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On 1-Sarvate–Beam designs

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In loving memory of Professor Ralph G. Stanton.

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ABSTRACT

The solution to a set theory exercise, "Partition the set of positive integers $\{1, 2, \ldots, v\}$ into k subsets such that the sum of the elements in each subset is $v(v+1)/(2k)$ whenever $v(v+1)/(2k)$ is an integer", gives a construction of non-simple 1-SB designs. This raises a natural question of the existence of simple 1-SB designs. We show that the necessary conditions for the existence of simple 1-SB designs for block sizes 2, 3, 4, 5 and 6 are sufficient. Moreover, the technique exhibited in the proof can be applied to block sizes greater than $k = 6$. We also show that simple t-SB$(v, t+1)$, 2-SB$(v, 3)$ and 2-SB$(v, 4)$ designs do not exist for any positive integers v and t.

A natural question, "Can we obtain a construction for simple 1-SB designs similar to Billington's classical construction of simple 1-designs for any block size k?", remains open.

1. Introduction

Given positive integers v and k such that $k \mid \frac{v(v+1)}{2}$ and $k < v$, is it possible to partition the v-set $\{1, 2, \ldots, v\}$ into k subsets, $P_1, P_2, \ldots, P_k$, such that the sum of the elements of each subset or part of the partition is $\frac{v(v+1)}{2k}$?

A solution to this problem can be used to construct a collection of k-subsets, called blocks, of the v-set with an interesting property as described below.

First construct a $k \times \frac{v(v+1)}{2k}$ array using the partition $\{P_1, P_2, \ldots, P_k\}$: for $i = 1$ to k, write (in the ith row) each element j in P_i, j times. Then construct a k-set using the entries in each column to obtain a collection of $\frac{v(v+1)}{2k}$ k-sets, called blocks.

Note that in this collection of blocks each element i from 1 to v occurs exactly i times.

Example 1. Given $v = 6$, and $k = 3$, the set $\{1, 2, 3, 4, 5, 6\}$ can be partitioned into $\{1, 6\}$, $\{3, 4\}$ and $\{5, 2\}$ where the sum of the elements in each part is 7. We construct the corresponding 3 by 7 array from the partition

1666666
3334444
5555522

and then construct the blocks,

$$\{1, 3, 5\}, \{6, 3, 5\}, \{6, 3, 5\}, \{6, 4, 5\}, \{6, 4, 5\}, \{6, 4, 2\}, \{6, 4, 2\}.$$

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Note that each element $i \in \{1, 2, 3, 4, 5, 6\}$ occurs in exactly i blocks.

Furthermore, this is an example of a 1-SB design. (Specifically, it is a *non-simple* strict 1-SB(6, 3).)

Definition 1. A t-SB(v, k) design is a collection, B, of k-subsets of a v-set such that each t-subset of V occurs with different frequencies. In a *strict* t-SB design, for each i, $1 \le i \le \binom{v}{t}$, there is exactly one t-subset which occurs in i blocks. A strict t-SB(v, k) design is *simple* if all blocks are distinct.

Hence, in a simple design, no k-subset is used more than once as a block.

In this paper, we will focus mainly on strict non-simple and strict simple 1-SB designs except in Section 3, where we show that strict simple 2-SB$(v, 3)$, strict simple 2-SB$(v, 4)$ and strict simple t-SB$(v, t + 1)$ designs for $t \ge 1$ do not exist for any positive integer v. The case of $t \ge 2$ is studied by many mathematicians such as Ralph Stanton [10,14,11,13,12], Sarvate and Beam [8], Hein and Li [5], Bradford et al. [1], Dukes [3], Dukes et al. [4], Ma et al. [6], Moolsombut and Hemakul [7] and Chan and Sarvate [2].

Definition 2. A 1-SB(v, k) design is a collection, B, of k-subsets of a v-set such that each element of V occurs in a distinct number of blocks. In a strict 1-SB design, for each i, $1 \le i \le v$, there is exactly one element which occurs in i blocks.

The next section gives the necessary conditions for the existence of simple strict 1-SB designs.

2. Necessary conditions

Sarvate and Beam [9] proved the following:

Theorem 1. *A strict t-SB(v, k) exists only if*

$$\binom{k}{t} \Bigg| \frac{\binom{v}{t}\left(\binom{v}{t}+1\right)}{2}$$

The above theorem yields:

Corollary 1. *The necessary condition for the existence of a strict 1-SB(v, k) is*

$$k \Big| \frac{v(v+1)}{2}.$$

Corollary 2. *The necessary condition for the existence of a simple strict 1-SB(v, k) is*

$$\binom{v}{k} \ge \frac{v(v+1)}{2k}.$$

Proof. By Corollary 1, we need $\frac{v(v+1)}{2k}$ blocks. Therefore for a simple strict 1-SB(v, k) to exist, $\binom{v}{k} \ge \frac{v(v+1)}{2k}$. □

3. Simple t-SB(v, k) for $t > 1$

Theorem 2. *A simple strict t-SB$(v, t + 1)$ does not exist for $t \ge 1$.*

Proof. As the design, if it exists, is strict, some t-set must occur $\binom{v}{t}$ times. Suppose the t-set $\{1, 2, 3, \ldots, t\}$ occurs $\binom{v}{t}$ times. There are $v - t$ other elements in the v-set which can be included in the t-set $\{1, 2, 3, \ldots, t\}$ to get $v - t$ sets of size $t + 1$. A $(t + 1)$-set can occur at most once as a block in a simple design; hence $\binom{v}{t}$ must be less than $v - t$, i.e.,

$$\frac{v!}{(v-t)!t!} < v - t$$

implies that $v(v - 1)(v - 2) \ldots (v - t + 1) < (v - t)(t)(t - 1)(t - 2) \ldots (2)(1)$.

Note that this is impossible as $v - t + 1 > v - t$, $v > t$, $v - 1 > t - 1, \ldots$, and $v - t + 2 = v - (t - 2) > 2$. Hence, a simple strict t-SB$(v, t + 1)$ does not exist. □

For $t = 2$ and any positive integer $k \ge 5$, a similar argument based on the fact that there is a pair which occurs $\binom{v}{2}$ times, and hence $\binom{v}{2} \le \binom{v-2}{k-2}$, cannot rule out the existence of a simple 2-SB(v, k) for any k in general. For $k = 3$, as $\binom{v}{2} = v(v - 1)/2 \le v - 2$ is not true for any positive integer v, we can claim that a simple 2-SB$(v, 3)$ does not exist. For $k = 4$, the same conclusion can be reached as it is not possible that $\binom{v}{2} \le \binom{v-2}{2}$. On the other hand, there is no contradiction from this argument for a 2-SB(10, 5) or a 2-SB(15, 5), so these designs may exist. However, we have:

Theorem 3. *A simple strict 2-SB$(v, 3)$ and a simple strict 2-SB$(v, 4)$ do not exist for any v.*

4. Constructions for non-simple 1-SB(v, k)

As we know, one can construct a non-simple 1-SB(v, k) design for any k by partitioning the set $\{1, 2, \ldots, v\}$ into k subsets such that the sum of each subset is $\frac{v(v+1)}{2k}$. Here is a general recursive method for obtaining a desired partition (each part with constant sum) of $\{1, 2, \ldots, v+2k\}$ from a partition of $\{1, 2, \ldots, v\}$ satisfying the constant sum property. This general recursive construction is useful for proving that the necessary conditions are sufficient for the existence of strict 1-SB designs.

A recursive construction for a partition: Suppose $v(v+1) \equiv 0 \pmod{2k}$; if a partition, say $\{P_1, P_2, \ldots, P_k\}$, of $\{1, 2, \ldots, v\}$, exists, then $\{P_i \cup \{v + 2k - (i-1), v+i\} | i = 1, 2, \ldots, k\}$ gives a partition of $\{1, 2, \ldots, v, v+1, \ldots, v+2k\}$ where the parts of the partition satisfy the sum property.

Therefore, a general procedure for proving that the necessary conditions are sufficient for the existence of non-simple 1-SB(v, k) for any k is: first, find the solutions for $v(v+1) \equiv 0 \pmod{2k}$ in terms of mod $2k$, and then construct 1-SB designs for the smallest possible values of v for each of the solutions/cases.

4.1. Non-simple 1-SB(v, k) where k is a prime or prime power

From the necessary condition that the number of blocks $\frac{v(v+1)}{2k}$ should be an integer, there are four cases to consider:

- $v \equiv 0 \pmod 2$ and $v \equiv 0 \pmod k$: In this case, $v \equiv 0 \pmod{2k}$ and the smallest example to construct is for $v = 2k$ a 1-SB$(2k, k)$.
- $v \equiv 0 \pmod 2$ and $v \equiv (k-1) \pmod k$: In this case, $v \equiv (k-1) \pmod{2k}$ and the smallest example to construct is for $v = 3k-1$ a 1-SB$(3k-1, k)$.
- $v \equiv 1 \pmod 2$ and $v \equiv 0 \pmod k$: In this case, $v \equiv k \pmod{2k}$ and the smallest example to construct is for $v = 3k$ a 1-SB$(3k, k)$.
- $v \equiv 1 \pmod 2$ and $v \equiv (k-1) \pmod k$: In this case, $v \equiv 2k-1 \pmod{2k}$ and the smallest example to construct is for $v = 2k$ a 1-SB$(4k-1, k)$.

Hence we need to construct a partition for each of the four base cases: for $v = 2k$, $3k-1$, $3k$ and $4k-1$.

For $v = 2k$, the partition is $\{\{i, v+1-i\} | i = 1, 2, \ldots, k\}$. The required sum for each part is $v + 1 = 2k + 1$.

For $v = 3k - 1$, we need to consider two subcases:

Subcase 1. $k = 4t + 1$, $v = 12t + 2$. These $3t + 1$ parts of the required partition: $\{\{6t+1, 12t+2\}, \{6t+2, 12t+1\}, \ldots, \{9t+1, 9t+2\}\}$, along with the next t parts, $\{i, i+1, i+2, 6t-(i-1), 6t-i, 6t-(i+1)\}$, $i = 1, 4, 7, \ldots, 3t-2$, give the required $k = 4t+1$ parts of the partition.

Subcase 2. $k = 4t+3$, $v = 12t+8$. Here the partition is obtained as follows: First, the $3t+2$ sets $\{\{6t+4, 12t+8\}, \{6t+5, 12t+7\}, \ldots, \{9t+5, 9t+7\}\}$, along with $\{3t+3, 6t+3, 9t+6\}$, give $3t+3$ parts of the partition. Next observe that the elements of the $3t$ subsets, $\{i, 6t+4-i\}$ for $i = 2, 3, \ldots, 3t$ and $\{1, 3t+1, 3t+2\}$, each add up to $6t+4$. Combine three sets for the remaining t parts of the partition to get $4t+3$ parts of the required partition. Note that the required sum, $(3k-1)3k/2k$, for each part is $3(3k-1)/2 = 3(6t+4)$.

For $v = 3k$, we need to consider two subcases, $k = 4t+1$ and $k = 4t+3$.

Subcase 1. $k = 4t+1$ and $v = 12t+3$; the partition is $\{\{1, 2, 3, 6t+1, 6t, 6t-1\}, \{4, 5, 6, 6t-2, 6t-3, 6t-4\}, \ldots, \{3t-2, 3t-1, 3t, 3t+2, 3t+3, 3t+4\}, \{3t+1, 6t+2, 9t+3\}, \{6t+3, 12t+3\}, \{6t+4, 12t+2\}, \ldots, \{9t+2, 9t+4\}\}$. The required sum, $(3k+1)3k/2k$, for each part is $3(3k+1)/2 = 18t+6$.

Subcase 2. $k = 4t+3$ and $v = 12t+9$; the partition is $\{\{6t+6, 12t+9\}, \{6t+7, 12t+8\}, \ldots, \{9t+7, 9t+8\}, \{1, 2, 6t+3, 6t+4, 6t+5\}, \{i, i+1, i+2, 6t+5-i, 6t+5-(i+1), 6t+5-(i+2)\}, i \in \{3, 6, \ldots, 3t\}\}$. The sum of each part is $3(6t+5)$ as required.

For $v = 4k-1$, the partition is $\{\{1, 4k-2, 4k-1\}, \{i-1, i, 4k-i, 4k-(i+1)\}, i \in \{3, 5, \ldots, 2k-1\}\}$. The required sum, $(4k-1)4k/2k$, for each part is $2(4k-1)$.

Even the case of block size 2 is interesting as there is no simple 1-SB$(v, 2)$ (Corollary 2). For this case we obtained the existence of strict 1-SB$(v, 2)$ designs in two ways: first by a recursive method and then with another recursive method in such a way that the designs have the minimum number of repeated blocks. For $k = 3$, we use a recursive method for the partition of $\{1, 2, \ldots, v+3\}$ from the partition of $\{1, 2, \ldots, v\}$.

4.2. 1-SB(v, k) for $k = 2$

For the existence of a 1-SB$(v, 2)$, we need $v(v+1) \equiv 0 \pmod 4$. Solving for v, we get $v \equiv 0, 3 \pmod 4$ when $k = 2$ (Corollary 1).

The following two examples are the designs for the smallest v and $k = 2$ when $v \equiv 0 \pmod 4$ and $v \equiv 3 \pmod 4$. (Note that a simple strict 1-SB$(3, 2)$ does not exist.)

Example 2. A non-simple strict 1-SB$(4, 2)$ design with blocks (as columns):

12333
24444

Example 3. A non-simple strict 1-SB(7, 2) design with blocks (as columns):

12244445666666
33355557777777

We can recursively construct non-simple strict 1-SB(v, 2) for all possible v's by using the above two examples as the base and using a recursive construction given below:

Assume 1-SB(v, 2) exists. Then, the blocks of 1-SB(v, 2), together with the block $\{v+4, v+3\}$ $v+3$ times, $v+1$ times the block $\{v+2, v+1\}$, and the block $\{v+4, v+2\}$ once, form a 1-SB($v+4$, 2).

Hence, we have:

Theorem 4. *A non-simple strict* 1-*SB*(v, 2) *exists for all* $v \equiv 0, 3 \pmod 4$.

Since simple 1-SB(v, 2) does not exist, a natural question is "What is the minimal number of blocks which need to be repeated?".

Theorem 5. *A non-simple strict* 1-*SB*(v, 2) *must contain at least* $\lceil \frac{v}{4} \rceil$ *repeated blocks.*

Proof. Without loss of generality, let i occur i times in the design for $i \in \{1, 2, \ldots, v\}$. Since there are $v-1$ elements other than v, v comes together with them once. Now, we must repeat at least one block with v; to minimize repetition, let $\{v, v-1\}$ be the repeated block. Since $v-1$ has occurred twice, we only need $v-1$ to occur $v-3$ times. Fortunately, there are exactly $v-3$ elements that we can place together with $v-1$ to avoid block repetition. Then from $v-2$ up to $\lfloor \frac{v}{2} \rfloor$, every alternate element starting with $v-3$ requires one repeating block as we continue this procedure. □

The following examples of 1-SB(v, 2)'s contain exactly $\lceil \frac{v}{4} \rceil$ repeated blocks.

Example 4. A 1-SB(7, 2) with repeated blocks {7, 6} and {5, 4}.

77777776666555
12345662345344

Example 5. A 1-SB(8, 2) with repeated blocks {8, 7} and {6, 5}.

888888887777766665
123456772345634554

Here is the general construction for the case of block size $k = 2$ with exactly $\lceil \frac{v}{4} \rceil$ repeated blocks:

Another recursive construction for $k = 2$: Suppose a 1-SB(v, 2) exists for v with blocks $b_1, \ldots, b_b$. Let $b_i + 2 = \{x + 2 : x \in b_i\}$. The collection of blocks $b_1 + 2, \ldots, b_b + 2$, blocks $\{v+4, i\}$ for $i = 1, 2, \ldots, v+3$, block $\{v+4, v+3\}$ and blocks $\{v+3, i\}$ for $i = 2, 3, \ldots, v+2$ gives the blocks for 1-SB($v+4$, 2).

Note that the blocks $b_1+2, \ldots, b_b+2$ are the blocks of a 1-SB(v, 2) on elements $3, 4, \ldots, v+2$ with the same number of repeated blocks used in $b_1, \ldots, b_b$. Also, the number of repeated blocks for the newly constructed 1-SB($v+4$, 2) is increased by exactly 1 as the block $\{v+4, v+3\}$ is used twice in the construction.

4.3. Non-simple results for $k = 3$

From the necessary conditions for the existence of a 1-SB(v, 3), $v \equiv 0, 2 \pmod 3$.

Note that 1-SB(6, 3) is the smallest possible design for $v \equiv 0 \pmod 3$ as a 1-SB(3, 3) is impossible to construct. One possible partition for $v = 6$ is to partition the set {1, 2, 3, 4, 5, 6} into $P_1 = \{1, 6\}$, $P_2 = \{2, 5\}$, and $P_3 = \{3, 4\}$ where the sum of each part is $\frac{6\times 7}{6} = 7$. In fact, we used this partition to obtain the first example shown earlier.

For $v \equiv 2 \pmod 3$, we first consider $v = 5$. Using the general construction for non-simple 1-SB(v, k), we require the sum of each part to be $\frac{5\times 6}{6} = 5$. The only possible partition for $v = 5$ is to partition the set {1, 2, 3, 4, 5} into sets {1, 4}, {2, 3}, and {5}.

Now we can recursively construct 1-SB($v+3$, 3) for $v \equiv 0, 2 \pmod 3$ from a 1-SB(v, 3) for all allowed v's as follows:

Recursive construction for $k = 3$: Let $\{P_1, P_2, P_3\}$ be the partition of $\{1, 2, \ldots, v\}$ satisfying the condition that the sum of the elements in each part is $\frac{v(v+1)}{2k}$. Without loss of generality let $1 \in P_1$. Then $\{P_1 - \{1\} \cup \{v+3\}, P_2 \cup \{1\} \cup \{v+1\}, P_3 \cup \{v+2\}\}$ gives the required partition for $\{1, 2, \ldots, v, v+1, v+2, v+3\}$.

Hence we have:

Theorem 6. *Necessary conditions* $v \equiv 0, 2 \pmod 3$, $v > 3$, *are sufficient for the existence of non-simple strict* 1-*SB*(v, 3).

4.4. Non-simple results for $k = 4$

The method used to construct non-simple 1-SB(v, 4) for $v \equiv 0, 7 \pmod 8$ is different from those of the previous two subsections as it depends directly on the general recursive method from the partition for the base designs. First we obtain the partitions $\{\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7\}\}$ and $\{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$ for the base designs 1-SB(7, 4) and 1-SB(8, 4) respectively. Then the partition $\{P_1 \cup \{v + 1, v + 8\}, P_2 \cup \{v + 2, v + 7\}, P_3 \cup \{v + 3, v + 6\}, P_4 \cup \{v + 4, v + 5\}\}$ of the set $\{1, 2, \ldots, v, v + 1, v + 2, \ldots, v + 8\}$ can be obtained from a valid partition $\{P_1, P_2, P_3, P_4\}$ of $\{1, 2, \ldots, v\}$ by the general recursive construction. Hence we get the following result.

Theorem 7. *The necessary conditions $v \equiv 0, 7 \pmod 8$ are sufficient for the existence of non-simple strict 1-SB$(v, 4)$.*

4.5. Non-simple results for $k = 5$

Recall that for the existence of a 1-SB(v, k) where k is a prime or a prime power and $k \geq 3$ the necessary condition that the number of blocks $\frac{v(v+1)}{2k}$ should be an integer implies four cases.

Hence, for $k = 5$, we need to construct base designs for four classes: $v \equiv 0, 4, 5, 9 \pmod{10}$. The smallest values of the v's in these cases are $v = 10$, 14, 15, and 9 respectively. Partitions $\{\{10, 1\}, \{9, 2\}, \{8, 3\}, \{7, 4\}, \{6, 5\}\}$, $\{\{14, 7\}, \{13, 8\}, \{12, 9\}, \{11, 10\}, \{1, 2, 3, 4, 5, 6\}\}$, $\{\{15, 9\}, \{14, 10\}, \{13, 11\}, \{12, 8, 4\}, \{1, 2, 3, 5, 6, 7\}\}$ and $\{\{9\}, \{8, 1\}, \{7, 2\}, \{6, 3\}, \{5, 4\}\}$ can be used for the base cases of $v = 10$, 14, 15 and 9 respectively.

Hence, as we can recursively construct a partition for $v + 2k = v + 10$ from v using the general recursive construction, we have:

Theorem 8. *Necessary conditions $v \equiv 0, 4, 5, 9 \pmod{10}$, $v > 5$, are sufficient for the existence of non-simple strict 1-SB$(v, 5)$.*

4.6. Non-simple results for $k = 6$

As $k = 6$ is not a prime, we cannot directly get the possible values of the v's. However, simple calculations yield that the necessary conditions on v for the existence of a 1-SB(v, k) are $v \equiv 0, 3, 8, 11 \pmod{12}$. Using the partitions $\{\{12, 1\}, \{11, 2\}, \{10, 3\}, \{9, 4\}, \{8, 5\}, \{7, 6\}\}$, $\{\{15, 5\}, \{14, 6\}, \{13, 7\}, \{12, 8\}, \{11, 9\}, \{1, 2, 3, 4, 10\}\}$, $\{\{20, 15\}, \{19, 16\}, \{18, 17\}, \{14, 13, 8\}, \{12, 10, 7, 6\}, \{1, 2, 3, 4, 5, 9, 11\}\}$ and $\{\{11\}, \{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}\}$ for the base cases of $v = 12, 15, 20$ and 11 respectively and the general recursive construction we can construct 1-SB$(v, 6)$ for $v \equiv 0, 3, 8, 11 \pmod{12}$, $v > 8$. Hence we have:

Theorem 9. *The necessary conditions $v \equiv 0, 3, 8, 11 \pmod{12}$, $v > 8$, are sufficient for the existence of a non-simple strict 1-SB$(v, 6)$.*

Note that for $v = 8$, 1-SB(8, 6) does not exist as the number of blocks, if the design exists, is 6, but by definition, it is also required that some element must appear eight times.

5. An interesting design and its application for a recursive construction

Let V be a set of $v = 2k - 2$ distinct elements. In this section we obtain a solution of the following question and apply it to get a recursive construction of simple strict 1-SB(v, k).

Is it possible to construct $k - 1$ distinct k-subsets of V such that the elements $2i$ and $2i - 1$ occur in exactly i subsets/blocks (not necessarily the same subsets/blocks)?

We show the existence of such designs, which can be called *generalized SB designs*, by a constructive proof using induction on k. We will call the entries of a k by $(k - 1)$ array at the locations $(2, k - 1)$, $(3, k - 2)$, ..., $(k, 1)$ the off-diagonal elements or off-diagonal entries of the array. Our solution for the required design can be given by a k by $(k - 1)$ array with the off-diagonal entries $\{k, k + 1, \ldots, 2k - 2\}$.

Example 6. Let $k = 3$. To construct a generalized SB design, we need elements 1 and 2 to occur with frequency 1 and elements 3 and 4 to occur with frequency 2. A solution is $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ and the corresponding array is

12
33
44

Note that the off-diagonal elements in this example are 3 and 4.

Example 7. A generalized SB design for $k = 4$.

123
344
555
666

Note that the off-diagonal entries are $\{4, 5, 6\}$.

Assume that a generalized SB design exists for some k given by a family of k-sets and the blocks can be arranged in a k by $k - 1$ array with the off-diagonal entries $\{k, k + 1, \ldots, 2k - 2\}$. Our aim is to construct a family of $(k + 1)$-sets with the required elements at the off-diagonal locations of the corresponding array. We strip the element at the off-diagonal location from each of the $k-1$ columns of the known array of k-sets, and write them as the kth column entries (to create kth partial set of our solution for $k+1$ where all the stripped sets are the first $k-1$ partial sets of the solution for $k+1$). Next we create a row with all entries $2k-1$ and then a row of all entries $2k$. In other words, here the union of the $k-1$ sets with $\{2k-1, 2k\}$ gives $k+1$ sets. We use the columns of the new array to get the required k sets of size $k+1$. The example below will help to explain the procedure.

Example 8. A generalized SB design for $k=5$.

1	2	3	4
3	4	5	5
5	6	6	6
7	7	7	7
8	8	8	8

are the four sets of size 5; then first note that at the off-diagonal locations we have the elements 5, 6, 7, 8, so we strip them and obtain the following array:

1	2	3	4	$*$
3	4	5	$*$	5
5	6	$*$	6	6
7	$*$	7	7	7
$*$	8	8	8	8

Essentially this array is giving us five sets of size 4:

1	2	3	4	5
3	4	5	6	6
5	6	7	7	7
7	8	8	8	8

Now we include a row of 9's and a row of 10's to obtain an array containing the required five sets for $k=6$ as columns.

1	2	3	4	5
3	4	5	6	6
5	6	7	7	7
7	8	8	8	8
9	9	9	9	9
10	10	10	10	10

Note that the new array also satisfies the condition that the first row contains $\{1, 2, 3\cdots, k-1=5\}$, and the next five rows contain the elements $k=6, 7, 8, 9$ and $10=2k-2$ respectively at the required locations. Therefore the array can be used in the construction for the next value, 7, recursively.

General recursive construction for 1-*SB*$(v+2k, k)$ *from* 1-*SB*(v, k) *for any* k: Suppose a simple 1-SB(v, k) exists. To construct a 1-SB$(v+2k, k)$, we follow the following procedure.

(1) Construct the blocks $\{v+2k, \dots, v+k+2, i\}$ for $i=1, 2, \dots, v+k, v+k+1$. These blocks are distinct, as the last element is different in each block. Similarly the next step gives distinct blocks as well as different blocks from this step.
(2) Construct the blocks $\{v+k+1, v+k, \dots, v+3, i\}$ for $i=2, 3, \dots, v+1, v+2$.
(3) Add 2 to every block of the simple 1-SB(v, k) so that we have the blocks of a 1-SB(v, k) on elements $3, \dots, v+1, v+2$. These blocks are distinct and different from those obtained in the earlier steps as they do not contain any element bigger than $v+2$.
(4) As for the requirement of a 1-design, elements $v+2k$ and $v+k+1$ need to occur $k-1$ times more, $v+2k-1$ and $v+k$ to occur $k-2$ times more, $v+2k-2$ and $v+k-1$ to occur $k-3$ times more, and so on, $v+k+3$ and $v+4$ to occur twice more, and $v+k+2$ and $v+3$ to occur once more. Essentially we need a design on $2k-2$ elements, namely, $\{v+3, v+4, \dots, v+2k\}$ such that a pair of elements appears once, a pair of elements appears twice, and so on, and a pair of elements appears $k-1$ times. This is exactly the kind of design with distinct blocks that we have constructed at the beginning of this section.

6. Existence of simple 1-SB(v, k) for small block sizes

In this section we prove that the necessary conditions are sufficient for the existence of simple strict 1-SB designs for small block sizes 3, 4, 5 and 6.

6.1. *Simple 1-SB*$(v, 3)$

Using Corollary 1, the necessary conditions for v are $v(v+1)\equiv 0 \pmod 6$ or $v\equiv 0, 2 \pmod 3$ for $k=3$.

Example 9. A simple 1-SB(6, 3) (*blocks as columns*):

6666665
5555444
1234233.

Example 10. A simple 1-SB(9, 3) (*blocks as columns*):

999999999877777
888888857566665
123456746423453.

The above example of the simple 1-SB(9, 3) is recursively constructed using the 1-SB(6, 3). In fact, the recursive construction (given below) can be used to construct a simple 1-SB($v + 3$, 3) from a simple 1-SB(v, 3).

Suppose $b_1, b_2, \ldots, b_b$ are the blocks of the known simple 1-SB(v, 3). First construct new blocks B_i from the blocks b_i by using $B_i = b_i + 1$, where $b_i + 1 = \{x + 1 : x \in b_i\}$. Note that the sets of all B_i's are the blocks of a simple 1-SB(v, 3) but the design is on $\{2, 3, \ldots, v + 1\}$ where the element i occurs $i - 1$ times, for $i = 2, 3, \ldots, v + 1$. Now construct blocks $C_i = \{v + 3, v + 2, i\}$ where $i = 1, 2, \ldots, v, v + 1$. Next, we construct three more blocks D_1, D_2, D_3 from any two blocks from $B_1, B_2, \ldots, B_b$, say B_1 and B_2. As the B_i's are distinct, there are three cases to consider. Suppose $|B_1 \cap B_2| = 2$ and $B_1 = \{a, b, x\}$, and $B_2 = \{a, b, y\}$; then $D_1 = \{v + 3, a, b\}$, $D_2 = \{v + 3, a, x\}$ and $D_3 = \{v + 2, b, y\}$. Suppose $|B_1 \cap B_2| = 1$ and $B_1 = \{a, b, x\}$, and $B_2 = \{a, c, y\}$; then $D_1 = \{v+3, a, b\}$, $D_2 = \{v+3, a, x\}$ and $D_3 = \{v+2, c, y\}$. Suppose $|B_1 \cap B_2| = 0$ and $B_1 = \{a, b, x\}$,and $B_2 = \{c, d, y\}$; then $D_1 = \{v + 3, a, b\}$, $D_2 = \{v + 3, c, x\}$ and $D_3 = \{v + 2, d, y\}$. Now it is easy to check that the blocks $D_1, D_2, D_3, B_3, B_4, \ldots, B_b, C_1, C_2, \ldots, C_{v+1}$ provide the required blocks of a simple 1-SB($v + 3$, 3).

For example, the simple 1-SB(6, 3) is used to construct the simple 1-SB(9, 3) by applying the steps from the method:

(1) Construct blocks $\{9, 8, i\}$ where $i = 1, 2, \ldots, 7$.
(2) Construct a 1-SB(6, 3) using elements 2, 3, 4, 5, 6, 7 by adding 1 to every block of the simple 1-SB(6, 3) on $\{1, 2, 3, 4, 5, 6\}$.
(3) Delete any two arbitrary blocks from the blocks constructed in Step (2), say {7, 5, 4} and {6, 5, 4}, and form three new blocks using them. Since we need 9 to occur nine times and 8 to occur eight times, we construct blocks {9, 5, 4}, {9, 7, 6}, and {8, 5, 4}.
(4) The blocks from (1), (2) (except {7, 5, 4} and {6, 5, 4}), and (3) give us a simple 1-SB(9, 3).

To complete the case of $k = 3$, we also need a design for the smallest allowed value of v satisfying $v \equiv 2 \pmod 3$.

Example 11. A simple 1-SB(8, 3) (*blocks as columns*):

888888887777
777666546653
654543435221

The recursive construction given above and the examples for $v = 6$ and $v = 8$ give the following result.

Theorem 10. *The necessary conditions $v \equiv 0, 2 \pmod 3$, $v > 5$, are sufficient for the existence of a simple strict* 1-*SB*(v, 3).

6.2. Simple 1-SB(v, 4)

Using Corollary 1, the necessary conditions are $v(v + 1) \equiv 0 \pmod 8$ or $v \equiv 0, 7 \pmod 8$. Hence, we give the following examples to be used as base cases in a recursive construction.

Example 12. A simple 1-SB(7, 4) (*blocks as columns*):

7777777
6666665
5555444
1234233

Example 13. A simple 1-SB(8, 4) (*blocks as columns*):

888888887
777777666
666555554
124234343

Suppose a simple 1-SB(v, 4) exists for $v = 8(t - 1) + 8$ or $v = 8(t - 1) + 7$ where $t \geq 1$. The blocks of a 1-SB($v + 8$, 4) can be constructed by the general recursive construction. Hence we have:

Theorem 11. *The necessary conditions $v \equiv 0, 7 \pmod 8$ are sufficient for the existence of a simple strict 1-SB$(v, 4)$.*

6.3. Simple 1-SB$(v, 5)$

Using Corollary 1, the necessary condition for the existence of a 1-SB$(v, 5)$ is $v(v+1) \equiv 0 \pmod{2 \cdot 5}$; this implies $v \equiv 0, 4, 5, 9 \pmod{10}$. Hence we need the following examples to apply the general recursive construction.

The first 1-SB$(v, 5)$ for $v \equiv 0 \pmod{10}$ is for $v = 10$.

Example 14. A simple 1-SB$(10, 5)$ (*blocks as columns*):

10	10	10	10	10	10	10	10	10	10	9
9	9	9	9	9	9	9	9	8	8	8
8	8	8	8	8	7	7	7	7	7	7
7	6	6	6	6	6	6	5	5	5	5
5	4	3	2	1	4	3	4	4	3	2

The first 1-SB$(v, 5)$ for $v \equiv 4 \pmod{10}$ is for $v = 14$ as the design for $v = 4$ is impossible to construct.

Example 15. A simple 1-SB$(14, 5)$ (*blocks as columns*):

14	14	14	14	14	14	14	14	14	14	14	14	14	14
13	13	13	13	13	13	12	12	12	12	12	12	12	12
11	11	11	11	10	11	11	11	10	10	10	11	10	10
9	9	9	9	9	9	8	8	8	8	8	8	8	8
6	5	4	3	3	2	6	5	6	5	4	3	2	1

13	13	13	13	13	13	13
12	12	12	12	11	11	11
10	10	10	9	10	9	9
7	7	7	7	7	7	7
6	5	4	6	6	5	4

The first 1-SB$(v, 5)$ for $v \equiv 5 \pmod{10}$ is for $v = 15$ as the design does not exist for $v = 5$.

Example 16. A simple 1-SB$(15, 5)$ (*blocks as columns*):

15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
14	14	14	14	14	13	13	13	13	13	13	13	13	13	13
12	12	12	12	12	12	11	11	11	11	11	11	11	11	11
10	10	10	9	9	9	9	9	9	9	9	9	8	8	8
7	6	5	7	6	7	6	5	4	3	2	1	6	5	4

14	14	14	14	14	14	14	14	14
13	13	13	12	12	12	12	12	12
11	11	10	10	10	10	10	10	10
8	8	7	8	8	7	7	8	7
3	2	4	6	5	6	5	4	3

The first 1-SB$(v, 5)$ for $v \equiv 9 \pmod{10}$ is for $v = 9$.

Example 17. A simple 1-SB$(9, 5)$ (*blocks as columns*):

999999999
888888887
777777666
666555545
123234434

We can use the above examples as base cases and construct all 1-SB$(v, 5)$'s for $v \equiv 0, 4, 5, 9 \pmod{10}$ by the general recursive construction. Hence we have the following result.

Theorem 12. *The necessary conditions $v \equiv 0, 4, 5, 9 \pmod{10}$, $v > 5$, are sufficient for the existence of a simple strict 1-SB$(v, 5)$.*

6.4. 1-SB(v, 6)

Using Corollary 1, the necessary conditions for the existence of a simple strict 1-SB$(v, 6)$ are $v(v+1) \equiv 0 \pmod{12}$ or $v \equiv 0, 3, 8, 11 \pmod{12}$. Here are the required examples for $k = 6$:

Example 18. A simple 1-SB(12, 6) for $v \equiv 0 \pmod{12}$:

12	12	12	12	12	12	12	12	12	12	12	12	11
11	11	11	11	11	11	11	11	11	11	10	10	10
10	10	10	10	10	10	10	9	9	9	9	9	9
9	9	9	8	8	8	8	8	8	8	8	7	7
7	7	7	7	7	6	6	6	6	6	6	5	5
5	4	3	5	4	5	4	4	3	2	1	3	2

Example 19. A simple 1-SB(15, 6) for $v \equiv 3 \pmod{12}$:

15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
14	14	14	14	14	14	14	14	14	13	13	13	13	13	13
13	13	12	12	12	12	12	12	12	12	12	12	12	12	11
11	11	11	11	11	10	10	10	10	10	10	10	10	10	9
9	9	9	9	8	8	8	8	8	8	8	8	7	7	7
6	5	6	5	6	6	5	4	3	6	5	4	5	4	4
					14	14	14	14	14					
					13	13	13	13	13					
					11	11	11	11	11					
					9	10	9	9	9					
					7	7	7	7	6					
					3	2	2	1	3					

Example 20. A simple 1-SB(20, 6) for $v \equiv 8 \pmod{12}$:

20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20
19	19	19	19	18	18	18	18	18	18	18	18	18	18	18	18	18	18
17	17	17	17	16	16	16	16	16	16	17	16	17	16	16	16	16	16
14	14	14	14	14	14	14	14	12	14	14	14	14	14	13	13	13	13
12	12	12	12	11	11	11	11	11	11	11	11	11	11	11	10	10	10
8	7	6	5	8	7	6	5	5	4	4	3	2	2	8	8	7	6

20	20	19	19	19	19	19	19	19	19	19	19	19	19	19	19	19
18	18	18	18	17	17	17	17	17	17	17	17	17	16	17	18	17
16	16	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15
13	13	13	13	13	13	13	13	13	12	12	12	14	12	12	12	12
10	10	10	10	10	10	10	9	9	9	9	9	9	9	9	9	8
5	1	8	7	8	7	6	8	7	7	6	5	4	4	3	3	6

Example 21. A simple 1-SB(11, 6) for $v \equiv 11 \pmod{12}$:

11	11	11	11	11	11	11	11	11	11	11
10	10	10	10	10	10	10	10	10	10	9
9	9	9	9	9	9	9	9	8	8	8
8	8	8	8	8	7	7	7	7	7	7
7	6	6	6	6	6	6	5	5	5	5
5	4	3	2	1	4	3	4	3	2	4

We can use the above examples as base cases and construct all 1-SB$(v, 6)$'s for $v \equiv 0, 3, 8, 11 \pmod{12}$ using the general recursive construction. Hence we have the following result.

Theorem 13. *The necessary conditions $v \equiv 0, 3, 8, 11 \pmod{12}$, $v > 8$, are sufficient for the existence of a simple strict 1-SB$(v, 6)$.*

There may be many ways to construct $k-1$ distinct sets of size k, where elements $v+2k$ and $v+(k+1)$ occur $k-1$ times, elements $v+2k-1$ and $v+k$ occur $k-2$ times, and so on, and elements $v+2k-(k-2) = k+2$ and 3 occur once. For example, another set of blocks for the above example with $k = 6$ is $\{\{v + 12, v + 7, v + 11, v + 6, v + 10, v + 9\}, \{v + 12, v + 7, v + 11, v + 6, v + 10, v + 4\}, \{v + 12, v + 7, v + 11, v + 6, v + 5, v + 10\}, \{v + 12, v + 7, v + 11, v + 5, v + 4, v + 9\}, \{v + 12, v + 7, v + 6, v + 5, v + 3, v + 8\}\}$. The difficulty arises in proving that the necessary conditions are sufficient for general k because the base values of v for which we need to construct "seed" designs do not seem to follow a pattern.

In conclusion, although the general recursive construction given in Section 5 paves a way to obtain simple 1-SB designs for any k, an elegant way to solve the existence problem of 1-SB designs for general k is desired and to the best of our knowledge a problem for simple 1-designs with non-constant replications is also untouched. Also the enumeration of all non-isomorphic designs, at least for small parameters, is also not discussed here, but is an interesting combinatorial problem as suggested by Professor Stanton in [13] for SB triple systems. Hence there is still a lot to do even for 1-designs!

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