The Characterization of the Derivatives for Linear Combinations of Post–Widder Operators in $L_p$

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1. INTRODUCTION AND MAIN RESULTS

It is well known that Post–Widder operators constitute the real inversion formula for the Laplace transform. Post–Widder operators are given by

$$P_n(f, x) = \frac{(n/x)^n}{(n-1)!} \int_0^\infty e^{-nu}u^{n-1}f(u) \, du, \quad x \in (0, \infty), \quad (1.1)$$

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where \( f \in L_p[0, \infty) \) \((1 \leq p < \infty)\) or \( f \in C[0, \infty)\). We will use for \( P_n(f, x) \) the combination \( P_n(f, x) \) given by \([1, \text{Chapter 9}]\)

\[
P_n(f, x) = \sum_{i=0}^{r-1} C_i(n) P_i(n, f, x), \quad x \in (0, \infty), \quad r \in N,
\]

where \( n \) and \( C_i(n) \) satisfy

(a) \( n = n_0 < \cdots < n_{r-1} \leq An; \)
(b) \( \sum_{i=0}^{r-1} C_i(n) \leq C; \)
(c) \( \sum_{i=0}^{r-4} C_i(n) = 1; \)
(d) \( \sum_{i=0}^{r-4} C_i(n) P_i((n-x)^k, x) = 0, \quad 1 \leq k \leq r - 1. \)

Concerning the approximation by linear combinations of Post–Widder operators, Ditzian and Totik \([1]\) proved direct and converse results for these operators in \( L_p \). Their main theorems show, for \( f \in L_p[0, \infty), \quad 1 \leq p < \infty \) (with \( C[0, \infty), \quad p = \infty \)) and \( \varphi(x) = x \), that

\[
\| P_n(f, x) - f(x) \|_p = C(n^{-\sigma}) \equiv \omega_\varphi^2(f, t)_p = C(t^{2\sigma}) \quad (0 < \varphi < r),
\]

where

\[
\omega_\varphi^2(f, t)_p = \sup_{0 < h, t, l \in N_0} \| A_{l, p}^2 f \|_{L_p(0, \infty)}, \quad \varphi(x) = x, \quad f \in L_p[0, \infty),
\]

and

\[
A_{l, p}^2 f(x) = \begin{cases} \sum_{j=0}^{2l} (-1)^j \binom{2l}{j} f(x + (r-j)h), & \text{for } x \geq rh \\ 0, & \text{otherwise.} \end{cases}
\]

The first aim of this paper is to prove new direct and converse results on weighted simultaneous approximation by the method of linear combinations of Post–Widder operators in \( L_p, \quad 1 \leq p \leq \infty \). Our results are stated as follows.

**Theorem 1.1.** Let \( f, f^{(i)} \in L_p[0, \infty), \quad 1 \leq p \leq \infty \) (with \( C[0, \infty), \quad p = \infty \)), \( l \in N_0, \quad r \in N \). Then

\[
\| \varphi A_{l, p}^2 f \|_{L_p(0, \infty)}^{2p} \leq C\{ \omega_\varphi^2(f^{(i)}, n^{-1/2})_{p, p} + n^{-r} \| \varphi f^{(i)} \|_p \}.
\]

Here

\[
\omega_\varphi^2(f^{(i)}, t)_{p, p} = \sup_{0 < h, t, l \in N_0} \| \varphi A_{l, p}^2 f^{(i)} \|_{L_p(0, \infty)}, \quad f^{(i)}, \varphi f^{(i)} \in L_p[0, \infty)
\]
is the weighted Ditkian–Totik modulus of smoothness which was shown in [1, Chap. 6] to be equivalent to the weighted K-functional defined by

\[
K_2^r(f^{(i)}, t^{2r})_{r, p} = \inf \{ \| \phi^i(f^{(i)} - g) \|_p + t^{2r} \| \phi^{i+2r}g^{2r} \|_p | \phi^i, \phi^{i+2r}g^{2r} \in L_p[0, \infty), 1 \leq p \leq \infty \}.
\]  

(1.5)

**Theorem 1.2.** Let \( f, f^{(i)}, \phi f^{(i)} \in L_p[0,\infty), 1 \leq p \leq \infty \) (with \( C[0,\infty) \)), for \( p = \infty \), \( l \in N_0 \), \( r \in N \), \( t > 0 \). Then

\[
K_2^r(f^{(i)}, t^{2r})_{r, p} \leq \| \phi^i(P_n(f, x) - f(x))^{(i)} \|_p + M(nt)^r K_2^r(n^{-2r})_{r, p}.
\]  

(1.6)

From Theorems 1.1 and 1.2 and Corollary 6.3.2 in [1], and the Berens–Lorentz Lemma [1, Chap. 9], we obtain

**Theorem 1.3.** Let \( f, f^{(i)}, \phi f^{(i)} \in L_p[0, \infty), 1 \leq p \leq \infty \) (with \( C[0,\infty) \)), for \( p = \infty \), \( l \in N_0 \), \( r \in N \), \( t > 0 \), and \( l/2 < \alpha < l/2 + r \). Then the following statements are equivalent.

\[
\| \phi^i(P_n(f, x - 1, x) - f(x))^{(i)} \|_p = O(n^{l/2-\alpha});
\]  

(1.7)

\[
\omega_2^r(f^{(i)}, t)_{r, p} = O(t^{2l-\alpha});
\]  

(1.8)

\[
\omega_2^{r+\alpha}(f, t)_{p} = O(t^{2\alpha}).
\]  

(1.9)

**Remark 1.4.** For \( l = 0 \), we obtain (1.3) mentioned above. Some ideas of our proof of Theorem 1.3 are from [5].

With the definition of (1.4), the Besov space of Ditkian–Totik type \( B^\alpha_\phi(L_p[0,\infty)) \) are defined for \( 0 < \alpha < m \), \( 1 \leq p \leq \infty \), and \( 0 < q \leq \infty \) as the set of all functions \( f \in L_p[0,\infty) \) for which

\[
|f|_{B^\alpha_\phi(L_p[0,\infty))} = \left( \int_0^\infty (t^{-\alpha} \omega_2^r(f, t)_{p})^q \frac{1}{t} dt \right)^{1/q}
\]  

(1.10)

is finite. Here, \( m \) is any integer larger than \( \alpha \). When \( q = \infty \), the usual change from integral to sup is made in (1.10). We define the following norms or quasi-norms for \( B^\alpha_\phi(L_p[0,\infty)) \):

\[
|f|_{B^\alpha_\phi(L_p[0,\infty))} = |f|_{L_p[0,\infty)} + |f|_{B^\alpha_\phi(L_p[0,\infty))}. 
\]  

(1.11)

These Besov spaces were defined for \( 1 \leq q \leq \infty \) and studied by Zhou [8] and also studied by several other authors [3, 4].

We note that when \( q < 1 \), (1.11) is not really a norm, it is only a quasi-norm, and that different values of \( m > \alpha \) result in norm or quasi-norm
which are equivalent. This is proved by establishing inequalities between the modulus of smoothness $\omega_m(f, t)_p$ and $\omega_{m+1}(f, t)_p$. A simple inequality is $\omega_{m+1}(f, t)_p \leq C \omega_m(f, t)_p$, which follows from [1, Chap. 4]. In the other direction, we have the Marchaud type inequality

$$\omega_m(f, t)_p \leq C \left\{ \int_0^t \frac{\omega_{m+1}(f, u)_p}{u^{m+1}} \, du + \| f \|_p \right\},$$

which was also proved in [1, Chap. 4]. Using these two inequalities for modulus $\omega_m(f, t)_p$ together with Hard inequality [1, Chap. 9], one shows that any two norms or quasi-norms given by (1.11) are equivalent provided that both $m$ satisfy $m > \alpha$.

Some papers [2, 6, 7] have characterized smoothness of the functions in $C[0, 1]$ by derivatives of Bernstein-type integral operators and also in $L_p$, $1 \leq p \leq \infty$ by derivatives of Bernstein–Durrmeyer operators. In the second part of this paper, by using of the commutative property of these operators, we will show that the derivatives of $P_n(f, x)$ can also be characterized in the Besov spaces defined as in (1.11).

**Theorem 1.5.** For $r \in \mathbb{N}$, $0 < \alpha < r$, $f \in L_p([0, \infty))$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$, the norms or quasi-norms

$$\| f \|_{B^r_p([0, \infty))},$$

$$\| \{(n+1)^r P_n, f(x) - f(x) \|_p \|_{L^1} \| f \|_p, (1.13)$$

and

$$\| (n^{r-q} \| \| P_{n,w} f(x) \|_p \|_{L^q} \| f \|_p \| (1.14)$$

are equivalent, where for a sequence $\{a_n\}_{n=1}^{\infty}$

$$\| \{a_n\} \|_q^r = \left\{ \left( \sum_{n=1}^{\infty} |a_n|^q n^{-q} \right)^{1/q} \right\}^{1/r}, \quad \text{for} \quad 0 < q < \infty$$

$$\sup_{n} |a_n|, \quad q = \infty.$$

**Remark 1.6.** For $q = \infty$ and $0 < \alpha < r$, we obtain the equivalent relations

$$\| P_n, f(x) - f(x) \|_p = C(n^{-\alpha}) \Rightarrow \omega_{2r}(f, t)_p$$

$$= C(t^{2\alpha}) \Rightarrow \| P_{n,2r} f \|_p = C(n^{t^{-\alpha}}),$$

which is also similar to results given in [2] for Bernstein–Durrmeyer operators and linear combinations of Bernstein–Durrmeyer operators.
Throughout this paper, $M$ and $C$ will always stand for positive constants which are dependent only on $p$, $q$, $r$, $l$, and $x$; their values may be different at different occurrences and $\varphi(x) = x$.

### 2. DEFINITIONS AND AUXILIARY RESULTS

For convenience we introduce the auxiliary operators given for $n \geq l + 1$, $l \in \mathbb{N}_0$, $f \in L_p(0, \infty)$ (1 $\leq p < \infty$), or $f \in C[0, \infty)$ by

$$P_{n,r,l}(f, x) = \sum_{i=0}^{r-1} C_i(n) \bar{P}_{n,r,l}(f, x), \quad x \in (0, \infty),$$

where

$$\bar{P}_{n,r,l}(f, x) = \frac{n^\lambda_r}{(n_l - 1)!} \int_0^\infty e^{-n^\lambda_r t} t^{l-1} f(tx) \, dt$$

$$= \frac{n^\lambda_r}{(n_l - 1)!} \int_0^\infty e^{-n^\lambda_r u} u^{n_l + l-1} f(u) \, du, \quad x \in (0, \infty). \quad (2.1)$$

It is easy to see that these operators too are bounded on the spaces $L_p[0, \infty)$, $1 \leq p \leq \infty$ (with $C[0, \infty)$, for $p = \infty$) and that the following four representations are also valid.

If $f, \varphi f \in L_p[0, \infty)$, $1 \leq p \leq \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in \mathbb{N}_0$, then

$$\varphi P_{n,r,l}(f, x) = P_{n,r,l}(\varphi f, x). \quad (2.2)$$

If $f, f^{(l)} \in L_p[0, \infty)$, $1 \leq p \leq \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in \mathbb{N}_0$, then

$$P_{n,r,l}^{(l)}(f, x) = P_{n,r,l}(f^{(l)}, x). \quad (2.3)$$

If $f, \varphi f \in L_p[0, \infty)$, $1 \leq p \leq \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in \mathbb{N}_0$, then

$$\varphi^{(l)} P_{n,r,l}(f, x) = P_{n,r,l}(\varphi^{(l)} f, x). \quad (2.4)$$

If $f, \varphi^{(2l)} f \in L_p[0, \infty)$, $1 \leq p \leq \infty$ (with $C[0, \infty)$, for $p = \infty$), $l \in \mathbb{N}_0$, then

$$\varphi^{(2l)} P_{n,r,l}^{(2l)}(f, x) = P_{n,r,l}(\varphi^{(2l)} f^{(2l)}, x). \quad (2.5)$$

For the proofs of our main theorems, we will need the following lemmas, which are of importance by themselves.
Lemma 2.1. If \( f, \varphi \in L_p[0, \infty), \; 1 \leq p < \infty \) (with \( C[0, \infty) \), for \( p = \infty \)), \( l \in \mathbb{N}_0 \), then
\[
\| \varphi^l P_{n, l}(f, x) \|_p \leq C \| \varphi \|_p. \tag{2.6}
\]

Proof. With (2.2), we have
\[
\| \varphi^l P_{n, l}(f, x) \|_p = \| P_{n, l} (\varphi f, x) \|_p \leq C \| \varphi \|_p.
\]

Lemma 2.2. If \( f, \varphi^{2r} f, \varphi^{2r+1} f^{2r} \in L_p(0, \infty), \; 1 \leq p < \infty \) (with \( C[0, \infty) \), for \( p = \infty \)), \( l \in \mathbb{N}_0 \), then
\[
\| \varphi^{2r+1} P_{n, l}^{(2r)} (f, x) \|_p \leq C \| \varphi^{2r+1} f^{2r} \|_p. \tag{2.7}
\]

Proof. Multiplying both sides in relation (2.5) by \( \varphi^l \), we obtain
\[
\varphi^{2r+1} P_{n, l}^{(2r)} (f, x) = \varphi^l P_{n, l} (\varphi^{2r+1} f^{2r}, x).
\]

Using (2.2), we have
\[
\| \varphi^{2r+1} P_{n, l}^{(2r)} (f, x) \|_p = \| P_{n, l} (\varphi^{2r+1} f^{2r}, x) \|_p,
\]
which implies
\[
\| \varphi^{2r+1} P_{n, l}^{(2r)} (f, x) \|_p \leq C \| \varphi^{2r+1} f^{2r} \|_p.
\]

Lemma 2.3. If \( f, \varphi f \in L_p[0, \infty), \; 1 \leq p < \infty \) (with \( C[0, \infty) \), for \( p = \infty \)), \( l \in \mathbb{N}_0 \), then
\[
\| \varphi^{2r+1} P_{n, l}^{(2r)} (f, x) \|_p \leq C n^r \| \varphi \|_p. \tag{2.8}
\]

Proof. By simple computation, we have
\[
\bar{P}_{n, l}(f, x) = \frac{1}{x^l} P_{n, l}(\varphi f, x), \tag{2.9}
\]
then
\[
\varphi^{2r+1} \bar{P}_{n, l}^{(2r)} (f, x) = \sum_{j=0}^{2r} C_{r, j} x^j \varphi(x)^{2r-j} P_{n, l}^{(2r-j)}(\varphi f, x).
\]

From [1, Chap. 9], it is easy to obtain
\[
P_{n, l}^{(2r-j)}(\varphi f, x) = \sum_{v=0}^{2r-j} Q_v(n, x) P_{n, l}((\cdot - x)^v \varphi f, x),
\]
where $Q_s(n_i, x) = \sum_{2r-1} C(v, r) n_i^{2r-1}$. Therefore

$$\varphi(x) |Q_s| C(n_i, x) \leq C \frac{n_i^{v+2}}{x^r}, \quad x > 0, \quad v = 0, 1, 2, 3, \ldots, 2r.$$

Thus, following the proof of Lemma 9.4.1 in [1], it is easy to complete the proof of Lemma 2.3.

**Lemma 2.4.** If $f, f, \varphi^{2r+1} \in L_p([0, \infty), 1 \leq p \leq \infty$ (with $C([0, \infty),$ for $p = \infty$), $f(x)$, then

$$\|\varphi^1 f, f, x - f(x)\|_p \leq C \phi_{r}\left\{\|f\|_p + \|f^{2r+1}\|_p\right\}. \tag{2.10}$$

**Proof.** With (2.2), we have

$$\phi^1(P_{n, r}, I(f, x)) = P_{n, r}(\phi^1 f, x) - (\phi^1 f)(x).$$

We expand $\phi f$ by the Taylor formula

$$(\phi f)(t) = \sum_{j=0}^{2r} \frac{(t-x)^j}{j!} (\phi f)^{(j)}(x) + R_{2r}(\phi f, t, x),$$

with the integral remainder

$$R_{2r}(\phi f, t, x) = \frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\phi f)^{(2r)}(v) \, dv.$$

We write

$$P_{n, r}(\phi f, x) = \sum_{j=0}^{2r-1} \frac{1}{j!} (\phi f)^{(j)}(x) P_{n, r}((\cdot - x)^j, x)$$

$$+ P_{n, r} \left( \frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\phi f)^{(2r)}(v) \, dv, x \right).$$

It follows from the definition of $P_{n, r}(f, x)$ that

$$P_{n, r}(\phi f, x) - (\phi f)(x) = \sum_{j=0}^{2r-1} \frac{1}{j!} (\phi f)^{(j)}(x) P_{n, r}((\cdot - x)^j, x)$$

$$+ P_{n, r} \left( \frac{1}{(2r-1)!} \int_x^t (t-v)^{2r-1} (\phi f)^{(2r)}(v) \, dv, x \right)$$

$$= I_1 + I_2.$$
Using Lemma 9.5.1 of [1], we have
\[ |I_1| \leq M^{1-r} \sum_{j=r}^{2r-1} \frac{1}{j!} |(\varphi')^{(j)}(x) \varphi(x)| \]
\[ \leq M^{1-r} \sum_{j=r}^{2r-1} \sum_{s=0}^{j} |\varphi^{(s+j)}(x) \varphi^{-(s-j)}(x)|. \]
Then
\[ \|I_1\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \sum_{s=0}^{j} \|\varphi^{(s+j)}\|_p \right) + \|\varphi\|_p. \]
Using that \( \|\varphi^{(s+j)}\|_p \leq C \|\varphi^{(s+j)}\|_p \) for \( j = 1, 2, \ldots \), we obtain
\[ \|I_1\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \right) \|\varphi\|_p. \]
To prove (2.10), it remains to show that for \( 1 \leq p < \infty \)
\[ \|I_2\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \right) \|\varphi\|_p. \]
From [1, Lemma 9.5.2], we have
\[ \|I_2\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \right) \|\varphi\|_p. \]
\[ \|I_2\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \right) \|\varphi\|_p. \]
thus we obtain in a way similar to before the estimate
\[ \|I_2\|_p \leq M^{1-r} \left( \sum_{j=r}^{2r-1} \right) \|\varphi\|_p. \]
The proof of Lemma 2.4 is complete.

Next we will prove the commutative property of these operators, which
is important for our purpose.

**Lemma 2.5.** For \( f(x) \in L_p(0, \infty), 1 \leq p < \infty \) (with \( C(0, \infty), \) for \( p = \infty \)),
\( m, n = 1, 2, 3, \ldots \), then
\[ P_{m, r}(P_{m, r}(f, \cdot), x) = P_{m, r}(P_{m, r}(f, \cdot), x), \quad x > 0. \]  
(2.11)
Proof. From the definition of linear combination for the Post–Widder operator $P_{n,r}(f, x)$, we need only to show that

$$P_{n,r}(f, x) = P_{m,s}(P_{n,r}(f, x), x), \quad m, n = 1, 2, 3, \ldots, \quad x > 0.$$ 

For $p = \infty$, 

$$P_{n,r}(f, x) = \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} P_{m,s}(f, u) du$$ 

$$= \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} \left( \int_0^\infty \frac{(m/v)^m}{(m-1)!} e^{-m/v} v^{m-1} f(v) dv \right) du.$$ 

Since 

$$\left| \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} \left| f(v) \right| dv \right| du \leq \left\| f \right\|_\infty,$$

thus, by using of Fubini’s theorem, we have 

$$P_{n,r}(f, x) = \int_0^\infty f(v) v^{m-1} dv \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} \left( \int_0^\infty \frac{(m/v)^m}{(m-1)!} e^{-m/v} v^{m-1} f(v) dv \right) du.$$ 

Let $v/u = w/x$, then 

$$\int_0^\infty f(v) v^{m-1} dv \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} \left( \int_0^\infty \frac{(m/v)^m}{(m-1)!} e^{-m/v} v^{m-1} f(v) dv \right) du$$ 

$$= \int_0^\infty f(v) v^{m-1} dv \int_0^\infty \frac{(n/u)^n}{(n-1)!} e^{-m/u} u^{n-1} \left( \int_0^\infty \frac{(m/v)^m}{(m-1)!} e^{-m/v} v^{m-1} \left( \frac{v}{w} \right)^{m-1} \right) dv$$ 

$$\times \frac{1}{(m-1)!} \left( \frac{mw}{xw} \right)^m e^{-m/w} \frac{xw}{w} dv$$ 

$$= \int_0^\infty f(v) v^{m-1} dv \int_0^\infty \frac{(m/w)^m}{(m-1)!} e^{-m/w} w^{m-1} (n/w)^n (n-1)! e^{-m/w} dw$$ 

$$= \int_0^\infty \frac{(m/w)^m}{(m-1)!} e^{-m/w} w^{m-1} P_{n,r}(f, w) dw$$ 

$$= P_{m,s}(P_{n,r}(f, x), x).$$
Thus we prove Lemma 2.5 for $p = \infty$. For $1 \leq p < \infty$, we define $C_0 = \{ f \in C[0, \infty), \text{supp} \ f \in [0, M] \text{ for some } M > 0 \}$. It is obvious that $C_0$ is dense in $L_p[0, \infty)$ for $1 \leq p < \infty$; therefore we need only to prove the result for $f \in C_0$, which is similar to the case of $p = \infty$. Then the proof of Lemma 2.5 is complete.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. The essential tool in this proof is the equivalence
\[ \omega_\phi^{(i)}(f^{(i)}, t, \phi, p) \sim K_\phi^{(i)}(f^{(i)}, t^{(i)}) \phi, p \]
and it has to be shown that
\[
\| \phi'(P_n, r(f, x) - f(x))^{(i)} \|_p \leq C_1 K_\phi^{(i)}(f^{(i)}, n^{-r}) \phi, p + n^{-r} \| \phi f^{(i)} \|_p. \tag{3.1}
\]

Now for every $g \in L_p[0, \infty)$ with $\| \phi' g \|_p$, $\phi^{2r} g^{(2r)} \in L_p[0, \infty)$, Lemma 2.1, Lemma 2.4 and (2.3) imply
\[
\| \phi'(P_n, r(f, x) - f(x))^{(i)} \|_p \leq \| \phi'(P_n, r(f^{(i)} - g, x)) \|_p + \| \phi'(f^{(i)}(x) - g(x)) \|_p
\]
\[
\leq C_1 \{ \| \phi'(f^{(i)}(x) - g(x)) \|_p
\]
\[
+ \| \phi'(P_n, r(g, x) - g(x)) \|_p
\]
\[
\leq C_1 \{ \| \phi'(f^{(i)}(x) - g(x)) \|_p
\]
\[
+ n^{-r} \| \phi^{2r} g^{(2r)} \|_p + n^{-r} \| \phi f \|_p. \tag{3.1}
\]

Taking here the infimum over all $g$ subject to the definition of the weighted $K$-functional gives the desired inequality (3.1). Then we complete the proof of Theorem 1.1.

Proof of Theorem 1.2. From the definition of the weighted $K$-functional in (1.5), Lemma 2.2 and Lemma 2.3, it is easy to prove theorem 1.2.

Proof of Theorem 1.5. We only prove the equivalence between (1.12) and (1.14) for $0 < q < \infty$, since the proof for the case $q = \infty$ is simple and the equivalence between (1.12) and (1.13) can be proved by using the same method as the proof of Theorem 4.1 of [3].

From [1, Chap. 9], it is easy to obtain
\[
\| \psi^{\psi} P_{n, r}(f, x) \|_p \leq M n^r \omega_\psi^{2r}(f, n^{-1/2} p), \quad 1 \leq p \leq \infty.
\]
Hence, for $0 < q < \infty$

$$
\sum_{n=1}^{\infty} (n^{\alpha + \gamma_2} \| \varphi^{2r} P^{2r}(f, x) \|_p)^{\frac{1}{q}} n
$$

$$
\leq M \left( \sum_{n=2}^{\infty} n^{\alpha - 1} \omega_2(f, n^{-1/2})^q + \| f \|_p^q \right)
$$

$$
\leq M \left( \sum_{n=2}^{\infty} \left( \frac{1}{n-1} \omega_2(f, t)^q \right) \frac{1}{t} dt + \| f \|_p^q \right)
$$

$$
\leq M \left( \int_0^{\infty} (t^{-2} \omega_2(f, t)^q \right) \frac{1}{t} dt + \| f \|_p^q \right).
$$

To prove the inverse part, we choose $2^k \in \mathbb{N}$, which will be determined later, and let \( \{n_k\}_{k \in \mathbb{N}} \) be a sequence of integers such that

\[ 2^{k-1} \leq n_k < 2^k \]

and

\[
\| \varphi^{2r} P^{2r}(n, f, x) \|_p = \min_{2^{k-1} \leq n < 2^k} \{ \| \varphi^{2r} P^{2r}(f, x) \|_p \}.
\] (3.2)

We now recall the Peetre $K$-functional

\[
K_{2r, \varphi}(f, t^2) = \inf_{g \in C_c(\mathbb{R})} (\| f - g \|_{L^p(0, \infty)} + t^2 \| \varphi^{2r} g^{(2)} \|_{L^p(0, \infty)}),
\]

\[ 1 \leq p \leq \infty, \] (3.3)

which was shown in [1, Chap. 2] to be equivalent to $\omega_2(f, t)_p$.

For $0 < q < \infty$, we have

\[
\int_0^{\infty} (t^{-2} K_{2r, \varphi}(f, t^2))^{\frac{q}{2}} \frac{dt}{t} \leq \frac{1}{1} \int_0^{\infty} (t^{-2} K_{2r, \varphi}(f, t^2))^{\frac{q}{2}} \frac{dt}{t} + M \| f \|_p^q
\]

$$
\leq M \left( \sum_{k=0}^{\infty} (2^{k-1} K_{2r, \varphi}(f, 2^{k-1}))^{\frac{q}{2}} \right).$$

We fix $m \in \mathbb{N}$ and let $U_k = 2^{k-1} K_{2r, \varphi}(P_{m, r}(f, x), 2^{k-1})$, and obtain by using Theorem 9.3.2, Lemma 9.7.2 of [1], and Lemma 2.5,

$$
U_k \leq 2^{k-1} \| P_{m, r}(f, x) - P_{m, r}(P_{m, r}(f, \cdot), x) \|_p
$$

$$
+ 2^{k(\alpha - 1)} \| \varphi^{2r} P^{2r}_m(P_{m, r}, f, \cdot, x) \|_p
$$
\[
\begin{align*}
&M^k r K_{2r,\varrho}(P_m, r(f, x), n_{k+2})_p + M^k r \|f\|_p \\
&+ M^k r \|q r P_m(2r, f, x)\|_p \\
&\leq M^k r U_{k+1} + M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p \\
&\leq (M^k r U_{k+1} + M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p) \\
&\times (\|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p).
\end{align*}
\]
Since
\[
U_{k+j} = \lambda^{k+j} r K_{2r,\varrho}(P_m, r(f, x), \lambda^{-k-j})_p \\
\leq \lambda^{k+j} r \|q r P_m(2r, f, x)\|_p.
\]
Taking \( \lambda \) to be big enough, then we have
\[
(M^k r U_{k+1} + M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p) \\
\leq (M^k r U_{k+1} + M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p).
\]
Taking \( \lambda \) to be big enough, then we have
\[
U_{k+j} = \lambda^{k+j} r K_{2r,\varrho}(P_m, r(f, x), \lambda^{-k-j})_p \\
\leq \lambda^{k+j} r \|q r P_m(2r, f, x)\|_p.
\]
Hence
\[
U_k \leq M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p.
\]
Note that for \( f \in L_p[0, \infty) \), \( 1 \leq p < \infty \), and \( C[0, \infty) \) for \( p = \infty \), we have
\[
\|P_m, r(f, x) - f(x)\|_p \to 0, \quad m \to \infty,
\]
and therefore
\[
\lambda^{k+j} r K_{2r,\varrho}(f, \lambda^{-k})_p \leq M^k r \|q r P_m(2r, f, x)\|_p + M^k r \|f\|_p.
\]
(3.4)
For \( 0 < q < \infty \), we choose \( 0 < \mu < \min\{1, q\} \), we have
\[
\begin{align*}
\sum_{k=0}^{\infty} (\lambda^{k+j} r K_{2r,\varrho}(f, \lambda^{-k}))_p &\leq M^k r \|f\|_p + \sum_{k=0}^{\infty} (M^k r \|q r P_m(2r, f, x)\|_p) + \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} (M^k r \|q r P_m(2r, f, x)\|_p)^q \right)^{1/q}.
\end{align*}
\]
Taking $\beta = x/2$ and $2 \gamma > M^2 r/x$, then the second term can be deduced by the Hölder inequality as

$$\left\{ \sum_{l=0}^{\infty} \left[ (M^{l-1} t^{\gamma} p_{n+1/l, r})^2 \right] \right\}^{q/p} \leq \left\{ \sum_{l=0}^{\infty} \left( n^{l-\beta} \| \phi^{2r} p_{n+1/l, r} \|_p \right)^q \right\}^{1/q}$$

$$\left\{ \sum_{l=0}^{\infty} \left[ (M^{l-1} t^{\gamma} p_{n+1/l, r})^2 \right] \right\}^{q/p} \leq C \left\{ \sum_{l=0}^{\infty} \left( n^{l-\beta} \| \phi^{2r} p_{n+1/l, r} \|_p \right)^q \right\}^{1/q}$$

Then

$$\sum_{k=0}^{\infty} (n^{l-k} K_{2r, \phi}^p (f, \lambda^{-k}))^q \leq C \sum_{k=0}^{\infty} (n^{l-k} K_{2r, \phi}^p (f, \lambda^{-k}))^q \leq C \sum_{l=0}^{\infty} \left( n^{l-\beta} \| \phi^{2r} p_{n+1/l, r} \|_p \right)^q + M \| f \|_p^q$$

$$\leq C \sum_{l=0}^{\infty} \left( n^{l-\beta} \| \phi^{2r} p_{n+1/l, r} \|_p \right)^q + M \| f \|_p^q$$

Therefore, we have

$$\left\{ \int_0^\infty (t^{-2x} K_{2r, \phi}^p (f, t^{2r}))^q \frac{dt}{t} \right\}^{1/q} \leq C \left\{ \sum_{n=1}^{\infty} \left( n^{l-\beta} \| \phi^{2r} p_{n+1/l, r} \|_p \right)^q + M \| f \|_p^q \right\}^{1/q}$$

Then we complete the proof of Theorem 1.5.
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