Europ. J. Combinatorics (1997) **18,** 59–64

A Family of One-regular Graphs of Valency 4

DRAGAN MARUŠIČ

A graph is said to be *one-regular* if its automorphism group acts regularly on the set of its arcs. A construction of an infinite family of one-regular graphs of valency 4 with vertex stabilizer Z_2^2 having a non-solvable group of automorphisms is given. The smallest graph in this family has 60 vertices.

1. INTRODUCTORY REMARKS

Throughout this paper, graphs are simple and undirected. Furthermore, all graphs and groups are assumed to be finite. For group-theoretic concepts not defined here, we refer the reader to [11, 14].

Given a group G and a generating subset Q of G such that $Q = Q^{-1}$ and $1 \notin Q$, the *Cayley graph* $Cay(G, Q)$ *of* G *relative to* Q has vertex set G and edges of the form $[g, gq]$, $g \in G, q \in Q$. Cayley graphs of cyclic groups are called *circulants.*

The action of a transitive permutation group G on a set V is said to be *regular* if $|G| = |V|$. Let X be a graph and let $\mathrm{Aut}\, X$ be its automorphism group. A subgroup $G \leqslant \mathrm{Aut}\, X$ is said to be *one-regular* if it acts regularly on the set of arcs of X. A graph X is said to be *vertex-transitive, edge-transitive, arc-transitive* and *one-regular* if $\mathrm{Aut}\, X$ is *vertex-transitive, edge-transitive, arc-transitive* and *one-regular,* respectively.

One-regular graphs are not rare. It may be inferred from the results proved in [2] and [7] that in any primitive action of S_n, other than actions on subsets and partitions, almost all the orbital graphs are one-regular. However, this fact alone is not enough to deduce an infinite family of one-regular graphs of bounded valency. The object of this paper is to give a construction of an infinite family of one-regular graphs with bounded valency; more precisely, with valency 4.

Since cycles of any length are one-regular graphs, the first interesting case is valency 3. In fact, quite a lot of work has been done on cubic one-regular graphs as part of a more general problem dealing with the investigation of a wider class of cubic arc-transitive graphs. The first paper along these lines is the greatly overlooked construction of a cubic one-regular graph due to Frucht [4]. Much later, a general construction for cubic one-regular graphs of girth 6 was proposed by Miller [10]. A concrete realization of this construction is given in [3], with a family of one-regular Cayley graphs of the dihedral groups D_{2p}, where $p \equiv 1 \pmod 6$ is a prime greater than or equal to 13. This construction can be generalized to all dihedral groups D_{2n}, $n > 7$, such that the Euler function $\phi(n) \equiv 1 \pmod 6$. This is done via a rather straightforward observation that a cubic graph X is one-regular iff its line graph is a vertex and edge but not arc-transitive graph of valency 4 and girth 3 (see [9]). A construction of such graphs of valency 4 is given in [1]. A further infinite family of cubic one-regular graphs is made up from finite alternating and symmetric groups of degree congruent to 1 modulo 6 (see [3]).

Regarding even valencies greater than 2, there is the classical construction of one-regular circulants on a prime number of vertices. Namely, let $p \geqslant 5$ be a prime and S be a proper multiplicative subgroup of Z_p^* of order $2d$. Then the circulant

0195-6698/97/010059 + 06 $25.00/0 ej950076

$Cay(Z_p, S)$ is a one-regular graph of valency $2d$ with the vertex stabilizer a cyclic group of order $2d$. This is seen using the well known Burnside theorem on transitive permutation groups of prime degree [11, p. 53], which states that a simply transitive permutation group of prime degree p may be identified with a proper subgroup of the group $\{x \to ax + b : a \in Z_p^*, b \in Z_p\}$, $(x \in Z_p)$ containing the group $\{x \to x + b : b \in Z_p\}$, $(x \in Z_p)$. With this notation the automorphism group of the graph $Cay(Z_p, S)$ above is precisely the group $\{x \to ax + b : a \in S, b \in Z_p\}$, $(x \in Z_p)$.

In this paper we are concerned with one-regular graphs of valency 4. The investigation of arc-transitive graphs of valency 4 has received considerable attention [5, 6, 8, 12, 13]. Despite that, one-regular graphs of valency 4 are still a rather untravelled field. To the author's best knowledge, the above circulant construction with $d = 2$ is the only one that generates one-regular graphs of valency 4. For example, no construction of one-regular graphs of valency 4 with a non-solvable group is known. The main purpose of this paper is to give a construction for an infinite family of one-regular graphs of valency 4 with vertex stabilizer Z_2^2 and a non-solvable group of automorphisms. In particular, for each alternating group A_n, $n \geqslant 5$ odd, we shall construct a Cayley graph with one-regular automorphism group $S_n \times Z_2$ (Theorem 2.3).

2. The Construction

Given a graph X and a 2-path $[u, v, w]$ in X, we let $\mathscr{C}(u, v, w)$ denote the set consisting of all possible lengths of cycles containing the 2-path $[u, v, w]$. The following lemma will prove useful later on.

Lemma 2.1. *Let X be a connected graph such that, for any two adjacent vertices $u, v \in V(X)$, the sets $\mathscr{C}(u, v, x)(x \in N(v)\backslash\{u\})$ are all distinct. Then no non-identity automorphism of X fixes two adjacent vertices.*

Proof. The proof is straightforward. Let $A = \operatorname{Aut} X$. Assume that u and v are adjacent in X and let $\alpha \in A_{u,v}$. By the assumption, we must have that α fixes the neighbors of u and those of v. Replacing first u by any other neighbor of v and then v by any other neighbor of u, the same argument as above gives us that α fixes all vertices at distance 2 from u and those at distance 2 from v. Continuing in this way, the connectedness of X implies that $\alpha = 1$. □

Let Q be a generating set of a group G such that $Q = Q^{-1}$ and $1 \notin Q$. By $\operatorname{Aut}(G, Q) = \{\alpha \in \operatorname{Aut} G : Q^\alpha = Q\}$, we denote the subgroup of all those automorphisms of G which fix Q. The proof of the following result is straightforward and is omitted.

Proposition 2.2. *Let $X = \operatorname{Cay}(G, Q)$. Then* $\operatorname{Aut} X \geqslant G \cdot \operatorname{Aut}(G, Q)$, where G is identified with its left regular representation.

We may now give the construction of an infinite family of one-regular graphs of valency 4 with a non-solvable automorphism group. The following is the main result of this paper.

Theorem 2.3. *Let $k \geqslant 2$ be a positive integer, $n = 2k + 1$ and $a, b \in S_n$, where $a = (0, 1, 2, \ldots, n-1)$ and $b = a^t$, with $t = (0, 1)$. Let $G = \langle a, b \rangle = A_n$. Then $X_n = \operatorname{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ is a one-regular graph of valency 4 and $\operatorname{Aut} X_n = S_n \times Z_2$.*

In order to prove Theorem 2.3, a few preliminary remarks are in order. We make the convention that the multiplication in S_n will be done from left to right. In other words, for $x, y \in S_n$ and $i \in \{0, 1, \ldots, n-1\}$, we have $xy(i) = y(x(i))$. Hereafter, the symbols a, b, G and X_n will have the same meaning as in the statement of Theorem 2.3 above.

The proof of the next lemma is omitted.

LEMMA 2.4. *Let r be the involution mapping according to the rule $r(i) = n - i + 1$, $i \in \{0, 1, \ldots, n-1\}$, with the addition taken modulo n. Then r normalizes both $\langle a \rangle$ and $\langle b \rangle$. More precisely, $a^r = a^{-1}$ and $b^r = b^{-1}$.*

Let $S = (s_1, s_2, \ldots, s_l)$, where each of the s_i is one of a, a^{-1}, b and b^{-1}. For integers j, h satisfying $1 \leqslant j \leqslant h \leqslant l$, let $\pi_{j,h}$ denote the product $s_j s_2 \cdots s_h$, and let $\pi_S = \pi_{1,l}$. Then S is called a *sequence* if, for all j, h such that $1 \leqslant j \leqslant h \leqslant l$, $\pi_{j,h} \neq 1$, with the possible exception that π_S may be equal to 1. For convenience, the notation for a sequence $S = (s_1, s_2, \ldots, s_l)$ will be shortened to $S = s_1 s_2 \cdots s_l$. By $l(S) = l$ we denote the *length* of S in this case. The *inverse sequence* of S is the sequence $S^{-1} = s_l^{-1} s_{l-1}^{-1} \cdots s_1^{-1}$. If $\pi_S = 1$ we say that S is a *relation* (in G). Of course, every relation gives rise to a cycle in X_n. We say that two sequences of equal length are *equivalent* if one may be obtained from the other by a finite series of transformations of the following four types: a cyclic rotation, taking the inverse sequence, interchanging a with b and a^{-1} with b^{-1}, or interchanging a with a^{-1} and b with b^{-1}. Note that the corresponding equivalence relation on sequences of a given length distinguishes between relations and non-relations. The fact that the transformations of the first two types preserve relations and non-relations is self-evident. As for the transformation of the third type, it corresponds to a conjugation by $t = (01)$ and so it preserves relations and non-relations. Similarly, it may be seen that the transformation of the fourth type corresponds to a conjugation by the involution r defined in Lemma 2.4.

LEMMA 2.5. *Sequences $s_1 s_2 \cdots s_{l-1} s_l$ and $s_l s_{l-1} \cdots s_2 s_1$ are equivalent.*

PROOF. This is implied by the fact that $s_l s_{l-1} \cdots s_2 s_1$ is obtained from $s_1 s_2 \cdots s_{l-1} s_l$ by first taking the inverse sequence and then applying the transformation of the fourth type which interchanges each term by its inverse. □

The following lemma is of crucial importance to the proof of Theorem 2.3.

LEMMA 2.6. *Let $k \geqslant 2$ be a positive integer, $n = 2k + 1$ and $a, b \in S_n$, where $a = (0, 1, 2, \ldots, n-1)$ and $b = a^t$, with $t = (0, 1)$. Let $c \geqslant 1$ and $S = a^{\varepsilon_1} b^{\varepsilon_2} \cdots a^{\varepsilon_{2c-1}} b^{\varepsilon_{2c}}$ be a relation of length $l \leqslant n$ in $G = \langle a, b \rangle$ such that $\varepsilon_1 > 0$. Then one of the following is true:*

(i) *$S = (ab)^k$ or*

(ii) *$\sum_{i=1}^{2c} \varepsilon_i = 0$.*

PROOF. As $\langle a \rangle \cap \langle b \rangle = 1$, the fact that S is a relation forces $c \geqslant 2$. For each $j \in \{1, \ldots, 2c\}$ let $e_j = \sum_{i=1}^{j} \varepsilon_i$. Set $P = \{e_j : e_j > 0\} \cup \{0\}$ and $N = \{|e_j| : e_j < 0\} \cup \{0\}$. Furthermore, let M and m be the maxima of the sets P and N, respectively. Clearly, $m + M \leqslant n$.

Let us first assume that $m + M \leqslant n - 3$. Then there exists an integer i such that $2 + m \leqslant i \leqslant n - 1 - M$. The choice of i forces $\pi_S(i) = a^{e_{2c}}(i) = i - e_{2c}$. But S is a relation and so $i + e_{2c} = i$, and thus n must divide e_{2c}. However, $\varepsilon_1 > 0$ and $e_{2c} \leqslant l \leqslant n$, and so we must have that $e_{2c} = \sum_{i=1}^{2c} \varepsilon_i$ is either 0 or n. The first possibility gives us (ii), whereas the latter implies $M = n$, contradicting the assumption that $m + M \leqslant n - 3$. We may therefore assume that

$$m + M \in \{n-2, n-1, n\}. \tag{1}$$

Suppose first that $\varepsilon_i > 0$ for each i. Then, of course, $m + M = l$ and we have essentially two different possibilities. Either $\varepsilon_i = 1$ for each i and thus $S = (ab)^c$, or S is equivalent to some $T = Rab^2$, where R is a sequence of length $l - 3$ with all positive exponents. Since $ab = (1, 3, 5, \ldots, n-2)(2, 4, 6, \ldots, n-1)$ the first possibility implies $c = k$ and $l(S) = 2k = n - 1$, giving us (i). The second possibility splits into three cases, depending on the length of S. Each of them leads to a contradiction. First, if $l(T) = l(S) = n$, then $\pi_R(2) = 0$, and since $ab^2(0) = 3$ we have $\pi_T(2) = 3$, contradicting the fact that S is a relation. Second, if $l(T) = l(S) = n - 1$, then $\pi_R(2) = 2k$, and since $ab^2(n-1) = 0$ we have $\pi_T(2) = 0$, a contradiction. Finally, if $l(T) = l(S) = n - 2$, then $\pi_R(2) = n - 2$ and, since $ab^2(n-2) = 1$, we have that $\pi_T(2) = 1$, a contradiction.

We may now assume that not all ε_i are positive integers. Combining the fact that $\varepsilon_1 > 0$ together with (1) we end up with only four different cases to consider. Each of them will lead to a contradiction.

Case 1: $\varepsilon_1 \in \{1, 2\}$ and $\varepsilon_i < 0$ for all other i. In this case, we have $M = \varepsilon_1$ and $m = l - 2\varepsilon_1$. Hence $m + M = l - \varepsilon_1 \in \{n-2, n-1, n\}$. Thus, for $\varepsilon_1 = 1$, we have that l is either n or $n - 1$ and, for $\varepsilon_1 = 2$, we have that $l = n$. Suppose first that $l(S) = n$ and $\varepsilon_1 = 1$. By computation, $\pi_S(n-2) = n - 1$ if $\varepsilon_{2c} = -1$ and $\pi_S(n-2) = 1$ if $\varepsilon_{2c} \leqslant -2$. Suppose next that $l(S) = n - 1$ and $\varepsilon_1 = 1$. By computation, $\pi_S(n-2) = 0$. Finally, suppose that $\varepsilon_1 = 2$ and $l = n$. If follows that $\pi_S(n-3) = 0$. Each of these contradicts the fact that S is a relation.

Case 2: $\varepsilon_1 = \varepsilon_{2c} = 1$ and $\varepsilon_i < 0$ for all other i. Here we have $M = 1$, $m = l - 3$ and so $M + m = l - 2$ and, since $l \leqslant n$, it follows that $l = n$. Then, by computation, $\pi_S(n-2) = 0$, a contradiction.

Case 3: $\varepsilon_{2c} \in \{-1, -2\}$ and $\varepsilon_i > 0$ for all other i. Assume first that $\varepsilon_{2c} = -1$. Then $M - l - 1$ and $m = 0$, and so l is either n or $n - 1$. In the first case $\pi_S(2) = 0$ if $\varepsilon_{2c-1} = 1$ and $\pi_S(2) = n - 1$ if $\varepsilon_{2c-1} \geqslant 2$. If $l = n - 1$, then it follows that $\pi_S(1) = n - 2$. Next, assume that $\varepsilon_{2c} = -2$. Then $M = l - 2$, $m = 0$ and so $l = n$. By computation, $\pi_S(1) = n - 3$. Again, none of these is possible, since S is a relation.

Case 4: $\varepsilon_j = -1$ for some $1 < j < 2c$ and $\varepsilon_i > 0$ for all other i. Here we have $M = l - 2$ and $m = 0$ and so $l = n$. It follows that $\pi_S(2) = 1$, a contradiction.

This completes the proof of Lemma 2.6. □

The following is an immediate consequence of Lemma 2.6.

Corollary 2.7. *Let $n = 2k + 1$, let $a = (0, 1, 2, \ldots, n-1)$ and $b = a^t$ with $t = (0, 1)$. Let $G = \langle a, b\rangle$. Then the only odd cycles in $X_n = \mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ of length at most n have length n and are generated by the relations $S = a^n$ and $S = b^n$.*

Next, we characterize cycles of length 6 in $X_n = \mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$.

Lemma 2.8. *Let $n = 2k + 1 \geqslant 7$, let $a = (0, 1, 2, \ldots, n-1)$ and $b = a^t$ with $t = (0, 1)$. Let $G = \langle a, b\rangle$. Then a 6-cycle in $X_n = \mathrm{Cay}(G, \{a, a^{-1}, b, b^{-1}\})$ arises from a relation equivalent to $(ab^{-1})^2$ or, if $n = 7$, also from a relation equivalent to $(ab)^2$.*

PROOF. Observe that

$$ab^{-1} = (1, 0, n-1) \tag{2}$$

and so $(ab^{-1})^2$ is a relation of length 6. Also, by Lemma 2.6, $(ab)^2$ is a relation of length 6 for $n = 7$. Let S be a sequence of length 6 non-equivalent to either of the above sequences. Then, using Lemmas 2.5 and 2.6, we have that, up to equivalence, S is one of the following sequences: a^3b^{-3}, $a^3b^{-1}a^{-1}b^{-1}$, $a^2b^{-2}a^{-1}b$, $a^2b^{-2}ab^{-1}$, $a^2ba^{-2}b^{-1}$, $abab^{-1}a^{-1}b^{-1}$ or $aba^{-1}b^{-1}ab^{-1}$. By computation, we have that in the first case $\pi_S(0) = n-1$, in the second case $\pi_S(0) = 2$, in the third case $\pi_S(1) = n-1$, in the fourth case $\pi_S(0) = n-1$, in the fifth case $\pi_S(0) = n-3$ and in the sixth case $\pi_S(0) = n-2$. Hence none of these sequences is a relation. This completes the proof of Lemma 2.8. □

We are now ready to prove Theorem 2.3.

PROOF OF THEOREM 2.3. Let $A = \mathrm{Aut}\, X_n$. Given any permutation x of $\{0, 1, \ldots, n-1\}$ let α_x denote the action of x on G by conjugation. Clearly, $\alpha_t \in \mathrm{Aut}\, G$. In fact, α_t interchanges a and b, and so $\alpha_t \in \mathrm{Aut}(G, \{a, a^{-1}, b, b^{-1}\})$. Hence, by Lemma 2.2, we have that $A \geqslant G \cdot \langle \alpha_t \rangle = \mathrm{Aut}\, G$, where G is identified with its left regular representation.

Let ρ_r denote the right multiplication by the involution r (defined in Lemma 2.4) on G. It is not difficult to check that both α_r and ρ_r are automorphisms of X_n; namely, for an arbitrary vertex v of X_n we have that, in view of Lemma 2.4, $N(\alpha_r(v)) = N(rvr) = \{rvra, rvra^{-1}, rvrb, rvrb^{-1}\} = \{rva^{-1}r, rvar, rvb^{-1}r, rvbr\} = \alpha_r(N(v))$. So $\alpha_r \in A$. Similarly, $\rho_r \in A$. Clearly, $\alpha_r \notin \mathrm{Aut}\, G$. Also, both α_t and α_r belong to the vertex stabilizer A_1. In fact, they commute as t and r do and hence $\langle \alpha_t, \alpha_r \rangle$ is isomorphic to Z_2^2. Clearly, the group $\langle \mathrm{Aut}\, G, \rho_r \rangle = \langle G, \alpha_t, \alpha_r \rangle$ is one-regular. Note that $\mathrm{Aut}\, G = S_n$ and, since ρ_x commutes with every element of $\mathrm{Aut}\, G$, we have that $\langle \mathrm{Aut}\, G, \rho_r \rangle = S_n \times Z_2$. In order to prove that X_n is one-regular with the automorphism group $S_n \times Z_2$, it remains to show that $A = \langle \mathrm{Aut}\, G, \rho_r \rangle$. This will be done by proving that $|A_1| = 4$ or, equivalently, by proving that $A_{1,a}$ is trivial. We shall for the most part rely on the use of Lemma 2.1. Let us analyse the structure of the sets $\mathscr{C}(a, 1, b)$, $\mathscr{C}(a, 1, b^{-1})$ and $\mathscr{C}(a, 1, a^{-1})$. By Corollary 2.7 it follows that $n \notin \mathscr{C}(a, 1, b) \cup \mathscr{C}(a, 1, b^{-1})$. Hence, as $n \in \mathscr{C}(a, 1, a^{-1})$, we have that

$$\mathscr{C}(a, 1, a^{-1}) \neq \mathscr{C}(a, 1, b),\ \mathscr{C}(a, 1, b^{-1}). \tag{3}$$

Suppose first that $n \geqslant 9$. Then Lemma 2.8 implies that $6 \in \mathscr{C}(a, 1, b)$ and $6 \notin \mathscr{C}(a, 1, b^{-1})$. Hence $\mathscr{C}(a, 1, b) \neq \mathscr{C}(a, 1, b^{-1})$. Combining this with (3) we have that, in view of Lemma 2.1, $A_{1,a}$ is trivial and so X_n is one-regular.

We are left with $n \in \{5, 7\}$. Suppose that $n = 5$. By Lemma 2.6(i), we have that $(ab)^2$ is a relation and so $4 \in \mathscr{C}(a, 1, b^{-1})$. In fact, $(ab)^2$ is the only relation of length 4; namely, by Lemma 2.6(ii), a relation S of length 4 different from $(ab)^2$ must be a conjugate of one of the following sequences: a^2b^{-2}, $aba^{-1}b^{-1}$, $ab^{-1}a^{-1}b$ and $ab^{-1}ab^{-1}$. But $\langle a \rangle$ and $\langle b \rangle$ have trivial intersection and so the first sequence is not a relation. Moreover, b does not normalize $\langle a \rangle$ and therefore the second and the third sequences are not relations. Finally, the last sequence is not a relation in view of (2). In particular, we may conclude that $4 \notin \mathscr{C}(a, 1, b)$. Hence $\mathscr{C}(a, 1, b) \neq \mathscr{C}(a, 1, b^{-1})$ for $n = 5$. This, together with (3), shows that, in view of Lemma 2.1, $A_{1,a}$ is trivial and so X_5 is one-regular.

It remains to settle the case $n = 7$. By Lemma 2.8, it follows that $6 \in \mathscr{C}(a, 1, b) \cap \mathscr{C}(a, 1, b^{-1})$. In particular, every edge of X_7 is contained on two 6-cycles. Assume X_7

is not one-regular. Then there exists $\gamma \in A_1$ fixing a and a^{-1} and interchanging b with b^{-1}. By Corollary 2.7, it follows that the collection of 7-cycles generated by the relations a^7 and b^7 forms an imprimitivity block system of A. We must therefore have that

$$\gamma(u)^{-1}\gamma(v) \in \{u^{-1}v, v^{-1}u\} \qquad \text{for each edge } [u, v] \text{ of } X_7. \tag{4}$$

Let $C_1 = (1, a, ab, aba, b^{-1}a^{-1}, b^{-1})$ and $C_2 = (1, a, ab^{-1}, ab^{-1}a, ba^{-1}, b)$ be the two 6-cycles containing the edge $[1, a]$. Clearly, γ interchanges C_1 and C_2 and so, in particular, γ interchanges ab^{-1} and ab. Similarly, considering the two 6-cycles on the edge $[1, a^{-1}]$, we obtain that γ interchanges $a^{-1}b$ and $a^{-1}b^{-1}$. Now γ must interchange the neighbors of ab with those of ab^{-1} and so, by (4), it follows that, in particular, γ interchanges ab^2 and ab^{-2}. Similarly, γ interchanges $a^{-1}b^2$ and $a^{-1}b^{-2}$. Observe that $a^2b^{-2} = (0, 1)(5, 6)$. Hence the sequence $(a^2b^{-2})^2$ generates an 8-cycle containing the 2-path $[1, a, ab^{-1}]$; namely, the cycle $W = (1, a, ab^{-1}, ab^{-2}, ab^{-2}a, a^{-1}b^2, a^{-1}b, a^{-1})$. Then $\gamma(W) = (1, a, ab, ab^2, \gamma(ab^{-2}a), a^{-1}b^{-2}, a^{-1}b^{-1}, a^{-1})$. It follows that $\gamma(ab^{-2}a)$ is a common neighbor of ab^2 and $a^{-1}b^{-2}$. Furthermore, in view of (4), it follows that $\gamma(ab^{-2}a) \in \{ab^2a, ab^2a^{-1}\} \cap \{a^{-1}b^{-2}a, a^{-1}b^{-2}a^{-1}\}$. But then one of the following three sequences would have to be a relation: a^2b^4, $(a^2b^2)^2$ or $a^2b^2a^{-2}b^2$. It is easily checked that this is not the case. This contradiction shows that X_7 is one-regular, concluding the proof of Theorem 2.3. □

Acknowledgements

This work was supported in part by the 'Ministrstvo za znanost in tehnologijo Slovenije', proj. no. J1-7035-94.

References

1. B. Alspach, D. Marušič, and L. Nowitz, Constructing graphs which are $\frac{1}{2}$-transitive, *J. Austral. Math. Soc. A,* **56** (1994), 391–402.
2. P. J. Cameron and W. M. Kantor, Random permutations: some group-theoretic aspects, *Combinatorics, Probability and Computing,* **2** (1993), 257–262.
3. M. D. E. Conder and C. E. Praeger, Remarks on path-transitivity in finite graphs, *Europ. J. Combin.,* **17** (1996), 371–378.
4. R. Frucht, A one-regular graph of degree three, *Can. J. Math.,* **4** (1952), 240–247.
5. A. Gardiner, An elementary classification of distance-transitive graphs of valency four, *Ars Combin.,* **19** (1985), 129–141.
6. A. Gardiner and C. E. Praeger, On 4-valent symmetric graphs, *Europ. J. Combin.,* **15** (1995), 375–381.
7. A. Gardiner and C. E. Praeger, A characterization of certain families of 4-valent symmetric graphs, *Europ. J. Combin.,* **15** (1995), 383–397.
8. M. W. Liebeck, C. E. Praeger and J. Saxl, A classification of the maximal subgroups of the symmetric and alternating groups, *J. Algebra,* **111** (1987), 365–383.
9. D. Marušič and M.-Y. Xu, A $\frac{1}{2}$-transitive graph of valency 4 with a non-solvable automorphism group, *J. Graph Theory,* submitted.
10. R. C. Miller, The trivalent symmetric graphs of girth at most six, *J. Combin. Theory,* **10** (1971), 163–182.
11. D. S. Passman, *Permutation Groups,* Benjamin, Menlo Park, California, 1968.
12. C. C. Sims, Graphs and finite permutation groups, *Math. Z.,* **95** (1967), 76–86.
13. C. C. Sims, Graphs and finite permutation groups II, *Math. Z.,* **103** (1968), 276–281.
14. H. Wielandt, *Finite Permutation Groups,* Academic Press, New York, 1964.

Received 28 May 1995 and accepted in revised form 9 October 1995

Dragan Marušič
IMFM, Oddelek za matematiko, Univerza v Ljubljani,
Jadranska 19, 61111 Ljubljana, Slovenija