Journal of Number Theory 129 (2009) 231–245

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Computation of Galois groups associated to the 2-class towers of some imaginary quadratic fields with 2-class group $C_2 \times C_2 \times C_2$

Harris Nover

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

ARTICLE INFO

Article history:
Received 5 September 2007
Revised 23 June 2008
Available online 12 September 2008
Communicated by Gebhard Böckle

MSC:
primary 11Y40
secondary 11R11, 11R32, 11R37, 11Y40, 12F10, 20D15

ABSTRACT

We describe a method for the explicit computation of a list of possibilities for the Galois group G of an unramified 2-class tower that combines the p-group generation algorithm with algorithms from explicit class field theory. We successfully applied this method to 19 of the 36 imaginary quadratic fields of absolute discriminant less than 20,000 that have 2-class group $(2, 2, 2)$, three negative prime discriminant factors in their discriminant, and whose 2-class towers have derived length at least 3. This is the only class of imaginary quadratic fields with 2-class group $(2, 2, 2)$ and three negative prime discriminant factors not entirely classified by recent work of Benjamin, Lemmermeyer and Snyder. Additionally, among the 19 are all such fields whose 2-class towers, if infinite, would provide improved upper bounds for the root discriminant problem. In each case we show that these 2-class towers are finite, and in fact write down for each a short list of candidate groups for the associated Galois groups. Some of these results are unconditional, while some require the Generalized Riemann Hypothesis.

1. Introduction

Let K be a number field and p a rational prime. An important object of study in number theory is $K^{un,p}$, the maximal unramified pro-p extension of K. Consider the p-class tower associated to K, $K = K_{0,p} \subseteq K_{1,p} \subseteq K_{2,p} \subseteq \cdots$, where $K_{i+1,p}$ is the Hilbert p-class field of $K_{i,p}$, that is, the maximal unramified abelian p-extension of $K_{i,p}$. Let $G = \mathrm{Gal}(K^{un,p}/K)$. Then the fields in the tower are the

E-mail address: nover@math.wisc.edu.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2008.06.007

fixed fields of the terms of the derived series of G, so the length of the tower equals the dervied length of G. As pro-p groups are pro-solvable, we further have that $\bigcup_i K_{i,p} = K^{un,p}$.

As class groups, and hence abelianizations of open subgroups of G, are finite, note that the tower is of finite length if and only if G is finite. While it is generally difficult to lay our hands computationally on a $K_{i,p}$ directly, combining group theory with explicit class field theory often will allow us partially to determine G.

In [3], Boston and Leedham-Green introduced a method to compute a short list of candidates for the Galois group associated to a tamely-ramified p-class tower of $\mathbb{Q}$. In [5], Bush modified that method to compute the unramified 2-class towers of certain imaginary quadratic fields of interest, all of whose class groups had 2-rank 2.

Bush was particularly interested in the field $\mathbb{Q}(\sqrt{-2379})$ due to its potential application to the problem of finding the minimal asymptotic root discriminant. For any number field L, define the root discriminant rd_L as $|\mathrm{Disc}_{L/\mathbb{Q}}|^{1/[L:\mathbb{Q}]}$. Let R_n be the minimal root discriminant over all number fields of degree n, and let $R = \liminf_{n\to\infty} R_n$. The Generalized Riemann Hypothesis implies that $R \geqslant 44.7$, and it is of interest to determine how tight this bound is. As all number fields in an unramified tower have the same root discriminant, if a number field K has an infinite unramified tower then $R \leqslant \mathrm{rd}_K$. The best upper bound for R at the moment is 82.2, due to Hajir and Maire [8]. Stark noted that if $\mathbb{Q}(\sqrt{-2379})$ had infinite 2-class tower, then the upper bound would be dramatically tightened. Hajir had earlier generalized a result of Koch showing that an imaginary quadratic field with 4-rank at least 3 has infinite 2-class tower [7], so it was natural to investigate an example of the 4-rank 2 case. However, Bush showed that this field in fact had finite 2-class tower.

The basic idea of both methods is to use explicit class field theory to compute the appropriate p-class groups of both K and its low degree extensions with appropriate ramification. These correspond to the abelianizations of G and its low index subgroups. One also has some information on the relation ranks of these groups from Galois cohomology. One then searches for G in the appropriate tree generated by O'Brien's p-group generation algorithm [9]. This tree contains, up to isomorphism, every p-group with exactly d generators, organized by increasing p-class, where H_0 and H_1 are connected if and only if $H_0/P_k(H_0)$ is isomorphic to H_1 and H_0 is of p-class $k+1$. By starting at the root, the elementary abelian p-group with d generators, and pruning the tree by the arithmetic data, one gets a list of candidates for G. If all groups below some p-class have been pruned away, then G must be on the list, and in particular G must be finite.

Since Bush's work, there has been interest in finding other imaginary quadratic fields with infinite 2-class towers but small root discriminants. Various heuristics suggested that the next most promising imaginary quadratic field was $\mathbb{Q}(\sqrt{-3135})$, but direct attempts to apply existing methods in this and other cases resulted in a combinatorial explosion in the tree.

The field $\mathbb{Q}(\sqrt{-3135})$ has also come up recently in another context. In [1], Benjamin, Lemmermeyer and Snyder made a study of imaginary quadratic fields with 2-class group isomorphic to $C_2 \times C_2 \times C_2$. They divided up such fields based on the diagram formed by the quadratic reciprocity relationships among the prime factors of the discriminant. Genus theory tells us that the discriminants of such fields will factor into 4 prime discriminants (treating ± 4 in the normal way). If three of these are positive, $G/[G', G]$ will have order 64, while if three are negative $G/[G', G]$ will have order 32. For each class of fields they computed exactly what $G/[G', G]$ is. For those classes of fields where $|G/[G', G]| = 32$, in all but one case knowledge of this quotient allowed them to explicitly determine G, which was then seen to have derived length 2. In the remaining case, $G/[G', G]$ is isomorphic to the group denoted by 32.033 in the Hall-Senior notation and by SmallGroup(32, 27) in Magma/GAP and the corresponding tower must have length at least 3. The field $\mathbb{Q}(\sqrt{-3135})$ has the smallest discriminant of any imaginary quadratic field in this class.

In [4], the author outlined his work on an improvement of the method of Boston and Leedham-Green and Bush that showed that the 2-class tower of $\mathbb{Q}(\sqrt{-3135})$ is finite of length 3. In this paper we expand on these results. Using the criterion developed in [1], one finds that there are 36 imaginary quadratic fields with 2-class group isomorphic to $C_2 \times C_2 \times C_2$, with $|G/[G', G]| = 32$, with 2-class tower of length at least 3, and with absolute discriminant less than 20,000. For 19 of these fields our method finds that the 2-class tower is finite of length 3. These 19 include in particular all such fields with root discriminant less than 90, so the 2-class tower of a field of this type will not provide an improved upper bound for the root discriminant. For a given field, the candidate groups are typically very similar, generally being of the same order and having the same quotients by non-trivial terms of the central and 2-central series.

Many of our results are conditional on the Generalized Riemann Hypothesis (GRH), as this hypothesis is often used to compute the class groups of some higher degree fields. In several cases though we can show unconditionally that the 2-class towers are finite, even if without GRH we cannot prune the list of candidate groups as thoroughly.

2. Mathematical background

As mentioned above, the algorithm combines class field theory and group theory. We first review some necessary mathematical background.

2.1. Some class field theory

Let K be a number field and $\mathrm{Cl}_p(K)$ the p-part of its class group. We say that L/K is a p-extension if $[L:K]$ is a power of p and L/K is Galois. Class field theory gives us a correspondence between unramified abelian p-extensions of K and subgroups of $\mathrm{Cl}_p(K)$. In particular, for $G = \mathrm{Gal}(K^{un,p}/K)$, we have that $\mathrm{Cl}_p(K) \cong G/G'$, where G' is the closure of the commutator subgroup of G.

Galois theory in fact tells us more. We have a one-to-one correspondence between open subgroups H of G and unramified p-power extensions L/K where the Galois closure of L/K is a p-extension. In this case we further have that $\mathrm{Cl}_p(L)$ is isomorphic to H/H'. By Burnside's basis theorem, we know that the p-rank of H/H' is the generator rank of H, and so this also tells us the generator rank of each such H.

In this paper we are interested in the case $p = 2$. It is not hard to show that a 2-power extension L/K has Galois closure a 2-extension if and only if there is a tower of intermediate fields $K = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n = L$ with $[L_{i+1} : L_i] = 2$ for all i. Consequently, the lattice of open subgroups of G with abelianizations attached corresponds to the lattice of unramified 2-power extensions of K reachable by repeated extensions of degree 2, with 2-class groups attached. This observation is at the heart of our algorithm.

In addition to the abelianization, and hence generator rank, of our subgroups, we also have some notion of their relator rank. In particular, we have

Proposition 1. *If K is a totally complex number field and $G = \mathrm{Gal}(K^{un,2}/K)$ has generator rank d and relator rank r, then we have that*

$$r - d \leqslant [K:\mathbb{Q}]/2.$$

Proof. This follows from [10]. □

2.2. Some p-group theory

Now we must review some group theory. Let G be a finite p-group. We define the lower p-central series[1] of G as $P_0(G) = G$ and $P_{k+1}(G) = [G, P_k(G)]P_k(G)^p$.

We say that G has p-class k if k is the smallest integer such that $P_k(G) = \{1\}$, that is, that $G/P_k(G) \cong G$. Note that $G/P_k(G)$ consequently has p-class the smaller of k and the p-class of G. If G is a pro-p group then the $P_k(G)$ are defined as the closures of the subgroups above. In either case, $P_1(G)$ is the Frattini subgroup of G, the intersection of the open maximal subgroups of G. We say that H is a descendant of G if $H/P_k(H) \cong G$ for some k, and we say that H is an immediate descendant if additionally the p-class of H is $k+1$. These definitions are well-motivated. Given a

[1] Some authors start with $P_1(G) = G$, and some of their other definitions will differ accordingly.

finite p-group G, the number of immediate descendants of G is finite, and an algorithm of O'Brien will compute them explicitly [9].

This leads us to the notion of the O'Brien tree. Fix a number of generators d. Then every p-group of generator rank d is a descendant of the elementary abelian p-group of rank d, say E. The root of our tree will be E. The children of E will be the immediate descendants of E, as computed by O'Brien's algorithm. Each of these vertices will have as children their immediate descendants, and so on. Every p-group of generator rank d will, up to isomorphism, appear exactly once on the corresponding infinite tree. Further, infinite pro-p groups with generator rank d correspond to infinite paths through the tree.

2.3. Inherited properties

From here onward we specialize to the case $p = 2$. Again let K be some number field and let G be $\mathrm{Gal}(K^{un,2}/K)$. What we should like to do is to gather arithmetic data from K and some of its extensions, and then prune our way down the O'Brien tree. To do that, we must know how our conditions on G imply conditions on the $G/P_k(G)$, or in other words how certain properties of groups imply other properties for their children or ancestors. Above we described two properties of G that are often directly computable, depending on the complexity of K and its unramified extensions: the lattice of low index subgroups of G with attached abelianizations and a bound on its relator rank given its generator rank. We now show how these properties, when known, impose requirements on $G/P_k(G)$.

We start by defining a partial order '$\preccurlyeq$' on abelian groups. We will write $A \preccurlyeq B$ exactly when A is realizable as a quotient of B. For a pro-2 group H, let $\mathrm{Ab}(H)$ denote its abelianization H/H'. Then we have

Proposition 2. *Let G be a finitely generated pro-2 group with generator rank d. Let v be 2-adic valuation. Then we have the following*:

(1) $\mathrm{Ab}(G/P_k(G)) \preccurlyeq \mathrm{Ab}(G)$ *for* $k \geqslant 1$, *with equality if* $k \geqslant v([G:G']) - d + 1$.
(2) *For $k \geqslant 1$, the lattices of subgroups of index at most 2 of $G/P_k(G)$ and of G are isomorphic. Further, if the index 2 subgroup $\overline{H} \leqslant G/P_k(G)$ lifts to $H \leqslant G$, then* $\mathrm{Ab}(\overline{H}) \preccurlyeq \mathrm{Ab}(H)$, *with equality if* $k \geqslant v([H:H']) - d + 2$.
(3) *For $k \geqslant 2$, the lattices of subgroups of index at most 4 of $G/P_k(G)$ and of G are isomorphic. Further, if the index 4 subgroup $\overline{M} \leqslant G/P_k(G)$ lifts to $M \leqslant G$, then* $\mathrm{Ab}(\overline{M}) \preccurlyeq \mathrm{Ab}(M)$, *with equality holding for k greater than or equal to $v([M:M']) + 3 - d$ for M is normal and for k greater than or equal to $2v([M:M']) - v([H:H']) + 4 - d$ otherwise, where H is an index 2 subgroup normalizing M.*

First we need a lemma.

Lemma 3. *Let M and N be subgroups of a group G with $N \trianglelefteq G$. If $N \leqslant M$, then* $\mathrm{Ab}(M/N) \preccurlyeq \mathrm{Ab}(M)$. *Further, if $N \leqslant M'$, then* $\mathrm{Ab}(M/N) = \mathrm{Ab}(M)$.

Proof. For any homomorphism φ, $\varphi(M') = \varphi(M)'$. Consequently,

$$(M/N)\big/(M/N)' \cong (M/N)\big/(M'N/N) \cong M/M'N \preccurlyeq M/M'.$$

If $N \leqslant M'$ then $M'N = M'$ and the second statement follows. $\square$

Proof of Proposition 2. For a finite index $N \leqslant G$, we start by finding an upper bound for the value of k such that $P_k(G) \leqslant N$. It suffices to consider the case where N is normal. Note that if $\theta : G \to G_0$ is a homomorphism, then $P_k(G)^\theta = P_k(G^\theta)$. So taking θ to be the canonical projection, we have that $P_k(G/N) = P_k(G)N/N$. Thus $P_k(G) \leqslant N$ if and only if $P_k(G/N) = N$. Also note that for pro-2 groups P, $P_k(P) \neq P_{k+1}(P)$ unless $P_k(P) = \{1\}$, so $[P_k(P) : P_{k+1}(P)] \geqslant 2$ if $P_k(P) \neq \{1\}$.

Assume $P_k(G) \nleqslant N$ for some k. Then

$$\begin{aligned}[G:N] &> \big[G/N : P_k(G/N)\big] \\ &= \big[G/N : P_1(G/N)\big]\big[P_1(G/N) : P_2(G/N)\big]\cdots\big[P_{k-1}(G/N) : P_k(G/N)\big] \\ &\geqslant \big[G/N : P_1(G/N)\big]\cdot 2^{k-1}.\end{aligned}$$

So $k \geqslant \nu([G:N]) - \nu([G/N : P_1(G/N)]) + 1$ implies that $P_k(G) \leqslant N$. Note that $[G/N : P_1(G/N)] = [G : P_1(G)] = 2^d$ if $N \leqslant P_1(G)$, and if not, it is at least 2, and so it suffices to have $k \geqslant \nu([G:N])$.

Now the proposition follows from a lattice isomorphism theorem and the lemma once we show that $P_k(G)$ is contained in the appropriate subgroups of G.

(1) We immediately have that $P_1(G)$ is contained in G. As $P_1(G)$ is the Frattini subgroup, $G/P_1(G)$ is elementary abelian, so $P_1(G) \geqslant G'$, yielding the second part.
(2) As $P_1(G)$ is the Frattini subgroup, it is the intersection of all open maximal subgroups, hence $P_1(G) \leqslant H$. As G is a pro-2 group, H is normal in G, so H' is as well, being characteristic in H. Further, $H/P_1(G)$ is a subgroup of the abelian group $G/P_1(G)$, so $P_1(G) \geqslant H'$, from which the formula for equality follows.
(3) If $M \trianglelefteq G$, then for $P_k(G) \leqslant M$ we only need that $k \geqslant \nu([G:M]) = 2$. So assume M is not normal. The core $M_G = \bigcap_{g\in G} M^g$ of M is normal in G, and is the kernel of the map from G into S_4, the symmetric group on 4 elements, induced by the coset action of G on M. As all 2-subgroups of S_4 have 2-class at most 2, $P_k(G) \leqslant M_G$ for $k \geqslant 2$.
Now we need to establish when $P_k(G) \leqslant M'$. If $M \trianglelefteq G$, then M' is as well. As $M' \leqslant G' \leqslant P_1(G)$, $P_k(G) \leqslant M'$ if $k \geqslant \nu([M:M']) + 3 - d$.
So assume M is not normal in G. Then the normalizer of M is H, and so also the normalizer of M' is H. Then M has one other conjugate, say M^g, and M' also has one conjugate, say M'^g, which is the commutator of M^g. The core of M' is $M' \cap M'^g$. We have $MM^g = H$, and so $M'M'^g \leqslant H'$. We then have by a lattice isomorphism theorem that

$$\begin{aligned}\big[G : M' \cap M'^g\big] &= [G:M']\big[M' : M' \cap M'^g\big] \\ &= [G:M']\big[M'M'^g : M'^g\big] \\ &\leqslant [G:M'][H':M'] \\ &= [G:M][M:M']\frac{[H:M']}{[H:H']}.\end{aligned}$$

In this case as well $M' \leqslant G' \leqslant P_1(G)$, so $P_k(G) \leqslant M' \cap M'^g \leqslant M'$ if $k \geqslant 2\nu([M:M']) - \nu([H:H']) + 4 - d$.

We note here that (1) and (2) go through for any p, and (3) goes through suitably modified based on estimates as to the indices of particular core subgroups. Also, further conditions can be developed for higher index subgroups. □

The relation rank of a descendant is also partly determined by its ancestors.

Proposition 4. *Let G be a finitely generated pro-p group. Then for $k \geqslant 1$, the difference between the ranks of the p-multiplicator and the nucleus of $G/P_k(G)$ is at most r, the relation rank of G.*

Proof. For a definition of the terms, see [9]. This is a generalization of a proposition in [6] inspired by a lemma in [3]. The proof is given in [4]. □

3. The method

We can now describe the method. For concreteness we describe the case where K is an imaginary quadratic field, which is the situation for all our examples. The method for this case has been fully implemented using Magma [2] and BASH scripts.

3.1. Gather lattice data

Start by computing the 2-class group $\mathrm{Cl}_2(K)$ of K. Currently the only general way we know to compute $\mathrm{Cl}_2(K)$ is to compute $\mathrm{Cl}(K)$ and take the 2-part. Utilizing explicit class field theory, use $\mathrm{Cl}_2(K)$ to construct all unramified extensions of K of degree 2. For each of these fields L we compute $\mathrm{Cl}_2(L)$ and compute their unramified degree 2 extensions F. Lastly, we check for and eliminate duplicates among the extensions F of the L's, and then compute $\mathrm{Cl}_2(F)$ for each F.

This information corresponds to the lattice of subgroups of index at most 4 in G with attached abelianizations, as described above. We call such a lattice an augmented lattice. We also have some information about normality: the index 2 subgroups are automatically normal, and the index 4 subgroups that are contained in three index 2 subgroups, corresponding to degree 4 extensions containing three degree 2 subextensions, are necessarily normal.

Class group computations are where some dependence on GRH may enter. Class group computations for higher degree and higher discriminant fields can become both memory and time intensive. As a result, calculating $\mathrm{Cl}_2(F)$ sometimes fails due to resource constraints. Class groups are computed by examining all ideals with norm up to some bound, and it is this bound that drives memory usage. A result of Bach states that under GRH this bound can be lowered, and Magma can be instructed to use this lower bound instead. As this lower bound was necessary to construct some of our augmented lattices, some of our results depend on GRH.

3.2. Pruning the tree

From Burnside's basis theorem, we know that the generator rank d of G is the generator rank of $\mathrm{Ab}(G) \cong \mathrm{Cl}_2(K)$, and so we know that G is a descendant of $\times_{i=1}^{d} C_2$. We construct and prune a subset of the O'Brien tree recursively, as follows.

Assume that the 2-group P, of 2-class $k-1$, has been determined to be a potential ancestor for G. Begin by using O'Brien's algorithm to construct the finite set D of immediate descendants of P, all of which will have 2-class k. Now for each $C \in D$ we want to check, first, if C is a possible ancestor of G, and, second, if C could be G. For each of these we have two conditions, a lattice condition and a cohomological condition.

First we compute the augmented lattice of subgroups of C of index at most 4. Then we attempt to match this lattice with our known augmented lattice of G. If k is large enough, Proposition 2 tells us that the lattices up to index 2 or 4 must match. Additionally, the abelianizations of groups in C must be realizable as quotients of the abelianizations of the matching groups in G, and depending on k some or all of these abelianizations must match exactly as well.

By combining Propositions 1 and 4, we also have that the p-multiplicator rank minus the nuclear rank must be at most $d+1$.

If C fails either of these conditions, we discard it. If it passes both, we recurse, repeating our procedure with C as our new P. Before we recurse, we check to see if C could in fact be G. First, if k was small enough that we did not check if the augmented index 4 lattice of C matched that of G exactly, we do so now. We also directly check that the relation rank of C is at most one more than the generator rank by computing $H^2(C, \mathbb{F}_2)$.

Further, as subgroups M of G correspond to number fields, Proposition 1 gives us an upper bound on $r(M) - d(M)$ in terms of the index $[G : M]$, and we can check if the resulting inequality is satisfied for all subgroups M of C of index up to some bound. If all these conditions are met, we add C to the list of candidates for G.

Often it is the case that the algorithm terminates, finding on any given path down the unpruned tree a P that has no descendants that can be ancestors of G. In this case we know that G must be among the list of potential candidates produced by the algorithm, and in particular that the 2-class tower is finite.

Of course, there is no guarantee that the pruned O'Brien tree will be finite. Certainly if G is infinite, the tree must be infinite, and this method will not yield G, though we do gain some information about possible quotients of G. Even if G is finite, the index 4 lattice data and cohomological criterion may not prune the O'Brien tree sufficiently. Using calculations similar to Proposition 2, we know that past a certain p-class every group in the tree encountered by the algorithm will pass our lattice criterion. As that is our strongest filter we expect paths from this point on will only terminate if the tree naturally does so. While this is not impossible, we only have so much time to continue looking.

Additionally, even if the pruned tree is finite, memory constraints may not allow us to discover this with the available resources. Although we carry out our search depth first to save memory, sometimes there is insufficient memory to calculate all the descendants of a given group.

3.3. *Narrowing the candidate list*

Assuming the tree recursion terminates successfully, we are left with a list of candidates for G. These candidates are necessarily similar, having the same augmented index 4 lattice as G.

We can now further narrow the list of candidates by selective computation of the class groups of certain unramified degree 8 extensions of K. We begin by computing the augmented index 8 lattices for each of our candidate groups. If all of these lattices are the same, then we stop. However, if they are non-isomorphic, then we can find an index 4 subgroup whose subgroups in the augmented index 8 lattice differ from one candidate group to another. There is some subtlety as to which index 4 subgroup in one candidate corresponds to a particular index 4 subgroup in another, but this can be dealt with.

We then go back to our arithmetic lattice of unramified 2-extensions and 2-class groups and determine which degree 4 extensions of K might correspond to the index 4 subgroup whose subgroups in the augmented index 8 lattices differ. For each of these fields, we compute its degree 2 extensions and their class groups. With the computing resources available to us, the computation of these class groups, which are of fields of degree 16 over $\mathbb{Q}$, always requires GRH. After getting this data, we have illuminated some of the index 8 lattice of G, chosen in such a way that it will eliminate some candidates. We then repeat the process until all remaining candidates share the same augmented index 8 lattice.

4. Results

We ran the algorithm on the first 36 imaginary quadratic fields with 2-class group elementary abelian of rank 3 and $G/[G', G]$ isomorphic to 32.033, which corresponded to a bound on the absolute discriminant of 20,000. Sometimes GRH was necessary to recover the lattice up to index 4, but often not. In 19 cases the pruning of the O'Brien tree was successful. For the remaining 17 fields, 7 times we ran out of memory when constructing the pruned O'Brien tree, while 10 times the augmented index 4 data appeared insufficient to constrain the tree.

In those cases where the tree was successfully pruned, it was always necessary to use GRH when calculating the class groups that were used in narrowing the candidate groups down to sharing a single augmented index 8 lattice.

We often found that two fields would have isomorphic augmented index 4 lattices. When this occurs they must necessarily have the same pruned O'Brien tree, and so we only needed to carry out this step once for such a family. Often, fields in this situation wound up having the same index 8 lattice data and so the same final candidate groups, but this did not always occur.

We summarize these results in Table 1. The table consists of the 36 imaginary quadratic fields mentioned above. For each, we indicated first whether GRH was necessary to recover the augmented index 4 lattice. As mentioned above, fields with the same index 4 data would have the same pruned O'Brien trees, and would also often have the same final set of candidates as well. Consequently, if a field is the first to have that particular index 4 lattice, we indicate the success of the tree pruning. If successful, we list the number of candidates after this stage. If unsuccessful, we put a 'w', for wide,

Table 1
A summary of results

Field	Index 4 GRH	Tree cands	Narrowed cands
$\mathbb{Q}(\sqrt{-3135})$	n	84	4
$\mathbb{Q}(\sqrt{-3864})$	n	38	4
$\mathbb{Q}(\sqrt{-3876})$	n	76	4
$\mathbb{Q}(\sqrt{-6132})$	y	$\mathbb{Q}(\sqrt{-3876})$	4
$\mathbb{Q}(\sqrt{-6216})$	y	$\mathbb{Q}(\sqrt{-3135})$	$\mathbb{Q}(\sqrt{-3135})$
$\mathbb{Q}(\sqrt{-6279})$	y	$\mathbb{Q}(\sqrt{-3135})$	$\mathbb{Q}(\sqrt{-3135})$
$\mathbb{Q}(\sqrt{-8148})$	y	w	
$\mathbb{Q}(\sqrt{-8772})$	y	d	
$\mathbb{Q}(\sqrt{-9044})$	y	$\mathbb{Q}(\sqrt{-3876})$	$\mathbb{Q}(\sqrt{-6132})$
$\mathbb{Q}(\sqrt{-9492})$	y	d	
$\mathbb{Q}(\sqrt{-9636})$	y	$\mathbb{Q}(\sqrt{-3876})$	$\mathbb{Q}(\sqrt{-3876})$
$\mathbb{Q}(\sqrt{-11508})$	y	w	
$\mathbb{Q}(\sqrt{-11748})$	y	w	
$\mathbb{Q}(\sqrt{-11928})$	y	d	
$\mathbb{Q}(\sqrt{-11935})$	y	$\mathbb{Q}(\sqrt{-11928})$	
$\mathbb{Q}(\sqrt{-12036})$	y	$\mathbb{Q}(\sqrt{-8772})$	
$\mathbb{Q}(\sqrt{-12243})$	y	w	
$\mathbb{Q}(\sqrt{-12264})$	y	84	8
$\mathbb{Q}(\sqrt{-13035})$	y	$\mathbb{Q}(\sqrt{-3135})$	$\mathbb{Q}(\sqrt{-3135})$
$\mathbb{Q}(\sqrt{-13640})$	y	d	
$\mathbb{Q}(\sqrt{-13668})$	y	w	
$\mathbb{Q}(\sqrt{-14007})$	y	$\mathbb{Q}(\sqrt{-11928})$	
$\mathbb{Q}(\sqrt{-14212})$	y	$\mathbb{Q}(\sqrt{-3876})$	4
$\mathbb{Q}(\sqrt{-14916})$	y	84	4
$\mathbb{Q}(\sqrt{-15252})$	y	d	
$\mathbb{Q}(\sqrt{-16296})$	y	$\mathbb{Q}(\sqrt{-3864})$	$\mathbb{Q}(\sqrt{-3864})$
$\mathbb{Q}(\sqrt{-16932})$	y	$\mathbb{Q}(\sqrt{-8772})$	
$\mathbb{Q}(\sqrt{-17043})$	y	84	4
$\mathbb{Q}(\sqrt{-17871})$	y	$\mathbb{Q}(\sqrt{-17043})$	$\mathbb{Q}(\sqrt{-17043})$
$\mathbb{Q}(\sqrt{-18084})$	y	$\mathbb{Q}(\sqrt{-3135})$	4
$\mathbb{Q}(\sqrt{-18088})$	y	d	
$\mathbb{Q}(\sqrt{-18312})$	y	$\mathbb{Q}(\sqrt{-3864})$	$\mathbb{Q}(\sqrt{-3864})$
$\mathbb{Q}(\sqrt{-18447})$	y	w	
$\mathbb{Q}(\sqrt{-18984})$	y	w	
$\mathbb{Q}(\sqrt{-19096})$	y	$\mathbb{Q}(\sqrt{-17043})$	$\mathbb{Q}(\sqrt{-17043})$
$\mathbb{Q}(\sqrt{-19572})$	y	$\mathbb{Q}(\sqrt{-3876})$	4

if we ran out of memory, or a 'd', for deep, if the tree seemed unconstrained. If a field has the same index 4 data as a prior field, then we list that field instead.

If the tree pruning was successful and this is the first field to narrow down to those particular candidates, we indicate how many candidates were left after the narrowing step. If those candidates had previously been narrowed down to, we list the first field where this occurred.

All of our candidate groups necessarily have a presentation with 3 generators and 4 relators. However, the few survivors after the narrowing step are generally quite similar, having the same immediate parent in the O'Brien tree, and so the most parsimonious presentation for all these groups at once is with power-conjugate (pc) presentations. In a pc presentation of a p-group P, where $n = v(|G|)$, we use n generators $x_1, \dots, x_n$. We then give relations where each LHS is either x_i^p or $x_i^{x_j}$ with $j > i$ and where each RHS is an expression in the x_k with $k \geqslant i$. Every p-group has such a presentation. For the interests of space, we do not list power relations where the RHS is trivial or conjugate relations where the generators commute.

We give the fields as $\mathbb{Q}(\sqrt{-D})$ where $D > 0$ is the absolute value of the discriminant, so D may be divisible by 4 but is otherwise square-free. As mentioned above, fields sharing the same augmented index 4 lattices necessarily have the same pruned trees and are often observed to have the same narrowed candidates, so we organize our results by shared augmented index 4 lattices.

4.1. $\mathbb{Q}(\sqrt{-3135})$, $\mathbb{Q}(\sqrt{-6216})$, $\mathbb{Q}(\sqrt{-6279})$, $\mathbb{Q}(\sqrt{-13035})$, $\mathbb{Q}(\sqrt{-18084})$

We can compute the index 4 lattice of $\mathbb{Q}(\sqrt{-3135})$ without GRH. The pruned O'Brien tree is finite with this data, yielding 84 possible candidate groups with a total of 24 possible index 8 lattices. All these candidate groups have derived length 3, 2-class either 6, 7, or 8, nilpotency class either 6, 7, or 8, and size either 4096, 8192, or 16384.

We then use GRH in the narrowing step to reduce to 4 possible candidates. For the other fields, GRH is necessary to get the augmented index 4 lattice. The resulting 84 candidates are narrowed by the partial index 8 data to the same 4 groups.

Proposition 5. *The* 2*-class tower of* $\mathbb{Q}\sqrt{-3135}$ *is finite of length* 3*. Additionally, under GRH, the* 2*-class towers of* $\mathbb{Q}(\sqrt{-6216})$, $\mathbb{Q}(\sqrt{-6279})$ *and* $\mathbb{Q}(\sqrt{-13035})$ *are as well, and the Galois groups for all four towers have size* 8192*, both* 2*-class and nilpotency class* 7*, and each is one of the* 4 *groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_7x_{12}, & x_3^2 = b, \\
x_4^2 = x_6x_{13}, & x_5^2 = x_8x_{10}, & x_6^2 = x_{13}, \\
x_7^2 = x_{11}x_{12}, & x_8^2 = x_{10}x_{13}, & x_9^2 = x_{12}, \\
x_{10}^2 = x_{13}, & x_{11}^2 = x_{13}, & x_2^{x_1} = x_2x_4, \\
x_3^{x_1} = x_3x_5, & x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_9x_{12}, \\
x_4^{x_3} = x_4x_7, & x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{11}x_{13}, \\
x_5^{x_3} = x_5x_8x_9x_{12}, & x_5^{x_4} = x_5x_9x_{12}x_{13}, & x_6^{x_1} = x_6x_{13}, \\
x_6^{x_2} = x_6x_{13}, & x_6^{x_3} = x_6x_{13}, & x_7^{x_1} = x_7x_9, \\
x_7^{x_2} = x_7x_{13}, & x_7^{x_4} = x_7x_{11}x_{12}, & x_8^{x_1} = x_8x_{10}, \\
x_8^{x_2} = x_8x_{11}, & x_8^{x_3} = x_8x_{10}x_{12}, & x_8^{x_4} = x_8x_{12}, \\
x_9^{x_1} = x_9x_{12}, & x_9^{x_2} = x_9x_{11}, & x_9^{x_4} = x_9x_{12}, \\
x_{10}^{x_1} = x_{10}x_{13}, & x_{10}^{x_2} = x_{10}x_{13}, & x_{10}^{x_3} = x_{10}x_{13}, \\
x_{11}^{x_1} = x_{11}x_{12}, & x_{11}^{x_2} = x_{11}x_{13}, & x_{11}^{x_4} = x_{11}x_{13}, \\
x_{12}^{x_2} = x_{12}x_{13}, & &
\end{array}$$

where a is either x_6x_{10} *or* $x_6x_{10}x_{13}$ *and b is either* $x_6x_7x_{10}x_{11}$ *or* $x_6x_7x_{10}x_{11}x_{13}$.

The field $\mathbb{Q}(\sqrt{-18084})$ shares the same index 4 lattice as the above fields, but narrows to something else.

Proposition 6. *Under GRH, the* 2*-class tower of* $\mathbb{Q}(\sqrt{-18084})$ *is finite of length* 3*. Further, the Galois group of this tower has size* 8192*, both* 2*-class and nilpotency class* 7*, and is one of the* 4 *groups given by the pc presentation*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_7x_9x_{10}x_{12}, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{10}, & x_7^2 = x_{11}, \\
x_8^2 = x_{10}x_{13}, & x_9^2 = x_{12}, & x_{10}^2 = x_{13},
\end{array}$$

$$
\begin{array}{lll}
x_{11}^2 = x_{13}, & x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, \\
x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_9x_{13}, & x_4^{x_3} = x_4x_7, \\
x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{11}x_{12}x_{13}, & x_5^{x_3} = x_5x_8, \\
x_5^{x_4} = x_5x_9x_{11}x_{12}x_{13}, & x_7^{x_1} = x_7x_9, & x_7^{x_2} = x_7x_{11}, \\
x_7^{x_3} = x_7x_{11}x_{13}, & x_7^{x_4} = x_7x_{11}x_{13}, & x_7^{x_5} = x_7x_{11}x_{13}, \\
x_8^{x_1} = x_8x_{10}, & x_8^{x_3} = x_8x_{10}, & x_9^{x_1} = x_9x_{12}, \\
x_9^{x_2} = x_9x_{11}, & x_9^{x_3} = x_9x_{11}, & x_9^{x_4} = x_9x_{12}, \\
x_9^{x_5} = x_9x_{12}, & x_{10}^{x_1} = x_{10}x_{13}, & x_{10}^{x_3} = x_{10}x_{13}, \\
x_{11}^{x_1} = x_{11}x_{12}, & x_{11}^{x_2} = x_{11}x_{13}, & x_{11}^{x_3} = x_{11}x_{13}, \\
x_{11}^{x_4} = x_{11}x_{13}, & x_{11}^{x_5} = x_{11}x_{13}, & x_{12}^{x_2} = x_{12}x_{13}, \\
x_{12}^{x_3} = x_{12}x_{13}, & &
\end{array}
$$

where a and b are each either x_6 or x_6x_{13}.

4.2. $\mathbb{Q}(\sqrt{-3864})$, $\mathbb{Q}(\sqrt{-16296})$, $\mathbb{Q}(\sqrt{-18312})$

The initial lattice for $\mathbb{Q}(\sqrt{-3864})$ was also recovered unconditionally. After pruning the tree we had 38 remaining candidates with a total of 11 different index 8 lattices. These groups all had derived length 3, size 4096 or 8192, 2-class 6 or 7, and nilpotency class 6 or 7. We then used GRH in the narrowing step to arrive at 4 final candidate groups.

The other two fields required GRH for the index 4 data, but narrowed to the same 4 final groups.

Proposition 7. *The* 2*-class tower of* $\mathbb{Q}(\sqrt{-3864})$ *is finite of length* 3*. Additionally, under GRH, the* 2*-class towers of* $\mathbb{Q}(\sqrt{-16296})$ *and* $\mathbb{Q}(\sqrt{-18312})$ *are also of length* 3*, and the Galois groups of the towers for all three fields have size* 4096*,* 2*-class and nilpotency class both* 7*, and each is one of the* 4 *groups given by the pc presentations*

$$
\begin{array}{lll}
x_1^2 = a, & x_2^2 = b, & x_3^2 = x_6x_9, \\
x_4^2 = x_6x_{12}, & x_5^2 = x_8x_{10}x_{12}, & x_6^2 = x_{11}, \\
x_7^2 = x_{10}x_{12}, & x_9^2 = x_{11}, & x_{10}^2 = x_{12}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_4^{x_1} = x_4x_6, \\
x_4^{x_2} = x_4x_6, & x_4^{x_3} = x_4x_7, & x_5^{x_1} = x_5x_8, \\
x_5^{x_2} = x_5x_7x_9x_{10}x_{11}x_{12}, & x_5^{x_3} = x_5x_8x_{10}, & x_5^{x_4} = x_5x_9, \\
x_6^{x_1} = x_6x_{11}, & x_6^{x_2} = x_6x_{11}, & x_6^{x_3} = x_6x_{10}x_{12}, \\
x_6^{x_5} = x_6x_{11}, & x_7^{x_1} = x_7x_9, & x_7^{x_2} = x_7x_{10}x_{12}, \\
x_7^{x_3} = x_7x_{10}, & x_7^{x_5} = x_7x_{10}x_{12}, & x_8^{x_1} = x_8x_{11}x_{12}, \\
x_8^{x_5} = x_8x_{12}, & x_9^{x_1} = x_9x_{11}, & x_9^{x_2} = x_9x_{11}, \\
x_9^{x_3} = x_9x_{10}, & x_9^{x_5} = x_9x_{11}, & x_{10}^{x_1} = x_{10}x_{11}, \\
x_{10}^{x_2} = x_{10}x_{12}, & x_{10}^{x_3} = x_{10}x_{12}, & x_{10}^{x_5} = x_{10}x_{12}, \\
x_{11}^{x_3} = x_{11}x_{12}, & &
\end{array}
$$

where a is x_6x_{10} or $x_6x_{10}x_{12}$ and b is x_8 or x_8x_{12}.

4.3. $\mathbb{Q}(\sqrt{-3876})$, $\mathbb{Q}(\sqrt{-6132})$, $\mathbb{Q}(\sqrt{-9044})$, $\mathbb{Q}(\sqrt{-9636})$, $\mathbb{Q}(\sqrt{-14212})$, $\mathbb{Q}(\sqrt{-19572})$

These 5 fields share the same augmented index 4 lattice, but can narrow to one of three sets of 4 final candidates each.

For $\mathbb{Q}(\sqrt{-3876})$, the initial lattice was computed unconditionally. The O'Brien tree again terminates and we are left with 76 candidate groups. All are of derived length 3, the possible sizes are 4096 and 8192 and the 2-class and nilpotency class are both 7 or 8.

The remaining fields all required GRH to get the index 4 data, but then narrowed. We therefore have

Proposition 8. *The* 2*-class tower of* $\mathbb{Q}(\sqrt{-3876})$ *is finite of length* 3*. Additionally, under GRH, so is the* 2*-class tower of* $\mathbb{Q}(\sqrt{-9636})$*, and the Galois groups of both towers have size* 4096*,* 2*-class and nilpotency class both* 7*, and each is one of the* 4 *groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_8, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{12}, & x_7^2 = x_{10}x_{11}x_{12}, \\
x_8^2 = x_{12}, & x_9^2 = x_{11}, & x_{10}^2 = x_{12}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_3^{x_2} = x_3x_{11}, \\
x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_{12}, & x_4^{x_3} = x_4x_7, \\
x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{10}, & x_5^{x_3} = x_5x_8x_{12}, \\
x_5^{x_4} = x_5x_9x_{10}, & x_7^{x_1} = x_7x_9, & x_7^{x_3} = x_7x_{10}x_{11}, \\
x_7^{x_4} = x_7x_{10}x_{11}, & x_7^{x_5} = x_7x_{10}x_{11}, & x_8^{x_1} = x_8x_{12}, \\
x_8^{x_3} = x_8x_{12}, & x_9^{x_1} = x_9x_{11}, & x_9^{x_2} = x_9x_{10}, \\
x_9^{x_3} = x_9x_{10}x_{11}x_{12}, & x_9^{x_4} = x_9x_{11}, & x_9^{x_5} = x_9x_{11}, \\
x_{10}^{x_1} = x_{10}x_{11}, & x_{10}^{x_2} = x_{10}x_{12}, & x_{10}^{x_4} = x_{10}x_{12}, \\
x_{10}^{x_5} = x_{10}x_{12}, & x_{11}^{x_2} = x_{11}x_{12}, & x_{11}^{x_3} = x_{11}x_{12},
\end{array}$$

where a is either x_6 *or* x_6x_{12} *and b is either* $x_6x_8x_{11}$ *or* $x_6x_8x_{11}x_{12}$*.*

Proposition 9. *Under GRH, the* 2*-class towers of both* $\mathbb{Q}(\sqrt{-6132})$ *and* $\mathbb{Q}(\sqrt{-9044})$ *have length* 3*, and their Galois groups have size* 4096*,* 2*-class and nilpotency class both* 7*, and each is one of the* 4 *groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_8x_9, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{11}x_{12}, & x_7^2 = x_{10}x_{11}x_{12}, \\
x_8^2 = x_{11}x_{12}, & x_9^2 = x_{11}, & x_{10}^2 = x_{12}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_3^{x_2} = x_3x_{11}, \\
x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_{12}, & x_4^{x_3} = x_4x_7, \\
x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{10}, & x_5^{x_3} = x_5x_8x_{11}x_{12}, \\
x_5^{x_4} = x_5x_9x_{11}, & x_7^{x_1} = x_7x_9, & x_7^{x_3} = x_7x_{10}x_{11}x_{12}, \\
x_7^{x_4} = x_7x_{10}x_{11}, & x_8^{x_1} = x_8x_{11}x_{12}, & x_8^{x_2} = x_8x_{10}x_{12}, \\
x_8^{x_3} = x_8x_{11}x_{12}, & x_8^{x_4} = x_8x_{11}, & x_9^{x_1} = x_9x_{11}, \\
x_9^{x_2} = x_9x_{10}, & x_9^{x_3} = x_9x_{11}, & x_9^{x_4} = x_9x_{11}, \\
x_{10}^{x_1} = x_{10}x_{11}, & x_{10}^{x_2} = x_{10}x_{12}, & x_{10}^{x_3} = x_{10}x_{12}, \\
x_{10}^{x_4} = x_{10}x_{12}, & x_{11}^{x_2} = x_{11}x_{12}, &
\end{array}$$

where a is either x_6 *or* x_6x_{12} *and b is either* $x_6x_8x_9x_{10}x_{11}$ *or* $x_6x_8x_9x_{10}x_{11}x_{12}$*.*

Proposition 10. *Under GRH, the* 2*-class tower of* $\mathbb{Q}(\sqrt{-14212})$ *is finite of length* 3*. The Galois group of the tower has size* 4096*, both* 2*-class and nilpotency class* 7*, and is one of the four groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_8, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{12}, & x_7^2 = x_{10}x_{11}x_{12}, \\
x_8^2 = x_{12}, & x_9^2 = x_{11}, & x_{10}^2 = x_{12}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_3^{x_2} = x_3x_{11}, \\
x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_{12}, & x_4^{x_3} = x_4x_7, \\
x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{10}, & x_5^{x_3} = x_5x_8x_{11}, \\
x_5^{x_4} = x_5x_9x_{10}, & x_7^{x_1} = x_7x_9, & x_7^{x_3} = x_7x_{10}x_{11}x_{12}, \\
x_7^{x_4} = x_7x_{10}x_{11}, & x_7^{x_5} = x_7x_{10}x_{11}, & x_8^{x_1} = x_8x_{12}, \\
x_8^{x_3} = x_8x_{12}, & x_9^{x_1} = x_9x_{11}, & x_9^{x_2} = x_9x_{10}, \\
x_9^{x_3} = x_9x_{10}x_{11}x_{12}, & x_9^{x_4} = x_9x_{11}, & x_9^{x_5} = x_9x_{11}, \\
x_{10}^{x_1} = x_{10}x_{11}, & x_{10}^{x_2} = x_{10}x_{12}, & x_{10}^{x_4} = x_{10}x_{12}, \\
x_{10}^{x_5} = x_{10}x_{12}, & x_{11}^{x_2} = x_{11}x_{12}, & x_{11}^{x_3} = x_{11}x_{12},
\end{array}$$

where a is either x_6 or x_6x_{10} and b is either x_6x_{10} or $x_6x_{10}x_{12}$.

Proposition 11. *Under GRH, the* 2*-class tower of* $\mathbb{Q}(\sqrt{-19572})$ *is finite of length* 3*. The Galois group of the tower has size* 4096*, both* 2*-class and nilpotency class* 7*, and is one of the four groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_8x_9, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{11}x_{12}, & x_7^2 = x_{10}x_{11}x_{12}, \\
x_8^2 = x_{11}x_{12}, & x_9^2 = x_{11}, & x_{10}^2 = x_{12}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_3^{x_2} = x_3x_{11}, \\
x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_{12}, & x_4^{x_3} = x_4x_7, \\
x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{10}, & x_5^{x_3} = x_5x_8, \\
x_5^{x_4} = x_5x_9x_{11}, & x_7^{x_1} = x_7x_9, & x_7^{x_3} = x_7x_{10}x_{11}, \\
x_7^{x_4} = x_7x_{10}x_{11}, & x_8^{x_1} = x_8x_{11}x_{12}, & x_8^{x_2} = x_8x_{10}x_{12}, \\
x_8^{x_3} = x_8x_{11}x_{12}, & x_8^{x_4} = x_8x_{11}, & x_9^{x_1} = x_9x_{11}, \\
x_9^{x_2} = x_9x_{10}, & x_9^{x_3} = x_9x_{11}, & x_9^{x_4} = x_9x_{11}, \\
x_{10}^{x_1} = x_{10}x_{11}, & x_{10}^{x_2} = x_{10}x_{12}, & x_{10}^{x_3} = x_{10}x_{12}, \\
x_{10}^{x_4} = x_{10}x_{12}, & x_{11}^{x_2} = x_{11}x_{12}, &
\end{array}$$

where a and b are each either x_6 or x_6x_{12}.

4.4. $\mathbb{Q}(\sqrt{-12264})$

No other fields shared this field's augmented index 4 lattice. GRH was necessary to obtain its index 4 data. The pruned O'Brien tree contained 84 possible candidates for the Galois group, which were narrowed to 8 remaining possibilities that all shared the same index 8 data.

Proposition 12. *Under GRH, the 2-class tower of $\mathbb{Q}(\sqrt{-12264})$ is finite of length 3, and the Galois group of the tower is of size 16384, 2-class and nilpotency class both 7, and is one of the eight groups given by the pc presentations*

$$
\begin{array}{lll}
x_1^2 = x_6 a_1, & x_2^2 = x_7 a_2, & x_3^2 = x_6 x_7 x_{11} x_{12} x_{13} a_3, \\
x_4^2 = x_6, & x_5^2 = x_8 x_{10}, & x_6^2 = a_4, \\
x_7^2 = x_{11} x_{13} a_5, & x_8^2 = x_{10} x_{12}, & x_9^2 = x_{13}, \\
x_{10}^2 = x_{12} x_{14}, & x_{11}^2 = x_{14}, & x_{12}^2 = x_{14}, \\
x_2^{x_1} = x_2 x_4, & x_3^{x_1} = x_3 x_5, & x_4^{x_1} = x_4 x_6, \\
x_4^{x_2} = x_4 x_6 x_9 x_{13} x_{14}, & x_4^{x_3} = x_4 x_7, & x_5^{x_1} = x_5 x_8, \\
x_5^{x_2} = x_5 x_7 x_9 x_{11} a_6, & x_5^{x_3} = x_5 x_8 x_9 x_{13} a_7, & x_5^{x_4} = x_5 x_9 x_{13} a_8, \\
x_6^{x_1} = x_6 a_9, & x_6^{x_2} = x_6 a_{10}, & x_6^{x_3} = x_6 a_{11}, \\
x_7^{x_2} = x_7 a_{12}, & x_7^{x_3} = x_7 a_{13}, & x_7^{x_1} = x_7 x_9, \\
x_7^{x_3} = x_7 x_{14}, & x_7^{x_4} = x_7 x_{11} x_{13}, & x_8^{x_1} = x_8 x_{10}, \\
x_8^{x_2} = x_8 x_{11} a_{14}, & x_8^{x_3} = x_8 x_{10} x_{13}, & x_8^{x_4} = x_8 x_{13}, \\
x_9^{x_1} = x_9 x_{13}, & x_9^{x_2} = x_9 x_{11}, & x_9^{x_4} = x_9 x_{13}, \\
x_{10}^{x_1} = x_{10} x_{12}, & x_{10}^{x_2} = x_{10} x_{14}, & x_{10}^{x_3} = x_{10} x_{12}, \\
x_{11}^{x_1} = x_{11} x_{13}, & x_{11}^{x_2} = x_{11} x_{14}, & x_{11}^{x_4} = x_{11} x_{14}, \\
x_{12}^{x_1} = x_{12} x_{14}, & x_{12}^{x_3} = x_{12} x_{14}, & x_{13}^{x_2} = x_{13} x_{14},
\end{array}
$$

where $(a_1, \ldots, a_{14})$ *is one of*

	Group							
	1	2	3	4	5	6	7	8
a_1	1	1	x_{14}	x_{14}	x_{12}	x_{12}	$x_{12}x_{14}$	$x_{12}x_{14}$
a_2	x_{12}	x_{12}	x_{12}	x_{12}	x_{13}	x_{13}	x_{13}	x_{13}
a_3	1	x_{14}	1	x_{14}	1	x_{14}	1	x_{14}
a_4	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_5	x_{14}	x_{14}	x_{14}	x_{14}	1	1	1	1
a_6	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_7	x_{14}	x_{14}	x_{14}	x_{14}	1	1	1	1
a_8	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_9	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_{10}	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_{11}	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_{12}	1	1	1	1	x_{14}	x_{14}	x_{14}	x_{14}
a_{13}	x_{14}	x_{14}	x_{14}	x_{14}	1	1	1	1
a_{14}	x_{14}	x_{14}	x_{14}	x_{14}	1	1	1	1

4.5. $\mathbb{Q}(\sqrt{-14916})$

We again needed GRH to get the augmented index 4 data, after which the pruned O'Brien tree contained 84 possible candidates for G. The narrowing stage brought this down to 4 groups, all sharing the same index 4 lattice.

Proposition 13. *Under GRH, the* 2*-class tower of* $\mathbb{Q}(\sqrt{-14916})$ *is finite of length* 3*. Further, the Galois group associated to the tower has size* 131072*,* 2*-class and nilpotency class both* 11*, and is one of the four groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_7x_9x_{10}x_{12}x_{16}, & x_3^2 = b, \\
x_4^2 = x_6, & x_5^2 = x_8x_{10}, & x_7^2 = x_{11}x_{14}x_{15}, \\
x_8^2 = x_{10}x_{17}, & x_9^2 = x_{12}, & x_{10}^2 = x_{17}, \\
x_{11}^2 = x_{13}, & x_{12}^2 = x_{14}, & x_{13}^2 = x_{15}, \\
x_{14}^2 = x_{16}, & x_{15}^2 = x_{17}, & x_2^{x_1} = x_2x_4, \\
x_3^{x_1} = x_3x_5, & x_4^{x_1} = x_4x_6, & x_4^{x_2} = x_4x_6x_9x_{14}x_{17}, \\
x_4^{x_3} = x_4x_7, & x_5^{x_1} = x_5x_8, & x_5^{x_2} = x_5x_7x_9x_{11}x_{12}x_{13}x_{16}x_{17}, \\
x_5^{x_3} = x_5x_8x_{12}, & x_5^{x_4} = x_5x_9x_{11}x_{12}x_{13}x_{16}x_{17}, & x_7^{x_1} = x_7x_9, \\
x_7^{x_2} = x_7x_{11}x_{15}, & x_7^{x_3} = x_7x_{11}x_{15}x_{17}, & x_7^{x_4} = x_7x_{11}x_{13}x_{14}x_{16}x_{17}, \\
x_7^{x_5} = x_7x_{11}x_{13}x_{14}x_{16}x_{17}, & x_8^{x_1} = x_8x_{10}, & x_8^{x_3} = x_8x_{10}, \\
x_9^{x_1} = x_9x_{12}x_{14}x_{16}, & x_9^{x_2} = x_9x_{11}, & x_9^{x_3} = x_9x_{11}, \\
x_9^{x_4} = x_9x_{12}x_{14}x_{16}, & x_9^{x_5} = x_9x_{12}x_{14}x_{16}, & x_{10}^{x_1} = x_{10}x_{17}, \\
x_{10}^{x_3} = x_{10}x_{17}, & x_{11}^{x_1} = x_{11}x_{12}, & x_{11}^{x_2} = x_{11}x_{13}x_{15}x_{17}, \\
x_{11}^{x_3} = x_{11}x_{13}x_{15}x_{17}, & x_{11}^{x_4} = x_{11}x_{13}x_{15}x_{17}, & x_{11}^{x_5} = x_{11}x_{13}x_{15}x_{17}, \\
x_{12}^{x_1} = x_{12}x_{14}x_{16}, & x_{12}^{x_2} = x_{12}x_{13}, & x_{12}^{x_3} = x_{12}x_{13}, \\
x_{12}^{x_4} = x_{12}x_{14}x_{16}, & x_{12}^{x_5} = x_{12}x_{14}x_{16}, & x_{13}^{x_1} = x_{13}x_{14}, \\
x_{13}^{x_2} = x_{13}x_{15}x_{17}, & x_{13}^{x_3} = x_{13}x_{15}x_{17}, & x_{13}^{x_4} = x_{13}x_{15}x_{17}, \\
x_{13}^{x_5} = x_{13}x_{15}x_{17}, & x_{14}^{x_1} = x_{14}x_{16}, & x_{14}^{x_2} = x_{14}x_{15}, \\
x_{14}^{x_3} = x_{14}x_{15}, & x_{14}^{x_4} = x_{14}x_{16}, & x_{14}^{x_5} = x_{14}x_{16}, \\
x_{15}^{x_1} = x_{15}x_{16}, & x_{15}^{x_2} = x_{15}x_{17}, & x_{15}^{x_3} = x_{15}x_{17}, \\
x_{15}^{x_4} = x_{15}x_{17}, & x_{15}^{x_5} = x_{15}x_{17}, & x_{16}^{x_2} = x_{16}x_{17}, \\
x_{16}^{x_3} = x_{16}x_{17}, & &
\end{array}$$

where a is either x_6 *or* x_6x_{17} *and b is either* $x_6x_{11}x_{12}$ *or* $x_6x_{11}x_{12}x_{17}$*.*

4.6. $\mathbb{Q}(\sqrt{-17043})$, $\mathbb{Q}(\sqrt{-17871})$, $\mathbb{Q}(\sqrt{-19096})$

These 3 fields have isomorphic augmented index 4 lattices, and all 3 require GRH to procure that information. After pruning the O'Brien tree, 84 candidates representing 10 different possible augmented index 8 lattices are left. These are narrowed to 4 remaining possibilities sharing a single index 8 lattice, which is the same for all 3 fields.

Proposition 14. *Under GRH, the* 2*-class towers of* $\mathbb{Q}(\sqrt{-17043})$*,* $\mathbb{Q}(\sqrt{-17871})$*, and* $\mathbb{Q}(\sqrt{-19096})$ *are all finite of length* 3*. Further, the Galois groups of these towers are all of size* 32768*,* 2*-class and nilpotency class both* 9*, and each is one of the four groups given by the pc presentations*

$$\begin{array}{lll}
x_1^2 = a, & x_2^2 = x_7x_{14}, & x_3^2 = b, \\
x_4^2 = x_6x_{13}x_{15}, & x_5^2 = x_8x_{10}, & x_7^2 = x_{11}x_{12}x_{13}x_{14}, \\
x_8^2 = x_{10}x_{14}x_{15}, & x_9^2 = x_{12}, & x_{10}^2 = x_{14}x_{15},
\end{array}$$

$$
\begin{array}{lll}
x_{11}^2 = x_{13}, & x_{12}^2 = x_{14}, & x_{13}^2 = x_{15}, \\
x_2^{x_1} = x_2x_4, & x_3^{x_1} = x_3x_5, & x_4^{x_1} = x_4x_6, \\
x_4^{x_2} = x_4x_6x_9x_{12}x_{13}x_{14}, & x_4^{x_3} = x_4x_7, & x_5^{x_1} = x_5x_8, \\
x_5^{x_2} = x_5x_7x_9x_{11}x_{15}, & x_5^{x_3} = x_5x_8x_9x_{14}, & x_5^{x_4} = x_5x_9x_{12}x_{14}x_{15}, \\
x_6^{x_1} = x_6x_{14}x_{15}, & x_6^{x_3} = x_6x_{15}, & x_6^{x_4} = x_6x_{15}, \\
x_7^{x_1} = x_7x_9, & x_7^{x_2} = x_7x_{15}, & x_7^{x_4} = x_7x_{11}x_{12}, \\
x_8^{x_1} = x_8x_{10}, & x_8^{x_2} = x_8x_{11}x_{13}, & x_8^{x_3} = x_8x_{10}x_{12}x_{14}, \\
x_8^{x_4} = x_8x_{12}, & x_9^{x_1} = x_9x_{12}x_{14}, & x_9^{x_2} = x_9x_{11}, \\
x_9^{x_4} = x_9x_{12}x_{14}, & x_{10}^{x_1} = x_{10}x_{14}x_{15}, & x_{10}^{x_2} = x_{10}x_{13}, \\
x_{10}^{x_3} = x_{10}x_{15}, & x_{10}^{x_4} = x_{10}x_{14}, & x_{11}^{x_1} = x_{11}x_{12}, \\
x_{11}^{x_2} = x_{11}x_{13}x_{15}, & x_{11}^{x_4} = x_{11}x_{13}x_{15}, & x_{12}^{x_1} = x_{12}x_{14}, \\
x_{12}^{x_2} = x_{12}x_{13}, & x_{12}^{x_4} = x_{12}x_{14}, & x_{13}^{x_1} = x_{13}x_{14}, \\
x_{13}^{x_2} = x_{13}x_{15}, & x_{13}^{x_4} = x_{13}x_{15}, & x_{14}^{x_2} = x_{14}x_{15},
\end{array}
$$

where a is either x_6x_{10} *or* $x_6x_{10}x_{15}$ *and b is either* $x_6x_7x_{10}x_{12}$ *or* $x_6x_7x_{10}x_{12}x_{15}$.

As one of our primary motivations was trying to identify imaginary quadratic fields with potentially infinite 2-class towers, we summarize some of our results in the following theorem.

Theorem 15. *Let K be an imaginary quadratic field with* 2*-class group isomorphic to* $C_2 \times C_2 \times C_2$. *Then if the factorization of the discriminant of K into* 4 *prime discriminants has three negative prime discriminant factors and K has an infinite* 2*-class tower, the root discriminant of K is at least* 78.3. *Further, under GRH, such a K must have root discriminant at least* 90.2.

Proof. By the results in [1], if K is such a field and has tower of derived length at least 3 and root discriminant less than 141, then it must appear on our table. However, we showed unconditionally that all such fields with absolute discriminant strictly less than 6132 have finite 2-class tower, and under GRH we showed that all such fields with absolute discriminant strictly less than 8148 also have finite 2-class tower. □

Acknowledgments

The author would like to thank Nigel Boston, for regular advice and stimulating discussions, as well as for suggesting the problem, and Michael Bush, for helpful comments on an earlier draft of this paper. The author is supported by the Office of Naval Research through an NDSEG fellowship.

References

[1] E. Benjamin, F. Lemmermeyer, C. Snyder, Imaginary quadratic fields with $\mathrm{Cl}_2(k) = (2, 2, 2)$, J. Number Theory 103 (2003) 38–70.
[2] W. Bosma, J.J. Cannon, Handbook of Magma Functions, School of Mathematics and Statistics, University of Sydney, 1996.
[3] N. Boston, C.R. Leedham-Green, Explicit computation of Galois p-groups unramified at p, J. Algebra 256 (2002) 402–413.
[4] N. Boston, H. Nover, Computing pro-p Galois groups, in: ANTS VII, in: Lecture Notes in Comput. Sci., vol. 4076, Springer, Berlin, 2006, pp. 1–10.
[5] M.R. Bush, Computation of Galois groups associated to the 2-class towers of some quadratic fields, J. Number Theory 100 (2003) 313–325.
[6] B. Eick, H. Koch, On maximal 2-extensions of $\mathbb{Q}$ with given ramification, in: Proc. St. Petersburg Math. Soc. XII (in Russian), in: AMS Translations Ser. 2, vol. 219, 2006 (English version).
[7] F. Hajir, On a theorem of Koch, Pacific J. Math. 176 (1996) 15–18.
[8] F. Hajir, C. Maire, Tamely ramified towers and discriminant bounds for number fields. II, J. Symbolic Comput. 33 (2002) 415–423.
[9] E.A. O'Brien, The p-group generation algorithm, J. Symbolic Comput. 9 (1990) 677–698.
[10] I.R. Shafarevich, Extensions with prescribed ramification points, IHES Publ. Math. 18 (1964) 71–95 (in Russian).