JOURNAL OF COMBINATORIAL THEORY, Series A 56, 223–238 (1991)

Harmonious Groups

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Communicated by Marshall Hall. Jr.

Received June 12, 1989

In this paper we introduce a method of sequencing the elements of a finite group that gives rise to a complete mapping of the group. Our definition was motivated by the concept of a harmonious graph invented by Graham and Sloane. Our concept has several connections to graph theory and as an application we complete the characterization of elegant cycles begun by Chang, Hsu, and Rogers. Our definitions are also variations of the notion of an R-sequenceable group first introduced by Ringel in his solution of the map coloring problem for all compact 2-dimensional manifolds except the sphere and expanded upon by Friedlander, Gordon, and Miller.

1. INTRODUCTION

A permutation ϕ of a group is a *complete mapping* if $x(x\phi) = y(y\phi)$ implies $x = y$. This concept was introduced by H. B. Mann [8] in 1942 in connection with the construction of orthogonal Latin squares. Applications to finite nets and to neofields were given by Bruck [1] and Paige [9], and the concept naturally arises in schemes for encoding numbers to detect

0097-3165/91 $3.00

errors (see [6]). M. Hall and Paige [7] showed that a necessary condition for a finite group of even order to have a complete mapping is that its Sylow 2-subgroup be non-cyclic and that this condition is sufficient for solvable groups. (For groups of odd order the identity mapping is a complete mapping.)

In this paper we introduce a method of sequencing the elements of a finite group that gives rise to complete mapping of the group. Our definition was motivated by the concept of a harmonious graph invented by Graham and Sloane [5]. Our concept has several connections to graph theory and as an application we complete the characterization of elegant cycles begun by Chang, Hsu, and Rogers [2]. Our definitions are also variations of the notion of an R-sequenceable group first introduced by Ringel in his solution of the map coloring problem for all compact 2-dimensional manifolds except the sphere [10] and expanded upon by Friedlander, Gordon, and Miller [3].

2. Definitions and Notation

Let G be a finite group. We say G is *harmonious* if the elements of G can be listed $g_1, g_2, ..., g_n$ so that $G = \{g_1 g_2, g_2 g_3, ..., g_{n-1} g_n, g_n g_1\}$. Analogously, letting $G^\#$ denote the set of non-identity elements of G, we say $G^\#$ is harmonious if there is a listing $g_1, g_2, ..., g_n$ of the elements of $G^\#$ such that $G^\# = \{g_1 g_2, g_2 g_3, ..., g_{n-1} g_n, g_n g_1\}$. In each case we call the list $g_1, g_2, ..., g_n$ a *harmonious sequence.* We observe that G is harmonious if and only if G has a complete mapping which is also a $|G|$-cycle. For example, if $g_1, g_2, ..., g_n$ is a harmonious sequence for G then $g_1 \to g_2$, $g_2 \to g_3$, $g_3 \to g_4, ..., g_n \to g_1$ is a complete mapping of G. Conversely, if ϕ is a complete mapping of G which is also a $|G|$-cycle then $e, e\phi, e\phi^2, ..., e\phi^{|G|-1}$ is a harmonious sequence for G (where e is the identity). We call such mappings *harmonious.*

For the purpose of comparison we give the following definitions. A connected graph with p vertices and $q \geqslant p$ edges is *harmonious* if it is possible to label the vertices x with distinct elements $f(x)$ of $\mathbf{Z}_q$ (the group of integers modulo q) in such a way that, when each edge xy is labeled with $(f(x)+f(y))$ modulo q, the resulting edge labels are distinct. A group G is *R-sequenceable* if there is a listing $g_1, g_2, ..., g_n$ of the elements of $G^\#$ such that $G^\# = \{g_1^{-1} g_2, g_2^{-1} g_3, ..., g_{n-1}^{-1} g_n, g_n^{-1} g_1\}$.

Harmonious groups can be given a graph-theoretic interpretation as follows. Let G be a finite group of order n and let K_n be the complete symmetric digraph with n vertices. Label the vertices with the elements of G and label the edge joining g_i to g_j with $g_i g_j$. Then the existence of a harmonious labeling for G is equivalent to the existence of a hamiltonian circuit in K_n such that each element of G occurs exactly once as an edge in the circuit. An analogous interpretation exists for $G^\#$.

We use I to denote the identity permutation; $g^x = x^{-1}gx$; $\mathrm{Syl}_2(G)$ is the set of all Sylow 2-subgroups of G; $D_n = \langle a, b \mid a^n = b^2 = e,\ a^b = a^{-1}\rangle$ (the dihedral group of order $2n$); $Q_n = \langle a,\ b \mid a^{2n} = b^4 = e,\ b^2 = a^n,\ a^b = a^{-1}\rangle$ (the quaternion group of order $4n$); G' is the commutator subgroup of G; $\mathrm{Aut}(G)$ is the automorphism group of G; $\mathrm{Inn}(G)$ is the inner automorphism group of G; $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. All other notation is standard.

3. Non-harmonious Groups

In this section we give several classes of groups that are not harmonious. We begin with a necessary condition for groups to possess complete mappings.

Theorem 3.1 (Paige [9]). *Let G be a group and $g_1, g_2, \ldots, g_n$ be a harmonious sequence for G or $G^\#$. Then the product $g_1 g_2 \cdots g_n$ must be in the commutator subgroup of G.*

Corollary 3.2. *If G is a group of even order and has a cyclic Sylow 2-subgroup, then G and $G^\#$ are not harmonious.*

Proof. Since G has a cyclic Sylow 2-subgroup, G has a normal 2-complement N [4, p. 257]. Let $g_1, \ldots, g_n$ be the elements of G, and let xN, x^2N, $x^3N, \ldots, x^{2^m}N$ be the elements of G/N. Then $g_1 g_2 \cdots g_n N = (xN\, x^2N\, x^3N \cdots x^{2^m}N)^{|N|} = (x^{2^{(m-1)}}N)^{|N|} = x^{2^{(m-1)}}N$, since $|N|$ is odd. Thus $g_1 g_2 \cdots g_n \notin N$. However, G/N is Abelian, so $G' \subseteq N$. Hence $g_1 g_2 \cdots g_n \notin G'$. By Theorem 3.1, G and $G^\#$ are not harmonious. ∎

We remark that all groups of order $2k$ where k is odd satisfy the hypothesis of Corollary 3.2.

Theorem 3.3. *Elementary Abelian 2-groups are not harmonious.*

Proof. If a product $gh = e$, then $g = h^{-1} = h$. ∎

4. Direct Products

Theorem 4.1. *If G and H are harmonious and H has odd order, then $G \times H$ is harmonious.*

Proof. Let $g_1, g_2, \ldots, g_m$ and $h_1, h_2, \ldots, h_n$ be harmonious sequences for G and H, respectively. Observe that since H has odd order, the mapping $h \to h^2$ is an injection of H. It follows that a harmonious sequence for $G \times H$ is $(g_1, h_1), (g_2, h_1), \ldots, (g_m, h_1); (g_1, h_2), (g_2, h_2), \ldots, (g_m, h_2); \cdots; (g_1, h_n), (g_2, h_n), \ldots, (g_m, h_n)$. ∎

THEOREM 4.2. *$\mathbf{Z}_n$ is harmonious if and only if n is odd. $\mathbf{Z}_n^{\#}$ is harmonious if and only if n is odd and greater than* 3.

Proof. In both instances necessity follows from Corollary 3.2. Now suppose n is odd and write $n = 2k+1$. A harmonious sequence for $\mathbf{Z}_n$ is $0, 1, 2, \ldots, 2k$.

For $\mathbf{Z}_n^{\#}$, n odd and greater than 3, we consider two cases. A harmonious sequence for $\mathbf{Z}_{4k+1}^{\#}$ is

$$2k+2,\ 2k+4, \ldots, 4k,\ 2, 4, 6, \ldots, 2k;$$

$$2k-1,\ 2k-3, \ldots, 1,\ 4k-1,\ 4k-3, \ldots, 2k+1.$$

A harmonious sequence for $\mathbf{Z}_{4k+3}^{\#}$ is

$$2k+2,\ 2k+4, \ldots, 4k+2,\ 2, 4, 6, \ldots, 2k;$$

$$2k+1,\ 2k-1, \ldots, 1,\ 4k+1,\ 4k-1, \ldots, 2k+3. \quad \blacksquare$$

In [2], Chang, Hsu, and Rogers defined a graph with q edges to be *elegant* if it is possible to label the vertices with distinct integers from 0 to q in such a way that when each edge xy is assigned the integer $(x+y)$ modulo $(q+1)$, the resulting edge labels are $1, \ldots, q$. Theorem 9 of their paper gave a partial characterization of the cycles that are elegant. Our Theorem 4.1 completes the characterization. In particular, our harmonious labeling of $\mathbf{Z}_{4k+3}^{\#}$ gives an elegant labeling of the cycle with $4k+2$ vertices that Chang, Hsu, and Rogers did only when $4k+3$ is prime. Our harmonious labeling of $\mathbf{Z}_{4k+1}^{\#}$ also gives a new elegant labeling of the cycle with $4k$ vertices.

As immediate consequences of Theorems 4.1, 4.2, and the fundamental theorem of finite Abelian groups, we have the following.

COROLLARY 4.3. *All non-trivial Abelian groups of odd order are harmonious.*

COROLLARY 4.4. *If the Sylow 2-subgroup of a finite Abelian group is harmonious then the group is harmonious.*

LEMMA 4.5. *Suppose K is a harmonious group of odd order, and there are harmonious sequences for both H and $H^{\#}$ that begin and end with the same term. Then $(H \times K)^{\#}$ is harmonious.*

Proof. Let $e = k_1, \ldots, k_n$ be a harmonious sequence for K; $h_1, \ldots, h_m$ and $\bar{h}_1, \ldots, \bar{h}_{m-1}$ be harmonious sequences for H and $H^{\#}$, where $h_1 = \bar{h}_1$ and $h_m = \bar{h}_{m-1}$. Then a harmonious sequence for $(H \times K)^{\#}$ is $(\bar{h}_1, k_1)$, $(\bar{h}_2, k_1), \ldots, (\bar{h}_{m-1}, k_1)$; (h_1, k_2), $(h_2, k_2), \ldots, (h_m, k_2); \cdots$; $(h_1, k_n), \ldots, (h_m, k_n)$. $\blacksquare$

5. Harmonious Group Extensions

In this section we develop a technique for constructing harmonious mappings of group extensions. We then use the results of Section 4 to show that several classes of solvable groups are harmonious.

Lemma 5.1. *Let G be a group and H a normal subgroup such that $K = G/H$ is harmonious and $K_1, K_2, ..., K_n$ is a harmonious sequence of K. Suppose that there is a k in the coset $K_1 K_2 \cdots K_n$ and complete mappings $\phi_1, \phi_2, ..., \phi_n$ of H such that the map $h \to h^k \phi_1 \phi_2 \cdots \phi_n$ is an $|H|$-cycle of H. Then G is harmonious.*

Proof. For $1 \leqslant i \leqslant n$, pick $k_i \in K_i$ such that $k_1 k_2 k_3 \cdots k_n = k$, let σ_i: $H \to H$ be conjugation by k_i, and let σ: $H \to H$ be conjugation by k. Note that if ϕ is a complete map of H and θ is an automorphism of H, then ϕ^θ (i.e., $\theta^{-1}\phi\theta$) is a complete map of H. For $1 \leqslant i \leqslant n$ let $\bar{\phi}_i = \phi_i^{\sigma_n^{-1}\sigma_{n-1}^{-1}\cdots\sigma_i^{-1}}$. We will now construct a harmonious mapping of G. Define ψ: $G \to G$ as follows (addition of subscripts is done modulo n):

$$(k_i h)\psi = (h\bar{\phi}_{i+1})k_{i+1} = k_{i+1}((h\bar{\phi}_{i+1})\sigma_{i+1}) \qquad \text{for} \quad 1 \leqslant i \leqslant n, h \in H.$$

Since each $\bar{\phi}_i$ is a permutation of H, ψ is a permutation of G. To see that ψ is a complete mapping, suppose that $k_i \cdot h \cdot (k_i \cdot h)\psi = k_j \cdot h'(k_j \cdot h')\psi$. Then

$$k_i \cdot h \cdot h\bar{\phi}_{i+1} \cdot k_{i+1} = k_j \cdot h' \cdot h'\bar{\phi}_{j+1} \cdot k_{j+1}.$$

So, $K_i K_{i+1} = K_j K_{j+1}$ and $i = j$ (since $K_1, K_2, ..., K_n$ is a harmonious sequence). Then $h = h'$, since $\bar{\phi}_{i+1}$ is a complete mapping. Therefore ψ is a complete mapping. Let $\bar{\psi}$: $H \to H$ be defined by

$$k_n \cdot h\bar{\psi} = (k_n \cdot h)\psi^n.$$

Since $K_i\psi = K_{i+1}$, if $\bar{\psi}$ is an $|H|$-cycle then it follows that ψ is a $|G|$-cycle and G is harmonious. But $\bar{\psi} = \bar{\phi}_1\sigma_1 \bar{\phi}_2\sigma_2 \cdots \bar{\phi}_n\sigma_n = \sigma\phi_1\phi_2\phi_3 \cdots \phi_n$ which is by supposition an $|H|$-cycle. ∎

Lemma 5.2. *Let G be a group and H a normal subgroup of G such that $K = G/H$ is harmonious and $K_1, K_2, ..., K_n$ is a harmonious sequence of K. If there is a k in $K_1 K_2 \cdots K_n$ that centralizes H and there are complete mappings $\phi_1, \phi_2, ..., \phi_n$ of H such that $\phi_1\phi_2 \cdots \phi_n$ is an $|H|$-cycle, then G is harmonious.*

Proof. The map $h \to h^k\phi_1\phi_2 \cdots \phi_n$ is an $|H|$-cycle, so all of the conditions of Lemma 5.1 are satisfied. ∎

LEMMA 5.3. *Let G be a group and H a normal subgroup such that $K=G/H$ is harmonious and $K_1, K_2, ..., K_n$ is a harmonious sequence of K. Then there is a k in the coset $K_1K_2 \cdots K_n$ that centralizes H in the following circumstances*:

(i) *K is Abelian.*

(ii) Out(H) *is Abelian.*

(iii) $|\mathrm{Out}(H)'|$ *is relatively prime to* $|K'|$.

(iv) *Every element of K' contains a k which centralizes H.*

Proof. First note that each of conditions (i) and (ii) implies (iii), so we assume (iii) holds. Let θ be the canonical homomorphism from G to Aut(H). Since $H\theta=\mathrm{Inn}(H)$ we have the induced homomorphism $\tilde{\theta}$: $K \to \mathrm{Out}(H)$. Since $K'\tilde{\theta} \subseteq \mathrm{Out}(H)'$, by (iii) of Lemma 5.3 we have that $K'\tilde{\theta}$ is trivial. Suppose $\tilde{k} \in K_i \in K'$. Then $\tilde{k}\theta$ is an inner automorphism of H, so there is some $h_0 \in H$ such that $h^{h_0}=h^{\tilde{k}}$ for all $h \in H$. Then the element $\tilde{k}h_0^{-1} \in K_i$ centralizes H, so condition (iv) holds. Now suppose that condition (iv) holds. By Theorem 3.1 $K_1K_2 \cdots K_n \in K'$, so there is a $k \in K_1K_2 \cdots K_n$ which centralizes H. ∎

LEMMA 5.4. *Let G be a group and H a normal subgroup of G such that $K=G/H$ is harmonious, with harmonious sequence $K_1, K_2, ..., K_n$. If there is a k in the coset $K_1K_2 \cdots K_n$ that centralizes H, then each of the following implies G is harmonious*:

(i) *$|H|$ is odd and H is harmonious.*

(ii) *$|K|$ is odd and H is Abelian and harmonious.*

(iii) *H is harmonious and $|H|$ and $|K|$ are relatively prime.*

Proof. First observe that by Lemma 5.2, we need only show that there are complete mappings $\phi_1, \phi_2, ..., \phi_n$ of H such that $\phi_1\phi_2 \cdots \phi_n$ is an $|H|$-cycle.

Suppose (i) holds. Then, since $|H|$ is odd, the map $h \to h^2$ is bijective, and so the identity map I is a complete mapping of H. Let $\phi_1=\phi_2= \cdots = \phi_{n-1}=I$, and let ϕ_n be a harmonious mapping of H. Then the product $\phi_1\phi_2 \cdots \phi_n=\phi_n$ is an $|H|$-cycle.

Next, suppose that (ii) is satisfied. Let ϕ be a harmonious map of H. Since H is Abelian, ϕ^{-1} is also a harmonious map of H. Let $\phi_1=\phi_3= \phi_5= \cdots =\phi_n=\phi$; $\phi_2=\phi_4= \cdots =\phi_{n-1}=\phi^{-1}$. Then $\phi_1\phi_2 \cdots \phi_n=\phi$ is an $|H|$-cycle.

Finally, suppose that (iii) is satisfied. Then let ϕ be a harmonious map of H and let $\phi_1=\phi_2= \cdots =\phi_n=\phi$. Then $\phi_1\phi_2 \cdots \phi_n=\phi^n$ is an $|H|$-cycle (since n and $|H|$ are relatively prime). ∎

THEOREM 5.5. *If G has odd order, then G is harmonious.*

Proof. We use induction on $|G|$. If G is Abelian, we are done by Corollary 4.3. Otherwise, assume that every group of odd order smaller than $|G|$ is harmonious. Let $H=G'$. By the Feit-Thompson Theorem, H is a proper subgroup of G, so by the induction hypothesis both H and G/H are harmonious. Since G/H is Abelian, condition (i) of Lemma 5.3 and (i) of Lemma 5.4 are met and G is harmonious by Lemma 5.4. ∎

THEOREM 5.6. *If H is a normal subgroup of G of odd order and G/H is harmonious and Abelian, then G is harmonious.*

Proof. By Theorem 5.5, H is harmonious, so condition (i) of Lemma 5.3 and condition (i) of Lemma 5.4 are met, and therefore G is harmonious. ∎

THEOREM 5.7. *If the dihedral group D_n is harmonious, then for all odd m, D_{nm} is harmonious. Likewise, if the generalized quaternion group Q_n is harmonious, then for all odd m, Q_{nm} is harmonious.*

Proof. Assume D_n is harmonious and let $G=\langle a,b \mid a^{nm}=b^2=e, a^b=a^{-1}\rangle \cong D_{nm}$. Let H be the subgroup generated by a^n. Since $\mathrm{Out}(H)$ is Abelian, condition (ii) of Lemma 5.3 is satisfied. Also $|H|$ is odd, H is normal in G, and $K=G/H\cong D_n$. So the hypothesis and condition (i) of Lemma 5.4 are satisfied and G is harmonious.

The same argument works if we substitute Q_n and Q_{nm} for D_n and D_{nm}. ∎

Our next result provides an infinite family of dihedral groups that are harmonious.

THEOREM 5.8. *D_{2^n} is harmonious for $n \geqslant 2$.*

Proof. Let $m=2^{n-2}$ and consider the $4\times 2m$ matrix

$$C=\begin{pmatrix} ba^{4m-1} & ba^{4m-2} & ba^{4m-3} & \cdots & ba^{3m} & ba^{3m-1} & ba^{3m-2} & \cdots & ba^{2m} \\ b & ba & ba^2 & \cdots & ba^{m-1} & ba^m & ba^{m+1} & \cdots & ba^{2m-1} \\ a^{2m-2} & a^{2m-4} & a^{2m-6} & \cdots & e & a^{2m-1} & a^{2m-3} & \cdots & a \\ a^{4m-2} & a^{4m-4} & a^{4m-6} & \cdots & a^{2m} & a^{4m-1} & a^{4m-3} & \cdots & a^{2m+1} \end{pmatrix}.$$

Let $\{n_i\}_{i=1}^{2m}=\{m, 1, m+2, 3, m+4, 5, ..., 2m, m+1, 2, m+3, 4, ..., m-2, 2m-1\}$, let $\{h_k\}_{k=1}^{8m}=\{c_{1,n_1}, c_{2,n_1}, c_{3,n_1}, c_{4,n_1}, c_{1,n_2}, c_{2,n_2}, ..., c_{3,n_{2m}}, c_{4,n_{2m}}\}$, and let C' be the $4\times 2m$ matrix with $c'_{ij}=h_k h_{k+1}$, where $c_{ij}=h_k$. Then we have

$$C'=\begin{pmatrix} a & a^3 & \cdots & a^{2m-3} & a^{2m-1} & a^{2m+1} & a^{2m+3} & \cdots & a^{4m-1} \\ ba^{2m-2} & ba^{2m-3} & \cdots & ba^m & ba^{m-1} & ba^{3m-1} & ba^{3m-2} & \cdots & ba^{2m} \\ a^{2m-4} & a^{2m-8} & \cdots & a^{2m+4} & a^{2m} & a^{2m-2} & a^{2m-6} & \cdots & a^{2m+2} \\ ba^{3m} & ba^{3m+1} & \cdots & ba^{4m-2} & ba^{2m-1} & ba^{4m-1} & b & \cdots & ba^{m-2} \end{pmatrix}.$$

and C' contains every element of D_{2^n} exactly once. Since C' was constructed to contain every product of the form $h_k h_{k+1}$ exactly once, $\{h_i\}$ is a harmonious sequence. ∎

COROLLARY 5.9. *If 4 divides n, D_n is harmonious.*

Proof. The result follows from Theorems 5.8 and 5.7. ∎

Theorems 5.8 and 5.7 together show that the dihedral groups $D_{2^n m}$ are harmonious when $n \geqslant 2$ and m is odd. Since D_6 is harmonious, the groups D_{6m} are harmonious for m odd. On the other hand, by Corollary 3.2, D_n is not harmonious when n is odd. Also, no harmonious sequence exists for Q_2 (by computer search), but one does exist for Q_4. For Q_n, where n is odd, no harmonious sequence exists by Corollary 3.2.

6. HARMONIOUS ABELIAN GROUPS

In this section we completely characterize the finite Abelian groups that are harmonious.

LEMMA 6.1. *The group $\mathbf{Z}_m \times \mathbf{Z}_2$ is harmonious if and only if m is even and greater than 2.*

Proof. Theorem 3.3 shows that $\mathbf{Z}_2 \times \mathbf{Z}_2$ is not harmonious. That $\mathbf{Z}_m \times \mathbf{Z}_2$ is not harmonious when m is odd follows from Corollary 3.2. For m even and greater than 2 we consider two cases.

Case 1. $\mathbf{Z}_{4n} \times \mathbf{Z}_2 \cong \langle a, b \mid a^{4n} = b^2 = e,\ ab = ba \rangle$.

A harmonious sequence is

$$e,\ a,\ a^2, ..., a^{2n}, ba^{2n}, ba^{2n-2},\ ba^{2n-4}, ..., b, a^{-1}, ba^{-1}, a^{-2},$$
$$ba^{-2}, a^{-3}, ..., a^{2n+1}, ba^{2n+1}, ba^{2n-1}, ba^{2n-3}, ba^{2n-5}, ..., ba.$$

(The products of consecutive terms are (in order)

$$a,\ a^3,\ a^5, ...,\ a^{-1},\ b,\ a^{-2},\ a^{-6},\ a^{-10}, ...,\ a^2,\ ba^{-1},\ ba^{-2},$$
$$ba^{-3}, ..., ba^2,\ e,\ a^{-4},\ a^{-8}, ...,\ a^4,\ ba.)$$

Case 2. $\mathbf{Z}_{4n+2} \times \mathbf{Z}_2 \cong \langle a, b \mid a^{4n+2} = b^2 = e, ab = ba \rangle$.

A harmonious sequence is

$$e, a^2, a, a^3, a^4, a^5, a^6, ..., a^{2n+1}, ba^{2n}, ba^{2n-2}, ba^{2n-4}, ..., ba^2, ba^3, ba^5, ba^7, ..., ba^{2n+1}, a^{2n+2}, ba^{2n+2}, a^{2n+3}, ba^{2n+3}, a^{2n+4}, ..., a^{-1}, ba^{-1}, ba, b.$$

(For $n > 1$, the products of consecutive terms are (in order)

$$a^2, a^3, a^4, a^7, a^9, a^{11}, a^{13}, ..., a^{-1}, ba^{-1}, a^{-4}, a^{-8}, a^{-12}, ..., a^6, a^5, a^8, a^{12}, a^{16}, ..., a^{4n}, ba, ba^2, ba^3, ba^4, ..., ba^{4n}, e, a, b.) \quad \blacksquare$$

LEMMA 6.2. *If G is Abelian, and G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by a harmonious group H, then G is harmonious.*

Proof. Let $H_1, ..., H_n$ be a harmonious labeling of the cosets of $\mathbf{Z}_2 \times \mathbf{Z}_2$ and choose an element h_i from each H_i. Then each $g \in G$ can be uniquely expressed as kh_i and as $k'h_{i'}h_{i'+1}$, where k and k' belong to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ subgroup of G.

Let a and b generate the subgroup $\mathbf{Z}_2 \times \mathbf{Z}_2$ of G.

Case 1. $|H| \equiv 0 \pmod 3$.

A harmonious listing is

$$eh_1, eh_2, ..., eh_n; bh_{n-1}, abh_{n-2}, ah_{n-3}, ..., bh_{n-(3k+1)}, abh_{n-(3k+2)}, ah_{n-(3k+3)}, ..., abh_1, ah_n; ah_1, abh_2, bh_3, ..., ah_{3k+1}, abh_{3k+2}, bh_{3k+3}, ..., bh_n; ah_{n-1}, bh_{n-2}, abh_{n-3}, ..., ah_{n-(3k+1)}, bh_{n-(3k+2)}, abh_{n-(3k+3)}, ..., bh_1, abh_n.$$

The products of adjacent terms are

$$eh_1h_2, eh_2h_3, ..., eh_{n-1}h_n; bh_{n-1}h_n; ah_{n-2}h_{n-1}, bh_{n-3}h_{n-2}, abh_{n-4}h_{n-3}, ..., ah_{n-(3k+2)}h_{n-(3k+1)}, bh_{n-(3k+3)}h_{n-(3k+2)}, abh_{n-(3k+4)}h_{n-(3k+3)}, ..., ah_1h_2, bh_nh_1; eh_nh_1; bh_1h_2, ah_2h_3, abh_3h_4, ..., bh_{3k+1}h_{3k+2}, ah_{3k+2}h_{3k+3}, abh_{3k+3}h_{3k+4}, ..., ah_{n-1}h_n; abh_{n-1}h_n; abh_{n-2}h_{n-1}, ah_{n-3}, h_{n-2}, bh_{n-4}, b_{n-3}, ..., abh_{n-(3k+2)}h_{n-(3k+1)}, ah_{n-(3k+3)}h_{n-(3k+2)}, bh_{n-(3k+4)}h_{n-(3k+3)}, ..., abh_1h_2, ah_nh_1, abh_nh_1.$$

Case 2. $|H| \equiv 1 \pmod 3$.

A harmonious listing is

$$eh_1,\ eh_2, \ldots,\ eh_n;\ bh_{n-1},\ ah_{n-2},\ abh_{n-3}, \ldots,\ bh_{n-(3k+1)},\ ah_{n-(3k+2)},\ abh_{n-(3k+3)}, \ldots,\ abh_1;\ ah_n;\ ah_1,\ bh_2,\ abh_3,\ ah_4, \ldots,\ bh_{3k+2},\ abh_{3k+3},\ ah_{3k+4}, \ldots,\ abh_{n-1};\ bh_n,\ ah_{n-1},\ abh_{n-2},\ bh_{n-3}, \ldots,\ ah_{n-(3k+1)},\ abh_{n-(3k+2)},\ bh_{n-(3k+3)}, \ldots,\ bh_1;\ abh_n.$$

The products of adjacent terms are

$$eh_1h_2,\ eh_2h_3, \ldots,\ eh_{n-1}h_n;\ bh_{n-1}h_n;\ abh_{n-2}h_{n-1},\ bh_{n-3}h_{n-2},\ ah_{n-4}h_{n-3}, \ldots,\ abh_{n-(3k+2)}h_{n-(3k+1)},\ bh_{n-(3k+3)}h_{n-(3k+2)},\ ah_{n-(3k+4)}h_{n-(3k+3)}, \ldots,\ bh_1h_2;\ bh_nh_1;\ eh_nh_1;\ abh_1h_2,\ ah_2h_3,\ bh_3h_4, \ldots,\ abh_{3k+1}h_{3k+2},\ ah_{3k+2}h_{3k+3},\ bh_{3k+3}h_{3k+4}, \ldots,\ ah_{n-2}h_{n-1};\ ah_{n-1}h_n,\ abh_{n-1}h_n,\ bh_{n-2}h_{n-1},\ ah_{n-3}h_{n-2}, \ldots,\ abh_{n-(3k+1)}h_{n-3k},\ bh_{n-(3k+2)}h_{n-(3k+1)},\ ah_{n-(3k+3)}h_{n-(3k+2)}, \ldots,\ ah_1h_2;\ ah_nh_1,\ abh_nh_1.$$

Case 3. $|H| \equiv 2 \pmod 3$.

A harmonious listing for $|H| > 5$ is

$$eh_1,\ eh_2, \ldots,\ eh_n;\ bh_{n-1},\ ah_{n-2},\ abh_{n-3}, \ldots,\ bh_{n-(3k+1)},\ ah_{n-(3k+2)},\ abh_{n-(3k+3)}, \ldots,\ ah_3;\ bh_2,\ abh_1,\ ah_n,\ ah_1,\ abh_2;\ bh_3,\ abh_4,\ ah_5, \ldots,\ bh_{3k},\ abh_{3k+1},\ ah_{3k+2}, \ldots,\ abh_{n-1};\ bh_n,\ ah_{n-1},\ abh_{n-2},\ bh_{n-3}, \ldots,\ abh_3;\ ah_2,\ bh_1,\ abh_n.$$

The products of adjacent terms are

$$eh_1h_2,\ eh_2h_3, \ldots,\ eh_{n-1}h_n;\ bh_{n-1}h_n;\ abh_{n-2}h_{n-1},\ bh_{n-3}h_{n-2},\ ah_{n-4}h_{n-3}, \ldots,\ abh_{n-(3k+2)}h_{n-(3k+1)},\ bh_{n-(3k+3)}h_{n-(3k+2)},\ ah_{n-(3k+4)}h_{n-(3k+3)}, \ldots,\ abh_3h_4;\ abh_2h_3,\ ah_1h_2,\ bh_nh_1,\ eh_nh_1,\ bh_1h_2,\ ah_2h_3;\ ah_3h_4,\ bh_4h_5,\ abh_5h_6, \ldots,\ ah_{3k}h_{3k+1},\ bh_{3k+1}h_{3k+2},\ abh_{3k+2}h_{3k+3}, \ldots,\ ah_{n-2}h_{n-1};\ ah_{n-1}h_n;\ abh_{n-1}h_n,\ bh_{n-2}h_{n-1},\ ah_{n-3}h_{n-2}, \ldots,\ abh_{n-(3k+1)}h_{n-3k},\ bh_{n-(3k+2)}h_{n-(3k+1)},\ ah_{n-(3k+3)}h_{n-(3k+2)}, \ldots,\ bh_3h_4,\ bh_2h_3;\ abh_1h_2,\ ah_nh_1,\ abh_nh_1.$$

If $|H| = 5$, $G \cong \mathbf{Z}_2 \times \mathbf{Z}_{10}$, which is harmonious by Lemma 6.1. ∎

LEMMA 6.3. *If G is an Abelian 2-group and G is neither cyclic nor elementary, then G is harmonious.*

Proof. The proof will proceed by induction on n, where $|G| = 2^n$. There is no such G for $n < 3$. For $n = 3$ or 4, $G \cong \mathbf{Z}_2 \times \mathbf{Z}_4$, $G \cong \mathbf{Z}_2 \times \mathbf{Z}_8$, $G \cong \mathbf{Z}_4 \times \mathbf{Z}_4$, or $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4$. Harmonious sequences for the first two cases are given in the proof of Lemma 6.1. A harmonious sequence for $\mathbf{Z}_4 \times \mathbf{Z}_4 \cong \langle a, b \mid a^4 = b^4 = e, ab = ba \rangle$ is a, a^3, a^3b^3, ab^3, a^2b^2, b^2, b^3, a^2b^3, a^3b, ab^2, e, ab, a^2b, a^3b^2, b, a^2. A harmonious sequence for $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4 \cong \langle a, b, c \mid a^2 = b^2 = c^4 = e, ab = ba, ac = ca, bc = cb \rangle$ is ac^3, ab, bc, c^3, abc^2, a, b, bc^2, abc, abc^3, e, c, ac, ac^2, c^2, bc^3.

For $n > 4$, by the induction hypothesis all non-cyclic, non-elementary Abelian 2-groups of order 2^{n-2} are harmonious. Since G is non-cyclic, it must be an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$. Assume G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by the cyclic group $\mathbf{Z}_{2^{n-2}}$. Then either $G \cong \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-1}}$, which is harmonious by Lemma 6.1, or $G \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_{2^{n-2}}$, which is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $\mathbf{Z}_2 \times \mathbf{Z}_{2^{n-3}}$, also harmonious by Lemma 6.1, in which case G is harmonious by Lemma 6.2.

Assume now that G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by the elementary group $(\mathbf{Z}_2)^{n-2}$. Since G is not elementary, $G \cong (\mathbf{Z}_2)^{n-2} \times \mathbf{Z}_4$ or $G \cong (\mathbf{Z}_2)^{n-4} \times \mathbf{Z}_4 \times \mathbf{Z}_4$ and these are extensions of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by $(\mathbf{Z}_2)^{n-4} \times \mathbf{Z}_4$ and $(\mathbf{Z}_2)^{n-5} \times \mathbf{Z}_2 \times \mathbf{Z}_4$, respectively, which are both harmonious by the induction hypothesis. Again, G is harmonious by Lemma 6.2.

Finally, if G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by a non-cyclic, non-elementary 2-group, G is harmonious by the induction hypothesis and Lemma 6.2. ∎

LEMMA 6.4. *If an Abelian group G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ by a harmonious group H, then G is harmonious.*

Proof. Since H is harmonious, there exists a harmonious sequence $K_1, ..., K_n$ of the cosets of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ in G. Choose $k_1, ..., k_n$ such that $k_i \in K_i$. Let a, b, c generate the $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ subgroup of G, and let ϕ be the permutation $(c, b, a, bc, ab, abc, ac)$ on $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Then a harmonious sequence is

$$\begin{aligned}
&k_1, k_2, ..., k_n, \phi^{n-2}(c)k_{n-1}, \phi^{n-3}(c)k_{n-2}, ..., \phi(c)k_2, ck_1,\\
&g_1k_n, bk_1, \phi(b)k_2, ..., \phi^{n-2}(b)k_{n-1}, g_2k_n,\\
&\phi^{n-2}(a)k_{n-1}, ..., \phi(a)k_2, ak_1, g_3k_n, bck_1, \phi(bc)k_2, ...,\\
&\phi^{n-2}(bc)k_{n-1}, g_4k_n, \phi^{n-2}(ab)k_{n-1}, ..., \phi(ab)k_2, abk_1,\\
&g_5k_n, abck_1, \phi(abc)k_2, ..., \phi^{n-2}(abc)k_{n-1}, g_6k_n,\\
&\phi^{n-2}(ac)k_{n-1}, ..., \phi(ac)k_{n-1}, ..., \phi(ac)k_2, ack_1, g_7k_n,
\end{aligned}$$

where the values of g_i are read from the following table:

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
$\|H\| \equiv 0 \pmod 7$	c	a	b	ab	bc	ac	abc
$\|H\| \equiv 1 \pmod 7$	ab	ac	c	b	abc	a	bc
$\|H\| \equiv 2 \pmod 7$	abc	ab	bc	ac	a	b	c
$\|H\| \equiv 3 \pmod 7$	abc	ac	bc	b	a	ab	c
$\|H\| \equiv 4 \pmod 7$	b	c	ac	ab	bc	a	abc
$\|H\| \equiv 5 \pmod 7$	b	c	ac	a	bc	ab	abc
$\|H\| \equiv 6 \pmod 7$	c	a	b	ac	bc	ab	abc

Verification is straight forward with the observation that ϕ is a complete mapping. ∎

LEMMA 6.5. *If G is Abelian with an elementary noncyclic Sylow 2-subgroup, and G is not a 2-group, then G is harmonious.*

Proof. Obviously, $G \cong (\mathbf{Z}_2)^n \times H$, where $n \geqslant 2$, H is Abelian, and $|H|$ is odd. Also, H is harmonious by Corollary 4.3. If n is even, $G \cong (\mathbf{Z}_2 \times \mathbf{Z}_2)^{n/2} \times H$, so G is harmonious by $n/2$ applications of Lemma 6.2. If n is odd, then $G \cong (\mathbf{Z}_2 \times \mathbf{Z}_2)^{(n-3)/2} \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2) \times H$, so G is harmonious by Lemma 6.4 and $(n-3)/2$ applications of Lemma 6.2. ∎

We now prove the main result of this paper.

THEOREM 6.6. *If G is a finite, non-trivial Abelian group, then G is harmonious if and only if G has a non-cyclic or trivial Sylow 2-subgroup and G is not an elementary 2-group.*

Proof. ($\Leftarrow$) If G is a 2-group the result follows from Lemma 6.3. Otherwise $G \cong H \times K$, where H is the Sylow 2-subgroup of G and $|K|$ is odd. If H is trivial, then G is harmonious by Corollary 4.3. If H is non-elementary, then H is harmonious by Lemma 6.3, since H is non-cyclic by hypothesis, and so G is harmonious by Theorem 4.1. If H is elementary, G is harmonious by Lemma 6.5.

($\Rightarrow$) This follows from Corollary 3.2 and Theorem 3.3.

7. $G^{\#}$ HARMONIOUS ABELIAN GROUPS

DEFINITION 7.1. If G and $G^{\#}$ have harmonious sequences $h_1, ..., h_n$ and $\bar{h}_1, ..., \bar{h}_{n-1}$, respectively, such that $h_1 = \bar{h}_1$ and $h_n = \bar{h}_{n-1}$, we say the sequences are *harmoniously-matched*, and G is a *harmoniously-matched group*.

Remark 7.2. If $h_1, ..., h_n$ is a harmonious sequence for an Abelian group, then

(i) $h_i, h_{i+1}, ..., h_n, h_1, ..., h_{i-1}$ is a harmonious sequence, $1 \leqslant i \leqslant n$, and

(ii) $h_n, h_{n-1}, ..., h_2, h_1$ is a harmonious sequence.

THEOREM 7.3. *$\mathbf{Z}_n$ is harmoniously-matched if and only if n is odd and $n \geqslant 5$.*

Proof. If n is even, $\mathbf{Z}_n^{\#}$ is not harmonious by Theorem 3.1. Also, $\mathbf{Z}_3^{\#}$ is clearly not harmonious. Now suppose $n = 4k+1$, $k \geqslant 1$. Applying Remark 7.2 to the sequence in Theorem 4.2 gives the following harmonious sequence for $\mathbf{Z}_n$:

$$2k,\ 2k+1, ..., 4k,\ 0,\ 1,\ 2, ..., 2k-1.$$

A harmoniously-matched sequence for $\mathbf{Z}_n^{\#}$ is

$$2k,\ 2k-2, ...,\ 4,\ 2,\ 4k,\ 4k-2, ...,\ 2k+4,\ 2k+2,\ 2k+1,$$
$$2k+3, ...,\ 4k-3,\ 4k-1,\ 1,\ 3, ...,\ 2k-3,\ 2k-1.$$

Finally, suppose $n = 4k+3$, $k \geqslant 1$. Again applying Remark 7.2 to Theorem 4.2, $\mathbf{Z}_n$ has the harmonious sequence

$$2k+1,\ 2k+2, ...,\ 4k+2,\ 0,\ 1, ...,\ 2k-1,\ 2k.$$

A harmoniously-matched sequence for $\mathbf{Z}_n^{\#}$ is

$$2k+1,\ 2k-1, ...,\ 1,\ 4k+1,\ 4k-1, ...,\ 2k+3,\ 2k+2,$$
$$2k+4, ...,\ 4k+2,\ 2,\ 4,\ 6, ...,\ 2k. \blacksquare$$

THEOREM 7.4. *$\mathbf{Z}_m \times \mathbf{Z}_2$ is hamoniously-matched if m is even and greater than 2.*

Proof. A harmonious sequence for $(\mathbf{Z}_{4n} \times \mathbf{Z}_2)^{\#}$ is

$$a,\ a^2,\ a^3, ...,\ a^{2n-1},\ a^{2n},\ ba^{2n-1},\ ba^{2n-3},\ ba^{2n-5}, ...,\ ba,\ b,$$
$$ba^2,\ ba^4, ...,\ ba^{2n},\ a^{2n+1},\ ba^{2n+1},\ a^{2n+2},\ ba^{2n+2}, ...,\ a^{4n-1},$$
$$ba^{4n-1}.$$

(The products of consecutive terms are, in order,

$$a^3,\ a^5, ...,\ a^{4n-1},\ ba^{4n-1},\ a^{4n-4},\ a^{4n-8}, ...,\ a^4,\ a,\ a^2,\ a^6, ...,$$
$$a^{4n-2},\ ba,\ ba^2,\ ba^3, ...\ ba^{4n-3},\ ba^{4n-2},\ b.)$$

Applying Remark 7.2 to this sequence and the sequence given in Theorem 4.2., both $\mathbf{Z}_{4n} \times \mathbf{Z}_2$ and $(\mathbf{Z}_{4n} \times \mathbf{Z}_2)^{\#}$ have harmonious sequences that begin with a^2 and end with a.

A harmonious sequence for $(\mathbf{Z}_{4n+2} \times \mathbf{Z}_2)^{\#}$ is

$$a,\ a^2,\ a^3, ..., a^{2n},\ a^{2n+1},\ ba^{2n},\ ba^{2n-2},\ ba^{2n-4}, ..., ba^2,\ b,\ ba,\ ba^3, ..., ba^{2n-1},\ ba^{2n+1},\ a^{2n+2},\ ba^{2n+2},\ a^{2n+3},\ ba^{2n+3}, ..., a^{4n+1},\ ba^{4n+1}.$$

(The products of consecutive terms are, in order,

$$a^3,\ a^5, ..., a^{4n+1},\ ba^{4n+1},\ a^{4n-2},\ a^{4n-6}, ..., a^6,\ a^2,\ a,\ a^4,\ a^8, ..., a^{4n},\ ba,\ ba^2,\ ba^3, ..., ba^{4n},\ b.)$$

Applying Remark 7.2 to this sequence and the sequence given in Lemma 6.1, both $\mathbf{Z}_{4n+2} \times \mathbf{Z}_2$ and $(\mathbf{Z}_{4n+2} \times \mathbf{Z}_2)^{\#}$ have harmonious sequences that begin with ba^{2n} and end with a^{2n+1}. ∎

LEMMA 7.5. *The group $\mathbf{Z}_3 \times \mathbf{Z}_3$ is harmoniously-matched.*

Proof. A harmonious sequence for $\mathbf{Z}_3 \times \mathbf{Z}_3$ is

$$ab,\ a^2b,\ b^2,\ ab^2,\ a^2b^2,\ e,\ a,\ a^2,\ b.$$

A harmonious sequence for $(\mathbf{Z}_3 \times \mathbf{Z}_3)^{\#}$ is

$$ab,\ a^2,\ a^2b,\ b^2,\ a^2b^2,\ a,\ ab^2,\ b. \quad ∎$$

THEOREM 7.6. *Suppose K is a group of odd order and H is a harmoniously-matched group. Then $(H \times K)^{\#}$ is harmoniously-matched.*

Proof. By Theorem 5.5, K is harmonious. Then applying Remark 7.2 to the sequence given in the proof of Lemma 4.5 we may obtain harmoniously-matched sequences beginning with (h_2, k_2) and ending with (h_1, k_2). ∎

COROLLARY 7.7. *All Abelian groups of odd order are harmoniously-matched except $\mathbf{Z}_3$.*

Proof. For elementary Abelian 3-groups, the result follows from Theorem 7.5 and Theorem 7.6. Otherwise, the result follows from Theorem 7.3 and Theorem 7.6. ∎

LEMMA 7.8. *If an Abelian group G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2$ by a harmoniously-matched group H, then G is harmoniously-matched.*

Proof. If $H \cong \mathbf{Z}_5$, then the result follows from Theorem 7.4. Assume then that $H \not\cong \mathbf{Z}_5$. Let $n = |G|/4$, and let $h_1, ..., h_n$ and $\bar{h}_1, ..., \bar{h}_{n-1}$ be harmoniously-matched sequences of H and $H^{\#}$. Let $g_1, ..., g_{4n}$ be the harmonious sequence of G constructed from $h_1, \ldots, h_n$ in the proof of Lemma 6.2. Then the sequence $\tilde{g}$, where

$$\tilde{g}_i = e\tilde{h}_i, \qquad 1 \leqslant i \leqslant n-1$$

$$\tilde{g}_i = g_{i+1}, \qquad n \leqslant i \leqslant 4n-1$$

is harmonious, and since $\tilde{g}_{n-1} = g_n$ and $\tilde{g}_n = g_{n+1}$, G is harmoniously-matched by Remark 7.2. ∎

THEOREM 7.9. *If an Abelian 2-group G is neither cyclic nor elementary, then G is harmoniously-matched.*

Proof. The proof is analogous to the proof of Lemma 6.3, using the corresponding theorems for harmoniously-matched groups instead of the theorems for harmonious groups. The group $(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_4)^{\#} \cong \langle a, b, c \mid a^2 = b^2 = c^4 = e,\ ab = ba,\ ac = ca,\ bc = cb \rangle$ has the harmonious sequence

$$ac^3,\ c^3,\ ab,\ b,\ bc,\ c,\ a,\ ac^2,\ abc^3,\ c^2,\ bc^2,\ abc,\ abc^2,\ ac,\ bc^3$$

which is harmoniously-matched with the sequence in the proof of Lemma 6.3. ∎

LEMMA 7.10. *If an Abelian group G is an extension of $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ by a harmoniously-matched group, then G is harmoniously-matched.*

Proof. The proof is analogous to the proof of Lemma 7.8, with the harmonious sequence constructed in Lemma 6.4 used in place of that constructed in Lemma 6.2. ∎

THEOREM 7.11. *If G is Abelian and has an elementary, non-cyclic Sylow 2-subgroup, and G is not a 2-group, then G is harmoniously-matched.*

Proof. $G \cong (\mathbf{Z}_2)^n \times H$, $|H|$ odd. If H is not isomorphic to $\mathbf{Z}_3$ or $n > 3$, then the proof is analogous to the proof of Lemma 6.5. Otherwise, G is either $(\mathbf{Z}_2)^2 \times \mathbf{Z}_3 \cong \mathbf{Z}_2 \times \mathbf{Z}_6$, which is done in Theorem 7.4, or $(\mathbf{Z}_2)^3 \times \mathbf{Z}_3 \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_6 \cong \langle a, b,\ c \mid a^2 = b^2 = c^6 = e,\ ab = ba,\ ac = ca,\ bc = cb \rangle$, which has the harmonious sequences

$$ab,\ c,\ ac^5,\ c^3,\ c^2,\ c^4,\ e,\ bc^4,\ c^5,\ abc^3,\ bc^2,\ ac^4,\ ac^3,\ bc,\ ac^2,$$
$$bc^3,\ bc^5,\ abc^4,\ b,\ abc,\ abc^2,\ a,\ abc^5,\ ac$$

and

$$ab,\ b,\ ac^5,\ bc,\ c,\ c^2,\ bc^2,\ abc^5,\ bc^5,\ ac^3,\ abc^2,\ abc^3,\ abc,\ ac^2,$$
$$a,\ bc^4,\ bc^3,\ c^3,\ abc^4,\ c^5,\ ac^4,\ c^4,\ ac.$$

The proof for $(\mathbf{Z}_2)^n \times \mathbf{Z}_3$, $n > 3$, can proceed as in the general case. ∎

COROLLARY 7.12. *If an Abelian group G has either a non-cyclic or trivial Sylow 2-subgroup, then $G^{\#}$ is harmonious, unless $G \cong \mathbf{Z}_3$.*

Proof. Follows from Corollary 7.7 and Theorems 7.9, 7.6, and 7.11. ∎

ACKNOWLEDGMENTS

The authors are grateful to David Moulton for contributing some results. The authors were supported by the National Science Foundation (grant number DMS 8709428) and the National Security Agency (grant number MDA 904-88-H-2027). The work was done at the University of Minnesota, Duluth.

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