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ON THE K-THEORY OF LOCAL FIELDS

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Introduction

In this paper we study the higher Quillen-K-groups of algebraically closed fields, local fields and real numbers. When k is an algebraically closed field we show that $K_i(k)$ is divisible, its torsion being zero when i is even, and equal to $W(n)$ when $i=2n-1$ (the nth Tate twist of the group W of roots of unity in k^*). This conjecture of Lichtenbaum was proved in [18] when k has positive characteristic, and we showed there that the case of one field of characteristic zero implies it for all fields. Here we prove the conjecture for the complex numbers (4.9), thus solving the general case. We also get a proof in positive characteristic (3.13) which does not use [13].

A crucial tool is the following result of O. Gabber [4], which extends the main idea of [18] and was also proved by H. Gillet and R. Thomason ("The K-theory of strict Hensel local rings and a Theorem of Suslin", these proceedings): when X is a smooth variety over a field $F, m\geq 1$ an integer invertible in $F, P:\mathrm{Spec}(F)\to X$ a rational point of X, and X_p^{h} the henselization of X at P, the natural morphism $K_*(F;\mathbb{Z}/m)\to K_*(X_p^{\mathrm{h}};\mathbb{Z}/m)$ between K-groups with coefficients is bijective.

When R is a henselian ring with valuation v of height one and residue field F, we show that, if $m\geq 1$ is invertible in F, the natural map $K_*(R;\mathbb{Z}/m)\to K_*(F;\mathbb{Z}/m)$ is bijective (Corollary 3.9; for a more general result cf. Theorem 3.6). The case of real numbers is studied in Theorem 4.9.

We work below mostly with K-theory with finite coefficients. We refer the reader to [12] for the details concerning homotopy groups with coefficients. Recall that for any space X one can define $\pi_i(X,\mathbb{Z}/m)$ for $i\geq 2$, and usually $\pi_2(X,\mathbb{Z}/m)$ is only a pointed set. However, if X is an H-space, the H-structure on X induces a group structure on $\pi_2(X,\mathbb{Z}/m)$. In this case one can also define $\pi_1(X,\mathbb{Z}/m)$ by the formula $\pi_1(X,\mathbb{Z}/m)=\pi_1(X)/m$. Finally, for any exact category $\mathscr{P}$, K-groups of $\mathscr{P}$ with coefficients are defined by the formula

$$K_i(\mathscr{P}, \mathbb{Z}/m) = \pi_{i+1}(BQ\mathscr{P}, \mathbb{Z}/m).$$

If $\mathscr{P}$ is the category $\mathscr{P}(A)$ of projective modules over the associative ring A, then $\Omega BQ\mathscr{P}$ coincides with $B\mathrm{GL}(A)^+$ (see [5]) and hence $K_i(A, \mathbb{Z}/m)$ also coincides with $\pi_i(BGL(A)^+, \mathbb{Z}/m)$ for $i \geq 2$.

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1. Auxiliary results

Lemma 1.1. *Suppose that A is a uniquely m-divisible Abelian group and P is a trivial A-module such that, for any $p \in P$, $m^N p = 0$ when N is sufficiently large. Then $H_i(A, P) = 0$ for $i \geq 1$.*

Proof. Denote by $\mathbb{Z}_m$ the localization of $\mathbb{Z}$ with respect to the multiplicative system $(1, m, m^2, \ldots)$. Since homology commutes with direct limits both in A and P we may suppose that A is a finitely generated $\mathbb{Z}_m$-module and P is a finitely generated $\mathbb{Z}$-module. In this case $m^N P = 0$ for N sufficiently large and, replacing m by m^N, we may suppose in addition that $mP = 0$. Suppose first that A is finite, consisting of n elements, then n is prime to m and the groups $H_i(A, P)$ $(i \geq 1)$ are zero, being killed both by m and n. Suppose now that $A = \mathbb{Z}_m$. Since

$$\mathbb{Z}_m = \varinjlim \left(\mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \cdots \right)$$

we see that $H_i(A, P) = \varinjlim H_i(\mathbb{Z}, P)$ is zero for $i \geq 2$. Furthermore $H_1(A, P) = H_1(A, \mathbb{Z}) \otimes P = A \otimes P$ is also zero. If $A = A_1 \oplus A_2$ and our statement is true both for A_1 and A_2, then it is also true for A as one sees immediately from the Hochschild–Serre spectral sequence $H_i(A_1, H_j(A_2, P)) \Rightarrow H_{i+j}(A, P)$. The general case follows from these remarks since every finitely generated $\mathbb{Z}_m$-module is a direct sum of a finite $\mathbb{Z}_m$-module and a free $\mathbb{Z}_m$-module.

Corollary 1.2. *Suppose that $\phi : A \to B$ is a homomorphism of Abelian groups such that both* Ker ϕ *and* Coker ϕ *are uniquely m-divisible. Then for any P as above the induced homomorphism $H_*(A, P) \to H_*(B, P)$ is bijective.*

Proof. It is sufficient to treat separately the cases when ϕ is surjective and when ϕ is injective. In each case the statement follows immediately from Lemma 1.1 and the Hochschild–Serre spectral sequence.

The following is a version of Ruth Charney's theorem [3].

Proposition 1.3. *Supose that I is a two-sided ideal in a ring R and m is an integer invertible in R/I. Then the natural action (by conjugation) of the group* $\mathrm{GL}(R)$ *on* $H_*(\mathrm{GL}(R,I),\mathbb{Z}/m)$ *is trivial.*

Proof. Denote $\mathrm{GL}_n(R,I)$ by Γ_n, $\mathrm{GL}(R,I)$ by Γ, and denote by Γ_n', Γ_n'' the following groups:

$$\Gamma_n'=\begin{pmatrix}\Gamma_n & I^n\\ 0 & 1\end{pmatrix},\qquad \Gamma_n''=\begin{pmatrix}\Gamma_n & R^n\\ 0 & 1\end{pmatrix}.$$

Consider the Hochschild–Serre spectral sequences corresponding to the group extensions:

$$1\to I^n\to\Gamma_n'\to\Gamma_n\to 1,$$

$$1\to R^n\to\Gamma_n''\to\Gamma_n\to 1.$$

Since the Abelian group $(R/I)^n$ is uniquely m-divisible we know from Corollary 1.2 that $H_*(I^n,\mathbb{Z}/m)\xrightarrow{\sim}H_*(R^n,\mathbb{Z}/m)$ and the comparison of these spectral sequences shows that the imbedding $\Gamma_n'\hookrightarrow\Gamma_n''$, is a homology isomorphism with $\mathbb{Z}/m$ coefficients. The group Γ_n'' contains elementary matrices $e_{i,n+1}(r)$ $(1\le i\le n)$ and the above remark shows that the action of $e_{i,n+1}(r)$ on $H_*(\Gamma_n',\mathbb{Z}/m)$ is trivial. Since $\Gamma_n\subset\Gamma_n'\subset\Gamma_{n+1}$ we deduce that the action of $e_{i,n+1}(r)$ on the image of $H_*(\Gamma_n,\mathbb{Z}/m)$ in $H_*(\Gamma,\mathbb{Z}/m)$ is also trivial. The same argument shows that the action of $e_{n+1,i}(r)$ on this image is also trivial. Since the matrices $e_{i,n+1}(r)$ and $e_{n+1,i}(r)$ generate the subgroup $E_{n+1}(R)\subset E_n(R)$, we see that the action of $E_n(R)$ on $\mathrm{Im}(H_*(\Gamma_n,\mathbb{Z}/m)\to H_*(\Gamma,\mathbb{Z}/m))$ is trivial. Taking the limit over n we deduce that the action of $E(R)$ on $H_*(\Gamma,\mathbb{Z}/m)$ is trivial. Finally for any $\alpha\in\mathrm{GL}_n(R)$ both

$$\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\in E(R)$$

act trivially on $\mathrm{Im}(H_*(\Gamma_n,\mathbb{Z}/m)\to H_*(\Gamma,\mathbb{Z}/m))$ and hence α also acts trivially on this image. Taking once again the limit over n, we get our assertion.

Lemma 1.4. *Suppose that X is a connected H-space and $p:X'\to X$ is a connected covering space of X with Galois group G (which is necessarily Abelian). Suppose further that G is uniquely m-divisible. Then the induced homomorphisms $H_*(X',\mathbb{Z}/m)\to H_*(X,\mathbb{Z}/m)$ and $\pi_*(X',\mathbb{Z}/m)\to\pi_*(X,\mathbb{Z}/m)$ are bijective.*

Proof. For any covering space $X'\to X$ the homomorphisms $\pi_i(X',\mathbb{Z}/m)\to\pi_i(X,\mathbb{Z}/m)$ are bijective for $i\ge 3$. Furthermore we have an exact sequence

$$0\to\pi_1(X')\to\pi_1(X)\to G\to 0$$

and since G is uniquely m-divisible we deduce that ${}_m\pi_1(X')\xrightarrow{\sim}{}_m\pi_1(X)$ and

$\pi_1(X')/m \xrightarrow{\sim} \pi_1(X)/m$. The universal coefficient theorem for homotopy groups (see [12]) shows immediately that $\pi_2(X', \mathbb{Z}/m) \xrightarrow{\sim} \pi_2(X, \mathbb{Z}/m)$ and $\pi_1(X', \mathbb{Z}/m) \xrightarrow{\sim} \pi_1(X, \mathbb{Z}/m)$.

It is well known and easy to prove [compare [16, p. 383]) that the action of G on X' is trivial up to homotopy (i.e. for any $g \in G$ the map $g: X' \to X'$ is homotopic to the identity map). This shows that the action of G on $H_*(X', \mathbb{Z}/m)$ is trivial. Now the spectral sequence $H_i(G, H_j(X', \mathbb{Z}/m)) \Rightarrow H_{i+j}(X, \mathbb{Z}/m)$ and Lemma 1.1 together prove the assertion concerning homology groups.

Proposition 1.5. *Suppose that $f: X \to Y$ is a map of connected H-spaces (we do not suppose that f respects the H-structure) and m is an integer such that both the kernel and the cokernel of the homomorphism $\pi_1(X) \to \pi_1(Y)$ are uniquely m-divisible. The following conditions are equivalent:*

(a) $f_*: \pi_*(X, \mathbb{Z}/m) \to \pi_*(Y, \mathbb{Z}/m)$ *is bijective.*

(b) $f_*: H_*(X, \mathbb{Z}/m) \to H_*(Y, \mathbb{Z}/m)$ *is bijective.*

Proof. Let $Y' \to Y$ be the covering space corresponding to the subgroup $\operatorname{Im}(\pi_1(X) \to \pi_1(Y))$. Then, fixing a basepoint in Y', there exists a unique lifting $f': X \to Y'$ of f and in view of Lemma 1.4 we may replace Y and f by Y' and f', thus reducing to the case when $\pi_1(X) \to \pi_1(Y)$ is surjective.

Let Y'' and X'' be the universal covering spaces for Y and X and let X' be the covering space of X corresponding to the subgroup $\operatorname{Ker}(\pi_1(X) \to \pi_1(Y))$. Fixing basepoints in X'' and Y'' we get unique lifings $f'': X'' \to Y''$ and $f': X' \to Y''$ of f. Consider the commutative diagram:

$$\begin{tikzcd}
X'' \arrow[r] \arrow[dr, "f''"'] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\
 & Y'' \arrow[r] & Y
\end{tikzcd}$$

Both X'/X and Y''/Y are Galois coverings with the same Galois group $G = \pi_1(Y)$ and f' is G-equivariant. In view of Lemma 1.4 we have the following equivalences: (a)$'' \Leftrightarrow$ (a)$'$, (b)$'' \Leftrightarrow$ (b)$'$. Furthermore, since X'' and Y'' are simply connected we know that (a)$'' \Leftrightarrow$ (b)$''$ [12, §3]. Thus it is sufficient to show that (a) $\Leftrightarrow$ (a)$'$ and (b) $\Leftrightarrow$ (b)$'$. Since $\pi_i(X', \mathbb{Z}/m) = \pi_i(X, \mathbb{Z}/m)$ and $\pi_i(Y'', \mathbb{Z}/m) = \pi_i(Y, \mathbb{Z}/m)$ when $i \geq 3$ we have to consider only π_1 and π_2. The groups $\pi_1(Y'', \mathbb{Z}/m) = \pi_1(Y'')/m$, $\pi_1(X', \mathbb{Z}/m) = \pi_1(X')/m$ are zero and the map

$$\pi_1(X, \mathbb{Z}/m) = \pi_1(X)/m \to \pi_1(Y, \mathbb{Z}/m) = \pi_1(Y)/m$$

is certainly an isomorphism. Since there is no m-torsion in $\pi_1(X')$ we know from the universal coefficient theorem [12, §1] that $\pi_2(X', \mathbb{Z}/m) = \pi_2(X')/m = \pi_2(X)/m$ and $\pi_2(Y'', \mathbb{Z}/m) = \pi_2(Y)/m$. Finally we have exact sequences of pointed sets

$$0 \to \pi_2(Y)/m \to \pi_2(Y, Z/m) \to {}_m\pi_1(Y) \to 0,$$

$$0 \to \pi_2(X)/m \to \pi_2(X, Z/m) \to {}_m\pi_1(X) \to 0.$$

Since X and Y are H-spaces these sequences are really exact sequences of groups and groups homomorphisms. This shows that the natural action of $\pi_2(Y)$ (resp. $\pi_2(X)$) on $\pi_2(Y, \mathbb{Z}/m)$ (resp. $\pi_2(X, \mathbb{Z}/m)$) (see [12, §1]] factors through the action of $\pi_2(Y)/m$ (resp. $\pi_2(X)/m$) and moreover this action is simply transitive on the fibers of the map $\pi_2(Y, \mathbb{Z}/m) \to {}_m\pi_1(Y)$ (resp. $\pi_2(X, \mathbb{Z}/m) \to {}_m\pi_1(X)$). From the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(X)/m & \longrightarrow & \pi_2(X, \mathbb{Z}/m) & \longrightarrow & {}_m\pi_1(X) & \longrightarrow & 0 \\
 & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
0 & \longrightarrow & \pi_2(Y)/m & \longrightarrow & \pi_2(Y, \mathbb{Z}/m) & \longrightarrow & {}_m\pi_1(Y) & \longrightarrow & 0
\end{array}$$

and the previous remarks we see that $\pi_2(X, \mathbb{Z}/m) \to \pi_2(Y, \mathbb{Z}/m)$ is an isomorphism if and only if $\pi_2(X)/m \to \pi_2(Y)/m$ is. Thus (a)$\Leftrightarrow$(a)$'$.

Replacing Y by the mapping cylinder of f we may suppose that f (and hence also f') is an imbedding. Condition (b) (resp. (b)$'$) is equivalent to the fact that $H_*(Y, X; \mathbb{Z}/m) = 0$ (resp. $H_*(Y'', X'; \mathbb{Z}/m) = 0$). The spectral sequence

$$H_i(G, H_j(Y'', X'; \mathbb{Z}/m)) \Rightarrow H_{i+j}(Y, X; \mathbb{Z}/m)$$

shows immediately that (b)$'\Rightarrow$(b). Suppose finally that $H_*(Y, X; \mathbb{Z}/m) = 0$ and let $h_j(Y'', X''; \mathbb{Z}/m)$ be the first nontrivial homology group of the pair (Y'', X'). The above spectral sequence shows that $H_0(G, H_j(Y'', X'; \mathbb{Z}/m)) = 0$. On the other hand we have an exact sequence of G-modules

$$H_j(Y'', \mathbb{Z}/m) \to H_j(Y'', X'; \mathbb{Z}/m) \to H_{j-1}(X', \mathbb{Z}/m)$$

and the action of G on the end terms of this sequence is trivial (since Y and X are H-spaces). This implies that $H_j(Y'', X'; \mathbb{Z}/m) = 0$, thus finishing the proof of the equivalence (b)$\Rightarrow$(b)$'$.

Corollary 1.6. *Let (R, I) be a henselian pair and m an integer invertible in R/I. The following conditions are equivalent:*

(a) *$K_*(R, \mathbb{Z}/m) \to K_*(R/I, \mathbb{Z}/m)$ is an isomorphism.*
(b) *$H_*(\mathrm{GL}(R), \mathbb{Z}/m) \to H_*(\mathrm{GL}(R/I), \mathbb{Z}/m)$ is an isomorphism.*
(c) *$H_*(\mathrm{GL}(R, I), \mathbb{Z}/m) = 0$.*

Proof. Since $I \subset \mathrm{Rad}(R)$, the homomorphism $K_0(R) \to K_0(R/I)$ is injective [1, ch. IX, §1]. Since (R, I) is a henselian pair one sees easily (compare [15, ch. XI, §2] that the map $\mathrm{Idemp}(M_n(R)) \to \mathrm{Idemp}(M_n(R/I))$ is surjective and hence $K_0(R) \to K_0(R/I)$ is also surjective. Thus $K_0(R) \xrightarrow{\sim} K_0(R/I)$ and condition (a) is equivalent to the fact that $\pi_*(B\mathrm{GL}(R)^+, \mathbb{Z}/m) \to \pi_*(B\mathrm{GL}(R/I)^+, \mathbb{Z}/m)$ is an isomorphism. Consider the map

$$\pi_1(BGL(R)^+)=K_1(R)\to K_1(R/I)=\pi_1(BGL(R/I)^+).$$

Since $I\subset \mathrm{Rad}(R)$, this map is surjective and its kernel coincides with the multiplicative group $1+I$. It follows immediately from our assumptions that this group is uniquely m-divisible and Proposition 1.5 shows that (a)$\Leftrightarrow$(b).

Consider now the Hochschild–Serre spectral sequence

$$H_p(\mathrm{GL}(R/I),H_q(\mathrm{GL}(R,I),\mathbb{Z}/m))\to H_{p+q}(\mathrm{GL}(R),\mathbb{Z}/m).$$

Proposition 1.3 shows that the action of $\mathrm{GL}(R/I)$ on $H_*(\mathrm{GL}(R,I),\mathbb{Z}/m)$ is trivial and the usual spectral sequence argument shows that (b)$\Leftrightarrow$(c).

Lemma 1.7. *Suppose that I is a nil ideal in a ring R (not necessarily commutative) and m is an integer invertible in R/I. Then (for any n) $\tilde{H}_*(\mathrm{GL}_n(R,I),\mathbb{Z}/m)=0$.*

Proof. We may suppose that I is finitely generated and hence nilpotent. We proceed by induction on m, where m is the least integer such that $I^m=0$. If $m=1$ we have nothing to prove. If $m=2$ the group $\mathrm{GL}_n(R,I)\cong M_n(I)$ is a uniquely m-divisible Abelian group and our assertion follows from Lemma 1.1. The general case follows from the Hochschild–Serre spectral sequence

$$H_p(\mathrm{GL}_n(R/I^2,I/I^2),H_q(\mathrm{GL}_n(R,I^2),\mathbb{Z}/m))\to H_{p+q}(\mathrm{GL}_n(R,I),\mathbb{Z}/m).$$

2. The universal homotopy construction

Throughout this section we fix a ring A an an integer m invertible in A. We suppose that A satisfies the following property:

Property 2.1. *Suppose that $X/\mathrm{Spec}\,A$ is a smooth affine scheme of finite type and $f:\mathrm{Spec}\,A\to X$ is a section. Denote by X_f^{h} the henselization of X along f* (see [15]). *Then the natural homomorphism $K_*(A,\mathbb{Z}/m)\to K_*(X_f^{\mathrm{h}},\mathbb{Z}/m)$ is bijective.*

Consider the smooth affine scheme $X_{n,i}=\mathrm{GL}_n\times\cdots\times\mathrm{GL}_n$ (i times) over $\mathrm{Spec}\,A$ and denote by $X_{n,i}^{\mathrm{h}}$ its henselization along the unit section. This scheme is affine and we denote by $O_{n,i}^{\mathrm{h}}$ its coordinate ring. $O_{n,i}^{\mathrm{h}}$ is an A-algebra and we have a canonical A-homomorphism $O_{n,i}^{\mathrm{h}}\to A$, the kernel $\mathcal{M}_{n,i}^{\mathrm{h}}$ of which lies in $\mathrm{Rad}(O_{n,i}^{\mathrm{h}})$.

For any group G we denote by $\tilde{C}_*(G,\mathbb{Z}/m)$ its reduced standard complex with coefficients $\mathbb{Z}/m$ (i.e. we replace $C_0(G,\mathbb{Z}/m)$ by zero).

Consider the following morphisms of schemes over $\mathrm{Spec}\,A$:

$$p_j^i:X_{n,i}\to X_{n,i-1}:\quad p_j^i(g_1\times\cdots\times g_i)=\begin{cases} g_2\times\cdots\times g_i & \text{if } j=0,\\ g_1\times\cdots\times g_jg_{j+1}\times\cdots\times g & \text{if } 1\le j\le i-1,\\ g_1\times\cdots\times g_{i-1} & \text{if } j=1.\end{cases}$$

These morphisms preserve the unit section and hence define morphisms $X_{n,i}^{\mathrm{h}}\to X_{n,i-1}^{\mathrm{h}}$ which we also denote by p_j^i. We use the notation $(p_j^i)^*$ for the induced homomorphisms $O^{\mathrm{h}}_{n,i-1} \to O^{\mathrm{h}}_{n,i}$, $\mathrm{GL}(O^{\mathrm{h}}_{n,i-1}, \mathscr{I}^{\mathrm{h}}_{n,i-1}) \to \mathrm{GL}(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i})$ and

$$\tilde{C}_*(\mathrm{GL}(O^{\mathrm{h}}_{n,i-1}, \mathscr{I}^{\mathrm{h}}_{n,i-1}), \mathbb{Z}/m) \to \tilde{C}_*(\mathrm{GL}(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i}), \mathbb{Z}/m).$$

We have the evident morphisms of schemes over $\operatorname{Spec} A$: $X^{\mathrm{h}}_{n,i} \to X_{n,i} \xrightarrow{\mathrm{pr}_k} \mathrm{GL}_n$ preserving the unit section and hence canonical matrices $\alpha_k \in \mathrm{GL}_n(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i})$. We denote by $u_{n,i} \in \tilde{C}_i(\mathrm{GL}(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i}), \mathbb{Z}/m)$ the chain $[\alpha_1, \dots, \alpha_i] \times (1 \bmod m\mathbb{Z})$ (we take $u_{n,0} = 0$).

Proposition 2.2. *There exist chains $c_{n,i} \in C_{i+1}(\mathrm{GL}(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i}), \mathbb{Z}/m)$ such that*

$$d(c_{n,i}) = u_{n,i} - \sum_{j=0}^{i} (-1)^j (p^i_j)^*(c_{n,i-1}).$$

Proof. Take $c_{n,0} = 0$ and suppose that $c_{n,0}, \dots, c_{n,i-1}$ are constructed. Since $d(u_{n,i}) = \sum_{j=0}^{i} (-1)^j (p^i_j)^*(u_{n,i-1})$ we see that

$$d\left(u_{n,i} - \sum_{j=0}^{i} (-1)^j (p^i_j)^*(c_{n,i-1})\right) = \sum_{j=0}^{i} (-1)^j (p^i_j)^*\left(\sum_{k=0}^{i-1} (-1)^k (p^{i-1}_k)^*(c_{n,i-2})\right)$$

$$= \sum_{j=0}^{i} \sum_{k=0}^{i-1} (-1)^{j+k} (p^{i-1}_k p^i_j)^*(c_{n,i-2}) = 0.$$

This shows that $u_{n,i} - \sum_{i=0}^{i} (-1)^j (p^i_j)^*(c_{n,i-1})$ is a cycle and hence a boundary since the group $\tilde{H}_i(\mathrm{GL}(O^{\mathrm{h}}_{n,i}, \mathscr{I}^{\mathrm{h}}_{n,i}), \mathbb{Z}/m)$ is zero in view of the Property 2.1 and Corollary 1.6. This gives the possibility to construct $c_{n,i}$ and thus to complete the inductive process.

Theorem 2.3. *Suppose that (R, I) is a henselian pair and R is an A-algebra. Then*

$$K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(R/I, \mathbb{Z}/m).$$

Proof. In view of Corollary 1.6 it is sufficient to show that $\tilde{H}_*(\mathrm{GL}(R, I), \mathbb{Z}/m) = 0$. To prove this we shall show that the imbedding $\tilde{C}_*(\mathrm{GL}_n(R, I), \mathbb{Z}/m) \to \tilde{C}_*(\mathrm{GL}(R, I), \mathbb{Z}/m)$ is null-homotopic. The group $C_i(\mathrm{GL}_n(R, I), \mathbb{Z}/m)$ $(i \geq 1)$ is a free $\mathbb{Z}/m$-module with basis $[\beta_1, \dots, \beta_i]$ where $\beta_j \in \mathrm{GL}_n(R, I)$. The matrices $\beta_1, \dots, \beta_i$ define a morphism $\operatorname{Spec} R \to X_{n,i}$ of schemes over $\operatorname{Spec} A$, which sends the closed subscheme $\operatorname{Spec}(R/I) \to \operatorname{Spec} R$ into the unit section of $X_{n,i}$. By the definition of the henselization this morphism factors uniquely through a morphism $\phi_\beta : \operatorname{Spec} R \to X^{\mathrm{h}}_{n,i}$ (also sending $\operatorname{Spec}(R/I)$ into the unit section). We define a homotopy operator $s : \tilde{C}_*(\mathrm{GL}_n(R, I), \mathbb{Z}/m) \to \tilde{C}_*(\mathrm{GL}(R, I), \mathbb{Z}/m)$ by means of the formula $s([\beta_1, \dots, \beta_i]) = (\phi_\beta)^*(c_{n,i})$ and we see immediately from properties of $c_{n,i}$ that s is the required null-homotopy.

Corollary 2.4. *If B is an A-algebra, then B also satisfies* 2.1.

Corollary 2.5. *If (R, I) is a henselian pair and R is an algebra over a field F, then*

$K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(R/I, \mathbb{Z}/m)$ *for any* m *prime to* char F.

This follows from Theorem 2.3 and the theorem of O. Gabber [4] stating that fields satisfy Property 2.1.

3. K-theory of henselian valuation rings

Let R be a henselian valuation ring with maximal ideal I, residue field $F = R/I$ and quotient field E. If char E = char F, then R is an algebra over a field and we know from Corollary 2.5 that $K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(F, \mathbb{Z}/m)$ for any m prime to char F. In this section we consider the case when char $E = 0$ and char $F = p > 0$. We denote by v the valuation of E associated with R and we denote by Γ the valuation group of v (thus Γ is a totally ordered group and v is a map $E \to \Gamma \cup \infty$). For every $\sigma \in \Gamma$ we denote by I_σ the R-submodule of E given by the formula $I_\sigma = \{x \in E : v(x) > \sigma\}$. If $\sigma \geq 0$, then I_σ is an ideal in R and I coincides with I_0. We use the notation $\hat{E}$ for the completion of E in the topology defined by v.

Lemma 3.1. *If L/E is a finite field extension, then there exists a unique extension of v on the field L.*

Proof. Denote by A the integral closure of R in L. It is known [2, ch. 6] that A is a semilocal ring and if $\mathfrak{M}_i$ are its maximal ideals, then $A_{\mathfrak{M}_i}$ are the valuation rings corresponding to extensions of v. On the other hand, since R is henselian, every semilocal integral R-algebra is a product of local rings [15, §1]. Since A is a domain we deduce that A is local, i.e. there is only one extension of v on L.

Corollary 3.2. *E is algebraically closed in $\hat{E}$.*

Proof. Let L/E be a finite field extension. The well known formula [2, ch. 6] $\hat{E} \otimes_E L = \prod_{w \mid v} L_w$ together with Lemma 3.1 shows that $\hat{E} \otimes_E L$ is a domain. Thus the extension $\hat{E}/E$ is regular [9] and hence E is algebraically closed in $\hat{E}$.

Lemma 3.3. *Consider the polynomial $p = T^n + \sum_{i=0}^{n-1} Y_i T^i \in E[Y_0, \dots, Y_{n-1}, T]$ and suppose that $y = (y_0, \dots, y_{n-1}) \in E^n$ and $t \in E$ are such that t is a simple root of $p(y, T)$. Then there exists an open neighborhood (for the topology defined by the valuation) $V(y) \subset E^n$ and a continuous function $u : V(y) \to E$ such that $u(y) = t$ and, for any $z \in V$, $p(z, u(z)) = 0$.*

Proof. If E is complete this follows immediately from the implicit function theorem (see [15, ch. §4]). In the general case we first construct $\hat{V}(y) \subset \hat{E}^n$ and $\hat{u} : \hat{V}(y) \to \hat{E}$ corresponding to $\hat{E}$ and then take $V(y) = \hat{V}(y) \cap E^n$, $\hat{u} = u|_{V(y)}$. The values of u lie in E in view of Corollary 3.2.

One deduces immediately from the previous result (see [15, ch. 6, §4]) the following.

Corollary 3.4. *Let X be a topological space and $x \in X$. Then the local ring of germs of continuous E-valued functions defined in a neighborhood of x is henselian.*

Consider the set $\mathrm{GL}_n(E) \times \cdots \times \mathrm{GL}_n(E)$ (i times) as a topological space with the topology defined by the valuation υ (the basis of neighborhoods of the unit in this topology consists of the sets $\mathrm{GL}_n(R, I_\sigma) \times \cdots \times \mathrm{GL}_n(R, I_\sigma)$, where $0 \le \sigma \in \Gamma$). Denote by $O_{n,i}^{\mathrm{cont}}$ the local ring of germs of continuous E-valued functions defined in a neighborhood of the unity $e \in \mathrm{GL}_n(E) \times \cdots \times \mathrm{GL}_n(E)$ and denote by $\mathcal{M}_{n,i}^{\mathrm{cont}}$ the maximal ideal of $O_{n,i}^{\mathrm{cont}}$. Every chain $c \in C_{i+1}(\mathrm{GL}_r(O_{n,i}^{\mathrm{cont}}, \mathcal{M}_{n,i}^{\mathrm{cont}}), \mathbb{Z}/m)$ defines a map of some neighborhood of $e \in \mathrm{GL}_n(E) \times \cdots \times \mathrm{GL}_n(E)$ to $C_{i+1}(\mathrm{GL}_r(E), \mathbb{Z}/m)$ which is continuous in the sense that for any $\sigma \ge 0$ there exists $\tau \ge 0$ such that c is defined in $\mathrm{GL}_n(R, I_\tau) \times \cdots \times \mathrm{GL}_n(R, I_\tau)$ and takes it to $C_{i+1}(\mathrm{GL}_r(R, I_\sigma), \mathbb{Z}/m)$. We use the same letter c to denote the natural $\mathbb{Z}/m$-linear extension of the previous map to a homomorphism $C_i(\mathrm{GL}_n(R, I_\tau), \mathbb{Z}/m) \to C_{i+1}(\mathrm{GL}_r(R, I_\sigma), \mathbb{Z}/m)$.

Proposition 3.5. *Let N and n be positive integers and $0 \le \tau \in \Gamma$. Then there exist $r \ge n$ (independent of τ) and $\sigma \ge \tau$* such that the imbedding $\mathrm{GL}_n(R, I_\sigma) \hookrightarrow \mathrm{GL}_r(R, I_\tau)$ *induces the trivial zero homomorphism on $\tilde{H}_i(-, \mathbb{Z}/m)$ with $0 \le i \le N$.*

Proof. Consider the algebraic variety $\mathrm{GL}_n \times \cdots \times \mathrm{GL}_n = X_{n,i}$ over E. The ring $O_{n,i}^{\mathrm{cont}}$ being henselian, we deduce that the evident morphism of schemes (over $\operatorname{Spec} E$) $\operatorname{Spec} O_{n,i}^{\mathrm{cont}} \to X_{n,i}$ factors through $\operatorname{Spec} O_{n,i}^{\mathrm{cont}} \to X_{n,i}^{\mathrm{h}} = \operatorname{Spec} O_{n,i}^{\mathrm{h}}$, thus giving a local homomorphism $O_{n,i}^{\mathrm{h}} \to O_{n,i}^{\mathrm{cont}}$. Denote by $c_{n,i}'$ the image of $c_{n,i} \in C_{i+1}(\mathrm{GL}(O_{n,i}^{\mathrm{h}}, \mathcal{M}_{n,i}^{\mathrm{h}}), \mathbb{Z}/m)$ in $C_{i+1}(\mathrm{GL}(O_{n,i}^{\mathrm{cont}}, \mathcal{M}_{n,i}^{\mathrm{cont}}), \mathbb{Z}/m)$. We can find $r \ge n$ such that all the chains $c_{n,i}'$ with $0 \le i \le N$ lie $C_{i+1}(\mathrm{GL}_r(O_{n,i}^{\mathrm{cont}}, \mathcal{M}_{n,i}^{\mathrm{cont}}), \mathbb{Z}/m)$ and then find $\sigma \ge \tau$ such that the $c_{n,i}'$ $(0 \le i \le N)$ are defined in $\mathrm{GL}_n(R, I_\sigma) \times \cdots \times \mathrm{GL}_n(R, I_\sigma)$ and sent to $C_{i+1}(\mathrm{GL}_r(R, I_\tau), \mathbb{Z}/m)$. This gives us a null homotopy (defined in degrees $\le N$) for the natural imbedding $\tilde{C}_*(\mathrm{GL}_n(R, I_\sigma), \mathbb{Z}/m) \hookrightarrow \tilde{C}_*(\mathrm{GL}_r(R, I_\tau), \mathbb{Z}/m)$ and proves the proposition.

Theorem 3.6. *For every $i \ge 0$ there exists $0 \le \sigma \in \Gamma$ such that the canonical homomorphism $H_i(\mathrm{GL}(R), \mathbb{Z}/m) \to H_i(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)$ is injective. Furthermore, for every $\sigma \ge 0$ there exists $\tau \ge \sigma$ such that the image of $H_i(\mathrm{GL}(R), \mathbb{Z}/m) \to H_i(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)$ coincides with the image of*

$$H_i(\mathrm{GL}(R/I_\tau), \mathbb{Z}/m) \to H_i(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m).$$

Proof. For every n and σ we have a Hochschild–Serre spectral sequence

$$\mathcal{E}(n, \sigma) : E_{p,q}^2 = H_p(\mathrm{GL}_n(R/I_\sigma), H_q(\mathrm{GL}_n(R, I_\sigma), \mathbb{Z}/m)) \Rightarrow H_{p+q}(\mathrm{GL}_n(R), \mathbb{Z}/m).$$

This spectral sequence defines a filtration on $H_k(\mathrm{GL}_n(R), \mathbb{Z}/m)$:

$$0=H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{-1,\sigma}\subset H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{0,\sigma}\subset\cdots\subset H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{k,\sigma}$$
$$=H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)$$

such that $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{p,\sigma}/H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{p-1,\sigma}=E^\infty_{p,k-p}$ and the kernel of $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)\to H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)$ coincides with $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{k-1,\sigma}$. If $\tau\geq\sigma$ and $r\geq n$, then we have a homomorphism of group extensions

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{GL}_n(R,I_\tau) \arrow[r] \arrow[d, hook] & \mathrm{GL}_n(R) \arrow[r] \arrow[d, hook] & \mathrm{GL}_n(R/I_\tau) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \mathrm{GL}_r(R,I_\sigma) \arrow[r] & \mathrm{GL}_r(R) \arrow[r] & \mathrm{GL}_r(R/I_\sigma) \arrow[r] & 0
\end{tikzcd}$$

and hence the induced homomorphism of spectral sequences $\mathscr{E}(n,\tau)\to\mathscr{E}(r,\sigma)$. In particular the canonical homomorphism $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)\to H_k(\mathrm{GL}_r(R),\mathbb{Z}/m)$ takes $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{p,\tau}$ to $H_k(\mathrm{GL}_r(R),\mathbb{Z}/m)_{p,\sigma}$.

3.6.1. *For every n and every $j<k$ there exists $\sigma\geq 0$ such that the image of $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{j,\sigma}$ in $H_k(\mathrm{GL}(R),\mathbb{Z}/m)$ is zero.*

Proof. We proceed by induction on j. The statement is evident for $j=-1$, so we may suppose that $j\geq 0$. Find $r\geq n$ as in Proposition 3.5 (taking $N=k$) and then $\sigma\geq 0$, corresponding to r and $j-1$. By the choice of r there exists $\tau\geq\sigma$ such that the homomorphism $H_q(\mathrm{GL}_n(R,I_\tau),\mathbb{Z}/m)\to H_q(\mathrm{GL}_r(R,I_\sigma),\mathbb{Z}/m)$ is zero when $1\leq q\leq k$. This shows that the homomorphisms $E^2_{p,q}(n,\tau)\to E^2_{p,q}(r,\sigma)$ are zero for $1\leq q\leq k$ and hence the homomorphisms $E^\infty_{p,q}(n,\tau)\to E^\infty_{p,q}(r,\sigma)$ $(1\leq q\leq k)$ are also zero. Thus the image of $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{j,\tau}$ in $H_k(\mathrm{GL}_r(R),\mathbb{Z}/m)$ is contained in $H_k\mathrm{GL}_r(R),\mathbb{Z}/m)_{j-1,\sigma}$ and hence its image in $H_k(\mathrm{GL}(R),\mathbb{Z}/m)$ is zero.

3.6.2. Since the rings R and R/I are local one knows ([18], [17]) that

$$H_k(\mathrm{GL}_n(-),\mathbb{Z}/m)=H_k(\mathrm{GL}(-),\mathbb{Z}/m)\quad\text{if } n\geq 2k+1.$$

This fact and 3.6.1 show that if $n\geq 2k+1$ and σ is sufficiently large, then $H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)_{k-1,\sigma}=0$ and hence

$$H_k(\mathrm{GL}(R),\mathbb{Z}/m)=H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)\hookrightarrow H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)$$
$$=H_k(\mathrm{GL}(R/I_\sigma),\mathbb{Z}/m).$$

3.6.3. The same spectral sequence defines a descending filtration on $H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)$: $H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)_i=E^i_{k,0}(n,\sigma)$ $(i\geq 2)$ and $\mathrm{Im}(H_k(\mathrm{GL}_n(R),\mathbb{Z}/m)\to H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m))$ coincides with $H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)_k=H_k(\mathrm{GL}_n(R/I_\sigma),\mathbb{Z}/m)_\infty$.

3.6.4. *For every n, j and every $\sigma \geq 0$ there exist $r \geq n$ (independent of φ) and $\tau \geq \sigma$ such that* $\mathrm{Im}(H_k(\mathrm{GL}_n(R/I_\tau), \mathbb{Z}/m) \to H_k(\mathrm{GL}_r(R/I_\sigma), \mathbb{Z}/m))$ *is contained in* $H_k(\mathrm{GL}_r(R/I_\sigma), \mathbb{Z}/m)_j$.

Proof. Once again we use induction on j. The statement is evident for $j=2$, so we may suppose that $2<j\leq k$. First find $r' \geq n$ which works for $j-1$, next find $r \geq r'$ as in Proposition 3.5 (for the data r', j). By the choice of r there exists $\tau' \geq \sigma$ such that the homomorphism

$$H_{j-1}(\mathrm{GL}_{r'}(R, I_{\tau'}), \mathbb{Z}/m) \to H_{j-1}(\mathrm{GL}_r(R, I_\sigma), \mathbb{Z}/m)$$

is zero. This implies that the homomorphism $E^2_{k-j,j-1}(r', \tau') \to E^2_{k-j,j-1}(r, \sigma)$ (and hence also $E^{j-1}_{k-j,j-1}(r', \tau') \to E^{j-1}_{k-j,j-1}(r, \sigma)$) is zero. From the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & E^j_{k,0}(r', \tau') \arrow[r] \arrow[d] & E^{j-1}_{k,0}(r', \tau') \arrow[r, "d^{j-1}"] \arrow[d] & E^{j-1}_{k-j,j-1}(r', \tau') \arrow[d] \\
0 \arrow[r] & E^j_{k,0}(r, \sigma) \arrow[r] & E^{j-1}_{k,0}(r, \sigma) \arrow[r, "d^{j-1}"] & E^{j-1}_{k-j,j-1}(r, \sigma)
\end{tikzcd}$$

we deduce that

$$\mathrm{Im}(H_k(\mathrm{GL}_{r'}(R/I_{\tau'}), \mathbb{Z}/m)_{j-1} \to H_k(\mathrm{GL}_r(R/I_\sigma), \mathbb{Z}/m)) \subset H_k(\mathrm{GL}_r(R/I_\sigma), \mathbb{Z}/m)_j.$$

Finally we can find $\tau \geq \tau'$ such that

$$\mathrm{Im}(H_k(\mathrm{GL}_n(R/I_\tau), \mathbb{Z}/m) \to H_k(\mathrm{GL}_{r'}(R/I_{\tau'}), \mathbb{Z}/m)) \subset H_k(\mathrm{GL}_{r'}(R/I_{\tau'}), \mathbb{Z}/m)_{j-1}.$$

3.6.5. Taking $n=2k+1$, $j=k$ in 3.6.4 and using once more the homology stability theorem we see that for any $\sigma \geq 0$ there exists $\tau \geq \sigma$ such that

$$\begin{aligned}
&\mathrm{Im}(H_k(\mathrm{GL}(R/I_\tau), \mathbb{Z}/m) \to H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)) \\
&= \mathrm{Im}(H_k(\mathrm{GL}_n(R/I_\tau), \mathbb{Z}/m) \to H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)) \\
&\subset H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)_k = \mathrm{Im}(H_k(\mathrm{GL}(R), \mathbb{Z}/m) \to H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)).
\end{aligned}$$

Corollary 3.7. *The inverse system* $\{H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m)\}$ *is Mittag-Leffler and*

$$H_k(\mathrm{GL}(R), \mathbb{Z}/m) = \varprojlim H_k(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m).$$

Corollary 3.8. *If R is a ring of integers in a non-Archimedean local field of characteristic zero, then the groups $H_k(\mathrm{GL}(R), \mathbb{Z}/m)$ are finite for any k, m.*

Corollary 3.9. *Suppose that the height of the valuation v is equal to one and m is prime to p, then $K_*(R, \mathbb{Z}/m) \xrightarrow{\sim} K_*(F, \mathbb{Z}/m)$.*

Proof. Since the height of υ is equal to one we see immediately that for any $\sigma \geq 0$, I/I_σ is a nil-ideal in R/I_σ and hence (see Lemma 1.7), $H_*(\mathrm{GL}(R/I_\sigma), \mathbb{Z}/m) \xrightarrow{\sim} H_*(\mathrm{GL}(F), \mathbb{Z}/m)$. This together with Corollary 3.7 shows that $H_*(\mathrm{GL}(R), \mathbb{Z}/m) \xrightarrow{\sim} H_*(\mathrm{GL}(F), \mathbb{Z}/m)$ and we finish the proof using Proposition 1.5.

Remark 3.10. Using induction on the height of υ one can easily generalize the previous result to the case of a valuation of arbitrary finite height.

Corollary 3.11. *Suppose that E is a henselian discretely valuated field with residue field F of characteristic p. For any $i > 0$ and any m prime to p we have a split exact sequence*

$$0 \to K_i(F, \mathbb{Z}/m) \to K_i(E, \mathbb{Z}/m) \xrightarrow{d} K_{i-1}(F, \mathbb{Z}/m) \to 0.$$

Proof. In view of Corollary 3.9 and the localization sequence we have only to show that (denoting by R the valuation ring of E) $K_i(F, \mathbb{Z}/m) = K_i(R, \mathbb{Z}/m) \to K_i(E, \mathbb{Z}/m)$ is split injective. Choose a prime element π of E and denote by $l(\pi)$ the corresponding element in $K_1(E) = E^*$, then the homomorphism

$$K_i(E, \mathbb{Z}/m) \xrightarrow{l(\pi)} K_{i+1}(E, \mathbb{Z}/m) \xrightarrow{d} K_i(F, \mathbb{Z}/m)$$

gives us the required splitting.

Proposition 3.12. *Suppose that F is an algebraically closed field of positive characteristic p and let E denote the algebraic closure of the quotient field E_0 of the ring $R_0 = W(F)$ of Witt vectors over F. For any m prime to p we have a canonical isomorphism $K_*(F, \mathbb{Z}/m) = K_*(E, \mathbb{Z}/m)$.*

Proof. If L/E_0 is a finite subextension of E/E_0, then L is a complete discretely valuated field with residue field F and we have an exact sequence

$$0 \to K_i(F, \mathbb{Z}/m) \to K_i(L, \mathbb{Z}/m) \xrightarrow{d} F_{i-1}(F, \mathbb{Z}/m) \to 0.$$

If $L' \supset L$ is another finite subextension of E/E_0, then we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_i(F, \mathbb{Z}/m) \arrow[r] \arrow[d, "="] & K_i(L, \mathbb{Z}/m) \arrow[r] \arrow[d] & K_{i-1}(F, \mathbb{Z}/m) \arrow[r] \arrow[d, "{[L':L]}"] & 0 \\
0 \arrow[r] & K_i(F, \mathbb{Z}/m) \arrow[r] & K_i(L', \mathbb{Z}/m) \arrow[r] & K_{i-1}(F, \mathbb{Z}/m) \arrow[r] & 0
\end{tikzcd}$$

Since for any L there exist finite extensions L'/L of arbitrary degree we deduce that the direct limit of the right hand side terms is zero and hence

$$K_i(E, \mathbb{Z}/m) = \varinjlim K_i(L, \mathbb{Z}/m) = K_i(F, \mathbb{Z}/m).$$

Corollary 3.13. *For any algebraically closed field F and any m prime to* char F *the group $K_i(F, \mathbb{Z}/m)$ is either zero (if i is odd) or isomorphic to $\mathbb{Z}/m$ (if i is even).*

Proof. It was shown in [18] that the group $K_i(F, \mathbb{Z}/m)$ (for F algebraically closed) can depend only on char F, and it follows from Proposition 3.12 that really it does not even depend on char F. Since by the work of Quillen [13] our statement is true for the algebraic closure of a finite field, we deduce that it is true for any algebraically closed field F.

Remark 3.14. We can avolid the use of Quillen's theorem since we compute the K-theory of complex numbers in the next section.

4. K-theory of Archimedan fields

If G is a topological group, then we use the notation BG^{top} to denote the classifying space of G considered as a topological group [7], reserving the notation BG for the classifying space of G considered as a discrete group.

Recall that for any discrete group G we have the following canonical model for BG. Denote by EG the geometric realization of the simplicial set whose p-simplices are $(p+1)$-tuples $\langle g_0, \dots, g_p \rangle$ of elements of G and face (resp. degeneracy) operators are given by omitting (resp. repeating) the corresponding element. This space is contractible and the evident action of G on EG is free, thus the space EG/G is a classifying space for G. In what follows, when speaking about BG we shall usually mean this canonical model. From the above description we see that BG is the geometric realization of the simplicial set whose p-simplices are p-tuples

$$[g_1, \dots, g_p] = \langle e, g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_p \rangle \bmod G$$

of elements of G and face and degeneracy operators are given by the formulae:

$$d_i([g_1, \dots, g_p]) = \begin{cases} [g_2, \dots, g_p] & \text{if } i = 0, \\ [g_1, \dots, g_i g_{i+1}, \dots, g_p] & \text{if } 1 \le i \le p-1, \\ [g_1, \dots, g_{p-1}] & \text{if } i = p, \end{cases}$$

$$s_i([g_1, \dots, g_p]) = [g_1, \dots, g_i, e, g_{i+1}, \dots, g_p].$$

Suppose now that G is a Lie group with finitely many connected components. Fix a left invariant Riemann metric on G and denote by G_ε the ε-ball with center at the unity of G. We denote by BG_ε the geometric realization of the simplicial set whose p-simplices are p-tuples $[g_1, \dots, g_p]$ of elements of G such that $G_\varepsilon \cap g_1 G_\varepsilon \cap \cdots \cap g_1 \cdots g_p G_\varepsilon \neq \emptyset$ and the face and degeneracy operators are the same as above.

Proposition 4.1. *If ε is small enough, then the sequence $BG_\varepsilon \to BG \to BG^{\mathrm{top}}$ is a fibration up to homotopy.*

Proof. Consider the universal principal G^{top}-fibration $G^{\mathrm{top}} \to EG^{\mathrm{top}} \to BG^{\mathrm{top}}$. For any topological space X we denote by $\operatorname{Sin} X$ its singular simplicial set [10]. Since functors Sin and geometric realization preserve fibrations we get a commutative diagram of fibrations:

$$\begin{tikzcd}
|\operatorname{Sin} G^{\mathrm{top}}| \arrow[r] \arrow[d] & |\operatorname{Sin} EG^{\mathrm{top}}| \arrow[r] \arrow[d] & |\operatorname{Sin} BG^{\mathrm{top}}| \arrow[d] \\
G^{\mathrm{top}} \arrow[r] & EG^{\mathrm{top}} \arrow[r] & BG^{\mathrm{top}}
\end{tikzcd}$$

and the vertical arrows in this diagram are homotopy equivalences [10]. The evident fiberwise action of the discrete group G on $|\operatorname{Sin} EG^{\mathrm{top}}|$ is free and factoring this action out we get a new fibration:

$$|\operatorname{Sin} G^{\mathrm{top}}|/G \to |\operatorname{Sin} EG^{\mathrm{top}}|/G \to |\operatorname{Sin} BG^{\mathrm{top}}|.$$

The space EG^{top} (and hence also $|\operatorname{Sin} EG^{\mathrm{top}}|$) is contractible and hence $|\operatorname{Sin} EG^{\mathrm{top}}|/G$ is homotopy equivalent to BG. Thus we have the fibration up to homotopy $|\operatorname{Sin} G^{\mathrm{top}}|/G \to BG \to BG^{\mathrm{top}}$ (compare [11]), where the first arrow corresponds to the principal G-fibration $|\operatorname{Sin} G^{\mathrm{top}}| \to |\operatorname{Sin} G^{\mathrm{top}}|/G$.

Suppose now that ε is small enough. Then G_ε is geodesically convex [6,§5.2] and hence every nonempty intersection $g_0G_\varepsilon \cap \cdots \cap g_pG_\varepsilon$ is contractible. Denote by $X.$ the simplicial topological space

$$X_p = \coprod_{g_0,\ldots,g_p} g_0G_\varepsilon \cap \cdots \cap g_pG_\varepsilon,$$

by $Y..$ the bisimplicial set $Y_{pq} = \operatorname{Sin}_p(X_q)$, by $(\operatorname{Sin} G^{\mathrm{top}})_\varepsilon$ the subobject of the simplicial set $\operatorname{Sin} G^{\mathrm{top}}$ consisting of singular simplices lying in some gG_ε and by E_ε the simplicial set whose p-simplices are $(p+1)$-tuples $\langle g_0,\ldots,g_p\rangle$ such that $g_0G_\varepsilon \cap \cdots \cap g_pG_\varepsilon \neq \emptyset$. We have the evident maps of bisimplicial sets

$$\begin{tikzcd}
 & Y.. \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\
(\operatorname{Sin} G^{\mathrm{top}})_\varepsilon & & E_\varepsilon
\end{tikzcd}$$

where we consider $(\operatorname{Sin} G^{\mathrm{top}})_\varepsilon$ (resp. E_ε) as a bisimplicial set trivial in the q (resp. p) -direction.

4.1.1. *ϕ is a homotopy equivalence.*

To prove this it is sufficient to show that for every p, $Y_{p.} \to (\operatorname{Sin} G^{\mathrm{top}})_{\varepsilon,p}$ is a homotopy equivalence. Denote $(\mathrm{Sin}\, G^{\mathrm{top}})_{\varepsilon, p}$ by T. By the very definition

$$T = \bigcup_{g \in G} T_g, \text{ where } T_g = \mathrm{Sin}_p(gG_\varepsilon) \text{ and } Y_{pq} = \coprod_{g_0, \dots, g_q} T_{g_0} \cap \cdots \cap T_{g_q}.$$

For every $t \in T$, put $G_t = \{g \in G : T_g \ni t\}$; then the fiber of $Y_{p.} \to T$ over t may be identified with the simplicial set whose p-simplices are $(p+1)$-tuples of elements of G_t. Thus all the fibers of $Y_{p.} \to T$are contractible and hence this map is a homotopy equivalence.

4.1.2. *ψ is a homotopy equivalence.*

This time it is sufficient to show that for every q, $Y_{.q} \to (E_\varepsilon)_q$ is a homotopy equivalence. For any $\langle g_0, \dots, g_q \rangle \in (E_\varepsilon)_q$, the fiber of $Y_{.q} \to (E_\varepsilon)_q$ over $\langle g_0, \dots, g_q \rangle$ coincides with $\mathrm{Sin}(g_0 G_\varepsilon \cap \cdots \cap g_q G_\varepsilon)$. It is contractible since $g_0 G_\varepsilon \cap \cdots \cap g_q G_\varepsilon$ is contractible.

4.1.3. *The imbedding* $(\mathrm{Sin}\, G^{\mathrm{top}})_\varepsilon \hookrightarrow \mathrm{Sin}\, G^{\mathrm{top}}$ *is a homotopy equivalence.*

This is well known – compare [16, ch. 4, §4].

4.1.4. Thus we have homotopy equivalences $|\mathrm{Sin}\, G^{\mathrm{top}}| \xleftarrow{\sim} |Y_{..}| \xrightarrow{\sim} |E_\varepsilon|$. The discrete group G acts freely on all these spaces and the maps above are G-equivariant. Factoring out this action of G we get a homotopy equivalence $BG_\varepsilon = |E_\varepsilon|/G \xrightarrow{\sim} |\mathrm{Sin}\, G^{\mathrm{top}}|/G$ and it is clear from the construction that the triangle

$$\begin{tikzcd} BG_\varepsilon \arrow[rr, "\sim"] \arrow[dr, hook] & & {|\mathrm{Sin}\, G^{\mathrm{top}}|/G} \arrow[dl] \\ & BG & \end{tikzcd}$$

is commutative up to homotopy.

Remark 4.2. In the above proof we have constructed (for small ε) a homotopy equivalence $BG_\varepsilon \xrightarrow{\sim} |\mathrm{Sin}\, G^{\mathrm{top}}|/G$. It is clear that for $\delta < \varepsilon$ the diagram

$$\begin{tikzcd} BG_\delta \arrow[rr, hook] \arrow[dr, "\sim"] & & BG_\varepsilon \arrow[dl, "\sim"] \\ & {|\mathrm{Sin}\, G^{\mathrm{top}}|/G} & \end{tikzcd}$$

is commutative up to homotopy and hence $BG_\delta \hookrightarrow BG_\varepsilon$ is also a homotopy equivalence.

Proposition 4.3. *Let k denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. If ε is small enough, then the imbedding $B\mathrm{GL}_n(k)_\varepsilon \hookrightarrow B\mathrm{GL}_n(k) \hookrightarrow B\mathrm{GL}(k)$ induces the zero homomorphism on $\tilde{H}_*(-, \mathbb{Z}/m)$.*

Proof. Denote by $O_{n,i}^{\text{cont}}$ the ring of germs of continuous functions $\mathrm{GL}_n(k)\times\cdots\times\mathrm{GL}_n(k)\to k$ defined in some neighborhood of the unity. The group $\mathrm{GL}_r(O_{n,i}^{\text{cont}})$ may be identified with the group of germs of continuous maps $\mathrm{GL}_n(k)\times\cdots\times\mathrm{GL}_n(k)\to\mathrm{GL}_r(k)$ defined in some neighborhood of the unity and hence every chain $c\in C_q(\mathrm{GL}_r(O_{n,i}^{\text{cont}}),\mathbb{Z}/m)$ defines a continuous map of some neighborhood of the unity $e\in\mathrm{GL}_n(k)\times\cdots\times\mathrm{GL}_n(k)$ to $C_q(\mathrm{GL}_r(k),\mathbb{Z}/m)$. Denote as usual by $O_{n,i}^{\mathrm{h}}$ the henselization of the variety $\mathrm{GL}_n\times\cdots\times\mathrm{GL}_n/k$ at the unity. The ring $O_{n,i}^{\text{cont}}$ being henselian [15], we get a canonical homomorphism $O_{n,i}^{\mathrm{h}}\to O_{n,i}^{\text{cont}}$ and we denote by $c_{n,i}^{\text{cont}}\in C_{i+1}(\mathrm{GL}(O_{n,i}^{\text{cont}}),\mathbb{Z}/m)$ the image under this homomorphism of the chain $c_{n,i}$ constructed in Section 2. For given $N>0$ we can find $\varepsilon>0$ such that the $c_{n,i}^{\text{cont}}$ are defined in $\mathrm{GL}_n(k)_\varepsilon\times\cdots\times\mathrm{GL}_n(k)_\varepsilon$ $(0\le i\le N)$. Thus for $0\le i\le N$ we get homomorphisms

$$s_i: C_i(B\mathrm{GL}_n(k)_\varepsilon,\mathbb{Z}/m)\to C_{i+1}(\mathrm{GL}(k),\mathbb{Z}/m)=C_{i+1}(B\mathrm{GL}(k),\mathbb{Z}/m)$$

and it is clear from the properties of $c_{n,i}$ that s is a null homotopy (defined in degrees $\le N$) for the canonical imbedding

$$\tilde{C}_*(B\mathrm{GL}_n(k)_\varepsilon,\mathbb{Z}/m)\hookrightarrow\tilde{C}_*(B\mathrm{GL}(k),\mathbb{Z}/m)=\tilde{C}_*(\mathrm{GL}(k),\mathbb{Z}/m).$$

Corollary 4.4. *With the same notations as above, the imbedding $B\mathrm{SL}_n(k)_\varepsilon\hookrightarrow B\mathrm{SL}_n(k)\hookrightarrow B\mathrm{SL}(k)$ induces the zero homomorphism on $\tilde{H}_*(-,\mathbb{Z}/m)$ for small enough ε.*

This follows immediately from Proposition 4.3 and the following.

4.4.1. For any field k the homomorphism $H_*(\mathrm{SL}(k))\to H_*(\mathrm{GL}(k))$ is split injective. The splitting is induced by the homomorphism $\mathrm{GL}(k)\to\mathrm{SL}(k)$ given by the formula

$$\alpha\to\begin{pmatrix}\det(\alpha)^{-1} & 0\\ 0 & \alpha\end{pmatrix}.$$

Corollary 4.5. *If ε is small enough, then $\tilde{H}_i(B\mathrm{SL}_n(k)_\varepsilon,\mathbb{Z}/m)=0$ for $0\le i\le(n-1)/2$.*

Proof. Consider the Serre spectral sequence of the fibration

$$B\mathrm{SL}_n(k)_\varepsilon\to B\mathrm{SL}_n(k)\to B\mathrm{SL}_n(k)^{\text{top}}.$$

This spectral sequence together with the fact (see [11]) that

$$H_*(B\mathrm{SL}_n(k),\mathbb{Z}/m)\to H_*(B\mathrm{SL}_n(k)^{\text{top}},\mathbb{Z}/m)$$

is onto shows that if i_0 is the least positive integer for which $H_{i_0}(B\mathrm{SL}_n(k)_\varepsilon,\mathbb{Z}/m)\neq 0$, then

$$H_{i_0}(B\mathrm{SL}_n(k)_\varepsilon,\mathbb{Z}/m)\hookrightarrow H_{i_0}(B\mathrm{SL}_n(k),\mathbb{Z}/m).$$

If $i\le(n-1)/2$, then the homology stability theorem ([18], [17]) shows that

$$H_i(BSL_n(k), \mathbb{Z}/m) = H_i(BSL(k), \mathbb{Z}/m)$$

and we deduce from Corollary 4.4 that

$$H_i(BSL_n(k)_\varepsilon, \mathbb{Z}/m) \to H_i(BSL_n(k), \mathbb{Z}/m)$$

is the zero homomorphism. Thus $i_0 > (n-1)/2$.

Corollary 4.6. *$BSL(k)^+ \to BSL(k)^{\text{top}}$ induces isomorphisms of homology and homotopy groups with finite coefficients.*

Proof. The statement concerning homology follows from 4.5. Since both spaces are simply connected this implies the homotopy statement [12, §3].

Corollary 4.7. *$BGL(k)^+ \to BGL(k)^{\text{top}}$ induces isomorphisms of homology and homotopy groups with finite coefficients.*

Proof. We have a commutative diagram of fibrations

$$\begin{array}{ccccc} BSL(k)^+ & \longrightarrow & BGL(k)^+ & \longrightarrow & Bk^* \\ \downarrow & & \downarrow & & \downarrow \\ BSL(k)^{\text{top}} & \longrightarrow & BGL(k)^{\text{top}} & \longrightarrow & Bk^{*\text{top}} \end{array}$$

The edge vertical arrows induce isomorphisms of homology and homotopy groups with finite coefficients, hence the same is true for the middle arrow.

Remark 4.7.1. The above fibrations are in fact trivial.

Corollary 4.8. *$BGL_n(k) \to BGL_n(k)^{\text{top}}$ induces isomorphisms on $H_i(-, \mathbb{Z}/m)$ for $i \leq n$.*

This follows from Corollary 4.6 since $H_i(GL_n(k)) = H_i(GL(k))$ for $0 \leq i \leq n$ and any infinite field k [19].

Theorem 4.9. *Modulo uniquely divisible groups the K-theory of the fields $\mathbb{R}$ and $\mathbb{C}$ are as displayed in Table 1 ($i > 0$).*

Table 1

i mod 8	0	1	2	3	4	5	6	7
$K_i(\mathbb{R})$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Q}/\mathbb{Z}$	0	0	0	$\mathbb{Q}/\mathbb{Z}$
$\downarrow$	0	inclusion	0	mult. by 2	0	0	0	iso.
$K_i(\mathbb{C})$	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$	0	$\mathbb{Q}/\mathbb{Z}$

Proof. Denote by Fk ($k = \mathbb{R}$ or $\mathbb{C}$) the homotopy fiber of the rational localization map $BGL(k)^{top} \to BGL(k)^{top} \otimes \mathbb{Q}$. It is clear that $Fk \to BGL(k)^{top}$ induces isomorphisms of homotopy groups with finite coefficients. Weibel [20] has proved that Fk is a retract of $BGL(k)^+$ in such a way that the composition $Fk \to BGL(k)^+ \to BGL(k)^{top}$ coincides with the natural map. This implies that $Fk \to BGL(k)^+$ induces isomorphisms of the homotopy groups with finite coefficients. Since Fk is a retract of $BGL(k)^+$ we deduce that

$$K_i(k) = \pi_i(BGL(k)^+) = \pi_i(Fk) \oplus (\text{uniquely divisible group}).$$

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