}$$

holds for every $a \in \mathbb{C}$.

Proof. Write

$$\Phi(a) := \int_0^1 \mathrm{e}^{ax} \, \mathrm{d}G(x).$$

It follows from (4.9) to (4.11) that

$$G\left(\frac{x}{3}\right) = \frac{1}{2} G(x), \quad G\left(\frac{2}{3} + \frac{x}{3}\right) = \frac{1}{2} + \frac{1}{2} G(x).$$

Hence

$$\begin{aligned}\Phi(a) &= \int_0^{1/3} \mathrm{e}^{ax}\,\mathrm{d}G(x) + \int_{2/3}^{1} \mathrm{e}^{ax}\,\mathrm{d}G(x) = \int_0^1 \mathrm{e}^{ax/3}\,\mathrm{d}G\left(\frac{x}{3}\right)\\ &\quad + \int_0^1 \mathrm{e}^{a\left(\frac{2}{3}+x/3\right)}\,\mathrm{d}G\left(\frac{2}{3}+\frac{x}{3}\right) = \frac{1}{2}\int_0^1 \mathrm{e}^{ax/3}\,\mathrm{d}G(x)\\ &\quad + \frac{1}{2}\int_0^1 \mathrm{e}^{a\left(\frac{2}{3}+x/3\right)}\,\mathrm{d}G(x)\\ &= \frac{1}{2}(1+\mathrm{e}^{a\frac{2}{3}})\Phi\left(\frac{a}{3}\right) = \mathrm{e}^{a/3}\cosh\left(\frac{a}{3}\right)\Phi\left(\frac{a}{3}\right).\end{aligned}$$

Consequently, by successive iteration we obtain

$$\Phi(a) = \exp\left(\frac{a}{3}+\frac{a}{9}+\cdots+\frac{a}{3^n}\right)\cosh\left(\frac{a}{3}\right)\cosh\left(\frac{a}{9}\right)\ldots\cosh\left(\frac{a}{3^k}\right)\Phi\left(\frac{a}{3^k}\right).$$

Since $\Phi(x) \to 1$ as $x \to 0$, the last formula implies (6.16). □

Observe that the function Φ, $\Phi(a)=\int_0^1 \mathrm{e}^{ax}\,\mathrm{d}G(x)$, is the exponential generating function of the moment sequence $\{m_n\}$.

The next result follows from (6.16) and shows that Fourier coefficients $\hat{\mu}_n$, $\hat{\mu}_n := \int_0^1 \mathrm{e}^{2\pi \mathrm{i} nx}\,\mathrm{d}\mu_C(x)$, of μ_C do not tend to zero as $n \to \infty$.

Proposition 6.17 (*Hille and Tamarkin [34]*). *For all integers n*,

$$\hat{\mu}_n = \mathrm{e}^{\pi \mathrm{i} n}\prod_{j=1}^{\infty}\cos\left(\frac{2\pi n}{3^j}\right).$$

Corollary 6.18 (*Hille and Tamarkin [34]*). *Let k be a positive integer and let* $n=3^k$. *Then*

$$\hat{\mu}_n = -\prod_{v=1}^{\infty}\cos\left(\frac{2\pi}{3^v}\right) = \text{const.} \neq 0.$$

Remark 6.19. The infinite product $\prod_{v=1}^{\infty}\cos\left(2\pi/3^v\right)$ converges absolutely because

$$\left|1-\cos\frac{2\pi}{3^v}\right| \leqslant 2\sin^2\left(\frac{\pi}{3^v}\right) < 2\,\frac{\pi^2}{9^v}$$

and hence $\prod_{v=1}^{\infty}\cos\left(2\pi/3^v\right) \neq 0$.

In the paper [56], Wiener and Wintner proved the more general result similar to Corollary 6.18. The study of the Fourier asymptotics of Cantor-type measures has been extended in [50–52,40,35].

Proposition 6.20. *The length of the arc of the curve* $y = G(x)$ *between the points* $(0, 0)$ *and* $(1, 1)$ *is* 2.

Remark 6.21. The detailed proof of this proposition can be found in [13]. In fact, this follows rather easily from 2.1.4. Probably the length of the curve $y=G(x)$ was first calculated in [34].

The next theorem is a refinement of Proposition 6.20.

Theorem 6.22. *Let $F:[0,1]\to\mathbb{R}$ be a continuous, increasing function for which $F(0)=0$ and $F(1)=1$. Then the following two statements are equivalent.*

6.22.1. *The length of the arc $y=F(x)$, $0\leqslant x\leqslant 1$, is 2.*
6.22.2. *The function F is singular.*

This theorem follows from the results of Pelling [46]. See also [19].

7. Some topological properties

There exists a simple characterization of the Cantor function up to a homeomorphism. If $\Phi: X\to Y$ and $F: X\to Y$ are continuous functions from topological space X to topological space Y, then F is said to be (topologically) *isomorphic* to Φ if there exist homeomorphisms $\eta: X\to X$ and $\psi: Y\to Y$ such that $F=\psi\circ\Phi\circ\eta$ [47, Chapter 4, Section 1]. If the last equality holds with ψ equal the identity function, then F and Φ will be called the Lebesgue equivalent functions (cf. [30, Definition 2.1]).

Proposition 7.1. *The Cantor function G is isomorphic (as a map from $[0,1]$ into $[0,1]$) to a continuous monotone function $q:[0,1]\to[0,1]$ if and only if the set of constancy L_g is everywhere dense in $[0,1]$ and*

$$g(\{0,1\})=\{0,1\}\subseteq[0,1]\setminus L_g. \tag{7.2}$$

Lemma 7.3. *Let A, B be everywhere dense subsets of $[0,1]$ and let $f: A\to B$ be an increasing bijective map. Then f is a homeomorphism and, moreover, f can be extended to a self-homeomorphism of the closed interval $[0,1]$.*

Proof. It is easy to see that

$$0=\lim_{\substack{x\to 0\\ x\in A}} f(x)=1-\lim_{\substack{x\to 1\\ x\in A}} f(x)$$

and

$$\lim_{\substack{x\to t\\ x\in A\cap[0,t]}} f(x)=\lim_{\substack{x\to t\\ x\in A\cap[t,1]}} f(x)$$

for every $t\in(0,1)$. Thus there exists the limit

$$\tilde{f}(t):=\lim_{\substack{x\to t\\ x\in A}} f(x),\quad t\in[0,1].$$

Obviously, $\tilde{f}$ is strictly increasing and $\tilde{f}(t) = f(t)$ for each $t \in A$. Suppose that

$$\tilde{f}^-(x_0) := \lim_{\substack{x \to x_0 \\ x \in [0, x_0)}} \tilde{f}(x) < \lim_{\substack{x \to x_0 \\ x \in (x_0, 1]}} \tilde{f}(x) := \tilde{f}^+(x_0)$$

for some $x_0 \in (0, 1)$. Since B is a dense subset of $[0, 1]$, there is $b_0 \in B$ such that

$$f(x_0) \neq b_0 \in (\tilde{f}^-(x_0), \tilde{f}^+(x_0)).$$

Let $b_0 = f(a_0)$, where $a \in A$. Now we have the following:

(i) $a_0 \neq x_0$, because $b_0 \neq \tilde{f}(x_0)$,
(ii) $a_0 \notin [0, x_0)$, because $b_0 > \tilde{f}^-(x_0)$,
(iii) $a_0 \notin (x_0, 1]$, because $b_0 < \tilde{f}^+(x_0)$

and this yields to a contradiction. Reasoning similarly, we can prove the continuity of $\tilde{f}$ for the points 0 and 1. Hence, $\tilde{f}$ is a continuous bijection of the compact set $[0, 1]$ onto itself and every such bijection is a homeomorphism. □

Proof of Proposition 7.1. It follows directly from the definition of isomorphic functions that (7.2) holds and

$$\mathrm{Clo}(L_g) = [0, 1], \tag{7.4}$$

if G is isomorphic to a continuous function $g : [0, 1] \to [0, 1]$.

Suppose now that $g : [0, 1] \to [0, 1]$ is a continuous monotone function for which (7.2) and (7.4) hold. We may assume, without loss of generality, that g is increasing. It follows from Lemma 3.6, that a set $[0, 1]\backslash L_g$ is compact and perfect. Moreover, (7.2) and (7.4) imply that $[0, 1]\backslash L_g$ is a nonempty nowhere dense subset of $[0, 1]$. Hence, there exists an increasing homeomorphism $\eta : [0, 1] \to [0, 1]$ such that

$$\eta(C) = [0, 1]\backslash L_g.$$

In fact, there is an order preserving homeomorphism $\eta^0 : C \to [0, 1]\backslash L_g$ [1, Chapter 4, Section 6, Theorem 25]. It can be extended on each complementary interval $J \in \mathscr{I}$ as a linear function. The resulting extension η is strictly increasing and maps $[0, 1]$ onto $[0, 1]$. Hence, by Lemma 7.3, η is a homeomorphism.

It is easy to see that a set of constancy of $g \circ \eta$ coincides with I°, the set of constancy of G. Set

$$g_{I^\circ} := \bigcup_{J \in \mathscr{I}} g(J), \quad G_{I^\circ} := \bigcup_{J \in \mathscr{I}} G(J),$$

($g(J)$ and $G(J)$ are one-point sets for every $J \in \mathscr{I}$).

The maps $\mathscr{I} \ni J \to g(J) \in g_{I^\circ}$ and $\mathscr{I} \ni J \to G(J) \in G_{I^\circ}$ are one-to-one and onto. Hence the map

$$\Psi^\circ : g_{I^\circ} \to G_{I^\circ}, \quad \Psi^\circ(g(J)) = G(J), \quad J \in \mathscr{I}$$

is a bijection. It follows from the definition of G and (7.4) that

$$\mathrm{Clo}(g_{I^\circ}) = \mathrm{Clo}(G_{I^\circ}) = [0, 1].$$

Moreover, since g and G are increasing functions, the function Ψ° is *strictly* increasing. Hence, by Lemma 7.3, Ψ° can be extended to a homeomorphism $\psi : [0, 1] \to [0, 1]$. It is easy to see that

$$G(x) = \psi(g(\eta(x)))$$

for every $x \in I^\circ$. Since I° is an everywhere dense subset of $[0, 1]$, the last equality implies that

$$G = \psi \circ g \circ \eta.$$

Thus g and G are topologically isomorphic. □

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function whose total variation is finite. Then, by the theorem of Bruckner and Goffman [7], f is Lebesgue equivalent to some function with bounded derivative. As a corollary we obtain

Proposition 7.5. *There exists a differentiable function f with an uniformly bounded derivative f' such that G and f are Lebesgue equivalent.*

We now recall a notion of set of varying monotonicity [30, Definition 3.7].

Let $f : [a, b] \to \mathbb{R}$. A point $x \in (a, b)$ is called a point of *varying monotonicity* of f if there is no neighborhood of x on which f is either strictly monotonic or constant. We also make the convention that both a and b are points of varying monotonicity for every $f : [a, b] \to \mathbb{R}$.

Let us denote by K_f the set of points of varying monotonicity of f. Then, as was shown by Bruckner and Goffman in [7], $m_1(f(K_f)) = 0$ if and only if f is Lebesgue equivalent to some continuously differentiable function. For every homeomorphism $\psi : [0, 1] :\to [0, 1]$ we evidently have

$$K_{\psi \circ G} = C, \quad \psi(G(C)) = [0, 1].$$

Thus, we obtain.

Proposition 7.6. *If F is topologically isomorphic to the Cantor function G, then F is not continuously differentiable.*

Let G_C be the restriction of G to the Cantor set C. As it is easy to see, G_C is continuous, closed but not open. For example we have

$$G_C\left(\left[0, \tfrac{1}{2}\right) \cap C\right) = G\left(\left[0, \tfrac{1}{2}\right)\right) = \left[0, \frac{1}{2}\right].$$

However, G_C is weakly open in the following sence.

Proposition 7.7. *Let A be a nonempty subset of C. If the interior of A is nonempty in C, then the interior of $G_C(A)$ is nonempty in* $[0, 1]$, *or in other words*

$$(\mathrm{Int}_C\, A \neq \emptyset) \Rightarrow \left(\mathrm{Int}_{[0,1]}\, G(A) \neq \emptyset)\right). \tag{7.8}$$

Proof. Suppose that $A \subseteq C$ and $\mathrm{Int}_C\, A \neq \emptyset$. Since C° is a dense subset of C, there is an interval (a, b) such that both a and b are in C° and

$$A \supseteq (a, b) \cap C.$$

It is easy to see that

$$(x < t < y) \Rightarrow (G(x) < G(t) < G(y)) \tag{7.9}$$

for all x, y in $[0, 1]$ and every $t \in C^\circ$. Hence

$$(G(a), G(b)) = G((a, b) \cap C) \subseteq G(A) = G_C(A). \qquad \square \tag{7.10}$$

Recall that a subset B of topological space X is residual in X if $X \backslash B$ is of the first category in X.

Proposition 7.11. *If B is a residual* (*first category*) *subset of* $[0, 1]$, *then* $G_C^{-1}(B)$ *is residual* (*first category*) *in C.*

Proof. Let B be a residual subset of $[0, 1]$. Write

$$K_1 := [0, 1]\backslash C^\circ, \quad W := G^{-1}(B).$$

It is sufficient to show that $W \cap C^\circ$ is residual in C.

Since G is one-to-one on C°,

$$K_1 \cup C^\circ = [0, 1] = G(K_1) \cup G(C^\circ), \tag{7.12}$$

and

$$C^\circ \cap K_1 = \emptyset = G(C^\circ) \cap G(K_1). \tag{7.13}$$

Moreover, we have

$$W \cap C^\circ = G^{-1}(G(W) \cap G(C^\circ)) = G_C^{-1}(G(W) \cap G(C^\circ)). \tag{7.14}$$

It follows immediately from (7.12) and (7.13) that

$$G(W) \cap G(C^\circ) = G(W) \cap ([0, 1]\backslash G(K_1)).$$

The set $G(K_1)$ is countable. Thus $[0, 1]\backslash G(K_1)$ is residual and $G(W) \cap G(C^\circ)$ is residual too, as an intersection of residual sets. Hence, there is a sequence $\{O_n\}_{n=1}^\infty$ for which

$$G(W) \cap G(C^\circ) \supseteq \bigcup_{n=1}^{\infty} O_n \tag{7.15}$$

and each O_n is a dense open subset of [0, 1]. Since G_C is continuous on C°, (7.14) and (7.15) imply that

$$W \cap C^\circ \supseteq \bigcup_{n=1}^{\infty} G_C^{-1}(O_n),$$

where each $G_C^{-1}(O_n)$ is open in C. Suppose that we have

$$\mathrm{Int}_C(C \backslash G_C^{-1}(O_{n_0})) \neq \emptyset$$

for some positive integer n_0. Then by Proposition 7.7 we obtain

$$\mathrm{Int}_{[0,1]}([0,1] \backslash O_{n_0}) \neq \emptyset$$

contrary to the properties of $\{O_n\}_{n=1}^{\infty}$. Consequently $G_C^{-1}(O_n)$ is dense in C for each positive integer n. It follows that $W \cap C^\circ$ is residual in C. For the case where B is of the first category in [0, 1] the conclusion can be obtained by passing to the complement. □

Remark 7.16. In a special case this proposition was mentioned without proof in the work by Eidswick [21].

Proposition 7.17. *Let $B \subseteq C$. Then B is an everywhere dense subset of C if and only if $G_C(B)$ is an everywhere dense subset of* [0, 1].

Proof. Since G_C is continuous, the image $G_C(B)$ is a dense subset of [0, 1] for every dense B. Suppose that

$$\mathrm{Clo}_{[0,1]}(G_C(B)) = [0,1]$$

but

$$C \backslash \mathrm{Clo}_C(B) \neq \emptyset.$$

G_C is a closed map, hence

$$G_C(\mathrm{Clo}_C(B)) \supseteq \mathrm{Clo}_{[0,1]}(G_C(B)) = [0,1]. \tag{7.18}$$

As in the proof of Proposition 7.7 we can find a two-point set $\{a, b\} \subseteq C^\circ$ such that (7.10) holds with $A = C \backslash \mathrm{Clo}_C(B)$. Taking into account implication (7.9) we obtain that

$$G(x) \notin (G(a), G(b))$$

for every $x \in \mathrm{Clo}_C(B)$, contrary to (7.18). □

Remark 7.19. If (X, τ_1, τ_2) is a space with two topological structures τ_1 and τ_2, then one can prove that the condition

$$(\mathrm{Int}_{\tau_1}(A) \neq \emptyset) \Leftrightarrow (\mathrm{Int}_{\tau_2}(A) \neq \emptyset) \quad \forall A \subseteq X$$

is equivalent to

$$(\mathrm{Clo}_{\tau_1}(A) = X) \Leftrightarrow (\mathrm{Clo}_{\tau_2}(A) = X) \quad \forall A \subseteq X.$$

The formulations and proofs of Propositions 7.7, 7.10, 7.17 can be easily carried over to the general case of functions which are topologically isomorphic to G. In the rest of this section we discuss a new characterization for such functions.

We say that a subset A of $\mathbb{R}$ has the *Baire* property if there is an open set $U \subseteq \mathbb{R}$ such that the symmetric difference $A\Delta U$, $A\Delta U = (A\backslash U) \cup (U\backslash A)$, is of first category in $\mathbb{R}$.

Theorem 7.20. *The Cantor function G is isomorphic (as a map from $[0, 1]$ into $\mathbb{R}$) to a continuous monotone function $f : [0, 1] \to \mathbb{R}$ with $\{0, 1\} \subseteq ([0, 1]\backslash L_f)$ if and only if the inverse image $f^{-1}(A)$ has the Baire property for every $A \subseteq \mathbb{R}$.*

Proof. Let $f : [0, 1] \to \mathbb{R}$ be a function which is topologically isomorphic to G and let $A \subseteq \mathbb{R}$. Then it follows from Proposition 7.1 that the set of constancy L_f is everywhere dense in $[0, 1]$. Reasoning as in the proof of Proposition 2.5 we can prove that $f^{-1}(A)$ is the union of a F_σ subset of $[0, 1]$ with some nowhere dense set. Since the collection of all subsets of $[0, 1]$ having the Baire property forms σ-algebra [45, Theorem 4.3] we obtain the Baire property for $f^{-1}(A)$.

Suppose now that $f : [0, 1] \to \mathbb{R}$ is a continuous monotone function, $\{0, 1\} \subseteq ([0, 1]\backslash L_f)$, but f is not isomorphic to G. Then using Proposition 7.1 we see that there is an open interval $(a, b) \subseteq ([0, 1]\backslash L_f)$. Let B be subset of (a, b) which does not have the Baire property. It is easy to see that $B = f^{-1}(f(B))$. Thus, there exists a set A such that $f^{-1}(A)$ does not have a Baire property. □

Remark 7.21. The existence of subset of the reals not having the Baire property depends on the axiom of choice. In fact, from the axiom of determinateness it follows that every $A \subseteq \mathbb{R}$ is Lebesgue measurable (cf. Propositions 2.4, 2.5) and has the Baire property. See, for example, [37,38].

8. Dini's derivatives

We recall the definition of the Dini derivatives. Let a real-valued function F be defined on a set $A \subseteq \mathbb{R}$ and let x_0 be a point of A. Suppose that A contains some half-open interval $[x_0, a)$. The upper right Dini derivative D^+F of F at x_0 is defined by

$$D^+F(x_0) = \limsup_{\substack{x \to x_0 \\ x \in (x_0, a)}} \frac{F(x) - F(x_0)}{x - x_0}.$$

We define the other three extreme unilateral derivatives D_+F, D^-F and D_-F similarly. (see [6] for the properties of the Dini derivatives).

Let C be the Cantor ternary set and G the Cantor ternary function.

We first note that the upper Dini derivatives D^+G and D^-G are always $+\infty$ or 0.

We denote by M the set of points where G fails to have a finite or infinite derivative. It is clear that $M \subseteq C$.

Since G satisfies Banach's (T_1) condition (see Proposition 2.10), the following conclusion holds.

Proposition 8.1. *The set $G(M)$ has a (linear Lebesgue) measure zero.*

Proof. See Theorem 7.2 in Chapter IX of Saks [48]. □

Corollary 8.2. *Let $\mathscr{H}^{s_c}$ be a Hausdorff measure with $s_c = \lg 2/\lg 3$. Then we have*

$$\mathscr{H}^{s_c}(M) = 0.$$

Proof. This follows from Propositions 8.1 and 5.5. □

Write

$$W := \{x \in (0,1) : D_{-}G(x) = D_{+}G(x) = 0\}.$$

In the next theorem, we have collected some properties of Dini derivatives of G which follow from [43].

Proposition 8.3.

8.3.1. *The set $G(W)$ is residual in* $[0, 1]$.
8.3.2. *For each $\lambda \geqslant 0$ the set $\{x \in [0,1) : D_{+}G(x) = \lambda\}$ has the power of the continuum.*

Remark 8.4. However, the set

$$\{x \in [0,1) : D^{+}G(x) = \lambda\}$$

is void whenever $0 < \lambda < \infty$.

Corollary 8.5. *The set $W \cap C$ is residual in C.*

Proof. The proof follows from 8.3.1 and Proposition 7.11. □

Remark 8.6. This corollary was formulated without a proof in [21].

Let x be a point of C. Let us denote by $z_x(n)$ the position of the nth zero in the ternary representation of x and by $t_x(n)$ the position of the nth digit two in this representation.

The next theorem was proved by Eidswick in [21].

Theorem 8.7. *Let $x \in C^{\circ}$ and let*

$$\lambda_x := \liminf_{n\to\infty} \frac{3^{z_x(n)}}{2^{z_x(n+1)}}. \tag{8.8}$$

Then

$$\lambda_x \leqslant D_+G(x) \leqslant 2\lambda_x. \tag{8.9}$$

Furthermore, if $\lim_{n\to\infty} z_x(n+1) - z_x(n) = \infty$, *then* $D_+G(x) = \lambda_x$.
Similarly, if

$$\mu_x := \liminf_{n\to\infty} \frac{3^{t_x(n)}}{2^{t_x(n+1)}}, \tag{8.10}$$

then

$$\mu_x \leqslant D_-G(x) \leqslant 2\mu_x, \tag{8.11}$$

with the equality on the right if $\lim_{n\to\infty} t_x(n+1) - t_x(n) = \infty$.

Corollary 8.12. *If* $x \in C^\circ$, *then* $G'(x) = \infty$ *if and only if* $\lambda_x = \mu_x = \infty$.

Remark 8.13. This Corollary implies that the four Dini derivatives of G agree at $x_0 \in (0, 1)$ if and only if either $x \in I^\circ$ or $\lambda_x = \mu_x = \infty$.

Corollary 8.14. *If* $x \in C^\circ$, *then* $D_-G(x) = D_+G(x) = 0$ *if and only if*

$$\lambda_x = \mu_x = 0.$$

The following result is an improvement of 8.3.2.

Theorem 8.15 (*Eidswick [21]*). *For each* $\mu \in [0, \infty]$ *and* $\lambda \in [0, \infty]$, *the set*

$$\{z \in C : D_-G(z) = \lambda \quad \textit{and} \quad D_+G(z) = \mu\}$$

has the power of the continuum.

Corollary 8.16. *For each* $\mu \in [0, \infty]$ *and* $\lambda \in [0, \infty]$ *there exists a nonempty, compact perfect subset in* $\{z \in C : D_-G(z) = \lambda \textit{ and } D_+G(z) = \mu\}$.

Proof. Since G is continuous on $[0, 1]$, each of the Dini derivatives of G is in a Baire class two [6, Chapter IV, Theorem 2.2]. Hence, the set $\{z \in C : D_-G(z) = \lambda \text{ and } D_+G(z) = \mu\}$ is a Borel set. It is well-known that, if A is uncountable Borel set in a complete separable topological space X, then A has a nonempty compact perfect subset [39, Section 39, I, Theorem 0]. □

Recall that M is the set of points at which G fails to have a finite or infinite derivative and $s_c = \lg 2/\lg 3$. The next result was proved by Richard Darst in [14].

Theorem 8.17. *The Hausdorff dimension of* M *is* s_c^2, *or in other words*

$$\dim_{\mathscr{H}} M = s_c^2. \tag{8.18}$$

Remark 8.19. It is interesting to observe that the packing and box counting dimension are equal for M and C

$$\dim_{\mathscr{P}} M = \dim_{\mathscr{B}} M = \dim_{\mathscr{P}} C = \dim_{\mathscr{B}} C. \tag{8.20}$$

The proof can be found in [16], see also [25] and [41] for the similar question.
The following proposition was published in [14] with a short sketch of the proof.

Proposition 8.21. *The equality*

$$\dim_{\mathscr{H}} (G(M)) = s_c$$

holds.

An equality which is similar to (8.18) was established in [15] for more general Cantor sets. Some interesting results about differentiability of the Cantor functions in the case of fat symmetric Cantor sets can be found in [13,17]. The Hausdorff dimension of M was found by Morris [42] in the case of nonsymmetric Cantor functions.

9. Lebesgue's derivative

The so-called Lebesgue's derivative or first symmetric derivative of a function f is defined as

$$\mathrm{SD}\, f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}.$$

It was noted by J. Uher that the Cantor function G has the following curious property.

Proposition 9.1 (*Uher [55]*). *G has infinite Lebesgue's derivative*, $\mathrm{SD}\, G(x) = \infty$, *at every point $x \in C \setminus \{0, 1\}$, and* $\mathrm{SD}\, G(x) = 0$ *for all $x \in I^{\circ}$.*

Remark 9.2. It was shown by Buczolich and Laczkovich [9] that there is no symmetrically differentiable function whose Lebesgue's derivative assumes just two finite values. Recall also that if f is continuous and has a derivative everywhere (finite or infinite), then the range of f' must be a connected set.

From the results in [9] we obtain

Theorem 9.3. *For all sufficiently large b the inequality*

$$\limsup_{\Delta x \to 0+} \frac{G(x + b\Delta x) - G(x - b\Delta x)}{G(x + \Delta x) - G(x - \Delta x)} < b \tag{9.4}$$

holds for all $x \in C \setminus \{0, 1\}$.

Remark 9.5. It was shown in [9] that (9.4) is true for all $b \geqslant 81$ and false if $b = 4$. Moreover, as it follows from [9] (see, also [53, Theorem 7.32]) that Theorem 9.3 implies Proposition 9.1.

10. Hölder continuity, distortion of Hausdorff dimension, s_c-densities

Proposition 10.1. *The function G satisfies a Hölder condition of order* $s_c = \lg 2/\lg 3$, *the Hölder coefficient being not greater than* 1. *In other words*

$$|G(x) - G(y)| \leqslant |x - y|^{s_c} \tag{10.2}$$

for all $x, y \in [0, 1]$. *The constants* s_c *and* 1 *are the best possible in the sense that the inequality*

$$|G(x) - G(y)| \leqslant a|x - y|^{\beta}$$

does not hold for all x, y *in* $[0, 1]$ *if either* $\beta > s_c$ *or* $\beta = s_c$ *and* $a < 1$.

Proof. Using lemma from [32] we obtain the inequality

$$\left(\frac{1}{2}t\right)^{s_c} \leqslant G(t) \leqslant t^{s_c} \tag{10.3}$$

for all $t \in [0, 1]$. Since G is its first modulus of continuity (see Proposition 3.2), the last inequality implies (10.2). Moreover, it follows from (10.3) that s_c is a sharp Hölder exponent in (10.2). By a simple calculation we have from formula (1.2) that

$$G(3^{-n}) = 3^{-ns_c}$$

for all positive integers n. Thus, the Hölder coefficient 1 is the best possible in (10.2). □

Remark 10.4. It was first proved in [34] that G satisfies the Hölder condition with the exponent s_c and coefficient 2. R.E.Gilman erroneously claimed (without proof and for a more general case) that both constants s_c and 2 are the best possible [29]. In [32], a similar to (10.3) inequality does not link with the Hölder condition for G. Observe also that $\frac{1}{2}^{s_c}$ is the sharp constant for (10.3) as well as the constant 1. See Fig. 4.

Proposition 10.5. *Let x be an arbitrary point of C. Then we have*

$$\liminf_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x - y|} = s_c.$$

Proof. It suffices to show that

$$\liminf_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x - y|} \leqslant s_c.$$

Choosing $|x - y| = \frac{2}{3^n}$ for $n \to \infty$ we obtain the result. □

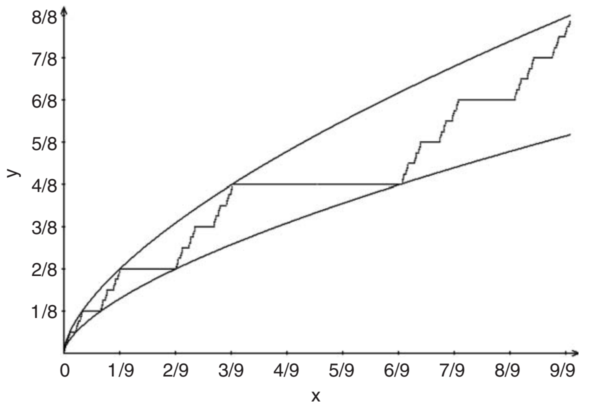

Fig. 4. The graph of G lies between two curves $y_1(x) = x^{s_c}$ and $y_2(x) = \left(\frac{1}{2}x\right)^{s_c}$.

Let x be a point of the Cantor ternary set C. Then x has a triadic representation

$$x = \sum_{m=1}^{\infty} \frac{\alpha_m}{3^m}, \tag{10.6}$$

where $\alpha_m \in \{0, 2\}$. Define a sequence $\{\mathscr{R}_x(n)\}_{n=1}^{\infty}$ by the rule

$$\mathscr{R}_x(n) := \begin{cases} \inf\{m - n : \alpha_m \neq \alpha_n, \ m > n\} & \text{if } \exists m > n : \alpha_m \neq \alpha_n, \\ 0 & \text{if } \forall m > n : \alpha_m = \alpha_n, \end{cases}$$

i.e., $\qquad \mathscr{R}_x(n) = 1 \iff (\alpha_n \neq \alpha_{n+1});$

$$\mathscr{R}_x(n) = 2 \iff (\alpha_n = \alpha_{n+1}) \ \& \ (\alpha_{n+1} \neq \alpha_{n+2})$$

and so on.

Theorem 10.7 (*Dovgoshey et al.* [20]). *For $x \in C$ we have*

$$\lim_{\substack{y \to x \\ y \in C}} \frac{\lg |G(x) - G(y)|}{\lg |x - y|} = s_c$$

if and only if

$$\lim_{n \to \infty} \frac{\mathscr{R}_x(n)}{n} = 0.$$

This theorem and a general criterion of constancy of linear distortion of Hausdorff dimension (see [20]) imply the following result.

Theorem 10.8. *There exists a set $M \subseteq C$ such that $\mathscr{H}^{s_c}(M) = 1$ and*

$$\dim_{\mathscr{H}}(G(A)) = \frac{1}{s_c}\dim_{\mathscr{H}}(A)$$

for every $A \subseteq M$.

Remark 10.9. In the last theorem we can take M as

$$M = G^{-1}(\mathrm{SN}_2),$$

where SN_2 is the set of all numbers from [0, 1] which are simply normal to base 2. See, for example, [44] for the definition and properties of simply normal numbers.

Let $\Theta^{*s_c}(t)$ and $\Theta_*^{s_c}(t)$ denote the upper and lower s_c-densities of μ_C at $t \in \mathbb{R}$, i.e.,

$$\Theta^{*s_c}(t) := \limsup_{\substack{x\to 0\\ x\neq 0}} \frac{|\hat{G}(t+x) - \hat{G}(t-x)|}{|2x|^{s_c}},$$

$$\Theta_*^{s_c}(t) := \liminf_{\substack{x\to 0\\ x\neq 0}} \frac{|\hat{G}(t+x) - \hat{G}(t-x)|}{|2x|^{s_c}}.$$

It can be shown that

$$\Theta^{*s_c}(t) < \Theta_*^{s_c}(t)$$

for every $t \in C$, i.e., s_c-density for measure μ_C does not exist at any t at the Cantor set C. However, the logarithmic averages s_c-density does exist for many fracfals including C.

Theorem 10.10. *For $\mathscr{H}^{s_c}$ almost all $x \in C$ the logarithmic average density*

$$A^{s_c}(x) := \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\hat{G}(x+\mathrm{e}^{-t}) - \hat{G}(x-\mathrm{e}^{-t})}{|2\mathrm{e}^{-t}|^{s_c}}\,\mathrm{d}t$$

exists with

$$A^{s_c}(x) = \frac{1}{2^{s_c}\log 2}\iint_{|x-y|\geqslant \frac{1}{3}} |x-y|^{-s_c}\,\mathrm{d}G(x)\,\mathrm{d}G(y) = 0,6234\ldots.$$

For the proof see [24, Theorem 6.6] and also [4].

The interesting results about upper and lower s_c-densities of μ_C can be found in [26]. For x with representation (10.6) we define $\hat{\tau}(x)$ and $\tau(x)$ as

$$\hat{\tau}(x) := \liminf_{k\to\infty} \sum_{m=1}^{\infty} \alpha_{m+k}3^{-m},$$

$$\tau(x) := \min(\hat{\tau}(x), \hat{\tau}(1-x)).$$

Theorem 10.11 (*Feng et al.* [26]). 10.11.1. *For all $x \in C$,*

$$\Theta_*^{s_c}(x) = (4 - 6\tau(x))^{-s_c}.$$

10.11.2. *For all* $x \in C$,

$$\Theta^{*s_c} = \begin{cases} 2^{-s_c} & \textit{if } x \in C^1, \\ \left(\frac{2}{3}\right)^{-s_c}(2+\tau(x))^{-s_c} & \textit{if } x \in C^\circ. \end{cases}$$

10.11.3. *For all* $x \in C^\circ$,

$$9(\Theta^{*s_c}(x))^{-1/s_c} + (\Theta_*^{s_c}(x))^{-1/s_c} = 16.$$

10.11.4.

$$\sup\{\Theta^{*s_c}(x) - \Theta_*^{s_c}(x) : x \in C\} = 4^{-s_c},$$

where the supremum can be attained at $\{x \in C^\circ : \tau(x) = 0\}$, *and*

$$\inf\{\Theta^{*s_c}(x) - \Theta_*^{s_c} : x \in C\} = \left(\frac{3}{2}\right)^{-s_c} - \left(\frac{5}{2}\right)^{-s_c},$$

where the infimum can be attained at $\left\{x \in C^\circ : \tau(x) = \frac{1}{4}\right\}$.

10.11.5. *For* $\mathscr{H}^{s_c}$*-almost all* $x \in C$,

$$\Theta_*^{s_c}(x) = 4^{-s_c}, \quad \Theta^{*s_c}(x) = 2.4^{-s_c}.$$

Remark 10.12. The general result similar to 10.11.5 can be found in the work Salli [49].

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References

[1] П.С. Александров, Введение в Теорию Множеств и Общую Топологию, Москва, Наука, 1977
P.S. Aleksandrov, Introduction to Set Theory and General Topology, Izdat, Nauka, Moskow, 1977.
[2] N.Bary, Memoire sur la representation finie des fonctions continues, Math. Ann. 103 (1930) 185–248, 598–653.
[3] N. Bary, D. Menchoff, Sur l'integrale de Lebesgue–Stieltjes et les fonctions absolumant continues, Ann. Math. Pura Appl. 5 (1928) 19–54.
[4] T. Bedford, A.M. Fisher, Analogues of the Lebesgue density theorem for fractal sets of reals and integer, Proc. London Math. Soc. 64 (1992) 95–124.
[5] Yu.M. Berezansky, Z.G. Sheftel, G.F. Us, Functional Analysis, vol. II, Birkhäuser Verlag, Basel, 1996.
[6] A.M. Bruckner, Differentiation of Real Functions. Lecture Notes in Mathematics, vol. 659, Springer, Berlin, 1978.
[7] A.M. Bruckner, J.L. Goffman, Differentiability through change of variables, Proc. Amer. Math. Soc. 61 (1976) 235–241.
[8] A.M. Bruckner, J.L. Leonard, On differentiable functions having an everywhere dense set of intervals of constancy, Cand. Math. Bull. 8 (1965) 73–76.
[9] Z. Buczolich, M. Laczkovich, Concentrated Borel measures, Acta. Math. Hungar. 57 (1991) 349–362.
[10] G. Cantor, De la puissance des ansembles parfaits de points, Acta Math. 4 (1884) 381–392; Reprinted G. Cantor, in: E. Zermelo (Ed.), Gezammelte Abhandlungen Mathematischen und Philosophischen Inhalts, Springer, New York, 1980.
[11] D.R. Chailice, A characterization of the Cantor function, Amer. Math. Monthly 98 (N3) (1991) 255–258.

[12] W.A. Coppel, An interesting Cantor set, Amer. Math. Monthly 90 (N7) (1983) 456–460.
[13] R.B. Darst, Some Cantor sets and Cantor functions, Math. Mag. 45 (1972) 2–7.
[14] R.B. Darst, The Hausdorff dimension of the nondifferentiability set of the Cantor function is $[\ln 2/\ln 3]^2$, Proc. Amer. Math. Soc. 119 (N1) (1993) 105–108.
[15] R.B. Darst, Hausdorff dimension of sets of non-differentiability points of Cantor functions, Math. Proc. Cambridge Philos. Soc. 117 (1995) 185–191.
[16] F.M. Dekking, W. Li, How smoth is a devil's staircase?, Fractals 11 (1) (2003) 101–107.
[17] T.P. Dence, Differentiable points of the generalized Cantor function, Rocky Mountain J. Math. 9 (2) (1979) 239–249.
[18] J. Doboš, The standard Cantor function is subadditive, Proc. Amer. Math. Soc. 124 (N11) (1996) 3425–3426.
[19] O. Dovgoshey, O. Martio, The arc length evaluation and Lebesgue decomposition, Reports in Mathematics, vol. 379, University of Helsinki, 2004.
[20] O. Dovgoshey, O. Martio, V. Ryazanov, M. Vuorinen, On linear distortion of Hausdorff dimension, Reports in Mathematics, vol. 366, University of Helsinki, 2003.
[21] J.A. Eidswick, A characterization of the nondifferentiability set of the Cantor function, Proc. Amer. Math. Soc. 42 (1) (1974) 214–217.
[22] G.C. Evans, Calculation of moments for a Cantor–Vitali function, Amer. Math. Monthly 64 (8) (1957) 22–27.
[23] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, Boca Raton, 1992.
[24] K. Falconer, Techniques in Fractal Geometry, Wiley, New York, 1997.
[25] K.J. Falconer, One-side multifractal analysis and points of non-differentiability of devil's staircases, Math. Proc. Cambridge Philos. Soc. 136 (1) (2004) 167–174.
[26] D.-J. Feng, S. Hua, Z.-Y. Wen, The pointwise densities of the cantor measure, J.Math. Anal. Appl. 250 (2000) 692–705.
[27] H.-J. Fisher, On the paper: Asymptotics for the moments of singular distributions, J. Approx. Theory 82 (3) (1995) 362–374.
[28] J.F. Fleron, A note on the history of the Cantor set and Cantor function, Math. Mag. 67 (1994) 136–140.
[29] R.E. Gilman, A class of functions continuous but not absolutely continuous, Ann. Math. (2) 33 (1932) 433–442.
[30] C. Goffman, T. Nishiura, D. Waterman, Homeomorphisms in Analysis, Mathematical Survey and Monographs, vol. 54, American Mathematical Society, Providence, RI, 1998.
[31] W. Goh, J. Wimp, Asymptotics for the moments of singular distributions, J. Approx. Theory 74 (3) (1993) 301–334.
[32] E.A. Gorin, B.N. Kukushkin, Integrals related to the Cantor function, St. Petersburg Math. J. 15 (3) (2004) 449–468.
[33] T. Hawkins, Lebesgue's Theory of Integration, Chelsea Publishing Co., Madison, WI, 1975.
[34] E. Hille, J.D. Tamarkin, Remarks on known example of a monotone continuous function, Amer. Math. Monthly 36 (1929) 255–264.
[35] T.-Y. Hu, K.-S. Lau, Fourier asymptotics of Cantor type measures at infinity, Proc. Amer. Math. Soc. 130 (9) (2002) 2711–2717.
[36] J.E. Hutchinson, Fractals and self–similarity, Indiana Univ. Math. J. 30 (5) (1981) 713–747.
[37] В.К. КаНоВеЙ АКсиоМа ВЫбора и АКсиоМа ДеМерМиНировaННосМи, МосКВа, НауКа, 1984 V.G. Kanoveı˘, The Axiom of Coice and the Axiom of Determinateness, Izdat, Nauka, Moskow, 1984.
[38] E.M. Kleinberg, Infinitary Combinatorics and the Axiom of Determinateness, Lecture Notes in Mathematics, vol. 612, Springer, Berlin, New York, 1977.
[39] K. Kuratowski, Topology, vol. 1, Academic Press, New York, London, 1966.
[40] K.S. Lau, J. Wang, Mean quadratic variations and Fourier asymptotics of self-similar measures, Monatsch Math. 115 (1993) 99–132.
[41] W. Li, D. Xiao, F.M. Dekking, Non-differentiability of devil's staircase and dimentios of subsets of Moran sets, Math. Proc. Cambridge Philos. Soc. 133 (2) (2002) 345–355.
[42] J. Morris, The Hausdorff dimension of the nondifferentiability set of a nonsymmetric Cantor function, Rocky Mountain J. Math. 32 (1) (2002) 357–370.
[43] A.P. Morse, Dini derivatives of continuous functions, Proc. Amer. Math. Soc. 5 (N1) (1954) 126–130.
[44] I. Niven, Irrational Numbers, Wiley, New York, 1956.
[45] J. Oxtoby, Measure and Category, Springer, New York, 1971.
[46] M.J. Pelling, Formular for the arc-length of curve in $\mathbb{R}^n$, Amer. Math. Monthly 84 (6) (1977) 465–467.
[47] В.А. РосЛИК, Д.Б. ФуКс, НахаЛЬНЫй Курс ТопоЛоιuu, ГеоМеМрихесКие ιЛaвЫ, МосКВа, НауКа, 1977 V.A. Rohlin, D.B. Fuks, A Begining Course in Topology: Geometrical Chapters, Izdat, Nauka, Moskow, 1977.
[48] S. Saks, Theory of the Integral, second revised ed, Dover Publications, New York, 1964.

[49] A. Salli, Upper density properties of Hausdorff measures on fractals, Ann. Acad. Sci. Fenn. Ser AI Math. Dissertationes 55 (1985).
[50] R. Strichartz, Fourier asymptotics of fractal measures, J. Funct. Anal. 89 (1990) 154–181.
[51] R. Strichartz, Self-similar measure and their Fourier transform I, Indiana Uni. Math. J. 39 (1990) 797–817.
[52] R. Strichartz, Self-similar measure and their Fourier transform II, Trans. Amer. Math. Soc. 336 (1) (1990) 335–361.
[53] B.C. Thomson, Symmetric Properties of Real Function, Marcel Dekker, New York, 1994.
[54] A.F. Timan, Theory of Approximation of Functions of a Real Variable, International Series of Monographs in Pure and Applied Mathematics, vol. 34, Pergamon Press, Oxford, 1963.
[55] J. Uher, Symmetric continuity implies continuity, Real Anal. Exchange 13 (1987/88) 36–38.
[56] N. Wiener, A. Wintner, On singular distributions, J. Math. Phys. 17 (1939) 233–346.
[57] N. Wiener, R.C. Young, The total variation of $g(x + h) - g(x)$, Trans. Amer. Math. Soc. 35 (1) (1933) 327–340.