A class of integral approximations for the factorial function

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A B S T R A C T

We introduce Stirling’s formula in a more general class of approximation formulas to extend the integral representation of Liu [Z. Liu, A new version of the Stirling formula, Tamsui Oxf. J. Math. Sci. 23 (4) (2007) 389–392]. Finally, an accurate approximation for the factorial function is established.

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1. Introduction and motivation

Stirling’s formula and its numerous versions have attracted great attention from many authors. It was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

\[ n! \approx \text{constant} \cdot \left( \frac{n}{e} \right)^n \sqrt{n} \]

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant \( \sqrt{2\pi} \):

\[ n! \approx \sqrt{2\pi n} \cdot \left( \frac{n}{e} \right)^n = \sigma_n, \tag{1.1} \]

when he was trying to give the normal approximation of the binomial distribution. Inspired by the formula (1.1), many authors are concerned with the problem of finding new, better formulas for approximation of the factorial function. A slightly more accurate approximation than Stirling’s formula was proposed by Mortici: [1]

\[ n! \approx \sqrt{\frac{2\pi}{e}} \left( \frac{n + 1}{e} \right)^{n+\frac{1}{2}} = \alpha_n \tag{1.2} \]

as an intermediate result for establishing some monotonicity properties of some functions involving the gamma function. However, Burnside’s approximation

\[ n! \approx \sqrt{2\pi} \left( \frac{n + \frac{1}{2}}{e} \right)^{n+\frac{1}{2}} = \beta_n, \tag{1.3} \]
is much better. It was first stated by Burnside [2], then rediscovered by Spouge [3]. For proofs and other details, see [2,1,4–13,3,14].

Mortici [1] introduced the approximations (1.1)–(1.3) in a general class of the form

\[ n! \approx \sqrt{2\pi e \cdot e^{-a}} \left( \frac{n + a}{e} \right)^{n+\frac{1}{2}} = \mu_n(a), \]

where \( a \in [0, 1] \). Notice that \( \mu_n(0) = \sigma_n, \mu_n(1/2) = \beta_n, \mu_n(1) = \alpha_n \).

One way to invent new, increasingly accurate approximations for the factorial function is to consider the remainder in known formulas as an incomplete convergent numeric series, or as an incomplete convergent improper integral.

Mortici [1] used the following representation of Hsu [15]:

\[ n! = \sqrt{2\pi n \cdot \left( \frac{n}{e} \right)^n \cdot \exp \left( \sum_{k=n}^{\infty} \sum_{j=2}^{\infty} \frac{j - 1}{2j(j+1)} \left( \frac{-1}{k} \right)^j \right)} \]

(1.4)

to characterize the class of series satisfying the relation

\[ n! = \sqrt{2\pi n \cdot \left( \frac{n}{e} \right)^n \cdot \exp \left( \sum_{k=n}^{\infty} a_k \right)}. \]

Very recently, Liu [16] used (1.4) to establish the following integral representation formula:

\[ n! = \sqrt{2\pi n \cdot \left( \frac{n}{e} \right)^n \cdot \exp \left( \int_n^{\infty} \frac{1 - \{x\}}{x} \, dx \right)} \]

(1.5)

associated with Stirling’s formula. Here and in the sequel, for every \( x \geq 0 \), \( \{x\} \) denotes the largest integer less than or equal to \( x \) and \( \lfloor x \rfloor = x - \{x\} \). Our main goal is to extend Liu’s formula as

\[ n! = \sqrt{2\pi e \cdot e^{-a}} \left( \frac{n + a}{e} \right)^{n+\frac{1}{2}} \exp \int_n^{\infty} \left( \frac{3}{2} - a - \{x\} \right) \frac{x + a}{a \lfloor x \rfloor + \lfloor x \rfloor} - \frac{1}{x} \right) \, dx, \]

(1.6)

where \( a \in [0, 1] \). Liu’s formula (1.5) is obtained for \( a = 0 \) in (1.6). We also have for \( a = 1 \) and \( a = 1/2 \)

\[ n! = \sqrt{2\pi e} \left( \frac{n + 1}{e} \right)^{n+\frac{1}{2}} \exp \int_n^{\infty} \frac{1 - \{x\}}{x + 1} \, dx, \]

and

\[ n! = \sqrt{2\pi} \left( \frac{n + \frac{1}{2}}{e} \right)^{n+\frac{1}{2}} \exp \int_n^{\infty} \left( 1 - \frac{\lfloor x \rfloor}{x + \frac{1}{2}} - \frac{\lfloor x \rfloor}{x (x + \lfloor x \rfloor)} \right) \, dx, \]

respectively, which are integral representations associated with approximation formulas (1.2) and (1.3).

As a consequence, we prove the following class of approximation formulas:

\[ n! \approx \sqrt{2\pi e} \cdot e^{-a} \left( \frac{n + 1 + a}{e} \right)^{n+\frac{1}{2}} \frac{1}{n + 1} \]

and then we indicate the values \( a = \left( 3 \pm \sqrt{3} \right) / 6 \) which provide the best results.

Finally, using a new trick, we establish the following approximation formula:

\[ n! \approx \sqrt{2\pi} \left( \frac{n^2 + 3n + \frac{13}{6}}{e^2} \right)^{n+\frac{3}{4}} \frac{1}{n + 1}, \]

whose superiority over other known formulas is proven using some numerical computations.
2. The main results

We start this section with the following:

**Theorem 2.1.** Let \( a \in (0, 1) \) be given. Then a locally integrable function \( f : [0, \infty) \to \mathbb{R} \) satisfies, for every integer \( n \geq 0 \), the relation

\[
  n! = \sqrt{2\pi} \cdot e^{-a} \left( \frac{n + a}{e} \right)^{n + \frac{1}{2}} \exp \int_{n}^{\infty} f(x) \, dx
\]  

(2.1)

if and only if for every integer \( n \geq 0 \), we have

\[
  \int_{n}^{n+1} f(x) \, dx = -1 - \ln(n + 1) + \left( n + \frac{3}{2} \right) \ln(n + 1 + a) - \left( n + \frac{1}{2} \right) \ln(n + a).
\]  

(2.2)

**Proof.** Relation (2.1) can be equivalently written as

\[
  \int_{n}^{\infty} f(x) \, dx = a - \ln \sqrt{2\pi} + n + \sum_{k=1}^{n} \ln k - \left( n + \frac{1}{2} \right) \ln(n + a).
\]  

(2.3)

Now, relation (2.2) follows by replacing \( n \) by \( n + 1 \) in (2.3) and subtracting the relations obtained.

In order to use an induction argument, it remains to show that the function \( f \) satisfying (2.2) also verifies (2.1) in the case \( n = 0 \), namely to show that

\[
  \int_{0}^{\infty} f(x) \, dx = a - \ln \sqrt{2\pi a}.
\]  

(2.4)

In this sense, using Eq. (2.2), for every integer \( n \geq 1 \), we have

\[
  \int_{0}^{n} f(x) \, dx = \sum_{k=0}^{n-1} \int_{k}^{k+1} f(x) \, dx = \sum_{k=0}^{n-1} \left( -1 - \ln(k + 1) + x_{k+1} - x_k \right),
\]

where \( x_k = \left( k + \frac{1}{2} \right) \ln(k + a) \). Hence,

\[
  \int_{0}^{n} f(x) \, dx = -n - \ln n! + \left( n + \frac{1}{2} \right) \ln(n + a) - \frac{1}{2} \ln a = \ln \left( \frac{(n + a)^{n + 1}}{n! \cdot e^{\sqrt{a}}} \right).
\]

Now, using the approximations \( \mu_n(a) \), we obtain

\[
  \lim_{n \to \infty} \int_{0}^{n} f(x) \, dx = \lim_{n \to \infty} \ln \left( \frac{(n + a)^{n + 1}}{e^{\sqrt{a}} \cdot n!} \right) = \lim_{n \to \infty} \ln \left( \frac{(n + a)^{n + 1}}{e^{\sqrt{a}} \cdot \sqrt{2\pi} \cdot e^{-a} \left( \frac{n + a}{e} \right)^{n + \frac{1}{2}}} \right) = a - \ln \sqrt{2\pi a},
\]

so (2.4) is justified and Theorem 2.1 is proved. \( \Box \)

Next we concentrate on finding a convenient function satisfying (2.2). In this sense, we have

\[
  -1 + \left( n + \frac{3}{2} \right) \ln \frac{n + 1 + a}{n + a} = - \int_{n}^{n+1} \frac{1}{x + a} \, dx + \left( n + \frac{3}{2} \right) \int_{n}^{n+1} \frac{1}{x + a} \, dx
\]

\[
  = - \int_{n}^{n+1} \frac{1}{x + a} \, dx + \int_{n}^{n+1} \frac{\lfloor x \rfloor + \frac{3}{2}}{x + a} \, dx = \int_{n}^{n+1} \frac{\frac{3}{2} - a - \lfloor x \rfloor}{x + a} \, dx
\]  

(2.5)

and, with the variable changes \( x = at \), then \( t = x - n \), we obtain

\[
  \ln \frac{n + a}{n} = \int_{0}^{a} \frac{1}{n + x} \, dx = \int_{0}^{1} \frac{1}{at + n} \, dt = \int_{n}^{n+1} \frac{a}{a(x - n) + n} \, dx
\]

\[
  = \int_{n}^{n+1} \frac{a}{a \lfloor x \rfloor + \lfloor x \rfloor} \, dx.
\]  

(2.6)
Furthermore,
\[
\ln \frac{n + a}{n + 1} = - \ln \frac{n + 1}{n} + \ln \frac{n + a}{n} = - \int_n^{n+1} \frac{dx}{x} + \int_n^{n+1} \frac{a}{a(x) + [x]}dx.
\] (2.7)

Now, from (2.5)–(2.7), it results that the function
\[
f(x) = \frac{3}{x + a} + \frac{a}{a(x) + [x]} - \frac{1}{x}
\]
satisfies (2.2) from Theorem 2.1, so the representation formula (1.6) holds.

The great advantage of the representation formula (1.6) is that it allows us to give new, stronger approximation formulas for the factorial function. In this sense, let us rewrite formula (2.1) in the equivalent form:
\[
n! = \sqrt{2\pi e} \cdot e^{-a} \left( \frac{n + a}{e} \right)^{n + \frac{1}{2}} \exp \int_n^{n+1} f(x)dx \cdot \exp \int_{n+1}^{\infty} f(x)dx.
\]

and using (2.2), we obtain
\[
n! = \sqrt{2\pi e} \cdot e^{-a} \left( \frac{n + 1 + a}{e} \right)^{n + \frac{3}{2}} \frac{1}{n + 1} \cdot \exp \int_{n+1}^{\infty} f(x)dx.
\]

In this way, we can state the approximation formula
\[
n! \approx \sqrt{2\pi e} \cdot e^{-a} \left( \frac{n + 1 + a}{e} \right)^{n + \frac{3}{2}} \frac{1}{n + 1} \] (2.8)

which is more accurate than \(\mu_n(a)\).

3. Approximating the big factorials

In this section, we concentrate on finding the values of the parameter \(a\) which provide the best approximations of the form (2.8).

In order to establish our results, we need the following result.

**Lemma 3.1.** If \((w_n)_{n \geq 1}\) is convergent to zero and if there exists the limit
\[
\lim_{n \to \infty} n^k(w_n - w_{n+1}) = l \in \mathbb{R},
\] (3.1)

with \(k > 1\), then there exists the limit
\[
\lim_{n \to \infty} n^{k-1}w_n = \frac{l}{k-1}.
\]

This lemma was first used by Mortici [1.4–13] to accelerate some convergences and to construct asymptotic expansions. For the proof and other details, see [7, 12]. Notice that in Lemma 3.1, the speed of convergence of the sequence \((w_n)_{n \geq 1}\) increases together with \(k\) satisfying (3.1).

Let us define the sequence \((w_n)_{n \geq 1}\) by the relations
\[
n! = \sqrt{2\pi e} \cdot e^{-a} \left( \frac{n + 1 + a}{e} \right)^{n + \frac{3}{2}} \frac{1}{n + 1} \cdot \exp w_n, \quad n \geq 1.
\]

Hence
\[
w_n - w_{n+1} = - \ln (n + 1) - \left( n + \frac{3}{2} \right) \ln \frac{n + 1 + a}{e} + \left( n + \frac{5}{2} \right) \ln \frac{n + 2 + a}{e} + \ln \frac{n + 1}{n + 2}.
\]

As we are interested in computing a limit of the form (3.1), we consider the expansion in a power series
\[
w_n - w_{n+1} = \left( \frac{1}{2} a^2 - \frac{1}{2} a + \frac{1}{12} \right) \frac{1}{n^2} - \left( \frac{2}{3} a^3 + a^2 - \frac{3}{2} a + \frac{1}{4} \right) \frac{1}{n^3} + O \left( \frac{1}{n^4} \right).
\]

Using this expansion and Lemma 3.1, we can readily state the following:
Theorem 3.1. (i) If \( a \neq \left( 3 \pm \sqrt{3} \right) / 6 \), then the rate of convergence of the sequence \( (w_n)_{n \geq 1} \) is \( n^{-1} \), since
\[
\lim_{n \to \infty} n w_n = \frac{1}{2} a^2 - \frac{1}{2} a + \frac{1}{12} \neq 0.
\]
(ii) If \( a = \left( 3 \pm \sqrt{3} \right) / 6 \), then the rate of convergence of the sequence \( (w_n)_{n \geq 1} \) is \( n^{-2} \), since
\[
\lim_{n \to \infty} n^2 w_n = \pm \frac{\sqrt{3}}{216}.
\]

Now we are able to define the new approximation formulas
\[
n! \approx \sqrt{2\pi e} \cdot e^{-\omega} \left( \frac{n + 1 + \omega}{e} \right)^{n+\frac{3}{2}} \frac{1}{n+1} =: \omega_n \quad (3.2)
\]
and
\[
n! \approx \sqrt{2\pi e} \cdot e^{-\zeta} \left( \frac{n + 1 + \zeta}{e} \right)^{n+\frac{3}{2}} \frac{1}{n+1} =: \zeta_n \quad (3.3)
\]
where \( \omega = \left( 3 - \sqrt{3} \right) / 6 \) and \( \zeta = \left( 3 + \sqrt{3} \right) / 6 \).

There are established in [1] the following approximation formulas:
\[
n! \approx \sqrt{2\pi e} \cdot e^{-\omega} \left( \frac{n + \omega}{e} \right)^{n+\frac{1}{2}} =: \tau_n \quad (3.4)
\]
and
\[
n! \approx \sqrt{2\pi e} \cdot e^{-\zeta} \left( \frac{n + \zeta}{e} \right)^{n+\frac{1}{2}} =: \eta_n, \quad (3.5)
\]
then numerical computations are made to show their superiority over other results, as
\[
n! \approx \frac{n^{n+1} e^{-n} \sqrt{2\pi}}{\sqrt{n-1/6}}
\]
(see Batir [17], Sandor and Debnath [18]) which is known to be more much more accurate than Stirling’s formula, or Burnside’s formula.

As we can see from the following table, our new formulas (3.2)–(3.3) improve on the recent formulas (3.4)–(3.5).

<table>
<thead>
<tr>
<th>n</th>
<th>n! − ( \tau_n )</th>
<th>n! − ( \eta_n )</th>
<th>n! − ( \omega_n )</th>
<th>n! − ( \zeta_n )</th>
</tr>
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<tr>
<td>10</td>
<td>277.08</td>
<td>251.02</td>
<td>230.03</td>
<td>210.19</td>
</tr>
<tr>
<td>20</td>
<td>4.7603 × 10^{13}</td>
<td>4.5252 × 10^{13}</td>
<td>4.3228 × 10^{13}</td>
<td>4.1189 × 10^{13}</td>
</tr>
<tr>
<td>100</td>
<td>7.4477 × 10^{151}</td>
<td>7.3711 × 10^{151}</td>
<td>7.3013 × 10^{151}</td>
<td>7.2269 × 10^{151}</td>
</tr>
<tr>
<td>250</td>
<td>4.1398 × 10^{485}</td>
<td>4.1226 × 10^{485}</td>
<td>4.1069 × 10^{485}</td>
<td>4.0900 × 10^{485}</td>
</tr>
</tbody>
</table>

Finally, we use the idea from [5] to construct a new approximation formula as the geometric mean of a lower approximation and an upper approximation of the factorial function. In [5], the following mean of the Stirling and Burnside results:
\[
n! \approx \sqrt{\sigma_n \beta_n^3} = \sqrt{\frac{2\pi}{e^{n+\frac{3}{2}}}} \left( n \left( n + \frac{1}{2} \right) \right)^{n+\frac{3}{2}} \frac{1}{n+1}
\]
is stated. It gives better results than those of Stirling and Burnside.

We introduce here the new approximation formula
\[
n! \approx \sqrt{\omega_n \kappa_n} = \sqrt{\frac{n^2 + 3n + \frac{13}{6}}{e^2}} \left( \frac{n+\frac{3}{2}}{n+1} \right)^{n+\frac{3}{2}} \frac{1}{n+1} =: \kappa_n
\]
which is more accurate than all formulas mentioned in this paper, as we can see below. The approximation $\kappa_n$ is also more accurate than Gosper’s formula

$$n! \approx 2\pi \left(1 + \frac{1}{6} \right) \left(\frac{n}{e}\right)^n = \gamma_n$$

(see [19]), rediscovered by Batir [20], which is slightly more accurate than $\omega_n$ and $\zeta_n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>20</th>
<th>100</th>
<th>250</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n! - \gamma_n$</td>
<td>239.18</td>
<td>$4.116 \times 10^{13}$</td>
<td>$6.4479 \times 10^{151}$</td>
<td>$3.5847 \times 10^{485}$</td>
</tr>
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<td>$n! - \kappa_n$</td>
<td>$9.9296 \times 10^{12}$</td>
<td>$1.0196 \times 10^{12}$</td>
<td>$3.7187 \times 10^{149}$</td>
<td>$8.4676 \times 10^{482}$</td>
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</tbody>
</table>

Finally, we are convinced that the method used in this paper is suitable for obtaining new accurate approximations of the factorial function.

Acknowledgements

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References

[10] C. Mortici, Optimizing the rate of convergence in some new classes of sequences convergent to Euler’s constant, Anal. Appl. (Singap.), manuscript (in press).
[14] J.R. Wilton, A proof of Burnside’s formula for $\log (x + 1)$ and certain allied properties of Riemann’s $\zeta$-function, Messenger Math. 52 (1922/1923) 90–93.