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Stability aspects of the traveling salesman problem based on k-best solutions

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Abstract

This paper discusses stability analysis for the Traveling Salesman Problem (TSP). For a traveling salesman tour which is known to be optimal with respect to a given instance (length vector) we are interested in determining the stability region, i.e. the set of all length vectors for which the tour is optimal. The following three subsets of the stability region are of special interest: (1) tolerances, i.e. the maximum perturbations of single edges; (2) tolerance regions which are subsets of the stability region that can be constructed from the tolerances; and (3) the largest ball contained in the stability region centered at the given length vector (the corresponding radius is known as the stability radius). It is well known that the problems of determining tolerances and the stability radius for the TSP are $\mathcal{NP}$-hard so that in general it is not possible to obtain the above-mentioned three subsets without spending a lot of computation time. The question addressed in this paper is the following: assume that not only an optimal tour is known, but also a set of k shortest tours ($k \geqslant 2$) is given. Then to which extent does this allow us to determine the three subsets in polynomial time? It will be shown in this paper that having k-best solutions can give the desired information only partially. More precisely, it will be shown that only some of the tolerances can be determined exactly and for the other ones as well as for the stability radius only lower and/or upper bounds can be derived. Since the amount of information that can be derived from the set of k-best solutions is dependent on both the value of k as well as on the specific length vector, we present numerical experiments on instances from the TSPLIB library to analyze the effectiveness of our approach. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

Motivated by the lecture of A.H.G. Rinnooy Kan at the CORS-TIMS-ORSA 1989 conference in Vancouver BC, the Quantitative Logistics Research Group at the

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University of Groningen, in cooperation with researchers of the Nijenrode University and the Polish Academy of Sciences, started in 1992 a research project on sensitivity analysis in case of $\mathcal{NP}$-hard problems. As yet, the main attention was focused on the Traveling Salesman Problem (TSP), since this problem can be considered as a sort of paradigm for $\mathcal{NP}$-hard problems. Other results of the cooperation have appeared in [24–27].

For a tour which is known to be optimal with respect to a given length vector, the *stability problem* is defined as the problem of determining the extent to which edge lengths can be changed while preserving the optimality of the given solution. The set of all length vectors for which the given tour is optimal is called the *stability region*. In this paper we focus our attention on the following three subsets of the stability region: *tolerances*, these being the maximum perturbations of single edges preserving the optimality of the given tour, *tolerance regions*, these being subsets of the stability region that can be constructed from the tolerances or their lower bounds, and the *stability radius*, this being the radius of the largest ball in the stability region with center in the original length vector.

Most of the concepts in stability analysis already have a long tradition in combinatorial optimization and integer linear programming. We refer to Geoffrion and Nauss [3] and Jenkins [6] for early surveys and to the recent (1997) book by Gal and Greenberg [2]. Another excellent reference to post-solution analysis in integer linear programming and combinatorial optimization is the annotated bibliography by Greenberg [5]. In linear programming, the interval determined by the upper and lower tolerances is known as the optimality preserving range of the objective function coefficients (cf. [29]). The computational complexity of the tolerance problem for a large class of 0/1 programming and combinatorial optimization problems (including the TSP) is considered by Van Hoesel and Wagelmans [23] and Ramaswamy and Chakravarti [16], respectively. There, it is shown that the tolerance problem is as hard as the optimization problem itself. Consequently, the tolerance problem for the TSP is $\mathcal{NP}$-hard. Approximate values of the tolerances for the TSP are considered by Libura [13]. For a survey on the tolerance problem for the TSP, we refer to the work of Van der Poort et al. [26].

One of the earliest papers on the stability region and radius is the one of Leontev [12] discussing the stability region and radius for the TSP using slightly different definitions. More recently, two comprehensive surveys are presented by Sotskov et al. [20, 21], discussing the stability region and radius of optimal and approximate solutions for various discrete optimization problems. Algorithms for determining the stability radius for job shop scheduling problems are designed and analyzed by Sotskov et al. [22], and necessary and sufficient conditions for the existence of an infinitely large stability radius are presented by Kravchenko et al. [10]. Algorithms for determining the stability radii for polynomially solvable combinatorial optimization problems are considered by Chakravarti and Wagelmans [1]. Subsets of the stability region for the TSP are considered by Jones [8, 9] where the behavior of different heuristics for the Euclidean TSP is analyzed when cities are moved in the plane. Finally, we mention that in linear programming, the stability radius is known as the maximum tolerance of the objective function coefficients and the corresponding stability ball is called the tolerance region (cf. [30–32]).

It is well known that the problems of determining Tolerances and the Stability Radius are $\mathcal{NP}$-hard. The question addressed in this paper is the following: assume that not only an optimal tour is known, but also a set of k shortest tours ($k \geqslant 2$) is given. Then which information with respect to the tolerances and the stability region can be determined in polynomial time? So, in other words, we assume that the k-best TSP (the problem of determining a set of k shortest tours) has already been solved and it is the objective of this paper to explore the possibility of using this information for finding the tolerances and the stability radius (and therefrom information with respect to the stability region). Similar approaches for zero-one programming and zero–one goal programming are studied by Piper and Zoltners [15] and Wilson and Jain [33], respectively. Algorithms for solving the k-best TSP are considered by Van der Poort et al. [24].

Obviously, if the decision maker's main goal is to determine the tolerances and the stability region, first solving the k-best TSP and then trying to determine the desired information from the k-best solutions is not the most effective way of obtaining this information. However, it is also obvious that the set of k-best solutions contains at least some of the information with respect to sensitivity values. In fact, the contribution of this paper is that if for one reason or another the set of k-best solutions has already been determined, our results show what additional work remains to be done. It should also be mentioned that one of the main reasons for conducting this analysis is to explore the relation between two sensitivity-related issues (k-best and stability) for an $\mathcal{NP}$-hard problem. In a subsequent paper [27] we will discuss the reversed question, i.e.: How can stability information be used to solve the k-best TSP?

In this paper it will be shown that, unfortunately, having k-best solutions can only give partial information about the tolerances and the stability radius. More precisely, it will be shown that only some of the tolerances can be determined exactly and that for the other ones lower bounds can be derived (see Section 3). Also, we show how to derive upper and lower bounds for the stability radius (Section 4.2). Furthermore, it will be shown how this information can be used to derive subsets of the stability region (see Sections 3 and 4). Obviously, the amount and value of information that can be derived from the set of k-best solutions depends both on the value of k as well as on the specific length vector. Therefore, in order to analyze the effectiveness of our approach, we present numerical experiments on instances from Reinelt's TSPLIB library [17].

2. Definitions and basic results

For $n \geqslant 3$ and $1 \leqslant m \leqslant \binom{n}{2}$, consider the graph $G=(V,E)$ with the set of vertices $V=\{1,\dots,n\}$ and the set of edges $E=\{e_1,\dots,e_m\} \subseteq \{\{i,j\}\colon\ i,j \in V,\ i \neq j\}$. The length of edge e is a real number denoted by $d(e)$. The vector $d=[d(e_1),\dots,d(e_m)]^{\mathrm{T}} \in \mathbb{R}^m$ is

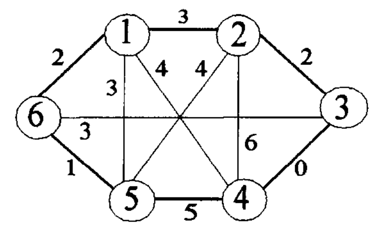

Fig. 1. The weighted graph (G,d).

called the *length vector* of the graph G and (G,d) a *weighted graph*. For any $S \subseteq E$, the length of S with respect to d is given by $L_d(S) := \sum_{e \in S} d(e)$. A *Hamiltonian tour* (a *tour* for short) in the graph G is a subset of E that forms a cycle containing each vertex in V exactly once. By $\mathscr{H}$ we denote the set of all tours in G. Using this notation, the TSP is the problem of finding a tour in the set $\arg\min\{L_d(H)\colon H \in \mathscr{H}\}$.

Let $1 \leqslant k \leqslant |\mathscr{H}|$. A set $\mathscr{H}(k) = \{H_{(1)}, \ldots, H_{(k)}\}$ of different tours in $\mathscr{H}$ satisfying

$$L_d(H_{(1)}) \leqslant L_d(H_{(2)}) \leqslant \cdots \leqslant L_d(H_{(k)}) \leqslant L_d(H) \quad \text{for all } H \in \mathscr{H} \setminus \mathscr{H}(k)$$

is called a *set of* k-best *tours*. The k-best TSP is defined as the problem of finding a set $\mathscr{H}(k)$ in $\mathscr{H}$ with respect to d. Observe that in general the set $\mathscr{H}(k)$ is not uniquely determined. In the extreme case, the so-called *constant TSP* [4], all tours have the same length and hence any subset of $\mathscr{H}$ with cardinality k is a set of k-best tours. The difference in length between a largest and a smallest tour in the set $\mathscr{H}(k)$ is denoted by L_k, i.e. $L_k := L_d(H_{(k)}) - L_d(H_{(1)})$. Observe that, unlike $\mathscr{H}(k)$, L_k is uniquely determined for given (G,d). Note that the set $\mathscr{H}(\lambda)$, with $\lambda := \max\{i \in \{1, \ldots, |\mathscr{H}|\}\colon L_i = 0\}$, is the set of all optimal tours. Obviously, $\mathscr{H}(\lambda)$ is uniquely determined for a given length vector. For any $\mathscr{S} \subseteq \mathscr{H}$, let $\bigcup \mathscr{S}$ denote the union of the sets in $\mathscr{S}$, and $\bigcap \mathscr{S}$ the intersection of the sets in $\mathscr{S}$. Using this notation, $\bigcap \mathscr{H}(k)$ denotes the set of edges that are contained in all tours of $\mathscr{H}(k)$ and $\bigcup \mathscr{H}(k)$ denotes the set of edges that are contained in at least one of the tours in $\mathscr{H}(k)$.

Throughout this paper we will use the following example.

Example. Fig. 1 shows a weighted graph (G,d) and all its tours are listed in order of nondecreasing length in Fig. 2. The set of 5-best tours is uniquely given by $\mathscr{H}(5) = \{H_{(1)}, H_{(2)}, \ldots, H_{(5)}\}$. Note that $L_5 = 4$, $\bigcap \mathscr{H}(5) = \{\{3,4\}\}$, and $\bigcup \mathscr{H}(5) = E$. $\square$

In the remainder of the paper we assume that, for a given integer k with $1 \leqslant k \leqslant |\mathscr{H}|$, the k-best TSP has been solved with respect to (G,d) and that $\mathscr{H}(k) = \{H_{(1)}, H_{(2)}, \ldots, H_{(k)}\}$.

Let $X = \{e_1, \ldots, e_{|X|}\}$ denote a nonempty subset of E. The *stability problem* is the problem of determining the maximum changes of the edge lengths in X while preserving the optimality of $H_{(1)}$. In the "simplest" case, X consists of only a single edge, and the corresponding problem is known as the tolerance problem [13]. More precisely, the *tolerance problem* for $e \in E$ is the problem of finding the maximum increase $u(e)$ and

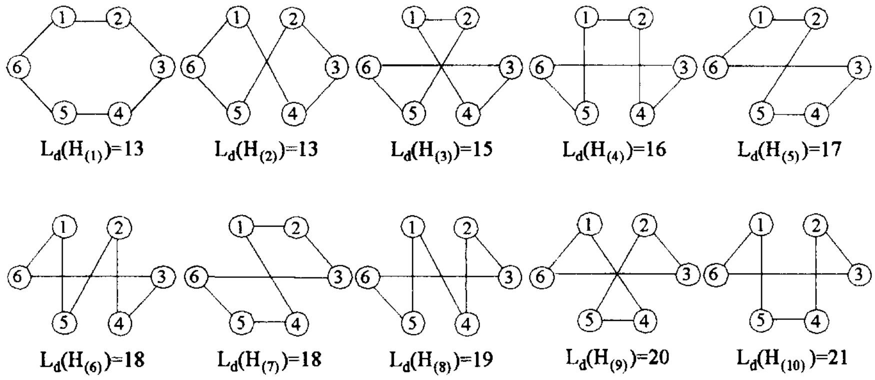

Fig. 2. All tours in (G,d) listed in order of nondecreasing length.

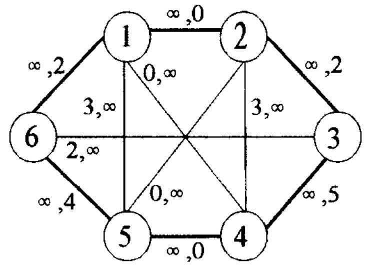

Fig. 3. Lower (first number) and upper (second number) tolerances.

the maximum decrease $l(e)$ in the edge length $d(e)$ preserving the optimality of $H_{(1)}$ under the assumption that the lengths of all other edges remain unchanged. The values $u(e)$ and $l(e)$ are called the *upper* and *lower tolerances* of edge e with respect to the $H_{(1)}$ and d. Note that these values may be infinite.

Example (continued). Fig. 3 gives the upper and lower tolerances with respect to $H_{(1)}$ for (G,d) in Fig. 1. For instance, $l(\{1,6\})=\infty$ and $u(\{1,6\})=2$.

Now consider the case that the length of several edges may be changed simultaneously, i.e. $|X|\geqslant 1$. In this case, it is of interest to have a description of the set of all length vectors for which the given solution of the TSP is optimal. The set of all such vectors is called the stability region. For any $c\in\mathbb{R}^m$, let $c_X\in\mathbb{R}^{|X|}$ denote the *restriction* of c to the edge set X defined by $c_X(e)=c(e)$ for each $e\in X$. The *stability region* with respect to the tour $H_{(1)}$ and the edge set X, denoted by $S(H_{(1)},X)$, is defined as the set of vectors $(d+\Delta)_X$ for which $H_{(1)}$ is an optimal solution in $(G,d+\Delta)$ and $\Delta\in\mathbb{R}^m$ satisfying $\Delta(e)=0$ for each $e\in E\backslash X$, i.e.

$$S(H_{(1)},X):=\left\{(d+\Delta)_X:\ \begin{array}{l}\Delta\in\mathbb{R}^m,\ \Delta(e)=0\quad\text{for each } e\in E\backslash X,\\ L_{d+\Delta}(H_{(1)})\leqslant L_{d+\Delta}(H)\quad\text{for all } H\in\mathscr{H}.\end{array}\right\}$$

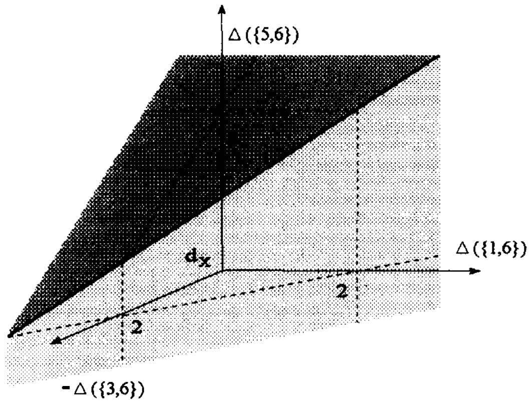

Fig. 4. The stability region $S(H_{(1)},X)$.

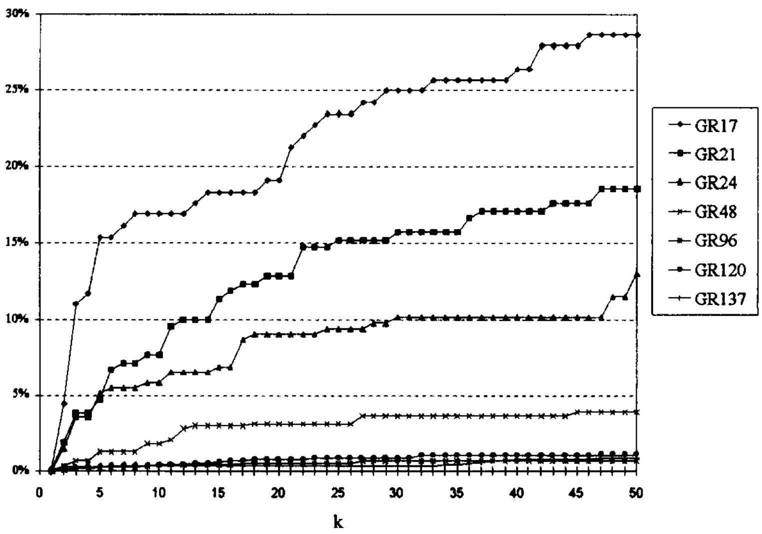

Fig. 5. Number of tolerances as percentage of the number of edges.

Considering the stability region for a single edge e in E just gives the upper and lower tolerances of edge e, i.e. if $e \in H_{(1)}$ then $-\infty = -l(e) \leqslant \Delta(e) \leqslant u(e)$ and $-l(e) \leqslant \Delta(e) \leqslant u(e) = \infty$ otherwise. Note that in Ward and Wendell [29] the stability region for linear programming is called the region of optimality.

Example (continued). Consider the edge set $X = \{\{1,6\},\{3,6\},\{5,6\}\}$ for (G,d) in Fig. 1. Note that $d_X = (2,3,1)^{\mathrm{T}}$. The conditions $L_{d+\Delta}(H_{(1)}) \leqslant L_{d+\Delta}(H)$ for all $H \in \mathscr{H}$

lead to the following stability region:

$$S(H_{(1)},X) = \left\{ \begin{pmatrix} 2 + \Delta(\{1,6\}) \\ 3 + \Delta(\{3,6\}) \\ 1 + \Delta(\{5,6\}) \end{pmatrix} : \begin{array}{l} \Delta(\{1,6\}) - \Delta(\{3,6\}) \leqslant 2, \\ \Delta(\{5,6\}) - \Delta(\{3,6\}) \leqslant 4. \end{array} \right\}$$

The stability region $S(H_{(1)},X)$ is shown in Fig. 4 using a coordinate system with origin d_X and axes corresponding to $\triangle(\{1,6\})$, $-\triangle(\{3,6\})$, and $\triangle(\{5,6\})$. Note that $S(H_{(1)},X)$ is described by two hyperplanes. For instance, the vector $d_X + [\Delta(\{1,6\}), \Delta(\{3,6\}), \triangle(\{5,6\})]^{\mathrm{T}} = d_X + [1,-1,1]^{\mathrm{T}}$ is contained in $S(H_{(1)},X)$.

Clearly, the stability region is a closed convex polyhedral set. More specifically, $S(H_{(1)},E)$ is the negative of the polar cone of the supporting cone at the extreme point $x^{H_{(1)}}$ of the Hamiltonian cycle polytope (see, e.g. Van der Poort [25]). Unfortunately, describing the TSP-polytope at a certain extreme point in terms of a (nonredundant) system of hyperplanes is not an "easy task", because this is more-or-less equivalent to finding the same kind of inequalities for the Hamiltonian cycle polytope, a topic that has been studied intensively in the literature; see e.g. [14]. In order to obtain some insights in the structure and usefulness of the stability region, we will concentrate on describing two types of interesting subsets of $S(H_{(1)},X)$, namely tolerance regions (see Section 4.1) and stability balls (see Section 4.2). The *tolerance region* with respect to X is defined as a subset of the stability region which can be derived from the tolerances of edges belonging to the set X. A *stability ball* is a ball in $S(H_{(1)},X)$ with center in d_X. We will show how the set of k-best tours can be used to describe both subsets.

3. Changing the length of a single edge

In this section we discuss how the set $\mathscr{H}(k)$ can be used to determine lower bounds and, in some cases, exact values of the tolerances $u(e)$ and $l(e)$. In [13], it has been shown that for each edge one of its tolerances is always infinite and the value of the other one can be calculated by determining the optimal value of an auxiliary instance of the TSP defined on a restricted set of tours. This is stated in the following theorem.

Theorem 1 (Libura [13]). *For $e \in H_{(1)}$ it holds that $l(e) = \infty$ and*

$$u(e) = \min\{L_d(H):\ H \in \mathscr{H},\ e \notin H\} - L_d(H_{(1)}).$$

For $e \in E \backslash H_{(1)}$ it holds that $u(e) = \infty$ and

$$l(e) = \min\{L_d(H):\ H \in \mathscr{H},\ e \in H\} - L_d(H_{(1)}).$$

The following corollary identifies the set of edges for which one of the tolerances is finite.

Corollary 2. $\{e \in E\colon u(e) < \infty$ *or* $l(e) < \infty\} = \bigcup \mathscr{H} \setminus \bigcap \mathscr{H}$.

Proof. It follows from Theorem 1 that $\{e \in E\colon u(e) < \infty \text{ or } l(e) < \infty\} = \{e \in H_{(1)}\colon u(e) < \infty\} \cup \{e \in E \setminus H_{(1)}\colon l(e) < \infty\}$. We have that $\{e \in H_{(1)}\colon u(e) < \infty\} = \{e \in H_{(1)}\colon \{H \in \mathscr{H}\colon e \notin H\} \neq \emptyset\} = \{e \in H_{(1)}\colon e \notin \bigcap \mathscr{H}\} = H_{(1)} \setminus \bigcap \mathscr{H}$. It can be shown in a similar way that $\{e \in E \setminus H_{(1)}\colon l(e) < \infty\} = \bigcup \mathscr{H} \setminus H_{(1)}$. Hence, we have that $\{e \in H_{(1)}\colon u(e) < \infty\} \cup \{e \in E \setminus H_{(1)}\colon l(e) < \infty\} = (H_{(1)} \setminus \bigcap \mathscr{H}) \cup (\bigcup \mathscr{H} \setminus H_{(1)}) = \bigcup \mathscr{H} \setminus \cap \mathscr{H}$, which completes the proof. □

From now on, we will only consider the upper tolerances for edges in $H_{(1)}$ and the lower tolerances for edges in $E \setminus H_{(1)}$, since according to Theorem 1 all other tolerances are infinite. Note that, for each $e \in E$, the situation $u(e), l(e) < \infty$ is impossible.

As a corollary of Theorem 1, we can use the set $\mathscr{H}(k)$ to obtain the following tolerances and lower bound. It follows that for all edges that are contained in at least one tour in $\mathscr{H}(k)$ but not in all, the tolerances do not depend on $\mathscr{H} \setminus \mathscr{H}(k)$. For the other edges, $\mathscr{H}(k)$ gives a lower bound on the finite tolerance.

Corollary 3. *For each* $e \in H_{(1)}$ *and* $2 \leqslant k \leqslant |\mathscr{H}|$ *it holds that*

$$\begin{aligned} u(e) &= \min\{L_d(H)\colon H \in \mathscr{H}(k),\ e \notin H\} - L_d(H_{(1)}) && \textit{if } e \in H_{(1)} \setminus \textstyle\bigcap \mathscr{H}(k), \\ u(e) &\geqslant L_k && \textit{otherwise.} \end{aligned}$$

For each $e \in E \setminus H_{(1)}$ *and* $2 \leqslant k \leqslant |\mathscr{H}|$ *it holds that*

$$\begin{aligned} l(e) &= \min\{L_d(H)\colon H \in \mathscr{H}(k),\ e \in H\} - L_d(H_{(1)}) && \textit{if } e \in \textstyle\bigcup \mathscr{H}(k), \\ l(e) &\geqslant L_k && \textit{otherwise.} \end{aligned}$$

The inequalities are strict when $\mathscr{H}(k)$ *is unique.*

Proof. We will prove the corollary only for the upper tolerances as the proof for the lower tolerances is similar. Let $e \in H_{(1)}$. Recall from Theorem 1 that $u(e) = \min\{L_d(H)\colon H \in \mathscr{H},\ e \notin H\} - L_d(H_{(1)})$. If $e \notin \bigcap \mathscr{H}(k)$ then there is a $H \in \mathscr{H}(k)$ such that $e \notin H$. Consequently, $\min\{L_d(H)\colon H \in \mathscr{H},\ e \notin H\} = \min\{L_d(H)\colon H \in \mathscr{H}(k),\ e \notin H\}$, because $L_d(H) \geqslant L_d(H_{(k)})$ for all $H \in \mathscr{H} \setminus \mathscr{H}(k)$. Hence, $u(e) = \min\{L_d(H)\colon H \in \mathscr{H}(k),\ e \notin H\} - L_d(H_{(1)})$. If $e \in \bigcap \mathscr{H}(k)$ then it follows from the definition of $\mathscr{H}(k)$ that $\min\{L_d(H)\colon H \in \mathscr{H},\ e \notin H\} \geqslant L_d(H_{(k)})$, and consequently $u(e) \geqslant L_d(H_{(k)}) - L_d(H_{(1)}) = L_k$. Finally, recall that if $\mathscr{H}(k)$ is unique then it holds for all $H \in \mathscr{H} \setminus \mathscr{H}(k)$ that $L_d(H) > L_d(H_{(k)})$, so that $u(e) > L_k$ for each $e \in \bigcap \mathscr{H}(k)$. □

As a direct consequence of Corollary 3, it is possible to determine the upper and lower tolerances for some of the edges in E. Let $E^{\mathrm{u}}_{\mathscr{H}(k)} := H_{(1)} \setminus \bigcap \mathscr{H}(k)$ and $E^{\mathrm{l}}_{\mathscr{H}(k)} := \bigcup \mathscr{H}(k) \setminus H_{(1)}$ denote the sets of edges in E for which Corollary 3 yields the exact values of the upper and lower tolerances, respectively. For the tolerances of all other edges in E, Corollary 3 gives a lower bound. Note that this lower bound is edge independent.

Example (continued). Consider $\mathscr{H}(5)$ in Fig. 1. In this case all lower tolerances can be determined exactly by using Corollary 3: $l(\{1,4\})=0$, $l(\{1,5\})=3$, $l(\{2,4\})=3$, $l(\{2,5\})=0$ and $l(\{3,6\})=2$. Also, all except one of the upper tolerances can be determined exactly: $u(\{1,2\})=0$, $u(\{1,6\})=2$, $u(\{2,3\})=2$, $u(\{4,5\})=0$ and $u(\{5,6\})=4$. Furthermore, since $\mathscr{H}(5)$ is unique, we have that $u(\{3,4\})>L_5=4$. Hence, $E^{\mathrm{u}}_{\mathscr{H}(5)}=H_{(1)}\backslash\{\{3,4\}\}$ and $E^{\mathrm{l}}_{\mathscr{H}(5)}=E\backslash H_{(1)}$. □

The number of tolerances that can be determined from a set of k-best tours is a nondecreasing function in k. Obviously, it is of interest to know the smallest value of k for which all finite tolerances can be determined from the set of k-best tours. In the following theorem this value is characterized in terms of the set $\mathscr{H}$.

Theorem 4. *Let* $k^*:=\min\{k\colon \bigcap\mathscr{H}(k)=\bigcap\mathscr{H}$ *and* $\bigcup\mathscr{H}(k)=\bigcup\mathscr{H}$ *for all* $\mathscr{H}(k)\subseteq\mathscr{H}\}$. *Then*

$$\{e\in E\colon u(e)<\infty \text{ or } l(e)<\infty\}=E^{\mathrm{u}}_{\mathscr{H}(k^*)}\cup E^{\mathrm{l}}_{\mathscr{H}(k^*)}.$$

Proof. Recall from Corollary 2 that $\{e\in E\colon u(e)<\infty \text{ or } l(e)<\infty\}=\bigcup\mathscr{H}\backslash\bigcap\mathscr{H}$. Take any $\mathscr{H}(k^*)\subseteq\mathscr{H}$. From the definition of k^*, it follows that $\bigcup\mathscr{H}\backslash\bigcap\mathscr{H}=\bigcup\mathscr{H}(k^*)\backslash\bigcap\mathscr{H}(k^*)$, which can be written as $(H_{(1)}\backslash\bigcap\mathscr{H}(k^*))\cup(\bigcup\mathscr{H}(k^*)\backslash H_{(1)})=E^{\mathrm{u}}_{\mathscr{H}(k^*)}\cup E^{\mathrm{l}}_{\mathscr{H}(k^*)}$. □

Example (continued). Consider (G,d) in Fig. 1. Note that $\mathscr{H}(7)=\{H_{(1)},\ldots,H_{(7)}\}$ is unique and that $\bigcup\mathscr{H}(7)=\bigcup\mathscr{H}$ and $\bigcap\mathscr{H}(7)=\bigcap\mathscr{H}$. For each $1\leqslant k\leqslant 6$, there is a $\mathscr{H}(k)\subseteq\mathscr{H}$ such that $\bigcup\mathscr{H}(k)\neq\bigcup\mathscr{H}$ or $\bigcap\mathscr{H}(k)\neq\bigcap\mathscr{H}$. Hence, $k^*=7$.

It follows from the definition of $\mathscr{H}(k)$ that $1\leqslant k^*\leqslant|\mathscr{H}|$. In the following theorem, it is shown that the existence of a polynomial algorithm for determining k^* in a Hamiltonian graph is very unlikely.

Theorem 5. *There is no polynomial algorithm for determining* k^* *in a Hamiltonian graph unless* $\mathscr{P}=\mathscr{NP}$.

Proof. Suppose that there is an algorithm, say A, with input a Hamiltonian graph G and length vector d, and output k^*. We use algorithm A to solve the Second Hamiltonian Cycle Problem, i.e. the problem of determining whether a graph G with a given tour H contains a second tour. Let $d(e):=0$ for all $e\in G$. We first consider the case that $\mathscr{H}=\{H\}$. Since $\bigcap\mathscr{H}(1)=H=\bigcap\mathscr{H}$ and $\bigcup\mathscr{H}(1)=H=\bigcup\mathscr{H}$, it follows that $k^*=1$. Now consider the case that $\mathscr{H}\neq\{H\}$. Then, by definition of k^*, it follows that $k^*>1$. Hence, we can use algorithm A to solve the Second Hamiltonian Cycle Problem. However, the Second Hamiltonian Cycle Problem is $\mathscr{NP}$-hard [7], so there is no polynomial algorithm for determining k^* unless $\mathscr{P}=\mathscr{NP}$. □

Note that k^* is determined by the weighted graph (G,d). The following theorem gives an upper bound for k^* in the complete graph K_n that is independent of d.

Theorem 6. *For the complete graph K_n ($n \geqslant 3$), it holds that $k^* \leqslant (n-2)!+1$ if $n \leqslant 5$ and $k^* \leqslant \frac{1}{2}(n-3)(n-2)!+1$ if $n \geqslant 5$. This bound is sharp.*

Proof. Take any $a \in \bigcup \mathscr{H} \backslash H_{(1)}$. The smallest value of k for which $a \in \bigcup \mathscr{H}(k)$ for all $\mathscr{H}(k) \subseteq \mathscr{H}$ is as large as possible when all tours not containing a are shorter than all tours containing a. For each $e \in E$, define $d(e) := 1$ if $e = a$ and $d(e) := 0$ otherwise. Clearly, $L_d(H) = 0$ for all $H \in \mathscr{H}$ with $a \notin H$ and $L_d(H) = 1$ for all $H \in \mathscr{H}$ with $a \in H$. We first determine the cardinality of the set $\{H \in \mathscr{H}\colon\ a \in H\}$. By contracting the edge a into a single vertex we obtain the complete graph K_{n-1} containing $\frac{1}{2}(n-2)!$ tours. Since a can be traversed in two directions, we have that $|\{H \in \mathscr{H}\colon\ a \in H\}| = 2 \cdot \frac{1}{2}(n-2)! = (n-2)!$. Hence, $|\{H \in \mathscr{H}\colon\ a \notin H\}| = |\mathscr{H}| - |\{H \in \mathscr{H}\colon\ a \in H\}| = \frac{1}{2}(n-1)! - (n-2)! = \frac{1}{2}(n-3)(n-2)!$ Consequently, $a \in \bigcup \mathscr{H}(k)$ for all $\mathscr{H}(k) \subseteq \mathscr{H}$ for $k > \frac{1}{2}(n-3)(n-2)!$. It can be shown in a similar way that, for any $a \in \bigcap \mathscr{H}$, $a \in \bigcap \mathscr{H}(k)$ for all $\mathscr{H}(k) \subseteq \mathscr{H}$ for $k > (n-2)!$. Consequently, if $n \leqslant 5$ then $\bigcap \mathscr{H}(k) = \bigcap \mathscr{H}$ and $\bigcup \mathscr{H}(k) = \bigcup \mathscr{H}$ for all $\mathscr{H}(k) \subseteq \mathscr{H}$ when $k \geqslant (n-2)!+1$, and if $n \geqslant 5$ then $\bigcap \mathscr{H}(k) = \bigcap \mathscr{H}$ and $\bigcup \mathscr{H}(k) = \bigcup \mathscr{H}$ for all $\mathscr{H}(k) \subseteq \mathscr{H}$ when $k \geqslant \frac{1}{2}(n-3)(n-2)!+1$. It follows from the constructed length vector that this bound is sharp. □

Since the lower bounds and the exact values of the tolerances obtained by applying Corollary 3 depend on both the value of k as well as on the specific length vector d and the number of cities n, we conclude this section by presenting the results of numerical experiments on instances from the TSPLIB library [17]. In order to reduce the amount of computational work, we have chosen to consider only the lower bounds and the exact values of the upper tolerances. For each of the selected instances, the k-best TSP is solved with k ranging from 1 to 50 and the upper tolerances that cannot be obtained from Corollary 3 are calculated by applying Theorem 1. The corresponding auxiliary instances are solved by applying the branch-and-bound program of Volgenant and Jonker [28] after modifying the length vector in an appropriate way in order to restrict the set of tours to not contain the edge for which the upper tolerance is calculated. The number of tolerances that is obtained as a percentage of the number of edges in the complete graph is shown in Fig. 5. Unfortunately, the average over all edges in $H_{(1)}$ of the lower bound as fraction of the exact values of the tolerances need not be a nondecreasing function in k as some of the edges obtain their exact tolerance value when k increases. We have therefore decided to analyze the lower bounds by considering the so-called "quality of the approximation" of the upper tolerance, being defined equal to 1 if the exact value of the tolerance has been determined and equal to $L_k/u(e)$ otherwise. Fig. 6 shows the average over all edges in $H_{(1)}$ of the quality of the approximations of the upper tolerances as a function of k. More detailed information on the sets $\mathscr{H}_{(k)}$ can be obtained from the authors.

It can be observed that the increase of the number of tolerances and the average quality of the approximations of the upper tolerances is the highest for small values of k. Actually, the growth of the number of tolerances and the average quality of the approximations of the upper tolerances slows down for increasing values of k. This

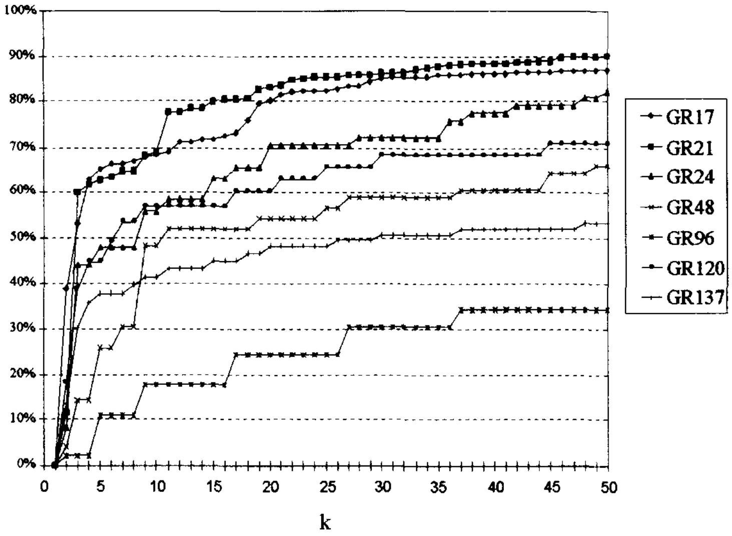

Fig. 6. Average quality of the approximations of the upper tolerances.

may suggest that it does not make much sense to increase the value of k too much. This phenomenon was already observed in Geoffrion and Nauss [3] for using k-best solutions in sensitivity analysis for 0/1-programming. Finally, note that for a given value of k the average quality of the approximations of the upper tolerances is mainly being influenced by the specific length vector and to a lesser degree by the number of cities.

4. Changing the length of several edges

In this section we consider subsets X of E consisting of several edges. It is shown that a set of k-best solutions can be used to construct subsets of the tolerance region and to give lower and upper bounds for the stability radius. We start with a basic result that plays an important role when considering subsets of the stability region. Recall that λ has been defined as the cardinality of the set of optimal solutions. The set of edges that are contained in some of the optimal tours, but not in all, is denoted by Z, i.e. $Z=\bigcup\mathscr{H}(\lambda)\setminus\bigcap\mathscr{H}(\lambda)$. Note that Z is nonempty when $\lambda\geqslant 2$. The following theorem gives a characterization of Z in terms of the tolerances and the set $\mathscr{H}(\lambda)$.

Theorem 7. *Let $H_{(1)}\in\mathscr{H}(\lambda)$, and for $e\in E$, let $u(e)$ and $l(e)$ denote the upper and lower tolerance of edge e with respect to $H_{(1)}$. Then*
1. $Z=E^{\mathrm{u}}_{\mathscr{H}(\lambda)}\cup E^{\mathrm{l}}_{\mathscr{H}(\lambda)}$, *and*
2. $Z=\{e\in E$: *either* $u(e)=0$ *or* $l(e)=0\}$.

Proof. (1) We have that $Z=\bigcup\mathscr{H}(\lambda)\setminus\bigcap\mathscr{H}(\lambda)=(\bigcup\mathscr{H}(\lambda)\setminus H_{(1)})\cup(H_{(1)}\setminus\bigcap\mathscr{H}(\lambda))=E^{\mathrm{l}}_{\mathscr{H}(\lambda)}\cup E^{\mathrm{u}}_{\mathscr{H}(\lambda)}$.

(2) We first show that $Z \subseteq \{e \in E$: either $u(e)=0$ or $l(e)=0\}$. Take any $e \in Z$. If $e \in E^{\mathrm{u}}_{\mathscr{H}(\lambda)}$ then it follows from Theorem 1 and Corollary 3 that $u(e)=0$ and $l(e)=\infty$. Similarly, if $e \in E^{\mathrm{l}}_{\mathscr{H}(\lambda)}$ then it follows from Theorem 1 and Corollary 3 that $l(e)=0$ and $u(e)=\infty$. Hence, $Z \subseteq \{e \in E$: either $u(e)=0$ or $l(e)=0\}$.

Now we show that also $Z \supseteq \{e \in E$: either $u(e)=0$ or $l(e)=0\}$. Take any $e \in H_{(1)}$ such that $u(e)=0$ and $l(e)=\infty$. Suppose, to the contrary, that $e \notin E^{\mathrm{u}}_{\mathscr{H}(\lambda)}$. Because L_λ is unique, we have from Corollary 3 that $u(e)>L_\lambda=0$, which is a contradiction. The case that $l(e)=0$ and $u(e)=\infty$ for $e \in E\backslash H_{(1)}$ can be proven in a similar way. Hence, $Z \supseteq \{e \in E$: either $u(e)=0$ or $l(e)=0\}$. This completes the proof. □

Example (continued). For (G,d) in Fig. 1, we have that $\lambda=2$ and $Z=E^{\mathrm{u}}_{\mathscr{H}(2)} \cup E^{\mathrm{l}}_{\mathscr{H}(2)} = \{\{1,2\},\{4,5\}\} \cup \{\{1,4\},\{2,5\}\}$.

4.1. Tolerance regions

Tolerance regions are subsets of the stability region that can be constructed from the tolerances. They are polyhedral sets, and their representations will be given in terms of extreme points and directions, and halfspaces. Moreover, it will be shown how the set of k-best solutions can be used to construct subsets of the tolerance regions.

Let $X_{\mathrm{f}} := X \cap (\bigcup \mathscr{H} \backslash \bigcap \mathscr{H})$ denote the subset of edges in X having one finite tolerance and let $X_\infty := X\backslash X_{\mathrm{f}}$ denote the subset of edges in X having no finite tolerance. Define $v_e, r_e \in \mathbb{R}^{|X|}$ for each $e \in X \cap H_{(1)}$ via their components $v_e(a)$, $r_e(a)$, $a \in X$, by

$$v_e(a) := \begin{cases} d(e)+u(e) & \text{if } a=e \text{ and } e \in X_{\mathrm{f}} \\ d(a) & \text{otherwise} \end{cases} \quad \text{and} \quad r_e(a) := \begin{cases} -1 & \text{if } a=e, \\ 0 & \text{otherwise,} \end{cases}$$

and for each $e \in X\backslash H_{(1)}$ and $a \in X$ by

$$v_e(a) := \begin{cases} d(e)-l(e) & \text{if } a=e \text{ and } e \in X_{\mathrm{f}} \\ d(a) & \text{otherwise} \end{cases} \quad \text{and} \quad r_e(a) := \begin{cases} 1 & \text{if } a=e, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $v_e = d_X$ for each $e \in X_\infty$. The *tolerance region* with respect to $H_{(1)}$ and X, denoted by $T(H_{(1)},X)$, is defined as the closure of the convex hull of the union of the halflines $\{v_e+\mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X_{\mathrm{f}}$, and the lines $\{d_X+\mu_e r_e\colon \mu_e \in \mathbb{R}\}$ for $e \in X_\infty$, i.e.

$$T(H_{(1)},X) := \operatorname{clconv}\Big(\bigcup_{e \in X_{\mathrm{f}}} \{v_e+\mu_e r_e\colon \mu_e \geqslant 0\} \bigcup_{e \in X_\infty} \{d_X+\mu_e r_e\colon \mu_e \in \mathbb{R}\}\Big),$$

with 'clconv' denoting the closure of the convex hull. Note that the closure of the convex hull of the (half) lines is really necessary as this convex hull is not closed by itself. The following theorem shows that a tolerance region is a subset of the stability region.

Theorem 8. *Let $H_{(1)} \in \mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Then $T(H_{(1)},X) \subseteq S(H_{(1)},X)$. If $|X|=1$ then $T(H_{(1)},X)=S(H_{(1)},X)$.*

Proof. Clearly, the halfline $\{v_e+\mu_e r_e\colon \mu_e \geqslant 0\}$, for $e \in X_f$, and the line $\{d_X+\mu_e r_e\colon \mu_e \in \mathbb{R}\}$, for $e \in X_\infty$, are subsets of the stability region $S(H_{(1)},X)$. Hence, the union over all halflines $\{v_e + \mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X_f$, and lines $\{d_X + \mu_e r_e\colon \mu_e \in \mathbb{R}\}$ for $e \in X_\infty$, belongs to the stability region. Since the stability region is a closed convex set, also closures of convex hulls of subsets belongs to it. Hence, the closure of the convex hull of the union of the halflines $\{v_e + \mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X_f$, and the lines $\{d_X + \mu_e r_e\colon \mu_e \in \mathbb{R}\}$ for $e \in X_\infty$, is a subset of $S(H_{(1)},X)$. In the case that X consists of a single edge, then $T(H_{(1)},X)$ is equal to the corresponding (half)line and hence $T(H_{(1)},X)=S(H_{(1)},X)$. □

Example (continued). Consider the edge set $X=\{\{1,6\},\{3,6\},\{5,6\}\}$ for (G,d) in Fig. 1. Note that $X_f=X$, $X_\infty=\emptyset$, and $d_X=(2,3,1)^T$. We have that $v_{\{1,6\}}=(4,3,1)^T$, $v_{\{3,6\}}=(2,1,1)^T$, $v_{\{5,6\}}=(2,3,5)^T$, $r_{\{1,6\}}=(-1,0,0)^T$, $r_{\{3,6\}}=(0,1,0)^T$, and $r_{\{5,6\}}=(0,0,-1)^T$. Then

$$T(H_{(1)},X)=\mathrm{clconv}\begin{pmatrix} \{(4-\mu_{\{1,6\}},3,1)^T\colon \mu_{\{1,6\}} \geqslant 0\}, \\ \{(2,1+\mu_{\{3,6\}},1)^T\colon \mu_{\{3,6\}} \geqslant 0\}, \\ \{(2,3,5-\mu_{\{5,6\}})^T\colon \mu_{\{5,6\}} \geqslant 0\} \end{pmatrix},$$

see Fig. 7. The halflines are indicated with bold arrows and the stability region is indicated with dashed lines (see also Fig. 4). Note that $S(H_{(1)},X)$ is larger than $T(H_{(1)},X)$.

In the following theorem the extreme points and directions of $T(H_{(1)},X)$ are established.

Theorem 9. *Let $H_{(1)} \in \mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Then, the following assertions hold.*
1. *$\{r_e\colon e \in X\} \cup \{-r_e\colon e \in X_\infty\}$ is the set of extreme directions of $T(H_{(1)},X)$.*
2. *If $X_\infty=\emptyset$ and $X \subseteq Z$ then d_X is the only extreme point of $T(H_{(1)},X)$.*
3. *If $X_\infty=\emptyset$ and $X \nsubseteq Z$ then $\{v_e\colon e \in X \setminus Z\}$ is the set of all extreme points of $T(H_{(1)},X)$.*

Proof. (1) This is obvious because $T(H_{(1)},X)$ is the closure of the convex hull of the union of halflines $\{v_e + \mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X_f$, and the lines $\{d_X + \mu_e r_e\colon \mu_e \in \mathbb{R}\}$ for $e \in X_\infty$, on the coordinate axes.

(2) Since for the edges e in Z either $u(e)=0$ or $l(e)=0$, it follows that all $|X|$ halflines have endpoint d_X. Hence, d_X is the only extreme point of $T(H_{(1)},X)$.

(3) $\{v_e\colon e \in X \setminus Z\}$ is the set of extreme points of $T(H_{(1)},X)$, since $T(H_{(1)},X)$ is the closure of the convex hull of $|X \cap Z|$ halflines with endpoint d_X, and the other $|X \setminus Z|$ halflines, which are all pairwise perpendicular, contain d_X in their interiors. □

As a direct consequence of Theorem 9, $T(H_{(1)},X)$ is equal to the set (see, e.g. [18, 19]).

$$\Bigg\{ (d+\Delta)_X\colon\ \Delta\in\mathbb{R}^m, \Delta(e)=0 \text{ for all } e\in E\backslash X,$$

$$d(a)+\Delta(a)=\sum_{e\in X_{\mathrm{f}}}\lambda_e v_e(a)+\sum_{e\in X_{\mathrm{f}}}\mu_e r_e(a) \text{ for all } a\in X_{\mathrm{f}}$$

$$\text{with } \lambda_e,\mu_e\geqslant 0 \text{ for all } e\in X_{\mathrm{f}} \text{ and} \sum_{e\in X_{\mathrm{f}}}\lambda_e=1\Bigg\}.$$

Furthermore, the following corollary gives necessary and sufficient conditions for $T(H_{(1)},X)$ being a pointed cone.

Corollary 10. *$T(H_{(1)},X)$ is a pointed cone if and only if $X_\infty=\emptyset$ and $|X\backslash Z|\leqslant 1$.*

Proof. Clearly, $T(H_{(1)},X)$ is a pointed cone if $X_\infty=\emptyset$ and either $|X\backslash Z|=1$ or $X\subseteq Z$. Moreover, if $T(H_{(1)},X)$ is a pointed cone then $X_\infty=\emptyset$ and $T(H_{(1)},X)$ has precisely one extreme point. This is the case when either $|X\backslash Z|=1$ or $X\subseteq Z$. □

In the following theorem the polyhedral representation of $T(H_{(1)},X)$ is revealed.

Theorem 11. *Let $H_{(1)}\in\mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Then $T(H_{(1)},X)$ is the set of vectors $(d+\Delta)_X$, with $\Delta\in\mathbb{R}^m$ and $\Delta(e)=0$ for all $e\in E\backslash X$, satisfying the following nonredundant system of inequalities*

$$\Delta(e)\leqslant u(e) \quad \textit{for } e\in X_{\mathrm{f}}\cap H_{(1)}, \tag{1a}$$

$$\Delta(e)\geqslant -l(e) \quad \textit{for } e\in X_{\mathrm{f}}\backslash H_{(1)}, \tag{1b}$$

$$\sum_{e\in Y\cap H_{(1)}}\frac{\Delta(e)}{u(e)}-\sum_{e\in Y\backslash H_{(1)}}\frac{\Delta(e)}{l(e)}\leqslant 1 \quad \textit{for } Y\subseteq X_{\mathrm{f}}\backslash Z, |Y|\geqslant 2. \tag{1c}$$

Note that, as to be expected, there are no constraints on the components of Δ corresponding to $e\in X_\infty$.

Proof. We first prove that any vector in $T(H_{(1)},X)$ satisfies (1). Take any $\Delta\in\mathbb{R}^m$, with $\Delta(e)=0$ for all $e\in E\backslash X$, such that $(d+\Delta)_X\in T(H_{(1)},X)$. As a consequence of Theorem 9, there exist $\lambda_e,\mu_e\geqslant 0$ for all $e\in X_{\mathrm{f}}$ with $\sum_{e\in X_{\mathrm{f}}}\lambda_e=1$ such that $d(a)+\Delta(a)=\sum_{e\in X_{\mathrm{f}}}\lambda_e v_e(a)+\sum_{e\in X_{\mathrm{f}}}\mu_e r_e(a)$ for all $a\in X_{\mathrm{f}}$. Then for each $e\in X_{\mathrm{f}}$, it follows from the definitions $v_e(e)$ and $r_e(e)$ that

$$\Delta(e)=\begin{cases}\lambda_e u(e)-\mu_e & \text{if } e\in X_{\mathrm{f}}\cap H_{(1)},\\ -\lambda_e l(e)+\mu_e & \text{if } e\in X_{\mathrm{f}}\backslash H_{(1)}.\end{cases}$$

Hence, for each $e\in X_{\mathrm{f}}\cap H_{(1)}$ we have that $\Delta(e)\leqslant u(e)$, because $\lambda_e\leqslant 1$ and $\mu_e\geqslant 0$. This proves (1a). Similarly, for each $e\in X_{\mathrm{f}}\backslash H_{(1)}$ it holds that $\Delta(e)\geqslant -l(e)$ which proves

(1b). In order to prove (1c), take any $Y \subseteq X_f \backslash Z$, $|Y| \geqslant 2$. Then we find that

$$\sum_{e \in Y \cap H_{(1)}} \frac{\Delta(e)}{u(e)} - \sum_{e \in Y \backslash H_{(1)}} \frac{\Delta(e)}{l(e)} = \sum_{e \in Y \cap H_{(1)}} \left(\lambda_e - \frac{\mu_e}{u(e)}\right) + \sum_{e \in Y \backslash H_{(1)}} \left(\lambda_e - \frac{\mu_e}{l(e)}\right)$$

$$= \sum_{e \in Y} \lambda_e - \sum_{e \in Y \cap H_{(1)}} \frac{\mu_e}{u(e)} - \sum_{e \in Y \backslash H_{(1)}} \frac{\mu_e}{l(e)} \leqslant 1,$$

since $\sum_{e \in Y} \lambda_e \leqslant 1$, $\sum_{e \in Y \cap H_{(1)}} \mu_e/u(e) \geqslant 0$, and $\sum_{e \in Y \backslash H_{(1)}} \mu_e/l(e) \geqslant 0$. This proves (1c), and hence $(d+\Delta)_X$ satisfies (1).

Take any $\Delta \in \mathbb{R}^m$ with $\Delta(e)=0$ for all $e \in E \backslash X$. We will show that if $(d+\Delta)_X$ satisfies Eq. (1) then there are for each $e \in X_f$ scalars λ_e and μ_e with $\lambda_e, \mu_e \geqslant 0$ and $\sum_{e \in X_f} \lambda_e = 1$ such that $d(a) + \Delta(a) = \sum_{e \in X_f} \lambda_e v_e(a) + \sum_{e \in X_f} \mu_e r_e(a)$ for all $a \in X_f$. Define

$$T := \{e \in X_f \cap H_{(1)}\colon\ \Delta(e) > 0\} \cup \{e \in X_f \backslash H_{(1)}\colon\ \Delta(e) < 0\}.$$

Note that $T = \emptyset$ when $X_f \subseteq Z$, because from (1) it follows that $\Delta(e) \leqslant u(e) = 0$ for each $e \in X_f \cap H_{(1)}$ and $\Delta(e) \geqslant -l(e) = 0$ for each $e \in X_f \backslash H_{(1)}$. We will show that the scalars λ_e and μ_e exist for each $e \in X_f$ by distinguishing the cases $T = \emptyset$ and $T \neq \emptyset$. We first consider the case that $T = \emptyset$. First assume that $X_f \cap H_{(1)} \neq \emptyset$, and let $a \in X_f \cap H_{(1)}$. Define

$$\lambda_e := \begin{cases} 1 & \text{if } e = a, \\ 0 & \text{if } e \in X_f \backslash \{a\} \end{cases} \quad \text{and} \quad \mu_e := \begin{cases} u(a) - \Delta(a) & \text{if } e = a, \\ -\Delta(e) & \text{if } e \in (X_f \cap H_{(1)}) \backslash \{a\}, \\ \Delta(e) & \text{if } e \in X_f \backslash H_{(1)}. \end{cases}$$

Then, $\sum_{e \in X_f} \lambda_e v_e(a) + \sum_{e \in X_f} \mu_e r_e(a) = d(a) + u(a) - (u(a) - \Delta(a)) = d(a) + \Delta(a)$, and for each $b \in X_f \backslash \{a\}$, it holds that $\sum_{e \in X_f} \lambda_e v_e(b) + \sum_{e \in X_f} \mu_e r_e(b) = d(b) + \mu_b r_b(b) = d(b) + \Delta(b)$. Clearly, $\sum_{e \in X_f} \lambda_e = 1$. Furthermore, since $\Delta(e) \leqslant 0$ for $e \in X_f \cap H_{(1)}$ and $\Delta(e) \geqslant 0$ for $e \in X_f \backslash H_{(1)}$, it follows that $\mu_e \geqslant 0$ for all $e \in X_f$. The case $X_f \cap H_{(1)} = \emptyset$ can be proven in a similar way.

Now consider the case that $T \neq \emptyset$. Let

$$\alpha := \left[\sum_{e \in T \cap H_{(1)}} \frac{\Delta(e)}{u(e)} - \sum_{e \in T \backslash H_{(1)}} \frac{\Delta(e)}{l(e)}\right]^{-1}.$$

Note that α is well defined, because $u(e) \geqslant \Delta(e) > 0$ for all $e \in T \cap H_{(1)}$ and $-l(e) \leqslant \Delta(e) < 0$ for all $e \in T \backslash H_{(1)}$. Moreover, it follows from (1) that $\alpha \geqslant 1$. Note that all parts of (1) are needed in order to prove that $\alpha \geqslant 1$, because the restrictions (1a) and (1b) are needed for the case that $|T|=1$. Let

$$\lambda_e := \begin{cases} \dfrac{\Delta(e)}{u(e)}\alpha & \text{if } e\in T\cap H_{(1)}, \\ -\dfrac{\Delta(e)}{l(e)}\alpha & \text{if } e\in T\backslash H_{(1)}, \\ 0 & \text{if } e\in X_{\mathrm{f}}\backslash T, \end{cases}$$

and

$$\mu_e := \begin{cases} |\Delta(e)|(\alpha-1) & \text{if } e\in T, \\ -\Delta(e) & \text{if } e\in (X_{\mathrm{f}}\cap H_{(1)})\backslash T, \\ \Delta(e) & \text{if } e\in (X_{\mathrm{f}}\backslash H_{(1)})\backslash T. \end{cases}$$

Clearly, $\sum_{e\in X_{\mathrm{f}}}\lambda_e=1$. Moreover, $\mu_e\geqslant 0$ for all $e\in T$, since $\alpha\geqslant 1$. Similarly, $\mu_e\geqslant 0$ for all $e\in X_{\mathrm{f}}\backslash T$. For each $a\in X_{\mathrm{f}}\backslash T$, it holds that $\sum_{e\in X_{\mathrm{f}}}\lambda_e v_e(a)+\sum_{e\in X_{\mathrm{f}}}\mu_e r_e(a)=d(a)+\mu_a r_a(a)=d(a)+\Delta(a)$. For each $a\in T\cap H_{(1)}$, it holds that $\Delta(a)>0$ and consequently $\sum_{e\in X_{\mathrm{f}}}\lambda_e v_e(a)+\sum_{e\in X_{\mathrm{f}}}\mu_e r_e(a)=d(a)+\lambda_a u(a)-\mu_a=d(a)+(\Delta(a)/u(a))\alpha u(a)-\Delta(a)(\alpha-1)=d(a)+\Delta(a)$. For each $a\in T\backslash H_{(1)}$, it holds that $\Delta(a)<0$, so that $\sum_{e\in X_{\mathrm{f}}}\lambda_e v_e(a)+\sum_{e\in X_{\mathrm{f}}}\mu_e r_e(a)=d(a)-\lambda_a l(a)+\mu_a=d(a)-(-(\Delta(a)/l(a))\alpha)l(a)-\Delta(a)(\alpha-1)=d(a)+\Delta(a)$. Hence, for all $\Delta\in\mathbb{R}^m$, with $\Delta(e)=0$ for all $e\in E\backslash X$, satisfying Eq. (1), it holds that $(d+\Delta)_X\in T(H_{(1)},X)$. Consequently, $T(H_{(1)},X)$ is the set of vectors $(d+\Delta)_X$, with $\Delta\in\mathbb{R}^m$ and $\Delta(e)=0$ for all $e\in E\backslash X$, satisfying Eq. (1).

Now, we will show that Eq. (1) is nonredundant. We first show that the restrictions (1a) are nonredundant. We may assume that $X_{\mathrm{f}}\cap H_{(1)}\neq\emptyset$. Take any $e^*\in X_{\mathrm{f}}\cap H_{(1)}$. We first consider the case that $e^*\in(X_{\mathrm{f}}\cap H_{(1)})\cap Z$. Define $\Delta\in\mathbb{R}^m$ by $\Delta(e^*):=u(e^*)+1$ and $\Delta(e):=0$ for all $e\in E\backslash\{e^*\}$. Clearly, Δ satisfies (1a)–(1c), except for $\Delta(e^*)\leqslant u(e^*)$. Now consider the case that $e^*\in(X_{\mathrm{f}}\cap H_{(1)})\backslash Z$. Define $\Delta\in\mathbb{R}^m$, with $\Delta(e)=0$ for all $e\in E\backslash X$, by

$$\Delta(e) := \begin{cases} 2u(e) & \text{if } e=e^*, \\ -u(e) & \text{if } e\in (X_{\mathrm{f}}\cap H_{(1)})\backslash\{e^*\}, \\ l(e) & \text{if } e\in X_{\mathrm{f}}\backslash H_{(1)}. \end{cases}$$

Clearly, Δ satisfies (1a) and (1b), except for $\Delta(e^*)\leqslant u(e^*)$. Take any $Y\subseteq X_{\mathrm{f}}\backslash Z$ such that $|Y|\geqslant 2$. If $e^*\in Y$ then

$$\sum_{e\in Y\cap H_{(1)}}\frac{\Delta(e)}{u(e)}-\sum_{e\in Y\backslash H_{(1)}}\frac{\Delta(e)}{l(e)}=2+\sum_{e\in(Y\cap H_{(1)})\backslash\{e^*\}}(-1)-\sum_{e\in Y\backslash H_{(1)}}(1)$$

$$=2-\sum_{e\in Y\backslash\{e^*\}}1=3-|Y|\leqslant 1,$$

so that Δ satisfies (1c). The case that $e^*\notin Y$ can be proven similarly and is omitted here. This proves that the restrictions (1a) are nonredundant. It can be shown in a

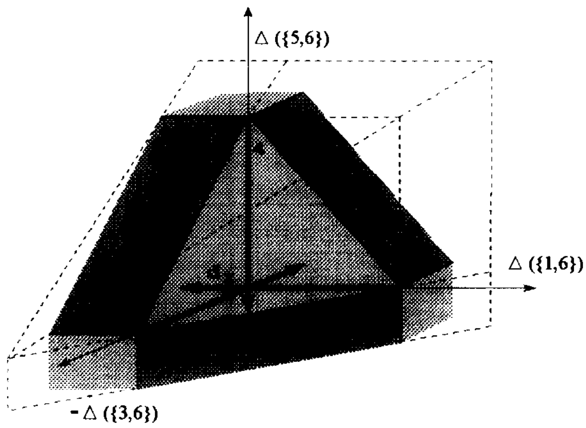

Fig. 7. The tolerance region $T(H_{(1)},X)$.

similar way that also the restrictions (1b) are nonredundant. Finally, we will show that the restrictions (1c) are nonredundant. Take any $Y^* \subseteq X_f \backslash Z$ such that $|Y^*| \geqslant 2$. Define $\Delta \in \mathbb{R}^m$, with $\Delta(e) = 0$ for all $e \in E \backslash X$, by

$$\Delta(e) := \begin{cases} \frac{u(e)}{|Y^*|-1} & \text{if } e \in Y^* \cap H_{(1)}, \\ -\frac{u(e)}{|Y^*|-1} & \text{if } e \in (H_{(1)} \cap X_f) \backslash Y^*, \\ -\frac{l(e)}{|Y^*|-1} & \text{if } e \in Y^* \backslash H_{(1)}, \\ \frac{l(e)}{|Y^*|-1} & \text{if } e \in X_f \backslash (H_{(1)} \cup Y^*). \end{cases}$$

Clearly, Δ satisfies the restrictions (1a) and (1b). For any $Y \subseteq X_f \backslash Z$, $|Y| \geqslant 2$, it holds that

$$\sum_{e \in Y \cap H_{(1)}} \frac{\Delta(e)}{u(e)} - \sum_{e \in Y \backslash H_{(1)}} \frac{\Delta(e)}{l(e)} = \frac{|Y \cap Y^*| - |Y \backslash Y^*|}{|Y^*| - 1}.$$

We distinguish the cases (a) $Y^* \not\subseteq Y$, (b) $Y^* \subset Y$, and (c) $Y^* = Y$. The reader can easily verify that Δ satisfies the restrictions (1c) in the cases (a) and (b) and violates the restrictions (1c) in case (c). This proves that (1) is nonredundant. □

Example (continued). Consider the tolerance region $T(H_{(1)},X)$ depicted in Fig. 7. Recall that $d_X = (2,3,1)^T$, $X = X_f$, $X_\infty = \emptyset$, and $X \cap Z = \emptyset$. Theorem 11 gives the following system of inequalities:

$$\Delta(\{1,6\}) \leqslant 2, \quad \Delta(\{5,6\}) \leqslant 4, \quad \Delta(\{3,6\}) \geqslant -2,$$
$$\tfrac{1}{2}\Delta(\{1,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1, \quad \tfrac{1}{2}\Delta(\{1,6\}) + \tfrac{1}{4}\Delta(\{5,6\}) \leqslant 1,$$
$$\tfrac{1}{4}\Delta(\{5,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1, \quad \tfrac{1}{2}\Delta(\{1,6\}) + \tfrac{1}{4}\Delta(\{5,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1.$$

So far, we discussed the tolerance region $T(H_{(1)},X)$ assuming that all finite tolerances are known. However, as mentioned earlier the problem of determining tolerances for the TSP is $\mathcal{NP}$-hard. In the remainder of this section, we therefore consider subsets of $T(H_{(1)},X)$ that can be found by using the exact values and lower bounds of the tolerances that are obtained by applying Corollary 3 to the set $\mathscr{H}(k)$.

Define $v_e^k \in \mathbb{R}^{|X|}$ for each $e \in X \cap H_{(1)}$ and $a \in X$ by

$$v_e^k(a) := \begin{cases} d(e) + \min\{u(e), L_k\} & \text{if } a = e, \\ d(a) & \text{otherwise,} \end{cases}$$

and for each $e \in X \backslash H_{(1)}$ and $a \in X$ by

$$v_e^k(a) := \begin{cases} d(e) - \min\{l(e), L_k\} & \text{if } a = e, \\ d(a) & \text{otherwise.} \end{cases}$$

Note that $\min\{u(e), L_k\} = u(e)$ if $e \in E^{\mathrm{u}}_{\mathscr{H}(k)}$ and $\min\{u(e), L_k\} = L_k$ otherwise. Similarly, we have that $\min\{l(e), L_k\} = l(e)$ if $e \in E^{\mathrm{l}}_{\mathscr{H}(k)}$ and $\min\{l(e), L_k\} = L_k$ otherwise. The tolerance region that can be determined from the set $\mathscr{H}(k)$, denoted by $T_k(H_{(1)},X)$, is defined as the closure of the convex hull of the union of the halflines $\{v_e^k + \mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X$, i.e.

$$T_k(H_{(1)},X) := \operatorname{clconv} \bigcup_{e \in X} \{v_e^k + \mu_e r_e\colon \mu_e \geqslant 0\}.$$

Note that the definition of $T_k(H_{(1)},X)$ does not contain the union of the lines $\{d_X + \mu_e r_e\colon \mu_e \in \mathbb{R}\}$ for $e \in X_\infty$, since the exact values and lower bounds of the tolerances obtained from the set $\mathscr{H}(k)$ are all finite. As a corollary of Theorem 8, we have the following result.

Corollary 12. *For each nonempty subset X of E, it holds that $T_k(H_{(1)},X) \subseteq T(H_{(1)},X) \subseteq S(H_{(1)},X)$. Moreover, for $k \geqslant k^*$, it holds that $T_k(H_{(1)},X) = T(H_{(1)},X)$ if and only if $X_\infty = \emptyset$.*

Proof. By definition $T_k(H_{(1)},X) \subseteq T(H_{(1)},X)$. Moreover, from Theorem 8, we have $T(H_{(1)},X) \subseteq S(H_{(1)},X)$. If $X_\infty = \emptyset$ then $T(H_{(1)},X)$ is the closure of the convex hull of only halflines. Since all finite tolerances are known for $k \geqslant k^*$, we have that $T_k(H_{(1)},X) = T(H_{(1)},X)$ for $k \geqslant k^*$. If $X_\infty \neq \emptyset$ then $T_k(H_{(1)},X) \neq T(H_{(1)},X)$ for $k \geqslant k^*$, because the halflines $\{d_X - \mu_e r_e\colon \mu_e \geqslant 0\}$ for $e \in X_\infty$ are missing in the definition of $T_k(H_{(1)},X)$. $\square$

As a corollary of Theorem 9, we have the following extreme points and directions of $T_k(H_{(1)},X)$.

Corollary 13. *For any nonempty subset X of E, the following assertions hold.*
1. *The set $\{r_e\colon e \in X\}$ is the set of extreme directions of $T_k(H_{(1)},X)$.*
2. *If $k \leqslant \lambda$ or $X \subseteq Z$ then d_X is the only extreme point of $T_k(H_{(1)},X)$.*
3. *If $k > \lambda$ and $X \nsubseteq Z$ then $\{v_e^k\colon e \in X \backslash Z\}$ is the set of all extreme points of $T_k(H_{(1)},X)$.*

Proof. The results for the case $k \leqslant \lambda$, i.e. $L_k = 0$, follow immediately from the case $X \subseteq Z$ in Theorem 9. The results for the case $k > \lambda$ follow also from Theorem 9 by taking the vectors v_e^k instead of v_e. □

As a corollary of Theorem 11, we also have the polyhedral representation of $T_k(H_{(1)}, X)$.

Corollary 14. *For any nonempty subset X of E, $T_k(H_{(1)}, X)$ is the set of vectors $(d + \Delta)_X$, with $\Delta \in \mathbb{R}^m$ and $\Delta(e) = 0$ for all $e \in E \setminus X$, satisfying the nonredundant system of inequalities*

$$\Delta(e) \leqslant 0 \text{ for } e \in X \cap H_{(1)} \quad \text{and} \quad \Delta(e) \geqslant 0 \text{ for } e \in X \setminus H_{(1)},$$

if $k \leqslant \lambda$, and
(1) $\Delta(e) \leqslant \min\{u(e), L_k\}$ for $e \in X \cap H_{(1)}$,
(2) $\Delta(e) \geqslant -\min\{l(e), L_k\}$ for $e \in X \setminus H_{(1)}$,
(3) $\sum_{e \in Y \cap H_{(1)}} \Delta(e)/\min\{u(e), L_k\} - \sum_{e \in Y \setminus H_{(1)}} \Delta(e)/\min\{l(e), L_k\} \leqslant 1$ for $Y \subseteq X \setminus Z, |Y| \geqslant 2$,
if $k > \lambda$.

Proof. The results for the case $k \leqslant \lambda$ follow from the case $X \subseteq Z$ in Theorem 11 by taking $u(e)$ and $l(e)$ equal to 0. The results for the case $k > \lambda$ follow also from Theorem 11 by letting $u(e) := \min\{u(e), L_k\}$ and $l(e) := \min\{l(e), L_k\}$. □

Example (continued). Consider the tolerance region $T_3(H_{(1)}, X)$ in Fig. 7 with $X = \{\{1,6\},\{3,6\},\{5,6\}\}$ and $L_3 = 2$. Note that $\min\{u(\{1,6\}), L_3\} = \min\{l(\{3,6\}), L_3\} = \min\{u(\{5,6\}), L_3\} = 2$. Define $v^3_{\{1,6\}} = (4,3,1)^{\mathrm{T}}$, $v^3_{\{3,6\}} = (2,1,1)^{\mathrm{T}}$, and $v^3_{\{5,6\}} = (2,3,3)^{\mathrm{T}}$ The extreme points and directions of $T_3(H_{(1)}, X)$ are $\{v^3_{\{1,6\}}, v^3_{\{3,6\}}, v^3_{\{5,6\}}\}$ and $\{r_{\{1,6\}}, r_{\{3,6\}}, r_{\{5,6\}}\}$, respectively. The geometrical description is as follows:

$$\Delta(\{1,6\}) \leqslant 2, \quad \Delta(\{5,6\}) \leqslant 2, \quad \Delta(\{3,6\}) \geqslant -2,$$

$$\tfrac{1}{2}\Delta(\{1,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1, \quad \tfrac{1}{2}\Delta(\{1,6\}) + \tfrac{1}{2}\Delta(\{5,6\}) \leqslant 1,$$

$$\tfrac{1}{2}\Delta(\{5,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1, \quad \tfrac{1}{2}\Delta(\{1,6\}) + \tfrac{1}{2}\Delta(\{5,6\}) - \tfrac{1}{2}\Delta(\{3,6\}) \leqslant 1.$$

Note that $T_3(H_{(1)}, X) \subset T(H_{(1)}, X)$ since $L_3 < u(\{5,6\})$. □

We conclude this section by mentioning that the size of $T_k(H_{(1)}, X)$ depends on the values of the finite tolerances and the lower bound L_k.

4.2. Stability balls

Any closed ball with center in d_X which is fully contained in the stability region is called a *stability ball*, and the radius of the largest stability ball is known as the *stability radius*. In this subsection, we give necessary and sufficient conditions on the set X such that the stability radius is positive. Furthermore, we show how the set $\mathscr{H}(k)$ can be used to give lower and upper bounds on the stability radius.

For any nonempty subset X of E and a given norm $||\cdot||$ in $\mathbb{R}^{|X|}$, the closed ball with center d_X and radius $\rho \in \mathbb{R}$ is defined by $K_\rho(d_X) := \{(d+\Delta)_X \in \mathbb{R}^{|X|}\colon\ ||\Delta_X|| \leqslant \rho\}$. If ρ is such that $K_\rho(d_X) \subseteq S(H_{(1)},X)$ then $K_\rho(d_X)$ is called a *stability ball* with respect to $H_{(1)}$ and X. The *stability radius* with respect to $H_{(1)}$ and X, denoted by $r_d(H_{(1)},X)$, is defined as the radius of the *largest* stability ball, i.e.

$$r_d(H_{(1)},X) := \sup\{\rho \in \mathbb{R}\colon\ K_\rho(d_X) \subseteq S(H_{(1)},X)\}.$$

A convenient norm to use is the l_∞-norm, because if this norm is used in $\mathbb{R}^{|X|}$ then the length of *each* edge in X may be increased or decreased by as much as the stability radius while preserving the optimality of the solution $H_{(1)}$.

In the case that X consists of only a single edge, say $X = \{e\}$ and $t = \min\{u(e), l(e)\}$, then any stability ball is the interval $[d(e) - t, d(e) + t]$ and $r_d(H_{(1)}, \{e\}) = t$. Since determining tolerances for the TSP is $\mathscr{NP}$-hard, it follows that computing the stability radius for the TSP is $\mathscr{NP}$-hard as well. Furthermore, if $X_\infty = X = \{e\}$ then $K_\rho(d_X) = \mathbb{R}$ and the stability radius is infinite. The case $X = E$ is considered in [12]. In the latter case the stability radius is frequently equal to zero (see also Corollary 16 below). In [12] the stability region is therefore defined in a different way; namely as the set of length vectors for which at least one of the tours in $\mathscr{H}(\lambda)$ is optimal. In this paper, we propose to handle the vanishing of the stability radius by choosing subsets X in such a way that the corresponding stability radius is positive.

In the following theorem the set Z is used to give necessary and sufficient conditions on the edge set X such that the stability radius $r_d(H_{(1)},X)$ is positive.

Theorem 15. *Let $H_{(1)} \in \mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Then $r_d(H_{(1)},X) > 0$ if and only if $X \cap Z = \emptyset$.*

Proof. (If) Assume that $X \cap Z = \emptyset$. By Theorem 7(2), there exists an $\varepsilon > 0$ such that $\min\{u(e), l(e)\} \geqslant \varepsilon$ for all $e \in X$. For each $e \in X$, let $\Delta^e \in \mathbb{R}^m$ be defined by $\Delta^e(e) = \varepsilon$ and $\Delta^e(a) = 0$ for all $a \in E \backslash \{e\}$. It follows directly from the definitions of the tolerances that $L_{d+\Delta^e}(H) \geqslant L_{d+\Delta^e}(H_{(1)})$ and $L_{d-\Delta^e}(H) \geqslant L_{d-\Delta^e}(H_{(1)})$ for all $H \in \mathscr{H}$. Hence, $(d - \Delta^e)_X, (d + \Delta^e)_X \in S(H_{(1)},X)$ for all $e \in X$. Let $B := \{(d - \Delta^e)_X\colon\ e \in X\} \cup \{(d + \Delta^e)_X\colon\ e \in X\}$. Since $S(H_{(1)},X)$ is a closed convex set, it follows that $\mathrm{conv}(B) \subseteq S(H_{(1)},X)$. Observe, that d_X may be expressed as a convex combination of the elements in B by taking all coefficients equal to $1/|B|$. This means that d_X is an interior point of $\mathrm{conv}(B)$, and that there is a $\rho > 0$ such that $K_\rho(d_X) \subseteq \mathrm{conv}(B)$. Hence, $r_d(H_{(1)},X) \geqslant \rho > 0$. The maximum size of ρ depends on the norm used. If, for example, the l_∞-norm is used, then it can be shown that the stability ball $K_{\varepsilon/|X|}(d_X)$ is contained in B.

(Only if) Assume that $\rho := r_d(H_{(1)},X) > 0$ and let $K_\rho(d_X)$ be the corresponding stability ball. It has to be shown that $X \cap Z = \emptyset$. Suppose, to the contrary, that there is an $e \in X \cap Z$. According to Theorem 7(1), either $e \in E^{\mathrm{u}}_{\mathscr{H}(\lambda)}$ or $e \in E^{\mathrm{l}}_{\mathscr{H}(\lambda)}$. Assume the first. Let $\varepsilon > 0$ be sufficiently small such that $(d + \Delta)_X \in K_\rho(d_X)$, with $\Delta \in \mathbb{R}^m$ defined by $\Delta(e) = \varepsilon$ and $\Delta(a) = 0$ for all $a \in E \setminus \{e\}$. Since $K_\rho(d_X) \subseteq S(H_{(1)}, X)$, it follows that $L_{d+\Delta}(H) - L_{d+\Delta}(H_{(1)}) \geqslant 0$ for all $H \in \mathscr{H}$. Consequently, we have $\min\{L_{d+\Delta}(H)\colon H \in \mathscr{H}, e \notin H\} - L_{d+\Delta}(H_{(1)}) \geqslant 0$, which can be rewritten, using the definition of Δ, as $\min\{L_d(H)\colon H \in \mathscr{H},\ e \notin H\} - L_d(H_{(1)}) \geqslant \varepsilon > 0$. However, since $e \in E^{\mathrm{u}}_{\mathscr{H}(\lambda)}$, it follows from Corollary 3 that $\min\{L_d(H)\colon H \in \mathscr{H},\ e \notin H\} - L_d(H_{(1)}) = 0$, which gives a contradiction. Consequently, $X \cap E^{\mathrm{u}}_{\mathscr{H}(\lambda)} = \emptyset$. It can be proven similarly that also $X \cap E^{\mathrm{l}}_{\mathscr{H}(\lambda)} = \emptyset$. □

As a special case of Theorem 15, we have the following result.

Corollary 16 (Leontev [12]). *$r_d(H_{(1)}, E) = 0$ if and only if there is more than one optimal tour.*

Proof. By Theorem 15 it is sufficient to show that $r_d(H_{(1)}, E) = 0$ if and only if $Z \neq \emptyset$. Since, by definition, $Z = \bigcup \mathscr{H}(\lambda) \setminus \bigcap \mathscr{H}(\lambda)$, it follows that $Z \neq \emptyset$ if and only if $\lambda > 1$. □

In the following theorem we give an expression for the stability radius $r_d(H_{(1)}, X)$ with respect to the l_∞-norm. Let $H' \otimes H'' := (H' \cup H'') \setminus (H'' \cap H')$ denote the symmetric difference of the tours $H', H'' \in \mathscr{H}$.

Theorem 17. *Let $H_{(1)} \in \mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Let r_d be defined with respect to the l_∞-norm in $\mathbb{R}^{|X|}$. Then*

$$r_d(H_{(1)}, X) = \min\left\{ \frac{L_d(H) - L_d(H_{(1)})}{|(H \otimes H_{(1)}) \cap X|}\colon H \in \mathscr{H}, (H \otimes H_{(1)}) \cap X \neq \emptyset \right\}. \tag{2}$$

Proof. We first consider the case that $(H \otimes H_{(1)}) \cap X = \emptyset$ for all $H \in \mathscr{H}$. It follows from the standard convention of the "min" operator that we have to prove that $r_d(H_{(1)}, X) = \infty$. We will therefore prove that in this case $S(H_{(1)}, X) = \mathbb{R}^{|X|}$ by showing that $l(e) = u(e) = \infty$ for all $e \in X$. For each $e \in X \cap H_{(1)}$, it holds that in this case $e \in \bigcap \mathscr{H}$, so that $u(e) = l(e) = \infty$ (cf. Corollary 2). Similarly, for each $e \in X \setminus H_{(1)}$, it holds in this case that $e \notin \bigcup \mathscr{H}$, so that $u(e) = l(e) = \infty$ (cf. Corollary 2). So $S(H_{(1)}, X) = \mathbb{R}^{|X|}$, and consequently $r_d(H_{(1)}, X) = \infty$.

Consider now the case that there is a $H \in \mathscr{H}$ such that $(H \otimes H_{(1)}) \cap X \neq \emptyset$. For ease of notation, let $\bar{\rho}$ be equal to the right-hand side of (2). We will prove that $r_d(H_{(1)}, X) \geqslant \bar{\rho}$ by showing that the ball $K_{\bar{\rho}}(d_X)$ is fully contained in $S(H_{(1)}, X)$. Take any $\Delta \in \mathbb{R}^m$ such that $(d + \Delta)_X \in K_{\bar{\rho}}(d_X)$. It has to be shown that $(d + \Delta)_X \in S(H_{(1)}, X)$, i.e. that $L_{d+\Delta}(H) - L_{d+\Delta}(H_{(1)}) \geqslant 0$ for all $H \in \mathscr{H}$. Since $(d + \Delta)_X \in K_{\bar{\rho}}(d_X)$, we have that $\|\Delta\|_\infty = \max\{|\Delta(e)|\colon e \in X\} \leqslant \bar{\rho}$ and consequently $-\bar{\rho} \leqslant \Delta(e) \leqslant \bar{\rho}$ for all $e \in X$.

Take any $H \in \mathscr{H}$. Then,

$$\begin{aligned} L_\Delta(H) - L_\Delta(H_{(1)}) &= L_\Delta(H \backslash H_{(1)}) - L_\Delta(H_{(1)} \backslash H) \\ &\geqslant -\overline{\rho}|(H \backslash H_{(1)}) \cap X| - \overline{\rho}|(H_{(1)} \backslash H) \cap X| \\ &= -\overline{\rho}|((H \backslash H_{(1)}) \cap X) \cup ((H_{(1)} \backslash H) \cap X)| \\ &= -\overline{\rho}|(H \otimes H_{(1)}) \cap X|. \end{aligned} \tag{3}$$

Hence

$$\begin{aligned} L_{d+\Delta}(H) - L_{d+\Delta}(H_{(1)}) &= L_d(H) - L_d(H_{(1)}) + L_\Delta(H) - L_\Delta(H_{(1)}) \\ &\geqslant L_d(H) - L_d(H_{(1)}) - \overline{\rho}|(H \otimes H_{(1)}) \cap X|. \end{aligned} \tag{4}$$

If $(H \otimes H_{(1)}) \cap X = \emptyset$ then $L_{d+\Delta}(H) - L_{d+\Delta}(H_{(1)}) \geqslant 0$. Consider, therefore, the case that $(H \otimes H_{(1)}) \cap X \neq \emptyset$. From the definition of $\overline{\rho}$, we have that

$$L_d(H) - L_d(H_{(1)}) - \overline{\rho}|(H \otimes H_{(1)}) \cap X| \geqslant 0. \tag{5}$$

Combining Eqs. (4) and (5) proves that $L_{d+\Delta}(H) - L_{d+\Delta}(H_{(1)}) \geqslant 0$. Hence, $K_{\overline{\rho}}(d_X) \subseteq S(H_{(1)}, X)$ and consequently $r_d(H_{(1)}, X) \geqslant \overline{\rho}$.

Now we will prove that $r_d(H_{(1)}, X) \leqslant \overline{\rho}$. Suppose, to the contrary, that $\rho' := r_d(H_{(1)}, X) > \overline{\rho}$. It will be shown that the ball $K_{\rho'}(d_X)$ is not fully contained in $S(H_{(1)}, X)$. Let $H' \in \mathscr{H}$ such that $(H' \otimes H_{(1)}) \cap X \neq \emptyset$ be a tour that minimizes the right-hand side in (2), i.e. $L_d(H') - L_d(H_{(1)}) = \overline{\rho}|(H' \otimes H_{(1)}) \cap X|$. Let $\Delta \in \mathbb{R}^m$ with $\Delta(e) = 0$ for $e \in E \backslash X$ be defined, for each $e \in X$, by

$$\Delta(e) := \begin{cases} \rho' & \text{if } e \in (H_{(1)} \backslash H') \cap X, \\ -\rho' & \text{if } e \in (H' \backslash H_{(1)}) \cap X, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(d + \Delta)_X \in K_{\rho'}(d_X)$, since $\|\Delta_X\|_\infty = \max\{|\Delta(e)|\colon e \in X\} \leqslant \rho'$. We will show that $L_{d+\Delta}(H') < L_{d+\Delta}(H_{(1)})$. Following the same steps as above (see (3) and (4)), we obtain that $L_{d+\Delta}(H') - L_{d+\Delta}(H_{(1)}) = L_d(H') - L_d(H_{(1)}) - \rho'|(H' \otimes H_{(1)}) \cap X|$. Using the definition of $\overline{\rho}$, and the assumption that $\rho' > \overline{\rho}$ we find that $L_d(H') - L_d(H_{(1)}) - \rho'|(H' \otimes H_{(1)}) \cap X|$ is smaller than $L_d(H') - L_d(H_{(1)}) - \overline{\rho}|(H' \otimes H_{(1)}) \cap X| = 0$, which proves that $L_{d+\Delta}(H') < L_{d+\Delta}(H_{(1)})$. But, this implies that $(d + \Delta)_X \notin S(H_{(1)}, X)$, and consequently we are arrived at a contradiction. This proves that $r_d(H_{(1)}, X) \leqslant \overline{\rho}$, and hence $r_d(H_{(1)}, X) = \overline{\rho}$. □

Example (continued). Note that $r_d(H_{(1)}, E) = 0$ in (G, d) of Fig. 1, because there are two shortest tours. As before, let $X = \{\{1,6\},\{3,6\},\{5,6\}\}$. Note that $(H_{(1)} \otimes H_{(i)}) \cap X = \emptyset$ for $i = 1, 2$ and $|(H_{(1)} \otimes H_{(i)}) \cap X| = 2$ for $i = 3, \ldots, 10$. Then

$$r_d(H_{(1)}, X) = \min\{\tfrac{1}{2}L_i\colon i = 3, \ldots, 10\} = \tfrac{1}{2}L_3 = 1.$$

Note that for large n it is computationally intractable to determine the stability radius by using Theorem 17 because this would imply that the complete set $\mathscr{H}$ needs to be considered. However, by evaluating subsets of $\mathscr{H}$, lower and upper bounds for $r_d(H_{(1)},X)$ can be derived. We will show how to use the set of k-best tours $\mathscr{H}(k)$ for this purpose. Define

$$R_d^k(H_{(1)},X) := \min\left\{\frac{L_d(H)-L_d(H_{(1)})}{|(H\otimes H_{(1)})\cap X|}:\ H\in\mathscr{H}(k),\ (H\otimes H_{(1)})\cap X\neq\emptyset\right\}.$$

The following lower and upper bounds for the stability radius $r_d(H_{(1)},X)$ can be obtained.

Theorem 18. *Let $H_{(1)}\in\mathscr{H}(\lambda)$ and let X be a nonempty subset of E. Let r_d be defined with respect to the l_∞-norm in $\mathbb{R}^{|X|}$. Then*

$$\min\left\{R_d^k(H_{(1)},X),\ \frac{L_k}{|H_{(1)}\cap X|+\min\{n,|X|\}}\right\}\leqslant r_d(H_{(1)},X)\leqslant R_d^k(H_{(1)},X).$$

Proof. The upper bound inequality is a direct consequence of the fact that $\mathscr{H}(k)\subseteq\mathscr{H}$. Consider now the lower bound on $r_d(H_{(1)},X)$. From Theorem 17, it follows that $r_d(H_{(1)},X)$ is equal to the minimum of $R_d^k(H_{(1)},X)$ and

$$\min\left\{\frac{L_d(H)-L_d(H_{(1)})}{|(H\otimes H_{(1)})\cap X|}:\ H\in\mathscr{H}\backslash\mathscr{H}(k),\ (H\otimes H_{(1)})\cap X\neq\emptyset\right\}.$$

For each $H\in\mathscr{H}\backslash\mathscr{H}(k)$, it holds that $L_d(H)-L_d(H_{(1)})\geqslant L_k$. Moreover, we have that $|(H\otimes H_{(1)})\cap X|\leqslant|H_{(1)}\cap X|+|H\cap X|\leqslant|H_{(1)}\cap X|+\min\{n,|X|\}$, which leads directly to the lower bound. □

Example (continued). We determine the lower and upper bounds of $r_d(H_{(1)},X)$ for $X=\{\{1,6\},\{3,6\},\{5,6\}\}$ using the set $\mathscr{H}(5)$. Note that $Z\cap X=\emptyset$, $(H_{(1)}\otimes H_{(i)})\cap X=\emptyset$ for $i=1,2$, and $|(H_{(1)}\otimes H_{(i)})\cap X|=2$ for $i=3,4,5$. The upper bound is given by

$$R_d^5(H_{(1)},X)=\min\left\{\frac{L_d(H)-L_d(H_{(1)})}{|(H\otimes H_{(1)})\cap X|}:\ H\in\{H_{(3)},H_{(4)},H_{(5)}\}\right\}$$

$$=\min\left\{\frac{2}{2},\frac{3}{2},\frac{4}{2}\right\}=1,$$

and the lower bound by

$$\min\left\{R_d^5(H_{(1)},X),\ \frac{L_5}{|H_{(1)}\cap X|+\min\{n,|X|\}}\right\}=\min\left\{1,\frac{4}{2+3}\right\}=\frac{4}{5}.$$

Hence, $\frac{4}{5}\leqslant r_d(H_{(1)},X)\leqslant 1$, which means that $H_{(1)}$ remains optimal when the length of any edge in X is increased or decreased simultaneously, by no more than $\frac{4}{5}$. We

Table 1
The stability radii for different sets X in $H_{(1)}\backslash Z$

Instance	$\lvert X\rvert=5$	$\lvert X\rvert=10$	$\lvert X\rvert=15$	$\lvert X\rvert=20$	$\lvert X\rvert=25$
GR17	0.75	0.6	0.4	—	—
GR21	1	1	1	0.5	—
GR24	2.33	1	1	1	—
GR48	0.25	0.25	0.25	0.25	0.25
GR96	1.2–5	0.6–4	0.4–4	0.3–2	0.2–2
GR120	1.75–54	0.9–3	0.6–1.5	0.45–1	0.25

also know that the optimality of $H_{(1)}$ can be destroyed by choosing an appropriate perturbation vector $\Delta \in \mathbb{R}^m$ with $\Delta(e)=0$ for all $e \in E\backslash X$ defined by $\Delta(e) := 1+\varepsilon$ for $e \in X \cap H_{(1)}$ and $\Delta(e) := 1-\varepsilon$ for all $e \in X\backslash H_{(1)}$, where ε is an arbitrarily small positive number. □

Since the lower and upper bounds of the stability radius obtained by applying Theorem 18 depend on both the value of k as well as on the specific length vector d and the number of cities n, we conclude this section by presenting the lower and upper bounds of the stability radius $r_d(H_{(1)},X)$ for a number of sets X and a number of instances from the TSPLIB library. Table 1 shows the values of the lower and upper bounds of the stability radius that can be obtained from $\mathscr{H}(200)$ for a number of rather arbitrary sets X in $H_{(1)}\backslash Z$. The sets X are taken as, respectively, the first 5, 10, 15, 20 and 25 edges in $H_{(1)}\backslash Z$. More detailed information on the sets $H_{(1)}$ and the applied ordering can be obtained from the authors. If the lower and upper bound are equal then the value of the stability radius is given.

It can be observed that the value of the stability radius and the convergence of the upper and lower bounds of the stability radius strongly depend on the specific choice of the set X and the length vector d. Clearly, it is difficult to say which choices of X give a fast convergence of the lower and upper bounds. We leave this as a subject for further research. Obviously, the stability radius and its upper bound do not increase when the set X is extended. The fact that $|X|$ increases does not mean that the upper bound becomes sharper, since the stability radius itself may decrease even faster. Finally, note that the lower bound of the stability radius is determined by the difference in tour length L_k. Consequently, it follows that the smaller the stability radius, or its upper bound, the faster is the convergence of the lower and upper bounds of the stability radius.

Fig. 8 shows the lower and upper bounds of the stability radius $r_d(H_{(1)},X)$ for GR120 as a function of k for sets X taken as, respectively, the first 5, 10, 15, 20, 25, and 30 edges in $H_{(1)}\backslash Z$. Again, it can be observed that the lower and upper bounds strongly depend on the specific sets X and their cardinality. It appears that for the determination of upper bounds small values of k suffice. However, the lower bound does hardly improve when the value of k is increased considerably from 100 to 200.

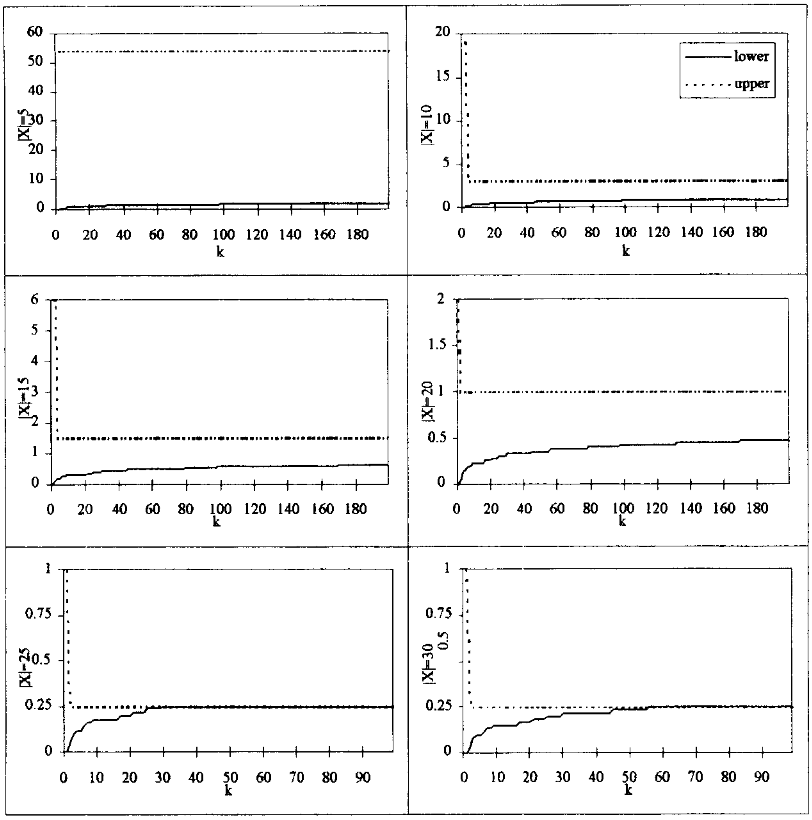

Fig. 8. The stability radius of GR120 for different sets X.

A similar fact was observed in Section 3 for the quality of the approximations of the upper tolerances.

5. Summary and conclusions

The objective of this paper was to explore the possibility of using a set of k-best solutions to determine information with respect to the tolerances and the stability radius, which in their turn provide subsets of the stability region. Our results can be summarized as follows. First, we have shown how a set of k-best tours can be used to obtain the exact tolerances for a subset of the edges as well as lower bounds for the remaining tolerances. Also, we have provided upper and lower bounds for the number of tolerances that can be found in this way. Second, it has been shown how to use the set of k-best tours to give necessary and sufficient conditions for the stability radius being positive, and to determine lower and upper bounds for the stability radius. Finally, it has been discussed how the earlier mentioned information can be used to construct subsets of the stability region. Where appropriate, both the representation in terms of extreme points and directions as well as the polyhedral representation of the subsets of the stability region have been given.

Obviously, the practical usefulness of the methods described in this paper depend on the amount and quality of the information that can be derived. Unfortunately, both the amount and the quality depend on the value of k as well as the specific instance at hand (i.e. the level of information obtained is data-specific). In order to get an idea about the practical usefulness we have conducted numerous numerical experiments for various values of k on a number of well-known instances in the TSPLIB library. It is observed that already for relatively low values of k "quite a lot" of sensitivity information can be obtained. We like to stress that once the set of k-best tours has been determined this information is obtained in polynomial time and therefore "inexpensive". However, it should also be clear that if the primary objective is to determine information about either the tolerances or the stability radius, first solving the k-best TSP is probably not the most effective way.

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