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Minimal cubature rules and polynomial interpolation in two variables

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Dedicated to my friend Péter Vértesi on the occasion of his 70th birthday

Abstract

Minimal cubature rules of degree $4n-1$ for the weight functions

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x,y) = |x+y|^{2\alpha+1}|x-y|^{2\beta+1}((1-x^2)(1-y^2))^{\pm\frac{1}{2}}$$

on $[-1,1]^2$ are constructed explicitly and are shown to be closely related to the Gaussian cubature rules in a domain bounded by two lines and a parabola. Lagrange interpolation polynomials on the nodes of these cubature rules are constructed and their Lebesgue constants are determined.

Keywords: Cubature; Gaussian; Minimal cubature; Polynomial interpolation; Two variables; Lebesgue constant

1. Introduction

Minimal cubature rules have the smallest number of nodes among all cubature rules of the same precision. Let W be a non-negative weight function on a domain $\Omega \subset \mathbb{R}^2$. For a positive integer s, a cubature rule of precision s with respect to W is a finite sum that satisfies

$$\int_\Omega f(x,y)W(x,y)\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{N} \lambda_k f(x_k,y_k), \quad \forall f \in \Pi_s^2, \tag{1.1}$$

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where Π_s^2 denotes the space of polynomials of degree at most s in two variables, and there exists at least one function f^* in Π_{s+1}^2 such that the Eq. (1.1) does not hold.

It is known that the number of nodes N of a cubature rule of degree s necessarily satisfies

$$N \ge \dim \Pi_{n-1}^2 = \frac{n(n+1)}{2}, \qquad s = 2n-1 \quad \text{or} \quad 2n-2 \tag{1.2}$$

(cf. [12,16]). A cubature rule of degree s with N attaining the lower bound in (1.2) is called Gaussian. Unlike quadrature rules in one variable, Gaussian cubature rules rarely exist. At the moment, they are known to exist only in two cases. The first case, discovered in [13], is for a family of weight functions that includes, in particular, $W_{\alpha,\beta,\pm\frac{1}{2}}$ defined by

$$W_{\alpha,\beta,\pm\frac{1}{2}}(u,v) = (1-u+v)^{\alpha}(1+u+v)^{\beta}(u^2-4v)^{\pm\frac{1}{2}} \tag{1.3}$$

on the domain $\Omega = \{(u,v) : 1+u+v > 0, 1-u+v > 0, u^2 > 4v\}$, bounded by two lines and a parabola. On the other hand, Gaussian cubature rules of degree $2n-1$ do not exist when W is centrally symmetric, that is, when W and its domain Ω are both symmetric with respect to the origin: $(-x,-y) \in \Omega$ whenever $(x,y) \in \Omega$ and $W(-x,-y) = W(x,y)$. For centrally symmetric weight functions and $s = 2n-1$, a stronger lower bound [10] for the number of nodes is given by

$$N \ge \dim \Pi_{n-1}^2 + \left\lfloor \frac{n}{2} \right\rfloor = \frac{n(n+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor. \tag{1.4}$$

A cubature rule that attains this lower bound is necessarily minimal. There are, however, only a couple of examples for which this lower bound is attained for all n, most notable being the product Chebyshev weight functions on the square.

In the present paper we shall show that the minimal cubature rules of degree $4n-1$ exist for a family of weight functions that includes, in particular,

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x,y) := |x+y|^{2\alpha+1}|x-y|^{2\beta+1}(1-x^2)^{\pm\frac{1}{2}}(1-y^2)^{\pm\frac{1}{2}}, \tag{1.5}$$

on $[-1,1]^2$ and, furthermore, there is a connection between these minimal cubature rules and Gaussian cubature rules associated with the weight function $W_{\alpha,\beta,\pm\frac{1}{2}}$. The weight functions (1.5) include the product Chebyshev weight functions (when $\alpha = \beta = \pm\frac{1}{2}$), for which the minimal cubature rules are known to exist and have been established in several different methods [1,9,11,20]. Our result shows that they can be deduced from the Gaussian cubature rules for $W_{-\frac{1}{2},-\frac{1}{2},\pm\frac{1}{2}}$ on Ω. Giving the fact that so few minimal cubature rules are known explicitly, this connection is rather surprising.

Cubature rules are closely related to interpolation by polynomials. Based on the nodes of a Gaussian cubature rule of degree $2n-1$, there is a unique Lagrange interpolation polynomial of degree $n-1$ which converges to f in L^2 norm as $n \to \infty$ [19]. On the nodes of the minimal cubature rule that attains (1.4), there is a unique Lagrange interpolation polynomial in an appropriate subspace of polynomials [20]. Furthermore, the interpolation polynomials based on the nodes of the minimal cubature rules for the product Chebyshev weight function $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$, studied in [21], has the Lebesgue constant of order $(\log n)^2$ [3], which is the minimal order of projection operators on $[-1,1]^2$ [18]. We shall discuss the Lagrange interpolations based on both the nodes of Gaussian cubature rules with respect to $W_{\alpha,\beta,\pm\frac{1}{2}}$ and the nodes of minimal cubature rules for (1.5) in this paper.

The paper is organized as follows. The next section is the preliminary, in which we recall basics on cubature rules and, in particular, the connection between cubature rules and interpolation polynomials, as well as basics on the orthogonal polynomials that will be needed in the paper. The Gaussian cubature rules for weight functions including $W_{\alpha,\beta,\pm\frac{1}{2}}$ and minimal cubature rules for $\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}$ are discussed in Sections 3 and 4, respectively. The interpolation polynomials based on the nodes of these cubature rules are treated in Sections 5 and 6, respectively.

2. Preliminary and background

Minimal cubature rules are closely connected to orthogonal polynomials and to polynomial interpolation. We recall the connections in this section and state necessary definitions and properties of the weight functions and their orthogonal polynomials that will be needed later in the paper.

2.1. Cubature, orthogonal polynomials and interpolation

Let W be a nonnegative weight function defined on a domain Ω in $\mathbb{R}^2$ that has all finite moments; that is, $\int_\Omega x_1^j x_2^k W(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 < \infty$ for all $j,k \in \mathbb{N}_0$. Then orthogonal polynomials of two variables with respect to W exist. Let $\mathcal{V}_n(W)$ denote the space of orthogonal polynomials of degree exactly n in two variables. Then

$$\dim \mathcal{V}_n(W) = n+1.$$

Assume that W is normalized so that $\int_\Omega W(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = 1$. A basis of $\mathcal{V}_n(W)$, denoted by $\{P_{k,n} : 0 \le k \le n\}$, is mutually orthogonal if

$$\int_\Omega P_{k,n}(x_1,x_2)P_{j,n}(x_1,x_2)W(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 = h_k\delta_{k,j}, \quad 0 \le k, j \le n,$$

where $h_k > 0$ and it is called orthonormal if $h_k = 1$ for $0 \le k \le n$. The reproducing kernel $K_n(W;\cdot,\cdot)$ of Π_n^2 in $L^2(W)$ is defined by

$$\int_\Omega K_n(W;x,y)p(y)W(y)\mathrm{d}y = p(x), \quad \forall p \in \Pi_n^2,$$

in which $x = (x_1,x_2)$ and $y = (y_1,y_2)$. If $P_{k,n}$ are orthonormal, then the reproducing kernel $K_n(W;\cdot,\cdot)$ of Π_n^2 in $L^2(W)$ is given by

$$K_n(W;x,y) = \sum_{m=0}^{n}\sum_{k=0}^{m} P_{k,m}(x)P_{k,m}(y). \tag{2.1}$$

Recall that a Gaussian cubature rule of degree $2n-1$, as in (1.1), has $\dim \Pi_{n-1}^2$ nodes. These nodes are necessarily common zeros of orthogonal polynomials in $\mathcal{V}_n(W)$, that is, zeros of all polynomials in $\mathcal{V}_n(W)$ ([12] and [5, Theorem 3.7.4.]).

Theorem 2.1. *Let $n \ge 1$. A Gaussian cubature rule of degree $2n-1$ exists if and only if its nodes are common zeros of orthogonal polynomials of degree n. Moreover, the weights λ_k of the*

Gaussian cubature rule are given by

$$\lambda_k = \left[K_{n-1}(W; (x_k, y_k), (x_k, y_k))\right]^{-1}, \quad 1 \le k \le N.$$

Unlike interpolation in one variable, polynomial interpolation in two variables may not exist for a set of distinct points. It does exists if the interpolation points are nodes of a Gaussian cubature rule [19].

Theorem 2.2. *Let $N = \dim \Pi^2_{n-1}$ and let $\{(x_k, y_k) : 1 \le k \le N\}$ be the nodes of a Gaussian cubature rule of degree $2n-1$. Then there is a unique interpolation polynomial, $L_n f$, of degree $n-1$ that satisfies*

$$L_n f(x_{k,n}, y_{k,n}) = f(x_{k,n}, y_{k,n}), \quad 1 \le k \le N.$$

Furthermore, this interpolation polynomial is given explicitly by

$$L_n f(x, y) = \sum_{k=1}^{N} f(x_k, y_k)\ell_k(x, y), \quad \ell_k(x, y) := \lambda_k K_{n-1}(W; (x, y), (x_k, y_k)).$$

For centrally symmetric weight functions, we consider the minimal cubature rules whose number of nodes attains the lower bound in (1.4). The nodes of such a cubature rule are common zeros of a subspace of $\mathcal{V}_n(W)$ [10].

Theorem 2.3. *Let $n \ge 1$. The minimal cubature rule of degree $2n-1$ that attains the lower bound (1.4) exists if and only if its nodes are common zeros of $\lfloor \frac{n+1}{2} \rfloor + 1$ orthogonal polynomials of degree n.*

Since the number of nodes of such minimal cubature rules is $N = \dim \Pi^2_{n-1} + \lfloor \frac{n}{2} \rfloor$, the polynomial that interpolates at the nodes of the cubature rule needs to be from a polynomial subspace of Π_n that has dimension N. An obvious candidate of this subspace is, by Theorem 2.3, the linear span of $\Pi^2_n \setminus \mathcal{I}_n$, where $\mathcal{I}_n := \{Q_{k,n} : k = 0, 1, \ldots, \lfloor \frac{n+1}{2} \rfloor\}$ denotes a set of orthonormal polynomials that vanish on the nodes of the minimal cubature rule. Let $\{P_{k,n} : 1 \le k \le \lfloor \frac{n}{2} \rfloor\}$ be the orthonormal basis of the orthogonal complement of $\mathcal{I}_n$ in $\mathcal{V}_n(W)$. Then $P_{k,n} \in \mathcal{V}_n(W)$ and none of $P_{k,n}$ vanishes on all nodes of the cubature rule. We define a subspace Π^*_n of Π^2_n by

$$\Pi^*_n := \Pi^2_{n-1} \cup \operatorname{span}\left\{P_{k,n} : 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor\right\}. \tag{2.2}$$

The weights $\lambda_{k,n}$ of the minimal cubature rule in Theorem 2.3 are given in the lemma below.

Lemma 2.4. *Let $P_{k,n}$ be as in (2.2). There exists a sequence of positive numbers $\{b_{k,n} : 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor\}$, uniquely determined, such that the kernel $K^*_n(\cdot, \cdot)$ defined by*

$$K^*_n(W; x, y) = K_{n-1}(W; x, y) + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} b_{k,n} P_{k,n}(x) P_{k,n}(y), \tag{2.3}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, satisfies

$$\lambda_{k,n} = \left[K^*_n(W; (x_k, y_k), (x_k, y_k))\right]^{-1}, \quad 1 \le k \le N. \tag{2.4}$$

This lemma was proved in [20] and the coefficients were shown to be determined by the matrix $[\mathcal{C}_n(P_{j,n} P_{k,n})]^n_{j,k=0}$ in [22], where $\{P_{0,n}, \ldots, P_{n,n}\}$ is an orthonormal basis of $\mathcal{V}_n(W)$

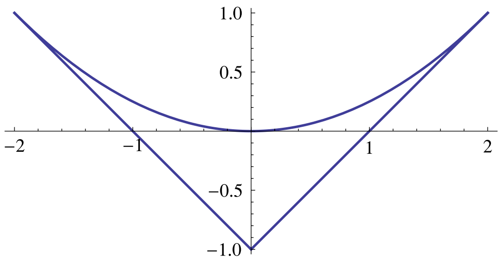

Fig. 1. Domain Ω.

and $\mathcal{C}_n f = \sum_{k=1}^{N} \lambda_k f(x_k, y_k)$ is the minimal cubature rule. The kernel $K_n^*(\cdot,\cdot)$ can also be used for the Lagrange interpolation polynomials based on the nodes of the minimal cubature rules, as stated in the following theorem [20].

Theorem 2.5. *Let W be a central symmetric weight function. Let $N = \dim \Pi_{n-1}^2 + \lfloor \frac{n}{2} \rfloor$ and let $\{(x_k, y_k) : 1 \le k \le N\}$ be the nodes of the minimal cubature rule of degree $2n-1$. Then there is a unique interpolation polynomial, $\mathcal{L}_n f$, in Π_n^* that satisfies*

$$\mathcal{L}_n f(x_{k,n}, y_{k,n}) = f(x_{k,n}, y_{k,n}), \quad 1 \le k \le N.$$

Furthermore, this interpolation polynomial is given explicitly by

$$\mathcal{L}_n f(x, y) = \sum_{k=1}^{N} f(x_k, y_k)\ell_k(x, y), \quad \ell_k(x, y) := \lambda_{k,n} K_n^*(W; (x, y), (x_k, y_k)),$$

where $\lambda_{k,n}$ are the weights of the cubature rule given in (2.4).

2.2. *Weight functions and orthogonal polynomials*

We define the weight functions for our Gaussian and minimal cubature rules. Our first weight function is defined on the domain

$$\Omega := \{(u, v) : 1 + u + v > 0, 1 - u + v > 0, u^2 > 4v\},$$

bounded by a parabola and two lines, as depicted in Fig. 1. Let w be a nonnegative weight function defined on $[-1, 1]$. We define

$$W_\gamma(u, v) := b_{w,\gamma} w(x) w(y) (u^2 - 4v)^\gamma, \quad (u, v) \in \Omega, \tag{2.5}$$

where the variables (x, y) and (u, v) are related by

$$u = x + y, \qquad v = xy \tag{2.6}$$

and $b_{w,\gamma}$ is the normalization constant such that $\int_\Omega W_\gamma(u, v) \mathrm{d}u\mathrm{d}v = 1$. In the case of the Jacobi weight function $w = w_{\alpha,\beta}$ defined by

$$w_{\alpha,\beta}(x) := (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1,$$

the weight function W_γ is denoted by $W_{\alpha,\beta,\gamma}$ and it is given by

$$W_{\alpha,\beta,\gamma}(u, v) := b_{\alpha,\beta,\gamma} (1 - u + v)^\alpha (1 + u + v)^\beta (u^2 - 4v)^\gamma, \quad (u, v) \in \Omega, \tag{2.7}$$

where $\alpha, \beta, \gamma > -1, \alpha+\gamma+\frac{1}{2} > -1$ and $\beta+\gamma+\frac{1}{2} > -1$ and [15, Lemma 6.1],

$$b_{\alpha,\beta,\gamma} := \frac{\sqrt{\pi}}{2^{2\alpha+2\beta+4\gamma+2}} \frac{\Gamma\left(\alpha+\beta+\gamma+\frac{5}{2}\right)\Gamma(\alpha+\beta+2\gamma+3)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\gamma+1)\Gamma\left(\alpha+\gamma+\frac{3}{2}\right)\Gamma\left(\beta+\gamma+\frac{3}{2}\right)}. \tag{2.8}$$

The weight function W_γ is related to $w(x)w(y)$ by the relation

$$\int_\Omega f(u,v)W_\gamma(u,v)\mathrm{d}u\mathrm{d}v = b_{w,\gamma}\int_\triangle f(x+y,xy)w(x)w(y)|x-y|^{2\gamma+1}\mathrm{d}x\mathrm{d}y, \tag{2.9}$$

where $\triangle := \{(x,y) : -1 < x < y < 1\}$. Since the integral in the right hand side has symmetric integrand in x, y, it is equal to half of the integral over $[-1,1]^2$. In particular, if c_w is the normalization constant of w on $[-1,1]$ so that $c_w \int_{-1}^1 w(x)\mathrm{d}x = 1$, then the normalization constant of $W_{-\frac{1}{2}}$ is given by $b_{w,-\frac{1}{2}} = 2c_w^2$.

The orthogonal polynomials with respect to $W_{\alpha,\beta,\gamma}$ were first studied by Koornwinder in [6] and further studied in [7,8,15]. They were applied to study cubature rules in [13]. In the case of $\gamma = \pm\frac{1}{2}$, the orthogonal polynomials with respect to $W_{\pm\frac{1}{2}}$ can be given explicitly. Let p_n denote the orthogonal polynomial of degree n with respect to w. Then an orthonormal basis with respect to $W_{-\frac{1}{2}}$ is given by

$$P_{k,n}^{(-\frac{1}{2})}(u,v) = \begin{cases} p_n(x)p_k(y) + p_n(y)p_k(x), & 0 \le k < n, \\ \sqrt{2}p_n(x)p_n(y), & k = n, \end{cases} \tag{2.10}$$

and an orthonormal basis with respect to $W_{\frac{1}{2}}$ is given by

$$P_{k,n}^{(\frac{1}{2})}(u,v) = \frac{p_{n+1}(x)p_k(y) - p_{n+1}(y)p_k(x)}{x-y}, \quad 0 \le k \le n, \tag{2.11}$$

both families are defined under the mapping (2.6). In the case of $W_{\alpha,\beta,\gamma}$ we denote the orthogonal polynomials by $P_{k,n}^{\alpha,\beta,\gamma}$. In particular, $P_{k,n}^{\alpha,\beta,\pm\frac{1}{2}}$ are expressible by the Jacobi polynomials.

Our second family of weight functions are defined on $[-1,1]^2$ by

$$\mathcal{W}_\gamma(x,y) := W_\gamma(2xy, x^2+y^2-1)|x^2-y^2|, \quad (x,y) \in [-1,1]^2, \tag{2.12}$$

where W_γ is the weight function in (2.5). In the case of $W_{\alpha,\beta,\gamma}$, it becomes

$$\mathcal{W}_{\alpha,\beta,\gamma}(x,y) := b_{\alpha,\beta,\gamma}4^\gamma|x-y|^{2\alpha+1}|x+y|^{2\beta+1}(1-x^2)^\gamma(1-y^2)^\gamma, \tag{2.13}$$

which includes (1.5). The $\mathcal{W}_\gamma$ is normalized if W_γ is because of the integral relation

$$\int_\Omega f(u,v)W_\gamma(u,v)\mathrm{d}u\mathrm{d}v = \int_{[-1,1]^2} f(2xy, x^2+y^2-1)\mathcal{W}_\gamma(x,y)\mathrm{d}x\mathrm{d}y. \tag{2.14}$$

The orthogonal polynomials with respect to $\mathcal{W}_\gamma$ can be expressed in terms of orthogonal polynomials with respect to W_γ [23]. For this paper we will only need a basis for $\mathcal{V}_{2n}(\mathcal{W}_\gamma)$, which consists of polynomials

$$
\begin{aligned}
{}_1Q_{k,2n}^{(\gamma)}(x,y) &:= P_{k,n}^{(\gamma)}(2xy, x^2+y^2-1), \quad 0 \le k \le n, \\
{}_2Q_{k,2n}^{(\gamma)}(x,y) &:= b_\gamma^{(1,1)}(x^2-y^2)P_{k,n-1}^{(\gamma),1,1}(2xy, x^2+y^2-1), \quad 0 \le k \le n-1,
\end{aligned}
\tag{2.15}
$$

where $P_{k,n-1}^{(\gamma),1,1}$ are orthonormal polynomials with respect to the weight function $(1-u+v)(1+u+v)W_\gamma(u,v)$ and $b_\gamma^{(1,1)}$ is a normalization constant for the weight function. In the case of $\mathcal{W}_{\alpha,\beta,\gamma}$, we denote the orthogonal polynomials by ${}_iQ_{k,2n}^{\alpha,\beta,\gamma}$, in which case, $P_{k,n-1}^{(\gamma),1,1} = P_{k,n-1}^{\alpha+1,\beta+1,\gamma}$ in (2.15). We will need explicit formulas for these polynomials when $\gamma = -1/2$, which we sum up in the following subsection. Further results on orthogonal polynomials with respect to W_γ can be found in [23].

2.3. *Jacobi polynomials and orthogonal polynomials for $\mathcal{W}_{\alpha,\beta,-\frac{1}{2}}$*

The Jacobi polynomials are orthogonal with respect to $w_{\alpha,\beta}$ and they are given explicitly by a hypergeometric function as

$$
P_n^{(\alpha,\beta)}(x,y) = \frac{(\alpha+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right) = l_n^{(\alpha,\beta)}x^n + \cdots,
$$

where $l_n^{(\alpha,\beta)}$ is the leading coefficient. By [17, (4.21.6)],

$$
l_n^{(\alpha,\beta)} = \frac{(n+a+b+1)_n}{2^n n!} \quad \text{and} \quad c_{\alpha,\beta} := \frac{\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}. \tag{2.16}
$$

The Jacobi polynomials satisfy the orthogonality conditions

$$
c_{\alpha,\beta}\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w_{\alpha,\beta}(x)\mathrm{d}x = h_n^{(\alpha,\beta)}\delta_{n,m},
$$

where

$$
h_n^{(\alpha,\beta)} := \frac{(\alpha+1)_n(\beta+1)_n(\alpha+\beta+n+1)}{n!(\alpha+\beta+2)_n(\alpha+\beta+2n+1)}. \tag{2.17}
$$

The reproducing kernel of $k_n^{(\alpha,\beta)}$ of the space of polynomials of degree at most n is given by, according to the Christoffel–Darboux formula,

$$
\begin{aligned}
k_n^{(\alpha,\beta)}(x,y) = {} & \frac{2(n+1)!(\alpha+\beta+2)_n}{(2n+\alpha+\beta+2)(\alpha+1)_n(\beta+1)_n} \\
& \times \frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y) - P_{n+1}^{(\alpha,\beta)}(y)P_n^{(\alpha,\beta)}(x)}{x-y}.
\end{aligned}
\tag{2.18}
$$

The Gaussian quadrature of degree $2n-1$ for the Jacobi weight is given by

$$
c_{\alpha,\beta}\int_{-1}^{1} f(x)w_{\alpha,\beta}(x)\mathrm{d}x = \sum_{k=1}^{n} \lambda_n^{(\alpha,\beta)} f(x_{k,n}), \quad \forall f \in \Pi_{2n-1}, \tag{2.19}
$$

where $x_{1,n}, \ldots, x_{n,n}$ are the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ and

$$
\lambda_n^{(\alpha,\beta)} = [k_n^{(\alpha,\beta)}(x_{k,n}, x_{k,n})]^{-1}.
$$

We denote the orthonormal Jacobi polynomials by $p_n^{(\alpha,\beta)}$. It follows readily that $p_n^{(\alpha,\beta)}(x) = (h_n^{(\alpha,\beta)})^{-\frac{1}{2}} P_n^{(\alpha,\beta)}(x)$. The following lemma will be needed in Section 6.

Lemma 2.6. *For $\alpha, \beta > -1$ and $m \ge 0$, define*

$$\widehat{h}_m := \sum_{k=1}^{n} \lambda_k^{(\alpha,\beta)} (1-x_k^2) \left[p_{n-1}^{(\alpha+1,\beta+1)}(x_k) \right]^2. \tag{2.20}$$

Then $\widehat{h}_m = c_{\alpha,\beta}/c_{\alpha+1,\beta+1}$ for $0 \le m \le n-2$, and

$$\widehat{h}_{n-1} = \frac{4(1+\alpha)(1+\beta)(1+\alpha+\beta+2n)}{(2+\alpha+\beta)(3+\alpha+\beta)(1+\alpha+\beta+n)}.$$

Proof. For $0 \le m \le n-2$, we can apply Gaussian quadrature and use the orthonormality of $p_m^{(\alpha+1,\beta+1)}$ to conclude, since $(1-x^2)w_{\alpha,\beta}(x) = w_{\alpha+1,\beta+1}(x)$,

$$\widehat{h}_m = c_{\alpha,\beta} \int_{-1}^{1} (1-x^2) \left[p_m^{(\alpha+1,\beta+1)}(x) \right]^2 w_{\alpha,\beta}(x) \mathrm{d}x = \frac{c_{\alpha,\beta}}{c_{\alpha+1,\beta+1}}.$$

For $m = n-1$, we cannot apply the Gaussian quadrature of degree $2n-1$ directly, since $(1-x^2)[p_m^{(\alpha+1,\beta+1)}(x)]^2$ has degree $2n$. However, by [17, (4.5.5)],

$$\begin{aligned}(1-x_k^2) P_{n-1}^{(\alpha+1,\beta+1)}(x_k) &= \frac{4(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x_k) \\ &\quad - \frac{4n(n+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{n+1}^{(\alpha,\beta)}(x_k),\end{aligned}$$

and by the three-term relation satisfied by $\{P_n^{(\alpha,\beta)}\}_{n\ge 0}$ [17, (4.5.1)],

$$\begin{aligned}&(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta) P_{n+1}^{(\alpha,\beta)}(x_k) \\ &\quad = -(n+\alpha)(n+\beta)(2n+\alpha+\beta+2) P_{n-1}^{(\alpha,\beta)}(x_k).\end{aligned}$$

From these two equations it follows that

$$(1-x_k^2) P_{n-1}^{(\alpha+1,\beta+1)}(x_k) = \frac{4(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x_k). \tag{2.21}$$

Denote the coefficient in front of $P_{n-1}^{(\alpha,\beta)}(x_k)$ in the above equation by D_n. Then, by the Gaussian quadrature and the orthogonality of the Jacobi polynomials,

$$\begin{aligned}\widehat{h}_{n-1} &= D_n \left[h_{n-1}^{(\alpha+1,\beta+1)} \right]^{-1} \sum_{k=1}^{n} \lambda_k^{(\alpha,\beta)} P_{n-1}^{(\alpha+1,\beta+1)}(x_k) P_{n-1}^{(\alpha,\beta)}(x_k) \\ &= D_n \left[h_{n-1}^{(\alpha+1,\beta+1)} \right]^{-1} c_{\alpha,\beta} \int_{-1}^{1} P_{n-1}^{(\alpha+1,\beta+1)}(x) P_{n-1}^{(\alpha,\beta)}(x) w_{\alpha,\beta}(x) \mathrm{d}x \\ &= D_n \left[h_{n-1}^{(\alpha+1,\beta+1)} \right]^{-1} c_{\alpha,\beta} \frac{l_{n-1}^{(\alpha+1,\beta+1)}}{l_{n-1}^{\alpha,\beta}} \int_{-1}^{1} \left[P_{n-1}^{(\alpha,\beta)}(x) \right]^2 w_{\alpha,\beta}(x) \mathrm{d}x\end{aligned}$$

$$= D_n \frac{h_{n-1}^{(\alpha,\beta)}}{h_{n-1}^{(\alpha+1,\beta+1)}} \frac{l_{n-1}^{(\alpha+1,\beta+1)}}{l_{n-1}^{\alpha,\beta}},$$

which simplifies, by (2.16) and (2.17), to the stated result for $\widehat{h}_{n-1}$. $\square$

In Section 6 we will need the explicit formula of orthogonal polynomials and reproducing kernels for the weight function $\mathcal{W}_{\alpha,\beta,-1/2}$, which we rename as

$$\mathcal{W}_{\alpha,\beta}(x,y) := \mathcal{W}_{\alpha,\beta,-1/2}(x,y) = 2c_{\alpha,\beta}^2 \frac{|x-y|^{2\alpha+1}|x+y|^{2\beta+1}}{\sqrt{1-x^2}\sqrt{1-y^2}}. \tag{2.22}$$

In this case the orthonormal polynomials of even degree in (2.15) are given in terms of Jacobi polynomials, which are

Proposition 2.7. *Let $\alpha, \beta > -1$. An orthonormal basis of $\mathcal{V}_{2n}(\mathcal{W}_{\alpha,\beta,-\frac{1}{2}})$ is given by, for $0 \le k \le n$ and $0 \le k \le n-1$, respectively,*

$$\begin{aligned} {}_1Q_{k,2n}^{(\alpha,\beta)}(\cos\theta,\cos\phi) &= p_n^{(\alpha,\beta)}(\cos(\theta-\phi))p_k^{(\alpha,\beta)}(\cos(\theta+\phi)) \\ &\quad + p_k^{(\alpha,\beta)}(\cos(\theta-\phi))p_n^{(\alpha,\beta)}(\cos(\theta+\phi)), \\ {}_2Q_{k,2n}^{(\alpha,\beta)}(\cos\theta,\cos\phi) &= \gamma_{\alpha,\beta}(x^2-y^2)\Big[p_{n-1}^{(\alpha+1,\beta+1)}(\cos(\theta-\phi))p_k^{(\alpha+1,\beta+1)}(\cos(\theta+\phi)) \\ &\quad + p_k^{(\alpha+1,\beta+1)}(\cos(\theta-\phi))p_{n-1}^{(\alpha+1,\beta+1)}(\cos(\theta+\phi))\Big], \end{aligned}$$

where ${}_1Q_{n,2n}^{(\alpha,\beta)}$ and ${}_2Q_{n,2n}^{(\alpha,\beta)}$ are multiplied by $\sqrt{2}/2$ and $\gamma_{\alpha,\beta} = c_{\alpha+1,\beta+1}/(\sqrt{2}c_{\alpha,\beta})$.

Denote the reproducing kernel of Π_n^2 with respect to $\mathcal{W}_{\alpha,\beta}$ by $\mathcal{K}_n^{\alpha,\beta}(\cdot,\cdot)$. By [23, Theorem 4.8], the kernel $\mathcal{K}_{2n-1}^{\alpha,\beta}$ is given explicitly by

$$\begin{aligned} \mathcal{K}_{2n-1}^{\alpha,\beta}(x,y) &= K_{n-1}^{\alpha,\beta}(s,t) + d_{\alpha,\beta}^{(1,1)}(x_1^2-x_2^2)(y_1^2-y_2^2)K_{n-2}^{\alpha+1,\beta+1}(s,t) \\ &\quad + d_{\alpha,\beta}^{(0,1)}(x_1+x_2)(y_1+y_2)K_{n-1}^{\alpha,\beta+1}(s,t) + d_{\alpha,\beta}^{(1,0)}(x_1-x_2)(y_1-y_2)K_{n-1}^{\alpha+1,\beta}(s,t), \end{aligned} \tag{2.23}$$

where $s = (2x_1x_2, x_1^2+x_2^2-1)$, $t = (2y_1y_2, y_1^2+y_2^2-1)$, $d_{\alpha,\beta}^{(i,j)} = c_{\alpha+i,b+j}^2/c_{\alpha,\beta}^2$ and, with $(x_1,x_2) = (\cos\theta, \cos\theta_2)$ and $(y_1,y_2) = (\cos\phi_1, \cos\phi_2)$,

$$\begin{aligned} K_n^{\alpha,\beta}(s,t) &:= \frac{1}{2}\Big[k_n^{\alpha,\beta}(\cos(\theta_1-\theta_2),\cos(\phi_1-\phi_2))k_n^{\alpha,\beta}(\cos(\theta_1+\theta_2),\cos(\phi_1+\phi_2)) \\ &\quad + k_n^{\alpha,\beta}(\cos(\theta_1-\theta_2),\cos(\phi_1+\phi_2))k_n^{\alpha,\beta}(\cos(\theta_1+\theta_2),\cos(\phi_1-\phi_2))\Big]. \end{aligned} \tag{2.24}$$

For orthonormal basis of odd degrees and the reproducing kernels of even degrees, as well as other results on them, see [23].

3. Gaussian cubature rules

In this section we consider Gaussian cubature rules for $W_{\pm\frac{1}{2}}$ on Ω and their transformations. We shall show that these rules can be transformed into minimal cubature rules for $\mathcal{W}_{\alpha,\beta,-1/2}$ on $[-1,1]^2$ in the next section. The first proof that Gaussian cubature rules exist for $W_{\pm\frac{1}{2}}$ was given in [13] via the structure matrices of orthogonal polynomials. Below is another proof that is of independent interest.

We start with the Gaussian quadrature rule for the integral against w on $[-1, 1]$,

$$c_w \int_{-1}^{1} f(x)w(x)\mathrm{d}x = \sum_{k=1}^{n} \lambda_k f(x_{k,n}), \qquad f \in \Pi_{2n-1}, \tag{3.1}$$

where Π_{2n-1} denotes the space of polynomials of degree $2n-1$ in one variable and c_w is the normalization constant so that $c_w \int_{-1}^{1} w(x)\mathrm{d}x = 1$. It is known that $\lambda_k > 0$ and $x_{k,n}$ are zeros of the orthogonal polynomial p_n with respect to w. When $w = w_{\alpha,\beta}$, the orthogonal polynomials are the Jacobi polynomials $P_n^{(\alpha,\beta)}$ and $x_{k,n}$, $1 \le k \le n$, are the zeros of $P_n^{(\alpha,\beta)}$. We define

$$u_{j,k} = u_{j,k,n} := x_{j,n} + x_{k,n} \quad \text{and} \quad v_{j,k} = v_{j,k,n} := x_{j,n} x_{k,n}. \tag{3.2}$$

Theorem 3.1. *For $W_{-\frac{1}{2}}$ on Ω, the Gaussian cubature rule of degree $2n-1$ is*

$$\int_{\Omega} f(u,v) W_{-\frac{1}{2}}(u,v)\mathrm{d}u\mathrm{d}v = 2 \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j f(u_{j,k}, v_{j,k}), \qquad f \in \Pi_{2n-1}^2, \tag{3.3}$$

where $\sum'$ means that the term for $j = k$ is divided by 2. For $W_{\frac{1}{2}}$ on Ω, the Gaussian cubature rule of degree $2n-3$ is

$$\int_{\Omega} f(u,v) W_{\frac{1}{2}}(u,v)\mathrm{d}u\mathrm{d}v = 2 \sum_{k=2}^{n} \sum_{j=1}^{k-1} \lambda_{j,k} f(u_{j,k}, v_{j,k}), \qquad f \in \Pi_{2n-3}^2, \tag{3.4}$$

where $\lambda_{j,k} = \lambda_j \lambda_k (x_{j,n} - x_{k,n})^2$.

Proof. The product of (3.1) is a cubature rule on $[-1, 1]^2$

$$c_w^2 \int_{[-1,1]^2} f(x,y)w(x)w(y)\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{n} \sum_{j=1}^{n} \lambda_k \lambda_j f(x_{k,n}, x_{j,n}), \tag{3.5}$$

which is exact for $f \in \Pi_{2n-1} \times \Pi_{2n-1}$, the space of polynomials of degree at most $2n-1$ in either x or y variable. Applying (3.5) on the symmetric polynomials $f(x+y, xy)$ and using the symmetry, we obtain

$$c_w^2 \int_{\triangle} f(x+y, xy)w(x)w(y)\mathrm{d}x\mathrm{d}y = \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j f(x_{k,n} + x_{j,n}, x_{k,n} x_{j,n})$$

if $f(x+y, xy)$ in $\Pi_{2n-1} \times \Pi_{2n-1}$. Under the change of variables $u = x + y$ and $v = xy$ and by (2.9), the above cubature becomes (3.3), since $\Pi_{2n-1} \times \Pi_{2n-1}$ becomes Π_{2n-1}^2 under the mapping $(x, y) \mapsto (u, v)$. It is easy to see that (3.3) has $\dim \Pi_{n-1}^2$ nodes, so that it is a Gaussian cubature rule.

To prove (3.4), we apply the product Gaussian cubature rule (3.5) on functions of the form $(x-y)^2 f(x+y, xy)$ for $f \in \Pi_{2n-2} \times \Pi_{2n-3}$ to get

$$\int_{\triangle} f(x+y, xy)(x-y)^2 w(x)w(y)\mathrm{d}x\mathrm{d}y = \sum_{k=2}^{n} \sum_{j=1}^{k-1} \lambda_j \lambda_k (x_{j,n} - x_{k,n})^2 f(u_{j,k}, v_{j,k}).$$

Since $(x-y)^2 w(x)w(y) = W_{\frac{1}{2}}(u,v)$ for $u = x + y$ and $v = xy$, the above cubature rule becomes (3.4) under $(x, y) \mapsto (u, v)$. $\square$

By Theorem 2.1, the nodes $\{(x_{k,n} + x_{j,n}, x_{k,n}x_{j,n}) : 1 \le j \le k \le n\}$ of the cubature rule (3.3) are common zeros of the orthogonal polynomials in $\{P_{0,n}^{(-\frac{1}{2})}, \ldots, P_{n,n}^{(-\frac{1}{2})}\}$, and the nodes $\{(x_{k,n} + x_{j,n}, x_{k,n}x_{j,n}) : 1 \le j \le k \le n-1\}$ of the cubature rules (3.4) are common zeros of $\{P_{0,n-1}^{(\frac{1}{2})}, \ldots, P_{n-1,n-1}^{(\frac{1}{2})}\}$. Formulated in the language of algebraic geometry, this states, for example, that the polynomial ideal $I = \langle P_{k,n}^{(-\frac{1}{2})}, \ldots, P_{n,n}^{(-\frac{1}{2})}\rangle$ has the zero-dimensional variety $V = \{(x_{k,n} + x_{j,n}, x_{k,n}x_{j,n}) : 1 \le j \le k \le n\}$.

We remark that the above procedure of deriving cubature rules for $W_{\pm\frac{1}{2}}$ on $[-1, 1]^2$ can be adopted for other types of cubature rules besides Gaussian cubature rules. In fact, instead of starting with the product Gaussian cubature rules for $w(x)w(y)$ on $[-1, 1]^2$ as in the proof of Theorem 3.1, we can start with a product cubature rule of other types. For example, we can start with a quadrature rule of degree $2n$ for w that has all nodes inside $[-1, 1]$, in which case an analogue of Theorem 3.1 was established in [13]. We can also start with a Gauss–Lobatto quadrature for w to get a cubature rule that has nodes also on the two linear branches of the boundary of Ω.

The Theorem 3.1 shows that Gaussian cubature rules exist for $W_{\pm\frac{1}{2}}$. An immediate question is if Gaussian cubature rules also exist for the weight function W_γ for $\gamma \ne \pm\frac{1}{2}$. The answer, however, is negative.

Theorem 3.2. *For $n \ge 1$, the Gaussian cubature rules do not exist for $W_{-1/2,-1/2,\gamma}$ if $\gamma \ne \pm 1/2$.*

Proof. It was shown in [15, (10.7)] that a basis of orthogonal polynomials of degree n with respect to $W_{-1/2,-1/2,\gamma}$ is given explicitly by

$$P_{k,n}^{-\frac{1}{2},-\frac{1}{2},\gamma}(2xy, x^2 + y^2 - 1) = P_{n+k}^{(\gamma,\gamma)}(x)P_{n-k}^{(\gamma,\gamma)}(y) + P_{n-k}^{(\gamma,\gamma)}(x)P_{n+k}^{(\gamma,\gamma)}(y),$$

where $0 \le k \le n$ and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n. It is easy to see that these polynomials do not have common zeros (considering, for example, $k = n$ first). Consequently, $\{P_{k,n}^{-\frac{1}{2},-\frac{1}{2},-\gamma}(x, y) : 0 \le k \le n\}$ does not have $\dim \Pi_{n-1}^2$ common zeros for $n \ge 1$. Hence, the Gaussian cubature rules do not exist according to Theorem 3.1. □

For what we will do in the following subsection, we make an affine change of variables $u = 2(s - t)$ and $v = 2s + 2t - 1$, which implies that the measure becomes $W_\gamma(u, v)\mathrm{d}u\mathrm{d}v = W_\gamma^*(s, t)\mathrm{d}s\mathrm{d}t$, where

$$\begin{aligned} W_\gamma^*(s, t) &:= 2b_w^\gamma 4^{\gamma+1} w(x)w(y)((1 - \sqrt{s})^2 - t)^\gamma ((1 + \sqrt{s})^2 - t)^\gamma \\ &\text{with } s = \frac{1}{4}(1 + x)(1 + y), t = \frac{1}{4}(1 - x)(1 - y), \end{aligned} \tag{3.6}$$

and the domain Ω becomes Ω^* defined by

$$\Omega^* := \{(s, t) : s \ge 0, t \ge 0, \sqrt{s} + \sqrt{t} \le 1\},$$

which is depicted in the right figure of Fig. 2. In the case of $w = w_{\alpha,\beta}$ the weight function W_γ^* becomes

$$W_{\alpha,\beta,\gamma}^*(s, t) = 2b_{\alpha,\beta,\gamma} 4^{\alpha+\beta+\gamma+1} s^\alpha t^\beta ((1 - \sqrt{s})^2 - t)^\gamma ((1 + \sqrt{s})^2 - t)^\gamma.$$

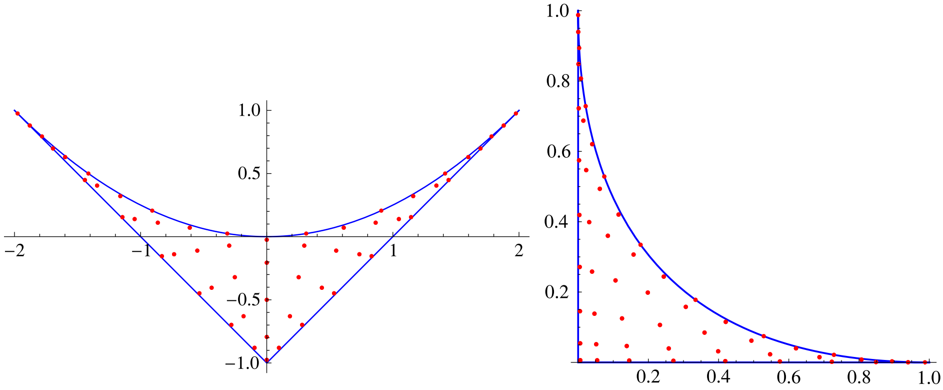

Fig. 2. Nodes of cubature rules of degree 19 for $W_{-\frac12,-\frac12,-\frac12}$ and $W^*_{-\frac12,-\frac12,-\frac12}$.

Since the affine transform does not change the strength of the cubature rules, the Gaussian cubature rules exist for the weight functions $W^*_{\pm\frac12}$. Let us denote by

$$x^*_{j,k} = x^*_{j,k,n} := \frac14 (1 + x_{j,n})(1 + x_{k,n}), \qquad y^*_{j,k} = y^*_{j,k,n} := \frac14 (1 - x_{j,n})(1 - x_{k,n}).$$

Corollary 3.3. *For $W^*_{-\frac12}$ on Ω^*, the Gaussian cubature rule of degree $2n-1$ is*

$$\int_{\Omega^*} f(s,t) W^*_{-\frac12}(s,t)\mathrm{d}s\mathrm{d}t = 2 \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j f(x^*_{j,k}, y^*_{j,k}), \quad f \in \Pi^2_{2n-1}. \tag{3.7}$$

For $W^*_{\frac12}$ on Ω^*, the Gaussian cubature rule of degree $2n-3$ is

$$\int_{\Omega^*} f(s,t) W^*_{\frac12}(s,t)\mathrm{d}s\mathrm{d}t = 2 \sum_{k=2}^{n} \sum_{j=1}^{k-1} \lambda_{j,k} f(x^*_{j,k}, y^*_{j,k}), \quad f \in \Pi^2_{2n-3}. \tag{3.8}$$

The nodes of the cubature ruled of degree 19 for $W_{-\frac12,-\frac12,-\frac12}$ and $W^*_{-\frac12,-\frac12,-\frac12}$ are depicted in the left and the right figures of Fig. 2, respectively.

4. Minimal cubature rules

Our goal in this section is to establish minimal cubature rules for the weight functions $\mathcal{W}_\gamma$ on $[-1,1]^2$. We shall do so by several transformations of the Gaussian cubature rules in the previous section.

First we recall the Sobolev theorem on invariant cubature rules [14]. A cubature rule in the form of (1.1) is invariant under a finite group G if the equality is unchanged under $f \mapsto \sigma f$, where $\sigma f(x) = f(x\sigma)$, for all $\sigma \in G$. The Sobolev theorem states that if a cubature is invariant under G then it is exact for a subspace $\mathcal{P}$ of polynomials if and only if it is exact for all polynomials in $\mathcal{P}$ that are invariant under G.

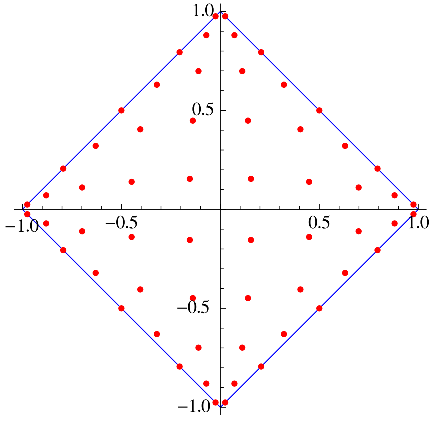

Fig. 3. Nodes of the minimal cubature rule of degree 19 for $U_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$.

We start from cubature rules for W_γ^* in Corollary 3.3 and make a change of variables $(s,t) \mapsto (u^2, v^2)$. The domain Ω^* becomes the triangle $T := \{(u,v) : u, v \ge 0, 1-u-v \ge 0\}$ and, since the weight function $W_\gamma^*(u^2,v^2)$ is even in both u and v, we extend it by symmetry to the rhombus R, depicted in Fig. 3,

$$R := \{(u,v) : -1 < u+v < 1, -1 < u-v < 1\}.$$

The change of variables has a Jacobian $dsdt = 4|uv|\mathrm{d}u\mathrm{d}v$. We define the weight function on R by

$$U_\gamma(u,v) := |uv| W_\gamma^*(u^2,v^2) = 2b_w^\gamma 4^{\gamma+1} w(x)w(y)|uv|((1-u)^2-v^2)^\gamma((1+u)^2-v^2)^\gamma,$$
$$\text{where } u = \frac{1}{2}\sqrt{1+x}\sqrt{1+y},\ v = \frac{1}{2}\sqrt{1-x}\sqrt{1-y}.$$

In the case of $W_{\alpha,\beta,\gamma}^*$, the corresponding weight function is

$$U_{\alpha,\beta,\gamma}(u,v) = 2b_{\alpha,\beta,\gamma} 4^{\alpha+\beta+\gamma+1} u^{2\alpha+1} v^{2\beta+1}((1-u)^2-v^2)^\gamma((1+u)^2-v^2)^\gamma. \qquad (4.1)$$

Under the change of variables $(s,t) \mapsto (u,v)$ and using the symmetry, the integrals are related by

$$\int_{\Omega^*} f(s,t)\mathrm{d}s\mathrm{d}t = 4\int_T f(u^2,v^2)uv\mathrm{d}u\mathrm{d}v = \int_R f(u^2,v^2)|uv|\mathrm{d}u\mathrm{d}v, \qquad (4.2)$$

from which it is easy to see that U_γ satisfies $\int_R U_\gamma(u,v)\mathrm{d}u\mathrm{d}v = 1$.

Directly from its definition, the weight function U_γ is evidently centrally symmetric. To state the cubature rules for U_γ, we introduce the notation $\theta_{k,n}$ by

$$x_{k,n} = \cos\theta_{k,n}, \quad k = 0, 1, \ldots, n.$$

Since w is supported on $[-1,1]$, the zeros of the orthogonal polynomial p_n are all inside $[-1,1]$, so that $0 < \theta_{k,n} < \pi$.

Theorem 4.1. *For $U_{-\frac{1}{2}}$ on the rhombus R, we have the minimal cubature rule of degree $4n-1$ with $\dim \Pi^2_{2n-1} + n$ nodes,*

$$\int_R f(u,v) U_{-\frac{1}{2}}(u,v) \mathrm{d}u \mathrm{d}v = \frac{1}{2} \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j \times \sum f\left(\pm \cos\frac{\theta_{j,n}}{2}\cos\frac{\theta_{k,n}}{2}, \pm \sin\frac{\theta_{j,n}}{2}\sin\frac{\theta_{k,n}}{2}\right), \quad f \in \Pi^2_{4n-1}, \tag{4.3}$$

where the innermost $\sum$ is a summation of four terms over all possible choices of signs. For $U_{\frac{1}{2}}$ on R, we have the minimal cubature rule of degree $4n-3$ with $\dim \Pi^2_{2n-3} + n$ nodes,

$$\int_R f(u,v) U_{\frac{1}{2}}(u,v) \mathrm{d}u \mathrm{d}v = \frac{1}{2} \sum_{k=2}^{n} \sum_{j=1}^{k-1} \lambda_{j,k} \times \sum f\left(\pm \cos\frac{\theta_{j,n}}{2}\cos\frac{\theta_{k,n}}{2}, \pm \sin\frac{\theta_{j,n}}{2}\sin\frac{\theta_{k,n}}{2}\right), \quad f \in \Pi^2_{4n-3}. \tag{4.4}$$

Proof. Changing variables $s = u^2$ and $t = v^2$ in (3.7) and applying (4.2), we obtain

$$\frac{1}{4}\int_R f(u^2,v^2) U_{-\frac{1}{2}}(u,v) \mathrm{d}u \mathrm{d}v = \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j f(x^*_{j,k}, y^*_{j,k})$$
$$= \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j f\left(\cos^2\frac{\theta_{j,n}}{2}\cos^2\frac{\theta_{k,n}}{2}, \sin^2\frac{\theta_{j,n}}{2}\sin^2\frac{\theta_{k,n}}{2}\right)$$

for all $f \in \Pi_{2n-1}$, where we have used the fact that $x^*_{j,k} = \cos^2\frac{\theta_{j,n}}{2}\cos^2\frac{\theta_{k,n}}{2}$, and $y^*_{j,k} = \sin^2\frac{\theta_{j,n}}{2}\sin^2\frac{\theta_{k,n}}{2}$, which follows from the definition of $x^*_{j,k}$ and $y^*_{j,k}$. The above cubature rule can be viewed as (4.3) applied to $f(x^2,y^2)$. Since $\{f(x^2,y^2) : f \in \Pi^2_{2n-1}\}$ consists of all polynomials in Π^2_{4n-1} that are invariant under the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, it implies, by the Sobolev theorem, cubature rule (4.3). Since none of the nodes of (3.7) are on the edges $s = 0$ or $t = 0$ of Ω^*, the number of nodes of cubature rule (4.3) is exactly

$$4 \dim \Pi^2_{n-1} = 2n(n+1) = \dim \Pi^2_{2n-1} + n = \dim \Pi^2_{2n-1} + \frac{2n}{2},$$

which attains the lower bound in (1.4). The proof of the cubature rule (4.4) is similar. □

The nodes of the cubature rules of degree 19 for $U_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$ are depicted in the right figure of Fig. 3.

As a final change of variables, we rotate the rhombus by 45° to the square $[-1,1]^2$. This amounts to a change of variables $u = (x+y)/2$ and $v = (x-y)/2$. The measure under this change of variables become $U_\gamma(u,v)\mathrm{d}u\mathrm{d}v = \mathcal{W}_\gamma(x,y)\mathrm{d}x\mathrm{d}y$,

$$\mathcal{W}_\gamma(x,y) = b_w^\gamma 4^\gamma w(\cos(\theta-\phi)) w(\cos(\theta+\phi)) |x^2-y^2| (1-x^2)^\gamma (1-y^2)^\gamma,$$
$$\text{where } x = \cos\theta,\ y = \cos\phi,\ (x,y) \in [-1,1]^2.$$

A simple computation shows that this is precisely the weight function defined in (2.12). In the case of $U_{\alpha,\beta,\gamma}$, the corresponding weight becomes $\mathcal{W}_{\alpha,\beta,\gamma}$ defined by

$$\mathcal{W}_{\alpha,\beta,\gamma}(x,y) = b_{\alpha,\beta,\gamma} 4^{\gamma} |x+y|^{2\alpha+1} |x-y|^{2\beta+1} (1-x^2)^{\gamma} (1-y^2)^{\gamma},$$

which is exactly (2.13). Since the strength of the cubature rules do not change under the affine change of variables, we then have minimal cubature formulas for $\mathcal{W}_{\alpha,\beta,\gamma}$. To state this cubature explicitly, let us define

$$s_{j,k} := \cos\frac{\theta_{j,n} - \theta_{k,n}}{2} \quad \text{and} \quad t_{j,k} := \cos\frac{\theta_{j,n} + \theta_{k,n}}{2}, \tag{4.5}$$

where $\theta_{k,n}$ is again the angular argument of the zeros $x_{k,n} = \cos\theta_{k,n}$ of p_n.

Theorem 4.2. *For $\mathcal{W}_{-\frac{1}{2}}$ on $[-1,1]^2$, we have the minimal cubature rule of degree $4n-1$ with $\dim \Pi^2_{2n-1} + n$ nodes,*

$$\int_{[-1,1]^2} f(x,y) \mathcal{W}_{-\frac{1}{2}}(x,y) \mathrm{d}x\mathrm{d}y = \frac{1}{2} \sum_{k=1}^{n} {\sum_{j=1}^{k}}' \lambda_k \lambda_j \big[f(s_{j,k}, t_{j,k}) + f(t_{j,k}, s_{j,k}) + f(-s_{j,k}, -t_{j,k}) + f(-t_{j,k}, -s_{j,k}) \big]. \tag{4.6}$$

For $\mathcal{W}_{\frac{1}{2}}$ on $[-1,1]^2$, we have the minimal cubature rule of degree $4n-3$ with $\dim \Pi^2_{2n-3} + n$ nodes,

$$\int_{[-1,1]^2} f(x,y) \mathcal{W}_{\frac{1}{2}}(x,y) \mathrm{d}x\mathrm{d}y = \frac{1}{2} \sum_{k=2}^{n} \sum_{j=1}^{k-1} \lambda_{j,k} \big[f(s_{j,k}, t_{j,k}) + f(t_{j,k}, s_{j,k}) + f(-s_{j,k}, -t_{j,k}) + f(-t_{j,k}, -s_{j,k}) \big], \tag{4.7}$$

where $\lambda_{j,k} = \lambda_j \lambda_k (\cos\theta_{j,n} - \cos\theta_{k,n})^2$.

In the case of the product Jacobi weight function $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},\pm\frac{1}{2}}$, these cubature rules were constructed in [11] and, more recently, in [9] via a completely different method. In all other cases these cubature rules are new. The nodes of the cubature rule of degree 35 for the weight function

$$\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}(x,y) = (1-x^2)^{-\frac{1}{2}} (1-y^2)^{-\frac{1}{2}},$$
$$\mathcal{W}_{0,0,-\frac{1}{2}}(x,y) = |x^2-y^2| (1-x^2)^{-\frac{1}{2}} (1-y^2)^{-\frac{1}{2}},$$

are depicted in the left and right figures in Fig. 4, respectively. The influence of the part $|x^2-y^2|$ in the weight function $\mathcal{W}_{0,0,-\frac{1}{2}}$ is clearly visible in comparing with the cubature rules for $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$.

By the relation (2.12) and the integral relation (2.14), we could arrive at cubature rules (4.6) and (4.7) from those in (3.3) and (3.4) by the mapping $(x,y) \mapsto (2xy, x^2+y^2-1)$, bypassing some of the middle steps. Our presentation, on the other hand, is more intuitive and provides, hopefully, a better explanation of the connection between the Gaussian cubature rules for W_γ and the minimal cubature rules for $\mathcal{W}_\gamma$.

We can also give a proof of Theorem 4.2 based on Theorem 2.3 by considering the common zeros of corresponding orthogonal polynomials, although a direct computation of the cubature

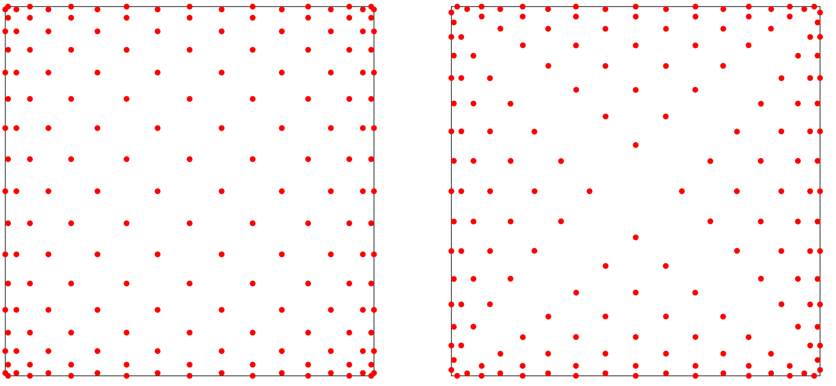

Fig. 4. Nodes of minimal cubature rules of degree 35 for $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}$ and $\mathcal{W}_{0,0,-\frac{1}{2}}$.

weights will not be easy. Recalling the orthogonal polynomials ${}_1Q_{k,2n}^{(\gamma)}$ defined in (2.15), the following corollary is an immediate consequence of Theorem 2.3.

Corollary 4.3. *The nodes of the minimal cubature rule* (4.6) *are the common zeros of orthogonal polynomials* $\{{}_1Q_{k,2n}^{\alpha,\beta} : 0 \le k \le n\}$ *in Proposition* 2.7. *And the nodes of the minimal cubature rule* (4.7) *are the common zeros of orthogonal polynomials* $\{{}_1Q_{k,2n-2}^{\alpha,\beta,\frac{1}{2}} : 0 \le k \le n-1\}$.

The relation (2.15) shows that the nodes of the minimal cubature rule (4.6) and the nodes of the Gaussian cubature rule (3.3) are related by a simple formula: if (s,t) is a node of the former, then $(2st, s^2+t^2-1)$ is a node of the latter; furthermore, the nodes $(s_{j,k}, t_{j,k})$, $(t_{j,k}, s_{j,k})$, $(-s_{j,k}, -t_{j,k})$, $(-t_{j,k}, -s_{j,k})$ of the former correspond to the same node $(2s_{j,k}t_{j,k}, s_{j,k}^2 + t_{j,k}^2 - 1)$ of the latter. This can also be verified directly by elementary trigonometric identities.

It should be pointed out that the Theorem 3.2 shows that the above construction does not work for the weight functions

$$\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},\gamma}(x,y) = (1-x^2)^\gamma(1-y^2)^\gamma$$

when $\gamma \neq \pm 1/2$. We cannot, however, conclude that the cubature rules of degree $4n-1$ that attain the lower bound (1.4) do not exist for these product Gegenbauer weight functions. In fact, examining the proof carefully shows that the procedures that we adopted could be reversed only if the cubature rules for $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},\gamma}$ satisfy certain properties. What we can conclude is then the following: If a cubature rule of degree $4n-1$ that attains the lower bound (1.4) exists for $\mathcal{W}_{-\frac{1}{2},-\frac{1}{2},\gamma}$, then either it is not invariant under the symmetry with respect to the diagonals $y = x$ and $y = -x$ of the rectangle $[-1,1]^2$ or some of its nodes are on these diagonals.

Finally, our procedure of deriving cubature rules for $\mathcal{W}_{\pm\frac{1}{2}}$ on $[-1,1]^2$ can be adopted for other type of cubature rules, such as cubature rules of even degree or Gauss–Lobatto type cubature rules, see the remark at the end of Section 3.1. In particular, if we start with a Gaussian–Lobatto quadrature for w, which has additional nodes at -1 and 1, then the resulted cubature rule for $\mathcal{W}_\gamma$ will have nodes on the diagonals of $[-1,1]^2$. Since they do not seem to have other features, we shall not pursue them further.

5. Lagrange interpolation and Gaussian cubature rules

Cubature rules are closely related to Lagrange interpolation polynomials, as stated in Section 2. In this section we consider Lagrange interpolation polynomials based on the zeros of the Gaussian cubature rules constructed in the Section 3.

The Lagrange interpolation polynomial based on the Gaussian cubature rule in Theorem 2.1 is given in Theorem 2.2. A more direct construction can be given however as follows.

Let $\{x_{k,n} : 1 \le k \le n\}$ be the zeros of the orthogonal polynomial p_n of degree n with respect to w on $[-1, 1]$, as in (3.1). The Lagrange interpolation polynomial (of one variable) of degree $< n$ based on these points is

$$I_n f(x) = \sum_{k=1}^{n} f(x_{k,n}) l_k(x), \quad l_k(x) := \frac{p_n(x)}{p_n'(x_{k,n})(x - x_{k,n})}. \tag{5.1}$$

Recall that $u_{j,k} = x_{j,n} + x_{k,n}$ and $v_{j,k} = x_{j,n} x_{k,n}$.

Theorem 5.1. *The unique Lagrange interpolation polynomial of degree $n-1$ based on the nodes of the Gaussian cubature rule (3.3) is given by*

$$L_n f(u, v) = \sum_{k=1}^{n} \sum_{j=1}^{k}{}' f(u_{j,k}, v_{j,k}) l_{j,k}(u, v),$$
$$\text{with } l_{j,k}(u, v) := l_j(x) l_k(y) + l_j(y) l_k(x),\ u = x + y,\ v = xy. \tag{5.2}$$

And the unique Lagrange interpolation polynomial of degree $n-2$ based on the nodes of the Gaussian cubature rule (3.4) is given by

$$L_n f(u, v) = \sum_{k=2}^{n} \sum_{j=1}^{k-1} f(u_{j,k}, v_{j,k}) l_{j,k}(u, v),$$
$$\text{with } l_{j,k}(u, v) := \frac{l_j(x) l_k(y) - l_j(y) l_k(x)}{x - y},\ u = x + y,\ v = xy. \tag{5.3}$$

Proof. A quick computation shows that if $0 \le j < k \le n$ and $0 \le p \le q \le n$, then

$$l_{j,k}(u_{p,q}, v_{p,q}) = l_j(x_{p,n}) l_k(x_{q,n}) + l_j(x_{q,n}) l_k(x_{p,n}) = \delta_{j,p}\delta_{k,q} + \delta_{k,p}\delta_{j,q} = \delta_{j,p}\delta_{k,q}.$$

If $0 \le j = k \le n$ and $0 \le p \le q \le n$, then

$$l_{j,k}(u_{p,q}, v_{p,q}) = 2 l_k(x_{p,n}) l_k(x_{q,n}) = 2\delta_{j,p}\delta_{k,q},$$

which proves (5.2). The proof of (5.3) is similar. □

The explicit formulas of $l_{j,k}$ can also be obtained from Theorem 2.2. In fact, as shown in [23, Theorem 3.1], the reproducing kernel $K_n^{(\pm\frac{1}{2})}(\cdot, \cdot) = K_n(W_{\pm\frac{1}{2}}; \cdot, \cdot)$ can be expressed in terms of the reproducing kernel

$$k_n(x, y) = k_n(w; x, y) := \sum_{k=0}^{n} p_k(x) p_k(y)$$

of one variable, where p_k are orthonormal polynomials with respect to w. Set

$$u := (u_1, u_2) = (x_1 + x_2, x_1 x_2) \quad \text{and} \quad v := (v_1, v_2) = (y_1 + y_2, y_1 y_2).$$

The reproducing kernel $K_n^{(-\frac{1}{2})}(\cdot,\cdot)$ for $W_{-\frac{1}{2}}$ is given by

$$K_n^{(-\frac{1}{2})}(u,v) = \frac{1}{2}\left[k_n(x_1,y_1)k_n(x_2,y_2) + k_n(x_2,y_1)k_n(x_1,y_2)\right], \tag{5.4}$$

and the reproducing kernel $K_n^{(\frac{1}{2})}(\cdot,\cdot)$ for $W_{\frac{1}{2}}$ is given by

$$K_n^{(\frac{1}{2})}(u,v) = \frac{k_{n+1}(x_1,y_1)k_{n+1}(x_2,y_2) - k_{n+1}(x_2,y_1)k_{n+1}(x_1,y_2)}{2(x_1-x_2)(y_1-y_2)}. \tag{5.5}$$

As an application of the explicit expression, we can estimate the uniform norm of the interpolation operator, often called the Lebesgue constant. For the interpolation polynomial $I_n f$ in (5.1), the Lebesgue constant $\|I_n\|_{C[-1,1]}$ satisfies

$$\|I_n\|_{C[-1,1]} = \max_{x\in[-1,1]} \sum_{k=1}^{n} |l_k(x)|.$$

Corollary 5.2. *The Lebesgue constant for $L_n f$ in (5.2) satisfies*

$$\|L_n\|_\infty \le 2\left(\|I_n\|_{C[-1,1]}\right)^2. \tag{5.6}$$

Proof. A standard argument shows that the Lebesgue constant for $\mathcal{L}_n f$ is given by

$$\|L_n\|_\infty = \max_{(u,v)\in\Omega} \sum_{k=1}^{n} {\sum_{j=1}^{k}}' |l_{j,k}(u,v)|.$$

Since $\ell_{j,k}(u,v) = \ell_{k,j}(u,v)$ by (5.2), a moment of reflection shows that

$$\sum_{k=1}^{n} {\sum_{j=1}^{k}}' |l_{j,k}(u,v)| = \sum_{k=1}^{n}\sum_{j=1}^{n} |l_j(x)l_k(y) + l_j(x)l_k(y)| \le 2\sum_{j=1}^{n}|l_j(x)| \sum_{j=1}^{n}|l_j(y)|,$$

from which the estimate (5.6) follows immediately. $\square$

Denote by $L_n^{\alpha,\beta} f$ the Lagrange interpolation polynomial based on the nodes of the Gaussian cubature rule of degree $2n-1$ for $W_{\alpha,\beta,-\frac{1}{2}}$.

Corollary 5.3. *Let $\alpha,\beta > -1$. The Lebesgue constant of $L_n^{\alpha,\beta} f$ satisfies*

$$\|L_n^{\alpha,\beta}\|_\infty = \mathcal{O}(1)\begin{cases} n^{2\max\{\alpha,\beta\}+1}, & \max\{\alpha,\beta\} > -1/2,\\ \log^2 n, & \max\{\alpha,\beta\} \le -1/2.\end{cases}$$

Proof. This follows from the previous corollary and the classical result on the Lagrange interpolation polynomials at the zeros of Jacobi polynomials [17]. $\square$

We expect that these estimates are sharp in the sense that the lower bound holds with the same order. For $\alpha = \beta = -1/2$, see [18].

6. Lagrange interpolation and minimal cubature rules

The relation between a minimal cubature rule and the Lagrange interpolation polynomial based on its nodes is stated in Theorem 2.5. In this section we discuss the Lagrange interpolation polynomials based on the nodes of the minimal cubature rules of degree $4n-1$ in Section 4.

In order to derive explicit formulas and discuss the Lebesgue constants, we shall limit our discussion to $\mathcal{W}_{\alpha,\beta,-\frac{1}{2}}$, which we renamed as $\mathcal{W}_{\alpha,\beta}$ at (2.22). An analogue discussion can be carried out for $\mathcal{W}_{\alpha,\beta,\frac{1}{2}}$.

6.1. *Construction of the interpolation polynomial*

Let X_n denote the set of nodes of the cubature formula $\mathcal{W}_{\alpha,\beta}$. The Lagrange interpolation polynomial based on X_n is given in Theorem 2.5, in which $x_{k,n} = x_{k,n}^{(\alpha,\beta)}$ are the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}$. The subspace Π_{2n}^* in (2.2) now takes the form

$$\Pi_{2n}^* := \Pi_{2n-1}^2 \cup \operatorname{span}\{{}_2Q_{k,2n}^{(\pm\frac{1}{2})} : 0 \le k \le n-1\}.$$

The interpolation polynomial in Π_{2n}^* is given in Theorem 2.5 in terms of a kernel of Π_{2n}^* defined by

$$\mathcal{K}_{2n}^*(x,y) := \mathcal{K}_{2n-1}^{\alpha,\beta}(x,y) + \sum_{k=0}^{n-1} b_{k,n}\, {}_2Q_{k,2n}^{\alpha,\beta}(x)\, {}_2Q_{k,2n}^{\alpha,\beta}(y), \tag{6.1}$$

where $b_{k,n}$ are certain positive numbers, $\mathcal{K}_{2n-1}^{\alpha,\beta}$ and ${}_2Q_{k,2n}$ are given explicitly in (2.23) and Proposition 2.7.

Although the cubature rule (4.6) of degree $4n-1$ for $\mathcal{W}_{-\frac{1}{2}}$ can be deduced from the Gaussian cubature rule (3.3) for $W_{-\frac{1}{2}}$, this deduction does not extend to interpolation polynomials, since each node of the cubature rule (3.3) corresponds to four nodes of the cubature rule (4.6). We have to work with the explicit formula given in Theorem 2.5, which we determine explicitly in the following theorem.

Theorem 6.1. *Let $x_{k,n} = \cos\theta_{k,n}$, $1 \le k \le n$, denote the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ and let $s_{j,k} := \cos\frac{\theta_{j,n}-\theta_{k,n}}{2}$ and $t_{j,k} := \cos\frac{\theta_{j,n}+\theta_{k,n}}{2}$. Set*

$$\mathbf{x}_{j,k}^{(1)} := (s_{j,k}, t_{j,k}), \qquad \mathbf{x}_{j,k}^{(2)} := (t_{j,k}, s_{j,k}),$$
$$\mathbf{x}_{j,k}^{(3)} := (-s_{j,k}, -t_{j,k}), \qquad \mathbf{x}_{j,k}^{(4)} := (-t_{j,k}, -s_{j,k}).$$

Then the Lagrange interpolation $\mathcal{L}_n^{\alpha,\beta} f$ in Π_n^* is given by

$$\begin{aligned}\mathcal{L}_n^{\alpha,\beta} f(x,y) = \sum_{k=1}^{n}\sum_{j=1}^{k} \Big[& f\left(\mathbf{x}_{j,k}^{(1)}\right)\ell_{j,k}^{(1)}(x,y) + f\left(\mathbf{x}_{j,k}^{(2)}\right)\ell_{j,k}^{(2)}(x,y) \\ & + f\left(\mathbf{x}_{j,k}^{(3)}\right)\ell_{j,k}^{(3)}(x,y) + f\left(\mathbf{x}_{j,k}^{(4)}\right)\ell_{j,k}^{(4)}(x,y)\Big],\end{aligned} \tag{6.2}$$

where the fundamental interpolation polynomials $\ell_{j,k}^{(i)}$ are given by

$$\ell_{j,k}^{(i)}(x,y) = \frac{1}{2}\lambda_j^{(\alpha,\beta)}\lambda_k^{(\alpha,\beta)}\mathcal{K}_{2n}^*\left((x,y), \mathbf{x}_{j,k}^{(i)}\right), \tag{6.3}$$

in which $\frac{1}{2}$ in the right hand side needs to be replaced by $\frac{1}{4}$ when $j = k$, and

$$\mathcal{K}_{2n}^*(x,y) = \mathcal{K}_{2n-1}^{\alpha,\beta}(s,t) + \frac{1+\alpha+\beta+n}{1+\alpha+\beta+2n} d_{\alpha,\beta}^{(1,1)}(x_1^2 - x_2^2)(y_1^2 - y_2^2)$$

$$\times \left[K_{n-1}^{\alpha+1,\beta+1}(s,t) - K_{n-2}^{\alpha+1,\beta+1}(s,t)\right] - \frac{n(1+\alpha+\beta+n)}{(1+\alpha+\beta+2n)^2}\,{}_2Q_{n-1,2n}(x)\,{}_2Q_{n-1,2n}(y), \tag{6.4}$$

where $s = (2x_1x_2, x_1^2 + x_2^2 - 1)$, $t = (2y_1y_2, y_1^2 + y_2^2 - 1)$, $\mathcal{K}_{2n-1}^{\alpha,\beta}(\cdot,\cdot)$ *and* $d_{\alpha,\beta}^{(1,1)}$ *are given in* (2.23) *and* $K_n^{\alpha,\beta}(\cdot,\cdot)$ *is given in* (2.24).

Proof. The formulas (6.2) and (6.3) are exactly those given in Theorem 2.5, specialized to the Jacobi case. It remains to establish the formula of (6.4), for which we need to determine the constants $b_{k,n}$ in (6.1).

Throughout this proof, we write $Q_{k,2n}(x,y) = {}_2Q_{k,n}^{(\alpha,\beta)}(x,y)$. By the explicit formula of $Q_{k,2n}$ in Proposition 2.7, it is easy to verify that

$$Q_{m,2n}\left(\mathbf{x}_{j,k}^{(1)}\right) = \gamma_{\alpha,\beta}\sqrt{1-x_j^2}\sqrt{1-x_k^2} \times \left[p_{n-1}^{(\alpha+1,\beta+1)}(x_k)p_m^{(\alpha+1,\beta+1)}(x_j) + p_{n-1}^{(\alpha+1,\beta+1)}(x_j)p_m^{(\alpha+1,\beta+1)}(x_k)\right], \tag{6.5}$$

and furthermore, since $Q_{m,2n}$ is symmetric in its variables,

$$Q_{m,2n}\left(\mathbf{x}_{j,k}^{(1)}\right) = Q_{m,2n}\left(\mathbf{x}_{j,k}^{(2)}\right) = -Q_{m,2n}\left(\mathbf{x}_{j,k}^{(3)}\right) = -Q_{m,2n}\left(\mathbf{x}_{j,k}^{(4)}\right). \tag{6.6}$$

Let us denote by $\mathcal{C}_n[f]$ the minimal cubature rule, that is,

$$\mathcal{C}_n[f] := \frac{1}{2}\sum_{k=1}^{n}{\sum_{j=1}^{k}}{}' \lambda_k^{(\alpha,\beta)}\lambda_j^{(\alpha,\beta)}\left[f\left(\mathbf{x}_{j,k}^{(1)}\right) + f\left(\mathbf{x}_{j,k}^{(2)}\right) + f\left(\mathbf{x}_{j,k}^{(3)}\right) + f\left(\mathbf{x}_{j,k}^{(4)}\right)\right].$$

By (2.4) and the fact that $\ell_{j,k}^{(i)}$ are the fundamental interpolation polynomials, we obtain

$$\mathcal{K}_{2n}^*\left(\mathbf{x}_{j,k}^{(1)}, \mathbf{x}_{j',k'}^{(1)}\right) = 2\left(\lambda_j^{(\alpha,\beta)}\lambda_k^{(\alpha,\beta)}\right)^{-1}\delta_{j,j'}\delta_{k,k'},$$

which implies immediately that

$$\mathcal{C}_n\left[\mathcal{K}_{2n}^*\left(\mathbf{x}_{j,k}^{(1)},\cdot\right)Q_{l,2n}\right] = Q_{l,2n}\left(\mathbf{x}_{j,k}^{(1)}\right). \tag{6.7}$$

On the other hand, using the formula of $\mathcal{K}_{2n}^*(\cdot,\cdot)$ in (6.1) shows that

$$\mathcal{C}_n\left[\mathcal{K}_{2n}^*\left(\mathbf{x}_{j,k}^{(1)},\cdot\right)Q_{l,2n}\right] = \mathcal{C}_n\left[K_{2n-1}^{\alpha,\beta}\left(\mathbf{x}_{j,k}^{(1)},\cdot\right)Q_{l,2n}\right] + \sum_{m=0}^{n-1} b_{m,n}\mathcal{C}_n\left[Q_{m,2n}Q_{l,2n}\right].$$

Since the cubature rule is of degree $4n-1$ and $Q_{l,2n}$ is an orthogonal polynomial of degree $2n$,

$$\mathcal{C}_n\left[K_{2n-1}^{\alpha.\beta}\left(\mathbf{x}_{j,k}^{(1)},\cdot\right)Q_{l,2n}\right] = 2c_{\alpha,\beta}^2\int_{[-1,1]^2} K_{2n-1}^{\alpha,\beta}\left(\mathbf{x}_{j,k}^{(1)},y\right)Q_{l,2n}(y)W_{\alpha,\beta}(y)\mathrm{d}y = 0.$$

Furthermore, since $Q_{m,2n}$ is symmetric in its variables, it follows from (6.6) that

$$\mathcal{C}_n\left[Q_{m,2n}Q_{l,2n}\right] = \sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_k^{(\alpha,\beta)}\lambda_j^{(\alpha,\beta)}Q_{m,2n}\left(\mathbf{x}_{j,k}^{(1)}\right)Q_{l,2n}\left(\mathbf{x}_{j,k}^{(1)}\right).$$

Recall the definition of $\widehat{h}_m$ defined in (2.20). By (6.6), the explicit formulas of $Q_{k,2n}$ and the Gaussian quadrature (2.19), it follows that

$$\mathcal{C}_n\left[Q_{m,2n}Q_{l,2n}\right] = 2\gamma_{\alpha,\beta}^2 \widehat{h}_{n-1}\widehat{h}_m \delta_{l,m}, \quad 0 \le l, m \le n-1.$$

Putting these formulas together, we have shown that

$$\mathcal{C}_n\left[\mathcal{K}_{2n}^*\left(\mathbf{x}_{j,k}^{(1)}, \cdot\right) Q_{l,2n}\right] = 2\gamma_{\alpha,\beta}^2 \widehat{h}_{n-1}\widehat{h}_m b_{l,n} Q_{l,2n}\left(\mathbf{x}_{j,k}^{(1)}\right).$$

Comparing with (6.7), it follows readily that $b_{l,n}^{-1} = 2\gamma_{\alpha,\beta}^2 \widehat{h}_{n-1}\widehat{h}_m$. Recalling that $\gamma_{\alpha,\beta} = c_{\alpha+1,\beta+1}/(\sqrt{2}c_{\alpha,\beta})$, applying Lemma 2.6 gives

$$b_{0,n} = \cdots = b_{n-2,n} = \frac{1+\alpha+\beta+n}{1+\alpha+\beta+2n}, \quad \text{and} \quad b_{n-1,n} = b_{0,n}^2.$$

The final step in verifying (6.4) uses the fact that

$$\sum_{k=0}^{n-1} Q_{k,2n}(x)Q_{k,2n}(y) = d_{\alpha,\beta}^{(1,1)}(x_1^2 - x_2^2)(y_1^2 - y_2^2)\left[K_{n-1}^{\alpha+1,\beta+1}(s,t) - K_{n-2}^{\alpha+1,\beta+1}(s,t)\right]$$

which can be verified using the explicit formulas of the quantities involved and the elementary trigonometric identity

$$2xy = \cos(\theta-\phi) + \cos(\theta+\phi), \qquad x^2 + y^2 - 1 = \cos(\theta-\phi)\cos(\theta+\phi), \tag{6.8}$$

see also Section 4 of [23]. This completes the proof. □

The above theorem gives a compact formula for the Lagrange interpolation polynomial based on the nodes of the minimal cubature rule with respect to $\mathcal{W}_{\alpha,\beta}$. In the case of $\alpha = \beta = -1/2$, the interpolation polynomials were introduced in [21] and they were studied numerically in [2]. The explicit formulas given in [21], however, takes a different form since the set of nodes were not divided into the four subsets as in (6.2) and a completely different formula for $\mathcal{K}_{2n}^*(\cdot,\cdot)$ was used.

6.2. Lebesgue constants of the interpolation operator

The Lebesgue constant of the interpolation operator $\mathcal{L}_n^{\alpha,\beta}$ is its operator norm $\|\mathcal{L}_n^{(\alpha,\beta)}\|_\infty$. Since

$$\|\mathcal{L}_n^{\alpha,\beta} f\|_\infty \le \|\mathcal{L}_n^{(\alpha,\beta)}\|_\infty \|f\|_\infty, \quad \forall f \in C[-1,1]^2,$$

the Lebesgue constant determines the convergence behavior of $\mathcal{L}_n^{\alpha,\beta} f$.

Lemma 6.2. *The Lebesgue constant of $\mathcal{L}_n^{\alpha,\beta} f$ in Proposition 6.1 satisfies*

$$\begin{aligned}\|\mathcal{L}_n^{\alpha,\beta}\|_\infty &= \frac{1}{4}\max\nolimits_{x\in[-1,1]^2} \sum_{k=1}^{n}{\sum_{j=1}^{k}}{}' \lambda_{j,k}\left[\left|\mathcal{K}_{2n}^*\left(x,\mathbf{x}_{j,k}^{(1)}\right)\right| + \left|\mathcal{K}_{2n}^*\left(x,\mathbf{x}_{j,k}^{(2)}\right)\right|\right. \\ &\quad \left. + \left|\mathcal{K}_{2n}^*\left(x,\mathbf{x}_{j,k}^{(3)}\right)\right| + \left|\mathcal{K}_{2n}^*\left(x,\mathbf{x}_{j,k}^{(4)}\right)\right|\right] \\ &\sim \max\nolimits_{x\in[-1,1]^2} \sum_{k=1}^{n}{\sum_{j=1}^{k}}{}' \lambda_{j,k}\left|\mathcal{K}_{2n}^*\left(x,\mathbf{x}_{j,k}^{(1)}\right)\right|.\end{aligned} \tag{6.9}$$

Proof. Recalling (4.5) and the definition of $\mathbf{x}_{j,k}^{(i)}$, it follows easily from the symmetry that $\|\mathcal{L}_n^{\alpha,\beta}\|_\infty$ is bounded above by 4 times of the quantity in the last expression and it is at least as big as the same quantity. $\square$

In order to deduce the order of the Lebesgue constant, we need to estimate, by the explicit formula at (6.4), several sums. We first deal with the easiest sum to be estimated. Let c denote a generic constant whose value may vary from line to line.

Lemma 6.3. *For $\alpha, \beta > -1$ and $x \in [-1, 1]$,*

$$\Lambda_Q := \sum_{k=1}^{n} \sum_{j=1}^{k} \lambda_k^{(\alpha,\beta)} \lambda_j^{(\alpha,\beta)} \left| {}_2Q_{n-1,2n}(x)\, {}_2Q_{n-1,2n}\left(x, \mathbf{x}_{j,k}^{(1)}\right) \right| \le c\, n^{2\max\{\alpha,\beta\}}.$$

Proof. We will need several well known estimates for the Jacobi polynomials and related quantities, all can be found in [17]. First we need

$$|p_n^{(\alpha,\beta)}(x)| \le c \left(\sqrt{1-x} + n^{-1}\right)^{-(\alpha+1/2)/2} \left(\sqrt{1+x} + n^{-1}\right)^{-(\beta+1/2)/2} \tag{6.10}$$

for $x \in [-1, 1]$. Using the fact that $\cos^2\theta - \cos^2\phi = \sin(\theta - \phi)\sin(\theta + \phi)$, it follows from the explicit expression of ${}_2Q_{n-1,n}(x)$ that

$$\left| {}_2Q_{n-1,2n}(x) \right| \le c \max_{-1 \le x \le 1} \left| \sqrt{1-x^2}\, p_{n-1}^{(\alpha,\beta)}(x) \right| \le c\, n^{2\max\{\alpha,\beta\}-1}, \quad x \in [-1, 1]^2.$$

Furthermore, we need the estimates

$$\lambda_{k,n}^{(\alpha,\beta)} = \left[k_n^{(\alpha,\beta)}(x_{k,n}, x_{k,n})\right]^{-1} \sim n^{-1} w_{\alpha,\beta}(x_{k,n}) \sqrt{1 - x_{k,n}^2}, \tag{6.11}$$

$$p_{n-1}^{(\alpha,\beta)}(x_{k,n}) \sim \left[w_{\alpha,\beta}(x_{k,n})\right]^{-1} (1 - x_{k,n})^{-1/4}. \tag{6.12}$$

From (6.12), it is not difficult to see (using (2.21), for example) that

$$\begin{aligned} \left| {}_2Q_{n-1,2n}\left(x, \mathbf{x}_{j,k}^{(1)}\right) \right| \sim{}& n \left[w_{\alpha,\beta}(x_{k,n})\right]^{-1/2} (1 - x_{k,n})^{-1/4} \\ & \times \left[w_{\alpha,\beta}(x_{j,n})\right]^{-1/2} (1 - x_{j,n})^{-1/4}. \end{aligned}$$

Consequently, since ${}_2Q_{n-1,2n}$ is symmetric in its variables, we see that

$$\Lambda_Q \le c\, n^{2\max\{\alpha,\beta\}} \left(\sum_{k=1}^{n} \lambda_k^{(\alpha,\beta)} \left[w_{\alpha,\beta}(x_{k,n})\right]^{-1/2} (1 - x_{k,n})^{-1/4} \right)^2 \le c\, n^{2\max\{\alpha,\beta\}}$$

as the sum is easily seen to be bounded upon using (6.11). $\square$

The other sums of $\|\mathcal{L}_n^{\alpha,\beta}\|_\infty$ cannot be deduced form the Lebesgue constant for the interpolation polynomial of one variable, as we did in Corollary 5.2, since there are four remaining sums by (6.4), and only one of them, the first one, is related directly to the fundamental interpolation polynomials of one variable. We can, however, reduce the proof to the estimate of several kernels in one variable. Let us define, for $i, j \ge 0$,

$$k_n^{(\alpha,\beta),i,j}(x, y) := (1-x)^{\frac{i}{2}} (1+x)^{\frac{j}{2}} (1-y)^{\frac{i}{2}} (1+y)^{\frac{j}{2}} k_n^{(\alpha+i,\beta+j)}(x, y).$$

Lemma 6.4. *Let $\alpha, \beta \ge -1/2$ and $i, j \ge 0$. Then*

$$|k_n^{(\alpha,\beta),i,j}(\cos\theta, \cos\phi)| \le c \frac{\left(\sin\frac{\theta}{2}\sin\frac{\phi}{2} + n^{-1}|\theta-\phi| + n^{-2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + n^{-1}|\theta-\phi| + n^{-2}\right)^{-\beta-\frac{1}{2}}}{|\theta-\phi| + n^{-1}}. \tag{6.13}$$

While (6.11) and (6.11) are classical, (6.13) is stated recently in [23, Lemma 5.3] and its proof follows from an estimate in [4]. The restriction $\alpha, \beta \ge -\frac{1}{2}$ instead of $\alpha, \beta > -1$ comes from the method used in [4]. For $i, j \ge 0$, let

$$\Lambda_n^{(i,j)}(x) := \sum_{k=1}^{n} \lambda_{k,n}^{(\alpha,\beta)} \left| k_n^{(\alpha,\beta),i,j}(x, x_{k,n}) \right|.$$

We will need the following result for our estimate of $\|\mathcal{L}_n^{\alpha,\beta}\|_\infty$.

Lemma 6.5. *Let $\alpha, \beta \ge -1/2$. For $i, j \ge 0$,*

$$\max_{x\in[-1,1]} \Lambda_n^{(i,j)}(x) = \mathcal{O}(1) \begin{cases} n^{\max\{\alpha,\beta\}+\frac{1}{2}}, & \max\{\alpha,\beta\} > -1/2, \\ \log n, & \max\{\alpha,\beta\} = -1/2. \end{cases} \tag{6.14}$$

Proof. We can assume $x \in [0, 1]$ and write $x = \cos\theta$. We consider $\alpha > -1/2$, the case $\alpha = -1/2$ is easier. Fix m such that $x_{m,n}$ is (one of) the closest zero to x. Then $1 \le m \le n/2+1$. We only consider the sum in $\Lambda_n^{(i,j)}$ for $1 \le k \le 2n/3$, the remaining part is easier since for $2n/3 < k \le n$, $|\theta - \theta_{k,n}| \sim 1$. If $k = m-1, m, m+1$, then by (6.11) and (6.13),

$$\lambda_{k,n}^{(\alpha,\beta)} \left| k_n^{(\alpha,\beta)}(x, x_{k,n}) \right| \le \frac{(\sin\theta_{k,n})^{\alpha+\frac{1}{2}}}{\left(\sin^2\frac{\theta_{k,n}}{2} + n^{-2}\right)^{\alpha+\frac{1}{2}}} \le cn^{\alpha+\frac{1}{2}}.$$

Using the fact that $|\theta - \theta_k| \sim |\theta_m - \theta|$, we have by (6.11) and (6.13)

$$\sum_{\substack{|k-m|>1 \\ 1\le k\le 2n/3}} \lambda_{k,n}^{(\alpha,\beta)} \left| k_n^{(\alpha,\beta)}(x, x_{k,n}) \right| \le cn^{\alpha+\frac{1}{2}} \sum_{\substack{|k-m|>1 \\ 1\le k\le 2n/3}} \frac{k^{\alpha+\frac{1}{2}}}{|k-m|(km+|k-m|)^{-\alpha-\frac{1}{2}}}.$$

The last sum can be shown to be bounded by dividing it into three sums over $1 \le k \le m/2$, $m/2 \le k \le 2m$ and $m \le k \le 2n/3$, respectively. Such estimates are rather standard affairs, we leave the details to the interested readers. □

For $i = j = 0$, the estimate (6.14) gives the order of the Lebesgue constant for the interpolation polynomials based on the zeros of Jacobi polynomials in one variable. The classical proof in [17], however, does not apply to the case of $(i, j) \ne (0, 0)$, since $\lambda_{k,n}^{(\alpha,\beta)} k_n^{(\alpha,\beta),i,j}(x, x_{k,n})$ does not always vanish at $x_{l,n}$ when $l \ne k$.

We are now ready to prove our result on the Lebesgue constant of $\mathcal{L}_n^{\alpha,\beta} f$.

Theorem 6.6. *Let $\alpha, \beta \ge -1/2$. The Lebesgue constant of the Lagrange interpolation polynomial $\mathcal{L}_n^{\alpha,\beta} f$ based on the nodes of the minimal cubature rule of degree $4n-1$ for $\mathcal{W}_{\alpha,\beta}$*

satisfies

$$\|\mathcal{L}_n^{\alpha,\beta}\|_\infty = \mathcal{O}(1) \begin{cases} n^{2\max\{\alpha,\beta\}+1}, & \max\{\alpha,\beta\} > -1/2, \\ (\log n)^2, & \max\{\alpha,\beta\} = -1/2. \end{cases} \tag{6.15}$$

Proof. Let $x_{k,n} = \cos\theta_{k,n}$ be the zeros of the Jacobi polynomial $p_n^{(\alpha,\beta)}$. We estimate $\|\mathcal{L}_n^{\alpha,\beta}\|_\infty$ in (6.9) by setting $x_1 = \cos\frac{\theta_1-\theta_2}{2}$ and $x_2 = \cos\frac{\theta_1+\theta_2}{2}$ and taking the maximum over $0 \le \theta_1, \theta_2 \le \pi$. It follows that $2x_1x_2 = \cos\theta_1 + \cos\theta_2$ and $x_1^2 + x_2^2 - 1 = \cos\theta_1\cos\theta_2$, and furthermore,

$$x_1 - x_2 = \sqrt{(1-\cos\theta_1)(1-\cos\theta_2)} \quad \text{and} \quad x_1 + x_2 = \sqrt{(1+\cos\theta_1)(1+\cos\theta_2)}.$$

Hence, recalling (4.5), it follows from (6.4), (2.24) and Lemma 6.3 that

$$\begin{aligned} \|\mathcal{L}_n^{\alpha,\beta}\|_\infty &= \mathcal{O}(1) \max\nolimits_{0\le\theta_1,\theta_2\le\pi} \sum_{k=0}^{n} {\sum_{j=1}^{k}}' \lambda_j\lambda_k \left[\left|J_{i,k}^{0,0}(\theta_1,\theta_2)\right| + \left|J_{j,k}n^{1,0}(\theta_1,\theta_2)\right| \right. \\ &\quad \left. + \left|J_{j,k}^{0,1}(\theta_1,\theta_2)\right| + \left|J_{j,k}^{1,1}(\theta_1,\theta_2)\right| \right] + \mathcal{O}(1) n^{2\max\{\alpha,\beta\}}, \end{aligned}$$

where $J_n^{i,j}$ are defined by

$$\begin{aligned} J_{j,k}^{i,j}(\theta_1,\theta_2) &= k_n^{(\alpha,\beta),i,j}(\cos\theta_1,\cos\theta_j) k_n^{(\alpha,\beta),i,j}(\cos\theta_2,\cos\theta_k) \\ &\quad + k_n^{(\alpha,\beta),i,j}(\cos\theta_1,\cos\theta_k) k_n^{(\alpha,\beta),i,j}(\cos\theta_2,\cos\theta_j). \end{aligned}$$

Hence, as in the proof of Corollary 5.2, we can reduce the estimate to the product of $\Lambda_n^{(i,j)}$, so that the desired result follows from (6.14). $\square$

In the case of $\alpha = \beta = -1/2$, the order of the Lebesgue constant was determined in [3] based on the explicit expression in [21]. In all other cases, the estimate (6.15) is new. One interesting question is if the result can be extended to the case of $\max\{\alpha,\beta\} < -\frac{1}{2}$. We expect that it can be and, furthermore, we believe that the order is $\|\mathcal{L}_n^{\alpha,\beta}\|_\infty = \mathcal{O}(1)(\log n)^2$ for $\max\{\alpha,\beta\} < -\frac{1}{2}$. Finally, we expect that this order and the one in (6.15) are sharp in the sense that the lower bound holds with the same order.

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