J. Differential Equations 251 (2011) 2164–2195

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Global well-posedness for Schrödinger equation with derivative in $H^{\frac{1}{2}}(\mathbb{R})$

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ARTICLE INFO

Article history:
Received 25 August 2010
Revised 30 May 2011

MSC:
primary 35Q55
secondary 47J35

Keywords:
Bourgain space
DNLS equation
Global well-posedness
I-method
Resonant decomposition

ABSTRACT

In this paper, we consider the Cauchy problem of the cubic nonlinear Schrödinger equation with derivative in $H^s(\mathbb{R})$. This equation was known to be the local well-posedness for $s \geqslant \frac{1}{2}$ (Takaoka, 1999 [27]), ill-posedness for $s < \frac{1}{2}$ (Biagioni and Linares, 2001 [1], etc.) and global well-posedness for $s > \frac{1}{2}$ (I-team, 2002 [10]). In this paper, we show that it is global well-posedness in the endpoint space $H^{\frac{1}{2}}(\mathbb{R})$, which remained open previously. The main approach is the third generation I-method combined with a new resonant decomposition technique. The resonant decomposition is applied to control the singularity coming from the resonant interaction.

1. Introduction

In this paper, we consider the Cauchy problem of the Schrödinger equation with derivative:

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\lambda\partial_x\big(|u|^2 u\big), & x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(0,x) = u_0(x) \in H^s(\mathbb{R}), \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the usual inhomogeneous Sobolev space of order s. It arises from describing the propagation of circularly polarized Alfvén waves in the magnetized plasma with a constant magnetic field (see [23,24,26]).

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2011.07.004

The local well-posedness for (1.1) is well understood. By the Fourier restriction norm in [3,4] and the gauge transformation in [16–18], Takaoka obtained the local well-posedness of (1.1) in $H^s(\mathbb{R})$ for $s \geqslant 1/2$ in [27]. This result was shown by Biagioni and Linares [1], Bourgain [5] and Takaoka [28] to be sharp in the sense that the flow map fails to be uniformly C^0 for $s < 1/2$.

The global well-posedness for (1.1) was also widely studied. In [25], Ozawa made use of two gauge transformations and the conservation of the Hamiltonian, and showed that (1.1) was globally well posed in $H^1(\mathbb{R})$ under the condition (1.2). In [28], Takaoka used Bourgain's "Fourier truncation method" [6,7] to obtain the global well-posedness in $H^s(\mathbb{R})$ for $s > \frac{32}{33}$, again under (1.2). In [9,10], I-team (Colliander–Keel–Staffilani–Takaoka–Tao) made use of the first, second generations of I-method to obtain the global well-posedness in $H^s(\mathbb{R})$, for $s > 2/3$ and $s > 1/2$, respectively. For other results, we refer to [14–19,25,29–31].

In this paper, we will combine the third generation of the I-method with the resonant decomposition to show the global well-posedness of (1.1) in $H^{\frac{1}{2}}(\mathbb{R})$. We think that the resonant decomposition technique here may also be used to study the global well-posedness of (1.1) in $H^{\frac{1}{2}}(\mathbb{T})$.

Theorem 1.1. *The Cauchy problem* (1.1) *is globally well posed in* $H^{\frac{1}{2}}(\mathbb{R})$ *under the assumption of*

$$\|u_0\|_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}. \tag{1.2}$$

The main approach, as described above, is the I-method. This method is based on the correction analysis of some modified energies and an iteration of local result. The first modified energy is defined as $E(Iu)$, for some smoothed out operator I (see (2.4)). Moreover, one can effectively add a "correction term" to $E(Iu)$. This gives the second modified energy $E_I^2(u)$, and allows us to better capture the cancellations in the frequency space. However, a further analogous procedure does not work. Since in this situation, a strong resonant interaction appears and this resonant interaction will make the related multiplier to be singular. More precisely, as shown in [10], we define the second modified energy by a 4-linear multiplier M_4, which will generate a 6-linear multiplier M_6 in the increment of the second modified energy. If we define the third modified energy naturally by the 6-linear multiplier σ_6 as

$$\sigma_6 = -\frac{M_6}{\alpha_6},$$

where $\alpha_6 = -i(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2)$, then α_6 vanishes in some large sets but M_6 does not. So it is not suitable to define the third modified energy in this way. Our argument is to decompose the multiplier M_6 into two parts: one is relatively small and another is non-resonant. The analogous way of resonant decomposition was previously used in [21,22]. However, it is of great complexity here and a dedicated multiplier analysis is needed in this situation. The resonant decomposition technical was also appeared previously in [2,8,13]. In particular, I-team [13] made use of the second generation "I-method", a resonant decomposition (in order to avoid the "orthogonal resonant interaction") and an "angularly refined bilinear Strichartz estimate" to obtain the global well-posedness of mass-critical nonlinear Schrödinger equation in dimension two.

Remark 1.1. Without loss of generality, we may take $\lambda = 1$ in (1.1) in the following context. Indeed, we may first assume that $\lambda > 0$, otherwise, we may consider $\bar{u}(x, -t)$ for instead. Then we may rescale the solution by the transformation

$$u(x,t) \to \frac{1}{\sqrt{\lambda}} u(x,t).$$

This deduces the general case to the case $\lambda = 1$.

Remark 1.2. For the global well-posedness, it is natural to impose the condition (1.2). Indeed, the solution of (1.1) (for $\lambda = 1$) enjoys the mass and energy conservation laws

$$M\big(u(t)\big) := \int \big|u(t)\big|^2\,dx = M(u_0), \tag{1.3}$$

and

$$H\big(u(t)\big) := \int \bigg[\big|u_x(t)\big|^2 + \frac{3}{2}\,\mathrm{Im}\big|u(t)\big|^2 u(t)\overline{u_x(t)} + \frac{1}{2}\big|u(t)\big|^6\bigg]\,dx = H(u_0). \tag{1.4}$$

By a variant gauge transformation

$$v(x,t) := e^{-\frac{3i}{4}\int_{-\infty}^{x}|u(y,t)|^2\,dy}u(x,t),$$

we have

$$\big\|v(t)\big\|_{L^2_x} = \big\|u(t)\big\|_{L^2_x},$$
$$H\big(u(t)\big) = \big\|v_x(t)\big\|^2_{L^2_x} - \frac{1}{16}\big\|v(t)\big\|^6_{L^6_x}.$$

Thus, the condition (1.2) guarantee the energy $H(u(t))$ to be positive via the sharp Gagliardo–Nirenberg inequality

$$\|f\|^6_{L^6} \leqslant \frac{4}{\pi^2}\|f\|^4_{L^2}\|f_x\|^2_{L^2}.$$

Remark 1.3. In [9], I-team obtained the increment bound N^{-1+} of the first generation modified energy, which leads to the global well-posedness in $H^s(\mathbb{R})$ for $s > 2/3$. In [10], the authors obtained the increment bound N^{-2+} of the second modified energy, which extend the exponent s to $s > 1/2$. In this paper, we will make use of the resonant decomposition to show the increment bound $N^{-5/2+}$ of the third generation modified energy, which allows us to extend the exponent s to $s = 1/2$.

The paper is organized as follows. In Section 2, we give some notations and state some preliminary estimates that will be used throughout this paper. In Section 3, we introduce the gauge transformation and transform (1.1) into another equation. Then we present the conservation law and define the modified energies. In Section 4, we establish the upper bound of the multipliers generated in Section 3. In Section 5, we obtain an upper bound on the increment of the third modified energy. In Section 6, we prove a variant local well-posedness result. In Section 7, we give a comparison between the first and third modified energy. In Section 8, we prove the main result.

2. Notations and preliminary estimates

We use $A \lesssim B$, $B \gtrsim A$ or sometimes $A = O(B)$ to denote the statement that $A \leqslant CB$ for some large constant C which may vary from line to line, and may depend on the data. When it is necessary, we will write the constants by $C_1(\cdot), C_2(\cdot), \ldots$ to see the dependency relationship. We use $A \sim B$ to mean $A \lesssim B \lesssim A$. We use $A \ll B$, or sometimes $A = o(B)$ to denote the statement $A \leqslant C^{-1}B$. The notation $a+$ denotes $a + \epsilon$ for any small ϵ, and $a-$ for $a - \epsilon$. $\langle\cdot\rangle = (1 + |\cdot|^2)^{1/2}$, $J_x^\alpha = (1 - \partial_x^2)^{\alpha/2}$. We use $\|f\|_{L^p_t L^q_x}$ to denote the mixed norm $(\int \|f(\cdot,t)\|^p_{L^q}\,dt)^{\frac{1}{p}}$. Moreover, we denote $\mathscr{F}_x$ to be the Fourier transformation corresponding to the variable x.

For $s, b \in \mathbb{R}$, we define the Bourgain space $X^{\pm}_{s,b}$ to be the closure of the Schwartz class under the norm

$$\|u\|_{X^{\pm}_{s,b}} := \left(\iint \langle\xi\rangle^{2s} \langle \tau \pm \xi^2 \rangle^{2b} \left| \hat{u}(\xi,\tau) \right|^2 d\xi\, d\tau \right)^{1/2}, \tag{2.1}$$

and we write $X_{s,b} := X^{+}_{s,b}$ in default. To study the endpoint regularity, we also need a slightly stronger space $Y^{\pm}_s$ (than $X^{\pm}_{s,\frac{1}{2}}$),

$$\|f\|_{Y^{\pm}_s} := \|f\|_{X^{\pm}_{s,\frac{1}{2}}} + \left\| \langle\xi\rangle^s \hat{f} \right\|_{L^2_\xi L^1_\tau}. \tag{2.2}$$

These spaces obey the embedding $Y^{\pm}_s \hookrightarrow C(\mathbb{R}, H^s(\mathbb{R}))$. Again, we write $Y_s := Y^{+}_s$. It motivates the space Z_s related to Duhamel term under the norm

$$\|f\|_{Z_s} := \|f\|_{X_{s,-\frac{1}{2}}} + \left\| \frac{\langle\xi\rangle^s \hat{f}}{\langle \tau + \xi^2 \rangle} \right\|_{L^2_\xi L^1_\tau}. \tag{2.3}$$

Let $s < 1$ and $N \gg 1$ be fixed, the Fourier multiplier operator $I_{N,s}$ is defined as

$$\widehat{I_{N,s} u}(\xi) = m_{N,s}(\xi) \hat{u}(\xi), \tag{2.4}$$

where the multiplier $m_{N,s}(\xi)$ is a smooth, monotone function satisfying $0 < m_{N,s}(\xi) \leqslant 1$ and

$$m_{N,s}(\xi) = \begin{cases} 1, & |\xi| \leqslant N, \\ N^{1-s}|\xi|^{s-1}, & |\xi| > 2N. \end{cases} \tag{2.5}$$

Sometimes we denote $I_{N,s}$ and $m_{N,s}$ as I and m respectively for short if there is no confusion.

It is obvious that the operator $I_{N,s}$ maps $H^s(\mathbb{R})$ into $H^1(\mathbb{R})$ for any $s < 1$. More precisely, there exists some positive constant C such that

$$C^{-1}\|u\|_{H^s} \leqslant \|I_{N,s}u\|_{H^1} \leqslant CN^{1-s}\|u\|_{H^s}. \tag{2.6}$$

Moreover, $I_{N,s}$ can be extended to a map (still denoted by $I_{N,s}$) from $X_{s,b}$ to $X_{1,b}$, which satisfies that for any $s < 1$, $b \in \mathbb{R}$,

$$C^{-1}\|u\|_{X_{s,b}} \leqslant \|I_{N,s}u\|_{X_{1,b}} \leqslant CN^{1-s}\|u\|_{X_{s,b}}.$$

Now we recall some well-known estimates in the framework of Bourgain space (see [10], for example). First, Strichartz's estimate gives us

$$\|u\|_{L^6_{xt}} \lesssim \|u\|_{X^{\pm}_{0,\frac{1}{2}+}}. \tag{2.7}$$

This interpolates with the identity

$$\|u\|_{L^2_{xt}} = \|u\|_{X_{0,0}},$$

to give

$$\|u\|_{L^q_{xt}} \lesssim \|u\|_{X^{\pm}_{0,\theta+}}, \quad \text{for } \theta \geqslant \frac{3}{2}\left(\frac{1}{2} - \frac{1}{q}\right). \tag{2.8}$$

Moreover, we have

$$\|f\|_{L^\infty_x L^\infty_t} \lesssim \|f\|_{Y_{\frac{1}{2}+}}. \tag{2.9}$$

Indeed, by Young's and Cauchy–Schwarz's inequalities, we have

$$\|f\|_{L^\infty_{xt}} \leqslant \|\hat{f}\|_{L^1_\xi L^1_\tau} \lesssim \left\|\langle\xi\rangle^{\frac{1}{2}+}\hat{f}\right\|_{L^2_\xi L^1_\tau}.$$

Lemma 2.1. *Let $f \in Y^{\pm}_s$ for any $s > 0$, then we have*

$$\|f\|_{L^6_{xt}} \lesssim \|f\|_{Y^{\pm}_s}. \tag{2.10}$$

Proof. We only consider Y_s-norm. By the dyadic decomposition, we write $f = \sum_{j=0}^{\infty} f_j$, for each dyadic component f_j with the frequency support $\langle\xi\rangle \sim 2^j$. Then, by (2.8) and (2.9), we have

$$\begin{aligned}
\|f\|_{L^6_{xt}} &\leqslant \sum_{j=0}^{\infty} \|f_j\|_{L^6_{xt}} \leqslant \sum_{j=0}^{\infty} \|f_j\|^{\theta}_{L^q_{xt}} \|f_j\|^{1-\theta}_{L^\infty_{xt}} \\
&\leqslant \sum_{j=0}^{\infty} \|f_j\|^{\theta}_{X_{0,\frac{1}{2}}} \|f_j\|^{1-\theta}_{Y_\rho} \lesssim \sum_{j=0}^{\infty} 2^{\rho(1-\theta)j} \|f_j\|_{Y_0},
\end{aligned}$$

where $\rho > \frac{1}{2}$, and we choose $q = 6-$ such that $\theta = 1-$. Choosing q close enough to 6 such that $s > \rho(1-\theta)$, then we have the conclusion by Cauchy–Schwarz's inequality. □

Moreover, interpolating between (2.9) and (2.10), we have

$$\|f\|_{L^q_{xt}} \lesssim \|f\|_{Y^{\pm}_{s_q}}, \tag{2.11}$$

for any $q \in (6, +\infty)$ and $s_q > \frac{1}{2}(1 - \frac{6}{q})$.

At last, we give some bilinear estimates. Define the Fourier integral operators $I^s_{\pm}(f, g)$ by

$$\widehat{I^s_{\pm}(f, g)}(\xi, \tau) = \int_\star m_{\pm}(\xi_1, \xi_2)^s \hat{f}(\xi_1, \tau_1)\hat{g}(\xi_2, \tau_2), \tag{2.12}$$

where $\int_\star = \int_{\xi_1+\xi_2=\xi,\, \tau_1+\tau_2=\tau} d\xi_1\, d\tau_1$, and

$$m_- = |\xi_1 - \xi_2|, \qquad m_+ = |\xi_1 + \xi_2|.$$

Then we have

Lemma 2.2. *For the Schwartz functions f, g, we have*

$$\big\|I_-^{\frac{1}{2}}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,\frac{1}{2}+}} \|g\|_{X^+_{0,\frac{1}{2}+}}, \tag{2.13}$$

$$\big\|I_-^{\frac{1}{2}}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^-_{0,\frac{1}{2}+}} \|g\|_{X^-_{0,\frac{1}{2}+}}, \tag{2.14}$$

$$\big\|I_+^{\frac{1}{2}}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,\frac{1}{2}+}} \|g\|_{X^-_{0,\frac{1}{2}+}}. \tag{2.15}$$

Proof. See [14,21] for example. □

When $s=0$, by (2.8) we have

$$\big\|I_\pm^0(f,g)\big\|_{L^2_{xt}} \leqslant \|f\|_{L^p_{xt}}\|g\|_{L^q_{xt}} \lesssim \|f\|_{X_{0,b+}}\|g\|_{X_{0,b'+}}, \tag{2.16}$$

where

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{2}, \qquad b=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{p}\right), \qquad b'=\frac{3}{2}\left(\frac{1}{2}-\frac{1}{q}\right),$$

that is, $b+b'=\frac{3}{4}$, and $b,b'\in[\frac{1}{4},\frac{1}{2}]$.

Interpolating between the results in Lemma 2.2 and (2.16) twice, we have

Corollary 2.1. *Let $I_\pm^s$ be defined by* (2.12)*, then for any $s\in[0,\frac{1}{2}]$,*

$$\big\|I_-^{s}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,b_1+}} \|g\|_{X^+_{0,b_2+}},$$

$$\big\|I_-^{s}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^-_{0,b_1+}} \|g\|_{X^-_{0,b_2+}},$$

$$\big\|I_+^{s}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,b_1+}} \|g\|_{X^-_{0,b_2+}},$$

where $b_1=\frac{1}{2}(1-s'+s)$, $b_2=\frac{1}{4}(2s'+1)$ for any $s'\in[s,\frac{1}{2}]$.

In this paper, we just need the following crude estimates:

$$\big\|I_-^{\frac{1}{2}-}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,\frac{1}{2}-}} \|g\|_{X^+_{0,\frac{1}{2}-}}, \tag{2.17}$$

$$\big\|I_-^{\frac{1}{2}-}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^-_{0,\frac{1}{2}-}} \|g\|_{X^-_{0,\frac{1}{2}-}}, \tag{2.18}$$

$$\big\|I_+^{\frac{1}{2}-}(f,g)\big\|_{L^2_{xt}} \lesssim \|f\|_{X^+_{0,\frac{1}{2}-}} \|g\|_{X^-_{0,\frac{1}{2}-}}. \tag{2.19}$$

Before the end of this section, we record the following forms of the mean value theorem, which are taken from [11]. To prepare for it, we state a definition: Let a and b be two smooth functions of real variables. We say that a is controlled by b if b is non-negative and satisfies $b(\xi) \sim b(\xi')$ for $|\xi| \sim |\xi'|$ and

$$a(\xi) \lesssim b(\xi), \qquad a'(\xi) \lesssim \frac{b(\xi)}{|\xi|}, \qquad a'' \lesssim \frac{b(\xi)}{|\xi|^2}.$$

Lemma 2.3. *If a is controlled by b and $|\eta|, |\lambda| \ll |\xi|$, then we have*

- (*Mean value theorem*)

$$\bigl|a(\xi+\eta) - a(\xi)\bigr| \lesssim |\eta| \frac{b(\xi)}{|\xi|}. \tag{2.20}$$

- (*Double mean value theorem*)

$$\bigl|a(\xi+\eta+\lambda) - a(\xi+\eta) - a(\xi+\lambda) + a(\xi)\bigr| \lesssim |\eta||\lambda| \frac{b(\xi)}{|\xi|^2}. \tag{2.21}$$

3. The Gause transformation, energy and the modified energies

3.1. Gauge transformation and conservation laws

First, we summarize some results presented in [9,10]. We start by recalling the gauge transformation used in [25] to improve the derivative nonlinearity presented in (1.1).

Definition 3.1. We define the nonlinear map $\mathcal{G} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\mathcal{G} f(x) := e^{-i\int_{-\infty}^{x} |f(y)|^2\, dy} f(x).$$

The inverse transformation $\mathcal{G}^{-1} f$ is then given by

$$\mathcal{G}^{-1} f(x) := e^{i\int_{-\infty}^{x} |f(y)|^2\, dy} f(x).$$

Set $w_0 := \mathcal{G} u_0$ and $w(t) := \mathcal{G} u(t)$ for all time t. Then (1.1) is transformed to

$$\begin{cases} i\partial_t w + \partial_x^2 w = -iw^2 \partial_x \bar{w} - \frac{1}{2}|w|^4 w, \quad w : \mathbb{R} \times [0,T] \mapsto \mathbb{C}, \\ w(0,x) = w_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.1}$$

In addition, the smallness condition (1.2) becomes

$$\|w_0\|_{L^2} < \sqrt{2\pi}. \tag{3.2}$$

Note that the transform $\mathcal{G}$ is a bicontinuous map from $H^s(\mathbb{R})$ to itself for any $s \in [0,1]$, thus the global well-posedness of (1.1) is equivalent to that of (3.1). Therefore, from now on, we focus our attention on (3.1) under the assumption (3.2).

Remark 3.1. For the equation without the derivative term in (3.1) (it is the focusing, mass-critical Schrödinger equation):

$$\begin{cases} i\partial_t w + \partial_x^2 w = -|w|^4 w, \\ w(0,x) = w_0(x), \quad x \in \mathbb{R},\ t \in \mathbb{R}, \end{cases}$$

it is global well-posedness below $H^{\frac{1}{2}}(\mathbb{R})$ with the mass less than that of the ground state. Indeed, in [21], the authors proved that it is global well-posedness in $H^s(\mathbb{R})$ for $s > \frac{2}{5}$. So the difficulty of Eq. (3.1) comes mainly from the derivative term.

Definition 3.2. For any $f \in H^1(\mathbb{R})$, we define the mass by

$$M(f) = \int |f|^2\, dx,$$

and the energy $E(f)$ by

$$E(f) := \int |\partial_x f|^2\, dx - \frac{1}{2}\,\mathrm{Im} \int |f|^2 f \partial_x \bar{f}\, dx.$$

By the gauge transformation and the sharp Gagliardo–Nirenberg inequality, we have (see [9] for details)

$$\|\partial_x f\|_{L^2} \leqslant C\big(\|f\|_{L^2}\big) E(f)^{\frac{1}{2}}, \tag{3.3}$$

for any $f \in H^1(\mathbb{R})$ such that $\|f\|_{L^2} < \sqrt{2\pi}$.

Moreover, the solution of (3.1) obeys the mass and energy conservation laws (see cf. [25]):

$$M\big(w(t)\big) = M(w_0), \qquad E\big(w(t)\big) = E(w_0). \tag{3.4}$$

3.2. Definition of n-linear functional

Let w be the solution of (3.1) throughout the following contents. For an even integer n and a given function $M_n(\xi_1, \ldots, \xi_n)$ defined on the hyperplane

$$\Gamma_n = \big\{(\xi_1, \ldots, \xi_n):\ \xi_1 + \cdots + \xi_n = 0\big\}, \tag{3.5}$$

we define the quantity

$$\Lambda_n\big(M_n; w(t)\big) := \int_{\Gamma_n} M_n(\xi_1, \ldots, \xi_n) \mathscr{F}_x w(\xi_1, t) \overline{\mathscr{F}_x w}(-\xi_2, t) \cdots \mathscr{F}_x w(\xi_{n-1}, t) \overline{\mathscr{F}_x w}(-\xi_n, t)\, d\xi_1 \cdots d\xi_{n-1}. \tag{3.6}$$

Then by (3.1) and a directly computation, we have

$$\begin{aligned}\frac{d}{dt}\Lambda_n\big(M_n; w(t)\big) = \Lambda_n\big(M_n\alpha_n; w(t)\big)\\ &- i\Lambda_{n+2}\left(\sum_{j=1}^{n} X_j^2(M_n)\xi_{j+1}; w(t)\right)\\ &+ \frac{i}{2}\Lambda_{n+4}\left(\sum_{j=1}^{n}(-1)^{j+1}X_j^4(M_n); w(t)\right),\end{aligned} \tag{3.7}$$

where

$$\alpha_n = i\sum_{j=1}^{n}(-1)^j\xi_j^2,$$

and

$$X_j^l(M_n) = M_n(\xi_1, \ldots, \xi_{j-1}, \xi_j + \cdots + \xi_{j+l}, \xi_{j+l+1}, \ldots, \xi_{n+l}).$$

Observe that if the multiplier M_n is invariant under the permutations of the even ξ_j indices, or of the odd ξ_j indices, then so is the functional $\Lambda_n(M_n; w(t))$.

Notations. In the following, we shall often write ξ_{ij} for $\xi_i + \xi_j$, ξ_{ijk} for $\xi_i + \xi_j + \xi_k$, etc. Also we write $m(\xi_i) = m_i$ and $m(\xi_i + \xi_j) = m_{ij}$, etc.

3.3. Modified energies

Define the first modified energy as

$$\begin{aligned}E_I^1\big(w(t)\big) &:= E\big(Iw(t)\big)\\ &= -\Lambda_2\big(\xi_1\xi_2 m_1 m_2; w(t)\big) + \frac{1}{4}\Lambda_4\big(\xi_{13} m_1 m_2 m_3 m_4; w(t)\big),\end{aligned} \tag{3.8}$$

where we have used the Plancherel identity and (3.6).

We define the second modified energy as

$$E_I^2\big(w(t)\big) := -\Lambda_2\big(\xi_1\xi_2 m_1 m_2; w(t)\big) + \frac{1}{2}\Lambda_4\big(M_4(\xi_1, \xi_2, \xi_3, \xi_4); w(t)\big), \tag{3.9}$$

where

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{m_1^2\xi_1^2\xi_3 + m_2^2\xi_2^2\xi_4 + m_3^2\xi_3^2\xi_1 + m_4^2\xi_4^2\xi_2}{\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2}. \tag{3.10}$$

Then by (3.7) (or see [10] for more details), we have

$$\frac{d}{dt}E_I^2\big(w(t)\big) = \Lambda_6\big(M_6; w(t)\big) + \Lambda_8\big(M_8; w(t)\big), \tag{3.11}$$

where

$$
\begin{aligned}
&M_6(\xi_1,\ldots,\xi_6)\\
&\quad := \beta_6(\xi_1,\ldots,\xi_6)\\
&\qquad - \frac{i}{72}\sum_{\substack{\{a,c,e\}=\{1,3,5\}\\ \{b,d,f\}=\{2,4,6\}}} \big(M_4(\xi_{abc},\xi_d,\xi_e,\xi_f)\xi_b + M_4(\xi_a,\xi_{bcd},\xi_e,\xi_f)\xi_c\\
&\qquad + M_4(\xi_a,\xi_b,\xi_{cde},\xi_f)\xi_d + M_4(\xi_a,\xi_b,\xi_c,\xi_{def})\xi_e\big), \qquad (3.12)\\
&M_8(\xi_1,\ldots,\xi_8)\\
&\quad := C_8\sum_{\substack{\{a,c,e,g\}=\{1,3,5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} \big(M_4(\xi_{abcde},\xi_f,\xi_g,\xi_h) + M_4(\xi_a,\xi_b,\xi_{cdefg},\xi_h)\\
&\qquad - M_4(\xi_a,\xi_{bcdef},\xi_g,\xi_h) - M_4(\xi_a,\xi_b,\xi_c,\xi_{defgh})\big) \qquad (3.13)
\end{aligned}
$$

for some constant C_8 and

$$\beta_6(\xi_1,\ldots,\xi_6) := -\frac{i}{6}\sum_{j=1}^{6}(-1)^j m_j^2\xi_j^2. \qquad (3.14)$$

Note that M_4, M_6, M_8 are invariant under the permutations of the even ξ_j indices, or of the odd ξ_j indices.

In order to consider the endpoint case, we also need to define the third modified energy. Before constructing it, we shall do some preparations. We adopt the notations that

$$|\xi_1^*| \geqslant |\xi_2^*| \geqslant \cdots \geqslant |\xi_6^*| \geqslant \cdots \geqslant |\xi_n^*|.$$

Moreover, by the symmetry of M_6, M_8 (and other multipliers defined later), we may restrict in Γ_n (defined in (3.5)) that

$$|\xi_1| \geqslant |\xi_3| \geqslant \cdots \geqslant |\xi_{n-1}|, \qquad |\xi_2| \geqslant |\xi_4| \geqslant \cdots \geqslant |\xi_n|.$$

Now we denote the sets

$$
\begin{aligned}
\Upsilon &= \big\{(\xi_1,\ldots,\xi_6)\in\Gamma_6:\ |\xi_1^*|\sim|\xi_2^*|\gtrsim N\big\},\\
\Omega_1 &= \big\{(\xi_1,\ldots,\xi_6)\in\Upsilon:\ |\xi_1|\sim|\xi_3|\gg|\xi_3^*| \text{ or } |\xi_2|\sim|\xi_4|\gg|\xi_3^*|\big\},\\
\Omega_2 &= \big\{(\xi_1,\ldots,\xi_6)\in\Upsilon:\ |\xi_1|\sim|\xi_2|\gtrsim N\gg|\xi_3^*|,\ |\xi_1|^{\frac12}|\xi_1+\xi_2|\gg|\xi_3^*|^{\frac32}\big\},\\
\Omega_3 &= \big\{(\xi_1,\ldots,\xi_6)\in\Upsilon:\ |\xi_3^*|\gg|\xi_4^*|\big\},
\end{aligned}
$$

and let

$$\Omega = \Omega_1\cup\Omega_2\cup\Omega_3.$$

Remark 3.2. M_4 is well controlled by α_4 since α_4 has a good factorization, see [10] or Lemma 4.5 below. However, in general, $|M_6|$ is not controlled by $|\alpha_6|$, this is the main difficulty lied in our problem. However, we exactly have (see Lemma 4.9 for the proof)

$$|M_6| \lesssim |\alpha_6|, \quad \text{for any } (\xi_1, \dots, \xi_6) \in \Omega.$$

For this reason, Ω is referred to the non-resonant set.

Rewrite (3.11) by

$$\frac{d}{dt} E_I^2\big(w(t)\big) = \Lambda_6\big(M_6 \cdot \chi_{\Gamma_6 \setminus \Omega}; w(t)\big) + \Lambda_6\big(M_6 \cdot \chi_\Omega; w(t)\big) + \Lambda_8\big(M_8; w(t)\big). \tag{3.15}$$

Now we are ready to define the third modified energy $E_I^3(w(t))$. Let

$$E_I^3\big(w(t)\big) = \Lambda_6\big(\sigma_6; w(t)\big) + E_I^2\big(w(t)\big), \quad \sigma_6 = -\frac{M_6}{\alpha_6} \cdot \chi_\Omega. \tag{3.16}$$

Then by (3.7) and (3.15), one has

$$\frac{d}{dt} E_I^3\big(w(t)\big) = \Lambda_6\big(M_6 \cdot \chi_{\Gamma_6 \setminus \Omega}; w(t)\big) + \Lambda_8\big(M_8 + \widetilde{M}_8; w(t)\big) + \Lambda_{10}\big(M_{10}; w(t)\big), \tag{3.17}$$

where M_6, M_8 are defined in (3.12), (3.13) respectively, and

$$\widetilde{M}_8 = -i \sum_{j=1}^{6} X_j^2(\sigma_6)\xi_{j+1}, \tag{3.18}$$

$$M_{10} = \frac{i}{2} \sum_{j=1}^{6} (-1)^{j+1} X_j^4(\sigma_6). \tag{3.19}$$

Remark 3.3. By the dyadic decomposition, we restrict that

$$\left|\xi_j^*\right| \sim N_j^*, \quad \text{for any } j = 1, 2, \dots.$$

Now we give some explanations about the construction of Ω_j. We keep in mind the denominator of σ_6,

$$\alpha_6 = -i\big(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2\big).$$

On one hand, for the non-resonant region, we expect $|\alpha_6|$ has a large lower bound in Ω. On the other hand, we expect that the multiplier M_6 has a small upper bound on the resonant region $\Gamma_6 \setminus \Omega$.

(a) By the definition of Ω_1, we have

$$|\alpha_6| \sim N_1^{*2}, \quad \text{for } (\xi_1, \dots, \xi_6) \in \Omega_1.$$

On the other hand, in $\Gamma_6 \setminus \Omega_1$, the following case is ruled out:

$$\xi_1^* = \xi_1, \qquad \xi_2^* = \xi_3; \quad \text{or} \quad \xi_1^* = \xi_2, \qquad \xi_2^* = \xi_4.$$

Therefore, to estimate $M_6 \cdot \chi_{\Gamma_6 \setminus \Omega}$, we only need to consider

$$\xi_1^* = \xi_1, \qquad \xi_2^* = \xi_2; \quad \text{or} \quad \xi_1^* = \xi_2, \qquad \xi_2^* = \xi_1.$$

This is carried out in Proposition 4.1 below.

(b) Now assume that we are in the situation: $|\xi_1| \sim |\xi_2| \gtrsim N \gg |\xi_3^*|$. We find that α_6 will not vanish if

$$|\xi_1 + \xi_2| \gg N_3^{*2}/N_1^*,$$

since in this case $|\alpha_6| \sim |\xi_1||\xi_1 + \xi_2|$. It is common to choose a lower bound of $|\xi_1 + \xi_2|$ between N_3^{*2}/N_1^* and N_3^*, and the choice of the bound will affect the bound of M_6 and $\widetilde{M}_8$. Generally (but not absolutely), a small lower bound of $|\xi_1 + \xi_2|$ gives a small upper bound of M_6, but it maybe lead to a large upper bound of $\widetilde{M}_8$. So, it appears important to make a suitable choice.

As shown in the definition of Ω_2, we choose a middle bound of

$$|\xi_1 + \xi_2| \gg \left|\xi_3^*\right|^{\frac{3}{2}}/|\xi_1|^{\frac{1}{2}}.$$

This leads to the upper bound of $M_6 \cdot \chi_{\Gamma_6 \setminus \Omega}, \widetilde{M}_8$ that if $|\xi_3^*| \ll N$, then

$$|M_6 \cdot \chi_{\Gamma_6 \setminus \Omega}| \lesssim N_1^{*\frac{1}{2}} N_3^{*\frac{1}{2}} N_4^*, \qquad |\widetilde{M}_8| \lesssim N_1^{*\frac{1}{2}} N_3^{*\frac{1}{2}}.$$

See Corollary 4.1 and Proposition 4.3 below.

(c) For the construction of Ω_3, we have two observations. On one hand, we can prove (see Lemma 4.9 below) that

$$|\alpha_6| \sim N_1^{*2}, \quad \text{for } (\xi_1, \ldots, \xi_6) \in \Omega_3.$$

On the other hand, it rules out the bad case

$$\left|\xi_3^*\right| \gtrsim N \gg \left|\xi_4^*\right|$$

in the resonant set $\Gamma_6 \setminus \Omega$. This case prevents us to give a better 6-linear estimate, see Proposition 5.1 below.

4. Upper bound of the multipliers: M_6, M_8, $\widetilde{M}_8$, M_{10}

The key ingredient to prove the almost conservation properties of the modified energies is to obtain the upper bounds of the multipliers introduced in Section 3. In this section, we will present a detailed analysis of the multipliers: M_6, M_8, $\widetilde{M}_8$, M_{10}.

4.1. An alternative description of the multipliers: M_6, M_8, $\widetilde{M}_8$

As a preparation of the next subsections, we rewrite the multipliers in a bright way by merging similar items.

Lemma 4.1. *For the multiplier M_6 defined in* (3.12)*, we have*

$$M_6 = \beta_6 + I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where β_6 is defined in (3.14) *and*

$$I_1 = C_6\big[M_4(\xi_3, \xi_{214}, \xi_5, \xi_6) + M_4(\xi_3, \xi_{216}, \xi_5, \xi_4) + M_4(\xi_3, \xi_{416}, \xi_5, \xi_2)\big]\xi_1,$$
$$I_2 = C_6\big[M_4(\xi_{123}, \xi_4, \xi_5, \xi_6) + M_4(\xi_{125}, \xi_4, \xi_3, \xi_6) + M_4(\xi_{325}, \xi_4, \xi_1, \xi_6)\big]\xi_2,$$
$$I_3 = C_6\big[M_4(\xi_1, \xi_{234}, \xi_5, \xi_6) + M_4(\xi_1, \xi_{236}, \xi_5, \xi_4) + M_4(\xi_1, \xi_{436}, \xi_5, \xi_2)\big]\xi_3,$$
$$I_4 = C_6\big[M_4(\xi_{143}, \xi_2, \xi_5, \xi_6) + M_4(\xi_{145}, \xi_2, \xi_3, \xi_6) + M_4(\xi_{345}, \xi_2, \xi_1, \xi_6)\big]\xi_4,$$
$$I_5 = C_6\big[M_4(\xi_1, \xi_{254}, \xi_3, \xi_6) + M_4(\xi_1, \xi_{256}, \xi_3, \xi_4) + M_4(\xi_1, \xi_{456}, \xi_3, \xi_2)\big]\xi_5,$$
$$I_6 = C_6\big[M_4(\xi_{163}, \xi_2, \xi_5, \xi_4) + M_4(\xi_{165}, \xi_2, \xi_3, \xi_4) + M_4(\xi_{365}, \xi_2, \xi_1, \xi_4)\big]\xi_6$$

for some constant C_6.

For M_8, we rewrite it as the following two formulations.

Lemma 4.2. *For the multiplier* M_8 *defined in* (3.13), *we have*

$$M_8 = J_1 + J_2 + J_3 + J_4 = J_1' + J_2' + J_3' + J_4', \tag{4.1}$$

where

$$J_1 = 2C_8' \sum_{\substack{\{a,c,e\}=\{3,5,7\}\\ \{b,d,f\}=\{4,6,8\}}} \big[M_4(\xi_{12abc}, \xi_d, \xi_e, \xi_f) - M_4(\xi_a, \xi_{12bcd}, \xi_e, \xi_f)\big],$$

$$J_2 = C_8' \sum_{\substack{\{a,c,e\}=\{3,5,7\}\\ \{b,d,f\}=\{4,6,8\}}} \big[M_4(\xi_{a2cbe}, \xi_d, \xi_1, \xi_f) - M_4(\xi_a, \xi_{b1dcf}, \xi_e, \xi_2)\big],$$

$$J_3 = C_8' \sum_{\substack{\{a,c,e\}=\{3,5,7\}\\ \{b,d,f\}=\{4,6,8\}}} \big[M_4(\xi_{1badc}, \xi_2, \xi_e, \xi_f) - M_4(\xi_1, \xi_{2abcd}, \xi_e, \xi_f)\big],$$

$$J_4 = 2C_8' \sum_{\substack{\{a,c,e\}=\{3,5,7\}\\ \{b,d,f\}=\{4,6,8\}}} \big[M_4(\xi_1, \xi_2, \xi_{abcde}, \xi_f) - M_4(\xi_1, \xi_2, \xi_a, \xi_{bcdef})\big],$$

$$J_1' = 2C_8' \sum_{\substack{\{a,c\}=\{5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} \big[M_4(\xi_{1b3da}, \xi_f, \xi_c, \xi_h) - M_4(\xi_a, \xi_{b1d3f}, \xi_c, \xi_h)\big],$$

$$J_2' = C_8' \sum_{\substack{\{a,c\}=\{5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} \big[M_4(\xi_{1badc}, \xi_f, \xi_3, \xi_h) + M_4(\xi_{3badc}, \xi_f, \xi_1, \xi_h)\big],$$

$$J_3' = -C_8' \sum_{\substack{\{a,c\}=\{5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} \big[M_4(\xi_1, \xi_{b3daf}, \xi_c, \xi_f) + M_4(\xi_3, \xi_{b1daf}, \xi_c, \xi_f)\big],$$

$$J_4' = 2C_8' \sum_{\substack{\{a,c,e\}=\{3,5,7\}\\ \{b,d,f\}=\{4,6,8\}}} \big[M_4(\xi_1, \xi_{badcf}, \xi_3, \xi_e) - M_4(\xi_1, \xi_b, \xi_3, \xi_{adcfe})\big]$$

for some constant C_8'.

For $\widetilde{M}_8$, we rewrite it as follows.

Lemma 4.3. *For the multiplier $\widetilde{M}_8$ defined in* (3.18), *we have*

$$\widetilde{M}_8 = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{R}_8, \tag{4.2}$$

where

$$\begin{aligned}
\tilde{J}_1 = \tilde{C}'_8\big[&\sigma_6(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8) + \sigma_6(\xi_3, \xi_{216}, \xi_5, \xi_4, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_3, \xi_{218}, \xi_5, \xi_4, \xi_7, \xi_6) + \sigma_6(\xi_3, \xi_{416}, \xi_5, \xi_2, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_3, \xi_{418}, \xi_5, \xi_2, \xi_7, \xi_6) + \sigma_6(\xi_3, \xi_{618}, \xi_5, \xi_2, \xi_7, \xi_4)\big]\xi_1, \\
\tilde{J}_2 = \tilde{C}'_8\big[&\sigma_6(\xi_{123}, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8) + \sigma_6(\xi_{125}, \xi_4, \xi_3, \xi_6, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_{127}, \xi_4, \xi_3, \xi_6, \xi_5, \xi_8) + \sigma_6(\xi_{325}, \xi_4, \xi_1, \xi_6, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_{327}, \xi_4, \xi_1, \xi_6, \xi_5, \xi_8) + \sigma_6(\xi_{527}, \xi_4, \xi_1, \xi_6, \xi_3, \xi_8)\big]\xi_2, \\
\tilde{J}_3 = \tilde{C}'_8\big[&\sigma_6(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8) + \sigma_6(\xi_1, \xi_{236}, \xi_5, \xi_4, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_1, \xi_{238}, \xi_5, \xi_4, \xi_7, \xi_6) + \sigma_6(\xi_1, \xi_{436}, \xi_5, \xi_2, \xi_7, \xi_8) \\
&+ \sigma_6(\xi_1, \xi_{438}, \xi_5, \xi_2, \xi_7, \xi_6) + \sigma_6(\xi_1, \xi_{638}, \xi_5, \xi_2, \xi_7, \xi_4)\big]\xi_3
\end{aligned}$$

for some constant $\tilde{C}'_8$, and

$$|\tilde{R}_8| \lesssim \max_{\Omega} |\sigma_6| \cdot \max\big\{|\xi_4|, \dots, |\xi_8|\big\}. \tag{4.3}$$

Next, we give the bounds of the multipliers one by one. From now on, we may assume by symmetry that

$$|\xi_1| \geqslant |\xi_2|$$

in the following analysis. Hence

$$\xi_1^* = \xi_1, \qquad \xi_2^* = \xi_2 \text{ or } \xi_3.$$

4.2. Known facts

In this subsection, we restate some results obtained in [10]. First, we have

Lemma 4.4. *(See [10].) If $N_1^* \ll N$, then we have*

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{2}(\xi_1 + \xi_3), \tag{4.4}$$

$$M_6(\xi_1, \dots, \xi_6) = 0, \qquad M_8(\xi_1, \dots, \xi_8) = 0. \tag{4.5}$$

Second, we present some estimates on the multipliers.

Lemma 4.5. *(See [10].) The following estimates hold*:

(1) $$\big|M_4(\xi_1, \xi_2, \xi_3, \xi_4)\big| \lesssim m_1^2 N_1^*; \tag{4.6}$$

(2) *If* $|\xi_1| \sim |\xi_3| \gtrsim N \gg |\xi_3^*|$, *then*

$$\left|M_4(\xi_1, \xi_2, \xi_3, \xi_4)\right| \lesssim m_1^2 N_3^*; \tag{4.7}$$

(3) *If* $|\xi_1| \sim |\xi_2| \gtrsim N \gg |\xi_3^*|$, *then*

$$M_4(\xi_1, \xi_2, \xi_3, \xi_4) = \frac{1}{2} m_1^2 \xi_1 + R(\xi_1, \xi_2, \xi_3, \xi_4), \quad \textit{for } |R| \lesssim N_3^*; \tag{4.8}$$

(4) *If* $|\xi_3^*| \gtrsim N$, *then*

$$\left|M_6(\xi_1, \dots, \xi_6)\right| \lesssim m_1^2 N_1^{*2}; \tag{4.9}$$

(5) *If* $|\xi_3^*| \ll N$, *then*

$$\left|M_6(\xi_1, \dots, \xi_6)\right| \lesssim N_1^* N_3^*. \tag{4.10}$$

4.3. An improvement upper bound of M_6

The estimates (4.9) and (4.10) are not enough for us to use, now we make some refinements.

Proposition 4.1. *For the multiplier M_6 defined in* (3.12)*, the following estimates hold*:

(1) *If* $\xi_2^* = \xi_2$, *then*

$$\left|M_6(\xi_1, \dots, \xi_6)\right| \lesssim N_1^* N_3^*. \tag{4.11}$$

(2) *Furthermore, if* $|\xi_1| \sim |\xi_2| \gtrsim N \gg |\xi_3^*|$, *then*

$$M_6(\xi_1, \dots, \xi_6) = -C_6 \xi_1 \xi_{12} + C_6' \left(m_2^2 \xi_2^2 - m_1^2 \xi_1^2\right) - C_6 m_1^2 \xi_1 \xi_{12} + O\left(N_3^{*2}\right), \tag{4.12}$$

where C_6 is the constant in Lemma 4.1 *and* $C_6' = \frac{1}{2} C_6 - \frac{i}{6}$.

Proof. For the sake of simplicity, we may assume that $C_6 = 1$. Moreover, for (4.11), we only consider the case $N_1^* \gg N_3^*$, otherwise it is contained in (4.9). Thus, we may assume that $|\xi_1| \sim |\xi_2| \gg |\xi_3^*|$ in (1).

Now we estimate (4.11) and (4.12) together. Note that

$$\beta_6 = -\frac{i}{6}\left(m_2^2 \xi_2^2 - m_1^2 \xi_1^2\right) + O\left(N_3^{*2}\right).$$

It suffices to estimate: $I_1, \dots, I_6$ by Lemma 4.1.

For I_1, I_2, by the definitions, we further divide them into three parts:

$$I_1 := I_{11} + I_{12} + I_{13}; \qquad I_2 := I_{21} + I_{22} + I_{23},$$

where

$$I_{11} := M_4(\xi_3, \xi_{214}, \xi_5, \xi_6)\xi_1, \qquad I_{12} := M_4(\xi_3, \xi_{216}, \xi_5, \xi_4)\xi_1,$$
$$I_{13} := M_4(\xi_3, \xi_{416}, \xi_5, \xi_2)\xi_1,$$
$$I_{21} := M_4(\xi_{123}, \xi_4, \xi_5, \xi_6)\xi_2, \qquad I_{22} := M_4(\xi_{125}, \xi_4, \xi_3, \xi_6)\xi_2,$$
$$I_{23} := M_4(\xi_{325}, \xi_4, \xi_1, \xi_6)\xi_2.$$

In order to estimate $I_1, \dots, I_6$, it is enough to prove the following three lemmas.

Lemma 4.6. *If $|\xi_1| \sim |\xi_2| \gg |\xi_3^*|$, then we have*

$$I_{13} + I_{23} = \frac{1}{2}\left(m_1^2\xi_1\xi_2 + m_2^2\xi_2^2\right) + O\left(N_3^{*2}\right). \tag{4.13}$$

Hence,

$$|I_{13} + I_{23}| \lesssim N_1^* N_3^*. \tag{4.14}$$

Proof. By the definition, we have

$$\begin{aligned} I_{13} &= M_4(\xi_3, \xi_{416}, \xi_5, \xi_2)\xi_1 \\ &= -\frac{m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416} + m_3^2\xi_3^2\xi_5 + m_5^2\xi_5^2\xi_3}{\alpha} \cdot \xi_1, \end{aligned}$$

where $\alpha = \xi_3^2 - \xi_{416}^2 + \xi_5^2 - \xi_2^2$. Similarly,

$$\begin{aligned} I_{23} &= M_4(\xi_{325}, \xi_4, \xi_1, \xi_6)\xi_2 \\ &= -\frac{m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325} + m_4^2\xi_4^2\xi_6 + m_6^2\xi_6^2\xi_4}{\alpha'} \cdot \xi_2, \end{aligned}$$

where $\alpha' = \xi_{325}^2 - \xi_4^2 + \xi_1^2 - \xi_6^2$. Note that $|\xi_1| \sim |\xi_2| \gg |\xi_3^*|$, we have

$$|\alpha|, \left|\alpha'\right| \sim N_1^{*2}. \tag{4.15}$$

Then,

$$\begin{aligned} I_{13} &= -\frac{m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416}}{\alpha} \cdot \xi_1 + O\left(N_3^{*2}N_4^*/N_1^*\right), \\ I_{23} &= -\frac{m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}}{\alpha'} \cdot \xi_2 + O\left(N_3^{*2}N_4^*/N_1^*\right), \end{aligned}$$

which yield that

$$\begin{aligned} I_{13} + I_{23} = &-\frac{m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416}}{\alpha} \cdot (\xi_1 + \xi_2) \\ &+ \xi_2 \cdot \left(\frac{m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416}}{\alpha} - \frac{m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}}{\alpha'}\right) \end{aligned}$$

$$+ O\big(N_3^{*2}N_4^*/N_1^*\big)$$
$$:= II_1 + \xi_2 \cdot II + O\big(N_3^{*2}N_4^*/N_1^*\big). \tag{4.16}$$

First, by the mean value theorem (2.20) and $m \leqslant 1$, we have

$$|II_1| \lesssim m_1^2|\xi_1+\xi_2||\xi_{1246}| \lesssim N_3^{*2}. \tag{4.17}$$

On the other hand, note that $\xi_{416} = -\xi_{325}$, then we have

$$\begin{aligned}
II &= \frac{m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416}}{\alpha} - \frac{m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}}{\alpha'} \\
&= \frac{1}{\alpha}\big(m_{416}^2\xi_{416}^2\xi_2 + m_2^2\xi_2^2\xi_{416} + m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}\big) \\
&\quad - \left(\frac{\alpha+\alpha'}{\alpha\alpha'}\right)\cdot\big(m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}\big) \\
&= \frac{1}{\alpha}\big(m_{416}^2\xi_{416}^2(\xi_1+\xi_2) + \xi_{416}\big(m_2^2\xi_2^2 - m_1^2\xi_1^2\big)\big) \\
&\quad - \left(\frac{\alpha+\alpha'}{\alpha\alpha'}\right)\cdot\big(m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}\big).
\end{aligned} \tag{4.18}$$

By the mean value theorem (2.20), we have

$$\frac{1}{\alpha} = \frac{1}{2\xi_1\xi_2} + O\big(N_3^*/N_1^{*3}\big),$$
$$m_{416}^2\xi_{416}^2(\xi_1+\xi_2) + \xi_{416}\big(m_2^2\xi_2^2 - m_1^2\xi_1^2\big) = O\big(N_1^{*2}N_3^*\big).$$

Thus,

$$\begin{aligned}
&\text{Term 1 of (4.18)} \\
&\quad = \frac{1}{2\xi_1\xi_2}\big(m_{416}^2\xi_{416}^2(\xi_1+\xi_2) + \xi_{416}\big(m_2^2\xi_2^2 - m_1^2\xi_1^2\big)\big) + O\big(N_3^{*2}/N_1^*\big) \\
&\quad = \frac{1}{2\xi_1\xi_2}\big(m_1^2\xi_1^2(\xi_1+\xi_2) + \xi_1\big(m_2^2\xi_2^2 - m_1^2\xi_1^2\big)\big) + O\big(N_3^{*2}/N_1^*\big) \\
&\quad = \frac{1}{2}\big(m_1^2\xi_1 + m_2^2\xi_2\big) + O\big(N_3^{*2}/N_1^*\big).
\end{aligned} \tag{4.19}$$

On the other hand, by the mean value theorem (2.20),

$$\big|\alpha+\alpha'\big| = |\alpha_6| \lesssim N_1^*N_3^*, \qquad \big|m_{325}^2\xi_{325}^2\xi_1 + m_1^2\xi_1^2\xi_{325}\big| = O\big(N_1^{*2}N_3^*\big).$$

Thus, by (4.15), we get

$$\text{Term 2 of (4.18)} \lesssim N_3^{*2}/N_1^*. \tag{4.20}$$

Combining (4.18), (4.19) with (4.20), we have

$$II = \frac{1}{2}\left(m_1^2\xi_1 + m_2^2\xi_2\right) + O\left(N_3^{*2}/N_1^*\right). \tag{4.21}$$

Inserting (4.17) and (4.21) into (4.16), we have the desired result. □

Lemma 4.7. *If $|\xi_1| \sim |\xi_2| \gg |\xi_3^*|$, then we have*

$$I_{11} + I_{12} + I_{21} + I_{22} \lesssim N_1^* N_3^*. \tag{4.22}$$

Furthermore, if $|\xi_3^*| \ll N$, we have

$$I_{11} + I_{12} + I_{21} + I_{22} = -\xi_1\xi_{12} + O\left(N_3^{*2}\right). \tag{4.23}$$

Proof. Since $|\xi_{12}| \lesssim N_3^*$, (4.22) follows from (4.6). Moreover, if $|\xi_{12}| \ll N$, then by (4.4), we have

$$I_{11} = I_{12} = \frac{1}{2}\xi_{35} \cdot \xi_1, \qquad I_{21} = I_{22} = -\frac{1}{2}\xi_{46} \cdot \xi_2,$$

which implies that

$$I_{11} + I_{12} + I_{21} + I_{22} = \xi_{35}\xi_1 - \xi_{46}\xi_2 = \xi_{12} \cdot \xi_{235} = -\xi_1\xi_{12} + O\left(N_3^{*2}\right).$$

This completes the proof of the lemma. □

Lemma 4.8. *If $|\xi_1| \sim |\xi_2| \gg |\xi_3^*|$, then we have*

$$|I_3 + I_4 + I_5 + I_6| \lesssim N_1^* N_3^*. \tag{4.24}$$

Furthermore, if $|\xi_3^*| \ll N$, then

$$I_3 + I_4 + I_5 + I_6 = -\frac{3}{2}m_1^2\xi_1\xi_{12} + O\left(N_3^{*2}\right). \tag{4.25}$$

Proof. (4.24) follows from (4.6). Now we consider the case $|\xi_3^*| \ll N$. By (4.8), we have

$$M_4(\overline{\xi_1}, \overline{\xi_2}, \xi_3, \xi_4) = \frac{1}{2}m_1^2\xi_1 + O\left(N_3^*\right), \tag{4.26}$$

where $\overline{\xi_1} + \overline{\xi_2} + \xi_3 + \xi_4 = 0$, $\overline{\xi_1} = \xi_1 + O(N_3^*)$, $\overline{\xi_2} = \xi_2 + O(N_3^*)$ and $|\overline{\xi_1}| \sim |\overline{\xi_2}| \gtrsim N \gg |\xi_3^*|$. Using (4.26), we obtain

$$\begin{aligned} I_3 + I_4 + I_5 + I_6 &= \frac{3}{2}m_1^2\xi_1(\xi_3 + \xi_4 + \xi_5 + \xi_6) + O\left(N_3^{*2}\right) \\ &= -\frac{3}{2}m_1^2\xi_1(\xi_1 + \xi_2) + O\left(N_3^{*2}\right). \end{aligned}$$

This completes the proof of the lemma. □

Now we finish the proof of Proposition 4.1. Indeed, (4.11) follows from (4.14), (4.22) and (4.24). While by (4.13) and (4.25), we have

$$\begin{aligned} & I_{13}+I_{23}+I_3+I_4+I_5+I_6 \\ & \quad =\frac{1}{2}\left(m_1^2\xi_1\xi_2+m_2^2\xi_2^2\right)-\frac{3}{2}m_1^2\xi_1(\xi_1+\xi_2)+O\left(N_3^{*2}\right) \\ & \quad =\frac{1}{2}\left(m_2^2\xi_2^2-m_1^2\xi_1^2\right)-m_1^2\xi_1(\xi_1+\xi_2)+O\left(N_3^{*2}\right). \end{aligned} \tag{4.27}$$

Therefore, (4.12) follows from (4.23) and (4.27).

Corollary 4.1. *If $|\xi_3^*| \ll N$, then we have*

$$\left|M_6(\xi_1,\ldots,\xi_6)\right| \lesssim N_1^{*\frac{1}{2}} N_3^{*\frac{1}{2}} N_4^* \quad \text{in } \Gamma_6\backslash\Omega. \tag{4.28}$$

Proof. In this situation, $\xi_2^* = \xi_2$ (see Remark 3.3(a)). Then by (4.12) and the mean value theorem (2.20), we have

$$\left|M_6(\xi_1,\ldots,\xi_6)\right| \lesssim |\xi_1||\xi_1+\xi_2| + N_3^{*2}.$$

Moreover, since $|\xi_1|^{\frac{1}{2}}|\xi_1+\xi_2| \lesssim |\xi_3^*|^{\frac{3}{2}}$ in $\Gamma_6\backslash\Omega$, we have

$$\left|M_6(\xi_1,\ldots,\xi_6)\right| \lesssim N_1^{*\frac{1}{2}} N_3^{*\frac{3}{2}}.$$

Then (4.28) follows by the fact that $N_3^* \sim N_4^*$ in $\Gamma_6\backslash\Omega_3$. □

4.4. An upper bound of M_8

Proposition 4.2.

$$\left|M_8(\xi_1,\ldots,\xi_8)\right| \lesssim N_1^*. \tag{4.29}$$

Furthermore, if $|\xi_3^*| \ll N$, then we have

$$\left|M_8(\xi_1,\ldots,\xi_8)\right| \lesssim N_3^*. \tag{4.30}$$

Proof. By (4.6), we have $|M_4(\xi_1,\xi_2,\xi_3,\xi_4)| \lesssim N_1^*$. Thus (4.29) follows. For (4.30), we split it into two cases.

Case 1. $\xi_2^* = \xi_2$. By (4.1), we have

$$M_8 = J_1 + J_2 + J_3 + J_4.$$

So it suffices to prove: $|J_1|, |J_2|, |J_3|, |J_4| \lesssim N_3^*$. First, J_1 follows immediately from $|\xi_1+\xi_2| \lesssim N_3^*$ and (4.6). While J_2 follows from (4.7) and J_3, J_4 follow from (4.26).

Case 2. $\xi_2^* = \xi_3$. Now we adopt the formulation:

$$M_8 = J_1' + J_2' + J_3' + J_4',$$

and it is necessary to prove: $|J_1'|, |J_2'|, |J_3'|, |J_4'| \lesssim N_3^*$. J_1' and J_2' are similar to J_1 and J_2. For J_3', we also use (4.26) to give

$$J_3' = C\big(m_1^2\xi_1 + m_3^2\xi_3\big) + O\big(N_3^*\big) = O\big(N_3^*\big),$$

where we used the mean value theorem (2.20). J_4' is similar to J_2. □

4.5. An upper bound of σ_6, $\widetilde{M}_8$

First, we prove that σ_6 is uniformly bounded in Ω, which implies that the set Ω is non-resonant.

Lemma 4.9. *In Ω, we have*

$$\big|\sigma_6(\xi_1, \dots, \xi_6)\big| \lesssim 1. \tag{4.31}$$

Particularly, in $\Omega_1 \cap \{|\xi_3^*| \ll N\}$, we have

$$\big|\sigma_6(\xi_1, \dots, \xi_6)\big| \lesssim N_3^*/N_1^*. \tag{4.32}$$

Proof. Recall that

$$\sigma_6 = -\frac{M_6}{\alpha_6} \cdot \chi_\Omega, \quad \alpha_6 = -i\big(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2 + \xi_5^2 - \xi_6^2\big).$$

In Ω_1, we have

$$\big|\alpha_6(\xi_1, \dots, \xi_6)\big| \sim N_1^{*2}.$$

This gives (4.32) by (4.10) and (4.31) by (4.9).

In Ω_2, we have

$$\big|\xi_1^2 - \xi_2^2\big| \sim |\xi_1||\xi_1 + \xi_2| \gg \big|\xi_3^*\big|^2,$$

which yields that

$$|\alpha_6| \sim |\xi_1||\xi_1 + \xi_2|. \tag{4.33}$$

While from (4.12) and the mean value theorem (2.20), we have

$$\big|M_6(\xi_1, \dots, \xi_6)\big| \lesssim |\xi_1||\xi_1 + \xi_2| + N_3^{*2} \lesssim |\xi_1||\xi_1 + \xi_2|.$$

This gives (4.31) in Ω_2.

In Ω_3, since $\xi_1^* \cdot \xi_2^* < 0$, $\xi_2^* \cdot \xi_3^* > 0$, it holds that

$$\big|\xi_1^*\big| = \big|\xi_2^*\big| + \big|\xi_3^*\big| + o\big(N_3^*\big).$$

We claim that

$$|\alpha_6| \gtrsim N_1^* N_3^*. \tag{4.34}$$

Indeed, for (4.34), we divide it into the following three cases:

(i) $\xi_2^* = \xi_2$, $\xi_3^* = \xi_3$; (ii) $\xi_2^* = \xi_2$, $\xi_3^* = \xi_4$; (iii) $\xi_2^* = \xi_3$, $\xi_3^* = \xi_2$.

If $\xi_2^* = \xi_2$, $\xi_3^* = \xi_3$, then we get

$$\begin{aligned}|\alpha_6| &= \big|\big(\xi_1^2 - \xi_2^2\big) + \xi_3^2 + \big(-\xi_4^2 + \xi_5^2 - \xi_6^2\big)\big| \\ &= \big(\xi_1^2 - \xi_2^2\big) + \xi_3^2 + o\big(|\xi_3|^2\big) \\ &= -\xi_1\xi_3 + \xi_3^2 + o\big(|\xi_1||\xi_3|\big) \\ &\sim |\xi_1||\xi_3|.\end{aligned}$$

If $\xi_2^* = \xi_2$, $\xi_3^* = \xi_4$, then we have

$$\begin{aligned}|\alpha_6| &= \big|\big(\xi_1^2 - \xi_2^2 - \xi_4^2\big) + \big(\xi_3^2 + \xi_5^2 - \xi_6^2\big)\big| \\ &= \big(\xi_1^2 - \xi_2^2 - \xi_4^2\big) + o\big(|\xi_4|^2\big) \\ &= \big(\big[|\xi_2| + |\xi_4| + o\big(|\xi_4|\big)\big]^2 - \xi_2^2 - \xi_4^2\big) + o\big(|\xi_4|^2\big) \\ &\sim |\xi_2||\xi_4|.\end{aligned}$$

If $\xi_2^* = \xi_3$, $\xi_3^* = \xi_2$, then we have

$$|\alpha_6| = \big(\xi_1^2 - \xi_2^2 + \xi_3^2\big) + o\big(|\xi_3|^2\big) \geqslant \xi_3^2 + o\big(|\xi_3|^2\big) \sim \xi_1^2.$$

This proves (4.34).

By (4.9) and (4.10), we have $|M_6(\xi_1, \ldots, \xi_6)| \lesssim N_1^{*2}$. Then (4.31) follows if $N_1^* \sim N_3^*$. Now we consider the other case: $N_1^* \gg N_3^*$. Thus we have: $\xi_2^* = \xi_2$ in $\Omega_3 \backslash \Omega_1$. Then (4.31) in $\Omega_3 \backslash \Omega_1$ follows from (4.11) and (4.34). □

Now we give the upper bound of $\widetilde{M}_8$.

Proposition 4.3.

$$\big|\widetilde{M}_8(\xi_1, \ldots, \xi_8)\big| \lesssim N_1^*. \tag{4.35}$$

Furthermore, if $|\xi_3^*| \ll N$, then we have

$$\big|\widetilde{M}_8(\xi_1, \ldots, \xi_8)\big| \lesssim N_1^{*\frac{1}{2}} N_3^{*\frac{1}{2}}. \tag{4.36}$$

Proof. Since $|\sigma_6| \lesssim 1$, we have (4.35). Now we turn to (4.36). By (4.2), we shall estimate: $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$. For this purpose, we divide it into two cases.

Case 1. $\xi_2^* = \xi_2$. Since $|\sigma_6| \lesssim 1$, we have $|\tilde{J}_3| \lesssim N_3^*$. Now we consider the other two parts. Since $\sigma_6 = 0$ for $|\xi_1^*| \ll N$, we know that the first, second, third terms of $\tilde{J}_1, \tilde{J}_2$ vanish. Therefore,

$$\begin{aligned}\widetilde{M}_8 = \tilde{C}_8'\big[&\sigma_6(\xi_3, \xi_{416}, \xi_5, \xi_2, \xi_7, \xi_8) + \sigma_6(\xi_3, \xi_{418}, \xi_5, \xi_2, \xi_7, \xi_6) \\ &+ \sigma_6(\xi_3, \xi_{618}, \xi_5, \xi_2, \xi_7, \xi_4)\big]\xi_1 + \tilde{C}_8'\big[\sigma_6(\xi_{325}, \xi_4, \xi_1, \xi_6, \xi_7, \xi_8) \\ &+ \sigma_6(\xi_{327}, \xi_4, \xi_1, \xi_6, \xi_5, \xi_8) + \sigma_6(\xi_{527}, \xi_4, \xi_1, \xi_6, \xi_3, \xi_8)\big]\xi_2 + O\big(N_3^*\big).\end{aligned} \tag{4.37}$$

By (4.32), each term is bounded by N_3^*.

Case 2. $\xi_2^* = \xi_3$. In this case, $|\tilde{J}_2| \lesssim N_3^*$, so we only need to estimate $\tilde{J}_1$, $\tilde{J}_3$. By permutating the terms in $\tilde{J}_1$, $\tilde{J}_3$, we may rewrite $\widetilde{M}_8$ as

$$\widetilde{M}_8 = \sum_{\substack{\{a,c\}=\{5,7\}\\ \{b,d,f,h\}=\{2,4,6,8\}}} \left[\sigma_6(\xi_3, \xi_{b1d}, \xi_a, \xi_f, \xi_c, \xi_h)\xi_1 + \sigma_6(\xi_1, \xi_{b3d}, \xi_a, \xi_f, \xi_c, \xi_h)\xi_3\right] + O\left(N_3^*\right).$$

As an example, we only consider

$$\sigma_6(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8)\xi_1 + \sigma_6(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8)\xi_3,$$

which equals to

$$III \cdot \xi_1 + O\left(N_3^*\right), \tag{4.38}$$

where

$$III := \sigma_6(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8) - \sigma_6(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8).$$

We first adopt some notations for short. We denote

$$A := M_6(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8); \qquad A' := M_6(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8),$$
$$B := \alpha_6(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8); \qquad B' := \alpha_6(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8).$$

Since

$$\Omega_2(\xi_3, \xi_{214}, \xi_5, \xi_6, \xi_7, \xi_8) = \Omega_2(\xi_1, \xi_{234}, \xi_5, \xi_6, \xi_7, \xi_8),$$

then by (4.31), (4.33) and the definition of Ω_2, we have

$$\left|\frac{A}{B}\right|, \left|\frac{A'}{B'}\right| \lesssim 1; \qquad |B|, \left|B'\right| \sim |\xi_{1234}||\xi_1| \gg N_1^{*\frac{1}{2}} N_3^{*\frac{3}{2}}. \tag{4.39}$$

Moreover,

$$III = \frac{A}{B} - \frac{A'}{B'} = \frac{1}{B}\left(A + A'\right) - \frac{A'}{B'} \cdot \frac{B + B'}{B}. \tag{4.40}$$

On one hand, by (4.12) and (4.39), we have

$$A + A' = C_6\xi_{1234} \cdot (2\xi_{2457} + \xi_{13}) - C_6\xi_{1234}\left(m_1^2\xi_1 + m_3^2\xi_3\right) + C_6'\left(m_{214}^2\xi_{214}^2 - m_3^2\xi_3^2 + m_{234}^2\xi_{234}^2 - m_1^2\xi_1^2\right) + O\left(N_3^{*2}\right).$$

Moreover, by the mean value theorem (2.20) in the second term and by the double mean value theorem (2.21) in the third term, we have

$$\left|A + A'\right| \lesssim m_1^2|\xi_{1234}||\xi_{24}| + N_3^{*2}. \tag{4.41}$$

Therefore, by (4.39) and (4.41), we have

$$\left|\frac{1}{B}\big(A+A'\big)\right| \lesssim m_1^2\frac{|\xi_{24}|}{|\xi_1|}+\frac{N_3^{*2}}{N_1^{*\frac{1}{2}}N_3^{*\frac{3}{2}}}$$
$$\lesssim N_3^*/N_1^*+N_3^{*\frac{1}{2}}/N_1^{*\frac{1}{2}}\lesssim N_3^{*\frac{1}{2}}/N_1^{*\frac{1}{2}}. \tag{4.42}$$

On the other hand,

$$\big|B+B'\big|=\big|\xi_1^2-\xi_{234}^2+\xi_3^2-\xi_{214}^2\big|+O\big(N_3^{*2}\big)=2|\xi_{1234}||\xi_{24}|+O\big(N_3^{*2}\big).$$

Therefore, by the similar estimates as those in (4.39) and (4.42), we have

$$\left|\frac{A'}{B'}\cdot\frac{B+B'}{B}\right|\lesssim N_3^{*\frac{1}{2}}/N_1^{*\frac{1}{2}}. \tag{4.43}$$

Inserting (4.42) and (4.43) into (4.40), we have

$$|III|\lesssim N_3^{*\frac{1}{2}}/N_1^{*\frac{1}{2}},$$

which together with (4.38) yields (4.36). □

5. An upper bound on the increment of $E_I^3(u(t))$

By the multilinear correction analysis, the almost conservation law of $E_I^3(u(t))$ is the key ingredient to establish the global well-posedness below the energy space. This is made up of the following 6-linear, 8-linear and 10-linear estimates.

Proposition 5.1. *For any $s\geqslant\frac{1}{2}$, we have*

$$\left|\int_0^\delta \Lambda_6\big(M_6\cdot\chi_{\Gamma_6\setminus\Omega};w(t)\big)\,dt\right|\lesssim N^{-\frac{5}{2}+}\|Iw\|_{Y_1}^6. \tag{5.1}$$

Proof. By (4.5), when $|\xi_1|,\dots,|\xi_6|\ll N$, we have $M_6=0$. Therefore, we may assume that $|\xi_1^*|\sim|\xi_2^*|\gtrsim N$. Note that

$$\big\|\chi_{[0,\delta]}(t)f\big\|_{X_{0,\frac{1}{2}-}}\lesssim\|f\|_{X_{0,\frac{1}{2}}}$$

(see Lemma 2.2 in [20] for example), (5.1) is reduced to

$$\left|\int\Lambda_6\big(M_6\cdot\chi_{\Gamma_6\setminus\Omega};w(t)\big)\,dt\right|\lesssim N^{-\frac{5}{2}+}\|Iw\|_{X_{1,\frac{1}{2}-}}\|Iw\|_{Y_1}^5.$$

But the 0+ loss is not essential by (2.17)–(2.19) and (2.8) for $q<6$, thus it will not be mentioned. By Plancherel's identity and $\widehat{\bar{f}}(\xi,\tau)=\bar{\hat{f}}(-\xi,-\tau)$, we only need to show that for any $f_j\in Y_0^+$, $j=1,3,5$ and $f_j\in Y_0^-$, $j=2,4,6$,

$$\int_{\Gamma_6\times\Gamma_6} \frac{M_6\cdot\chi_{\Gamma_6\backslash\Omega}(\xi_1,\ldots,\xi_6)\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_6}(\xi_6,\tau_6)}{\langle\xi_1\rangle m(\xi_1)\cdots\langle\xi_6\rangle m(\xi_6)}$$
$$\lesssim N^{-\frac{5}{2}+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_5\|_{Y_0^+}\|f_6\|_{Y_0^-}, \tag{5.2}$$

where $\Gamma_6\times\Gamma_6=\{(\xi,\tau)\colon\ \xi_1+\cdots+\xi_6=0,\ \tau_1+\cdots+\tau_6=0\}$, $\xi=(\xi_1,\ldots,\xi_6)$, $\tau=(\tau_1,\ldots,\tau_6)$. Now we divide it into four regions:

$$A_1=\{(\xi,\tau)\in(\Gamma_6\backslash\Omega)\times\Gamma_6\colon\ |\xi_2^*|\gtrsim N\gg|\xi_3^*|\},$$
$$A_2=\{(\xi,\tau)\in(\Gamma_6\backslash\Omega)\times\Gamma_6\colon\ |\xi_3^*|\gtrsim N\gg|\xi_4^*|\},$$
$$A_3=\{(\xi,\tau)\in(\Gamma_6\backslash\Omega)\times\Gamma_6\colon\ |\xi_4^*|\gtrsim N\gg|\xi_5^*|\},$$
$$A_4=\{(\xi,\tau)\in(\Gamma_6\backslash\Omega)\times\Gamma_6\colon\ |\xi_5^*|\gtrsim N\}.$$

In the following, we adopt the notation f_j^* to be one of f_j for $j=1,\ldots,6$ and satisfy $\widehat{f_j^*}=\widehat{f_j^*}(\xi_j^*,\tau_j)$.

Estimate in A_1. By the definition of Ω and (4.28), in $(\Gamma_6\backslash\Omega)\times\Gamma_6$, we have

$$|\xi_1|\sim|\xi_2|\gtrsim N\gg|\xi_3^*|,\quad\text{and}\quad |M_6\cdot\chi_{\Gamma_6\backslash\Omega}|\lesssim N_1^{*\frac{1}{2}}N_3^{*\frac{1}{2}}N_4^*.$$

Therefore, by (2.17)–(2.19), we have

$$\begin{aligned}
\text{LHS of (5.2)} &\lesssim N^{2s-2}\int_{A_1}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_6}(\xi_6,\tau_6)}{|\xi_1^*|^{2s-\frac{1}{2}}\langle\xi_3^*\rangle^{\frac{1}{2}}\langle\xi_5^*\rangle\langle\xi_6^*\rangle}\\
&= N^{2s-2}\int_{A_1}|\xi_1^*|^{-2s-\frac{1}{2}+}\langle\xi_3^*\rangle^{-\frac{1}{2}}\cdot\left(|\xi_1^*|^{\frac{1}{2}-}\widehat{f_1^*}\widehat{f_3^*}\right)\left(|\xi_2^*|^{\frac{1}{2}-}\widehat{f_2^*}\widehat{f_4^*}\right)\\
&\quad\cdot\left(\langle\xi_5^*\rangle^{-1}\widehat{f_5^*}\right)\left(\langle\xi_6^*\rangle^{-1}\widehat{f_6^*}\right)\\
&\lesssim N^{-\frac{5}{2}+}\left\|I_\pm^{\frac{1}{2}-}(f_1^*,f_3^*)\right\|_{L^2_{xt}}\left\|I_\pm^{\frac{1}{2}-}(f_2^*,f_4^*)\right\|_{L^2_{xt}}\\
&\quad\cdot\left\|J_x^{-1}f_5^*\right\|_{L^\infty_{xt}}\left\|J_x^{-1}f_6^*\right\|_{L^\infty_{xt}}\\
&\lesssim N^{-\frac{5}{2}+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_5\|_{Y_0^+}\|f_6\|_{Y_0^-},
\end{aligned}$$

where we use the relations that $|\xi_1^*\pm\xi_3^*|\sim|\xi_1^*|$ and $|\xi_2^*\pm\xi_4^*|\sim|\xi_1^*|$.

Estimate in A_2. Note that $A_2=\emptyset$ in $(\Gamma_6\backslash\Omega_3)\times\Gamma_6$, thus $M_6\cdot\chi_{\Gamma_6\backslash\Omega}=0$.

Estimate in A_3. By (4.9), we have

$$|M_6\cdot\chi_{\Gamma_6\backslash\Omega}|\lesssim m_1^2N_1^{*2}. \tag{5.3}$$

Therefore, by (2.17)–(2.19) and (2.10), we have

$$\begin{aligned}
\text{LHS of (5.2)} &\lesssim N^{2s-2}\int_{A_3}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_6}(\xi_6,\tau_6)}{|\xi_3^*|^s|\xi_4^*|^s\langle\xi_5^*\rangle\langle\xi_6\rangle}\\
&= N^{2s-2}\int_{A_1}\left|\xi_1^*\right|^{-\frac12+}\left|\xi_3^*\right|^{-s}\left|\xi_4^*\right|^{-s}\left\langle\xi_5^*\right\rangle^{-1}\cdot\left(\left|\xi_1^*\right|^{\frac12-}\widehat{f_1^*}\widehat{f_5^*}\right)\left(\left|\xi_2^*\right|^{0-}\widehat{f_2^*}\right)\\
&\quad\cdot\left(\left|\xi_3^*\right|^{0-}\widehat{f_3^*}\right)\left(\left|\xi_4^*\right|^{0-}\widehat{f_4^*}\right)\left(\left\langle\xi_6^*\right\rangle^{-1}\widehat{f_6^*}\right)\\
&\lesssim N^{-\frac52+}\left\|I_\pm^{\frac12-}\left(f_1^*,f_5^*\right)\right\|_{L^2_{xt}}\left\|J_x^{0-}f_2^*\right\|_{L^6_{xt}}\left\|J_x^{0-}f_3^*\right\|_{L^6_{xt}}\\
&\quad\cdot\left\|J_x^{0-}f_4^*\right\|_{L^6_{xt}}\left\|J_x^{-1}f_6^*\right\|_{L^\infty_{xt}}\\
&\lesssim N^{-\frac52+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_5\|_{Y_0^+}\|f_6\|_{Y_0^-},
\end{aligned}$$

where we use the fact that $|\xi_1^*\pm\xi_5^*|\sim|\xi_1^*|$ in this case.

Estimate in A_4. The worst case is $|\xi_j|\gtrsim N$ for any $j=1,\dots,6$, we only consider this case. Then by (5.3), (2.8) for $q=6-$ and (2.11) for $q=6+$, we have

$$\begin{aligned}
\text{LHS of (5.2)} &\lesssim N^{4s-4}\int_{A_4}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_6}(\xi_6,\tau_6)}{|\xi_3^*|^s|\xi_4^*|^s|\xi_5^*|^s|\xi_6^*|^s}\\
&\lesssim N^{-4+}\left\|f_1^*\right\|_{L^{6-}_{xt}}\cdots\left\|f_5^*\right\|_{L^{6-}_{xt}}\left\|J_x^{0-}f_6^*\right\|_{L^{6+}_{xt}}\\
&\lesssim N^{-4+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_5\|_{Y_0^+}\|f_6\|_{Y_0^-}.
\end{aligned}$$

This gives the proof of the proposition. □

Proposition 5.2. *For any $s\geqslant\frac12$, we have*

$$\left|\int_0^\delta \Lambda_8\left(M_8+\widetilde{M}_8;w(t)\right)dt\right|\lesssim N^{-\frac52+}\|Iw\|_{Y_1}^8. \tag{5.4}$$

Proof. When $|\xi_1|,\dots,|\xi_8|\ll N$, we have $M_8,\widetilde{M}_8=0$. Similar to (5.2), it suffices to show

$$\begin{aligned}
&\int_{\Gamma_8\times\Gamma_8}\frac{(M_8+\widetilde{M}_8)(\xi_1,\dots,\xi_8)\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_8}(\xi_8,\tau_8)}{\langle\xi_1\rangle m(\xi_1)\cdots\langle\xi_8\rangle m(\xi_8)}\\
&\quad\lesssim N^{-\frac52+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_7\|_{Y_0^+}\|f_8\|_{Y_0^-},
\end{aligned}\tag{5.5}$$

where $\Gamma_8\times\Gamma_8=\{(\xi_1,\dots,\xi_8,\tau_1,\dots,\tau_8)\colon\ \xi_1+\cdots+\xi_8=0,\ \ \tau_1+\cdots+\tau_8=0\}$. Now we divide it into three regions:

$$\begin{aligned}
B_1&=\left\{(\xi_1,\dots,\xi_8,\tau_1,\dots,\tau_8)\in\Gamma_8\times\Gamma_8\colon\ \left|\xi_1^*\right|\sim\left|\xi_2^*\right|\gtrsim N\gg\left|\xi_3^*\right|\right\},\\
B_2&=\left\{(\xi_1,\dots,\xi_8,\tau_1,\dots,\tau_8)\in\Gamma_8\times\Gamma_8\colon\ \left|\xi_3^*\right|\gtrsim N\gg\left|\xi_4^*\right|\right\},\\
B_3&=\left\{(\xi_1,\dots,\xi_8,\tau_1,\dots,\tau_8)\in\Gamma_8\times\Gamma_8\colon\ \left|\xi_4^*\right|\gtrsim N\right\}.
\end{aligned}$$

Estimate in B_1. By (4.30) and (4.36), we have

$$|M_8+\widetilde{M}_8|\lesssim N_1^{*\frac{1}{2}}N_3^{*\frac{1}{2}}.$$

Therefore, similar to the estimate in A_1 in Proposition 5.1, we have

$$\begin{aligned}
\text{LHS of (5.5)} &\lesssim N^{2s-2}\int\limits_{B_1}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_8}(\xi_8,\tau_8)}{|\xi_1^*|^{2s-\frac{1}{2}}\langle\xi_3^*\rangle^{\frac{1}{2}}\langle\xi_4^*\rangle\cdots\langle\xi_8^*\rangle}\\
&\lesssim N^{-\frac{5}{2}+}\big\|I_\pm^{\frac{1}{2}-}\big(f_1^*,f_3^*\big)\big\|_{L^2_{xt}}\big\|I_\pm^{\frac{1}{2}-}\big(f_2^*,f_4^*\big)\big\|_{L^2_{xt}}\big\|J_x^{-1}f_5^*\big\|_{L^\infty_{xt}}\cdots\big\|J_x^{-1}f_8^*\big\|_{L^\infty_{xt}}\\
&\lesssim N^{-\frac{5}{2}+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_7\|_{Y_0^+}\|f_8\|_{Y_0^-}.
\end{aligned}$$

Estimate in B_2. By (4.29) and (4.35), we have

$$|M_8+\widetilde{M}_8|\lesssim N_1^*. \tag{5.6}$$

Moreover, it satisfies that

$$\big|\xi_1^*\big|-\big|\xi_3^*\big|\sim\big|\xi_1^*\big|\quad\text{in } B_2.$$

Indeed, we have $|\xi_1^*|=|\xi_2^*|+|\xi_3^*|+o(N_3^*)$ (see the proof of Lemma 4.9 for more details). Therefore, similar to the estimate in B_1, we have

$$\begin{aligned}
\text{LHS of (5.5)} &\lesssim N^{3s-3}\int\limits_{B_2}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_8}(\xi_8,\tau_8)}{|\xi_1^*|^{2s-1}|\xi_3^*|^{s}\langle\xi_4^*\rangle\cdots\langle\xi_8^*\rangle}\\
&\lesssim N^{-3+}\big\|I_\pm^{\frac{1}{2}-}\big(f_1^*,f_3^*\big)\big\|_{L^2_{xt}}\big\|I_\pm^{\frac{1}{2}-}\big(f_2^*,f_4^*\big)\big\|_{L^2_{xt}}\\
&\quad\cdot\big\|J_x^{-1}f_5^*\big\|_{L^\infty_{xt}}\cdots\big\|J_x^{-1}f_8^*\big\|_{L^\infty_{xt}}\\
&\lesssim N^{-3+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_7\|_{Y_0^+}\|f_8\|_{Y_0^-}.
\end{aligned}$$

Estimate in B_3. We only consider the worst case: $|\xi_j|\gtrsim N$ for any $j=1,\dots,8$. By (5.6) and the similar estimates in A_4 in Proposition 5.1, we have

$$\begin{aligned}
\text{LHS of (5.5)} &\lesssim N^{8s-8}\int\limits_{B_3}\frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_8}(\xi_8,\tau_8)}{|\xi_1^*|^{2s-1}|\xi_3^*|^{s}\cdots|\xi_8^*|^{s}}\\
&\lesssim N^{-6+}\big\|f_1^*\big\|_{L^{6-}_{xt}}\cdots\big\|f_5^*\big\|_{L^{6-}_{xt}}\big\|J_x^{0-}f_6^*\big\|_{L^{6+}_{xt}}\big\|J_x^{-\frac{1}{2}-}f_7^*\big\|_{L^\infty_{xt}}\big\|J_x^{-\frac{1}{2}-}f_8^*\big\|_{L^\infty_{xt}}\\
&\lesssim N^{-6+}\|f_1\|_{Y_0^+}\|f_2\|_{Y_0^-}\cdots\|f_7\|_{Y_0^+}\|f_8\|_{Y_0^-}.
\end{aligned}$$

This gives the proof of the proposition. □

Proposition 5.3. *For any $s \geqslant \frac{1}{2}$, we have*

$$\left| \int_0^\delta \Lambda_{10}\big(M_{10}; w(t)\big)\, dt \right| \lesssim N^{-3+} \|Iw\|_{Y_1}^{10}. \tag{5.7}$$

Proof. When $|\xi_1|, \dots, |\xi_{10}| \ll N$, we have $M_{10} = 0$. Therefore, we may assume that $|\xi_1^*| \sim |\xi_2^*| \gtrsim N$. Moreover, by symmetry, we may assume $|\xi_1| \geqslant \dots \geqslant |\xi_{10}|$ again. Similar to (5.2), it suffices to show

$$\int_{\Gamma_{10}\times\Gamma_{10}} \frac{M_{10}(\xi_1, \dots, \xi_{10}) \widehat{f_1}(\xi_1, \tau_1) \cdots \widehat{f_{10}}(\xi_{10}, \tau_{10})}{\langle \xi_1 \rangle m(\xi_1) \cdots \langle \xi_{10} \rangle m(\xi_{10})}$$
$$\lesssim N^{-3+} \|f_1\|_{Y_0^+} \|f_2\|_{Y_0^-} \cdots \|f_9\|_{Y_0^+} \|f_{10}\|_{Y_0^-}, \tag{5.8}$$

where $\Gamma_{10} \times \Gamma_{10} = \{(\xi_1, \dots, \xi_{10}, \tau_1, \dots, \tau_{10}):\ \xi_1 + \dots + \xi_{10} = 0,\ \tau_1 + \dots + \tau_{10} = 0\}$. Now we divide it into two regions:

$$D_1 = \big\{(\xi_1, \dots, \xi_{10}, \tau_1, \dots, \tau_{10}) \in \Gamma_{10} \times \Gamma_{10}:\ |\xi_2| \gtrsim N \gg |\xi_3|\big\},$$
$$D_2 = \big\{(\xi_1, \dots, \xi_{10}, \tau_1, \dots, \tau_{10}) \in \Gamma_{10} \times \Gamma_{10}:\ |\xi_3| \gtrsim N\big\}.$$

Estimate in D_1. By Lemma 4.9, we have $|\sigma_6| \lesssim 1$ and thus

$$|M_{10}| \lesssim 1. \tag{5.9}$$

Similar to the estimates in A_1 in Proposition 5.1, we have

$$\begin{aligned}
\text{LHS of (5.8)} &\lesssim N^{2s-2} \int_{D_1} \frac{\widehat{f_1}(\xi_1, \tau_1) \cdots \widehat{f_{10}}(\xi_{10}, \tau_{10})}{|\xi_1|^s |\xi_2|^s \langle \xi_3 \rangle \cdots \langle \xi_{10} \rangle} \\
&\lesssim N^{-3+} \big\| I_-^{\frac{1}{2}-}(f_1, f_3) \big\|_{L^2_{xt}} \big\| I_-^{\frac{1}{2}-}(f_2, f_4) \big\|_{L^2_{xt}} \\
&\quad \cdot \big\| J_x^{-1} f_5 \big\|_{L^\infty_{xt}} \cdots \big\| J_x^{-1} f_{10} \big\|_{L^\infty_{xt}} \\
&\lesssim N^{-3+} \|f_1\|_{Y_0^+} \|f_2\|_{Y_0^-} \cdots \|f_9\|_{Y_0^+} \|f_{10}\|_{Y_0^-}.
\end{aligned}$$

Estimate in D_2. We only consider the worst case: $|\xi_j| \gtrsim N$ for any $j = 1, \dots, 10$. Thus by (5.9), and the similar estimates in B_3 in Proposition 5.2, we have

$$\begin{aligned}
\text{LHS of (5.8)} &\lesssim N^{10s-10} \int_{D_2} \frac{\widehat{f_1}(\xi_1, \tau_1) \cdots \widehat{f_{10}}(\xi_{10}, \tau_{10})}{|\xi_1|^s |\xi_2|^s |\xi_3|^s |\xi_4|^s \cdots |\xi_{10}|^s} \\
&\lesssim N^{-8+} \big\| f_1^* \big\|_{L^{6-}_{xt}} \cdots \big\| f_5^* \big\|_{L^{6-}_{xt}} \big\| J_x^{0-} f_6^* \big\|_{L^{6+}_{xt}} \\
&\quad \cdot \big\| J_x^{-\frac{1}{2}-} f_7^* \big\|_{L^\infty_{xt}} \cdots \big\| J_x^{-\frac{1}{2}-} f_{10}^* \big\|_{L^\infty_{xt}} \\
&\lesssim N^{-8+} \|f_1\|_{Y_0^+} \|f_2\|_{Y_0^-} \cdots \|f_9\|_{Y_0^+} \|f_{10}\|_{Y_0^-}.
\end{aligned}$$

This gives the proof of the proposition. □

6. A comparison between $E_I^1(w)$ and $E_I^3(w)$

In this section, we show that the third generation modified energy $E_I^3(w)$ is comparable to the first generation modified energy $E_I^1(w) = E(Iw)$. In Section 5, we have shown that $E_I^3(w)$ is almost conserved with a tiny increment. Then the result in this section forecasts that $E_I^1(w)$ is also almost conserved with a similar tiny increment (which will be realized in the next section). Now we state the result in this section.

Lemma 6.1. *Let $s \geqslant \frac{1}{2}$, then we have*

$$\left|E_I^3\big(w(t)\big) - E_I^1\big(w(t)\big)\right| \lesssim N^{0-}\big(\|Iw(t)\|_{H^1}^4 + \|Iw(t)\|_{H^1}^6\big). \tag{6.1}$$

Proof. By (3.8), (3.9) and (3.16), we have

$$E_I^3\big(w(t)\big) - E_I^1\big(w(t)\big) = \frac{1}{2}\Lambda_4\bigg(M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \frac{1}{2}\xi_{13}m_1m_2m_3m_4; w(t)\bigg) + \Lambda_6\big(\sigma_6; w(t)\big).$$

Therefore, it suffices to prove

$$\left|\Lambda_4\bigg(M_4(\xi_1,\xi_2,\xi_3,\xi_4) - \frac{1}{2}\xi_{13}m_1m_2m_3m_4; w(t)\bigg)\right| \lesssim N^{0-}\|Iw(t)\|_{H^1}^4, \tag{6.2}$$

and

$$\left|\Lambda_6\big(\sigma_6; w(t)\big)\right| \lesssim N^{0-}\|Iw(t)\|_{H^1}^6. \tag{6.3}$$

For (6.2), we refer to (32) in [10]. Now we turn to prove (6.3). By Plancherel's identity, it suffices to show

$$\int_{\Gamma_6} \frac{\sigma_6(\xi_1,\dots,\xi_6)\widehat{f_1}(\xi_1,t)\cdots\widehat{f_6}(\xi_6,t)}{\langle\xi_1\rangle m(\xi_1)\cdots\langle\xi_6\rangle m(\xi_6)} \lesssim N^{0-}\|f_1(t)\|_{L_x^2}\cdots\|f_6(t)\|_{L_x^2}. \tag{6.4}$$

We may assume that $|\xi_1| \geqslant |\xi_2| \geqslant \cdots \geqslant |\xi_6|$ by symmetry. Since $\sigma_6 = 0$ when $|\xi_j| \ll N$ for any $j = 1,\dots,6$, we may assume that $|\xi_1| \sim |\xi_2| \gtrsim N$. By Lemma 4.9, we have $|\sigma_6| \lesssim 1$. Note that

$$\langle\xi\rangle m(\xi) \gtrsim \langle\xi\rangle^s, \quad \text{for any } \xi \in \mathbb{R},$$

we have by Sobolev's inequality,

$$\begin{aligned}
\text{LHS of (6.4)} &\lesssim N^{-2+}\int_{\Gamma_6} \frac{\widehat{f_1}(\xi_1,\tau_1)\cdots\widehat{f_{10}}(\xi_{10},\tau_{10})}{\langle\xi_3\rangle^{s+}\cdots\langle\xi_6\rangle^{s+}} \\
&\lesssim N^{-2+}\|f_1(t)\|_{L_x^2}\|f_2(t)\|_{L_x^2}\big\|J_x^{-\frac{1}{2}-}f_3(t)\big\|_{L_x^\infty}\cdots\big\|J_x^{-\frac{1}{2}-}f_{10}(t)\big\|_{L_x^\infty} \\
&\lesssim N^{-2+}\|f_1(t)\|_{L_x^2}\cdots\|f_{10}(t)\|_{L_x^2}.
\end{aligned}$$

This gives the proof of the lemma. □

7. The proof of Theorem 1.1

7.1. A variant local well-posedness

In this subsection, we will establish a variant local well-posedness as follows.

Proposition 7.1. *Let $s \geqslant \frac{1}{2}$, then Cauchy problem* (3.1) *is locally well posed for the initial data w_0 satisfying $Iw_0 \in H^1(\mathbb{R})$. Moreover, the solution exists on the interval $[0, \delta]$ with the lifetime*

$$\delta \sim \|I_{N,s} w_0\|_{H^1}^{-\mu} \tag{7.1}$$

for some $\mu > 0$. Furthermore, the solution satisfies the estimate

$$\|I_{N,s} w\|_{Y_1} \lesssim \|I_{N,s} w_0\|_{H^1}. \tag{7.2}$$

Proof. By the standard iteration argument (see cf. [27]), it suffices to prove the multilinear estimates,

$$\big\|I(w_1 \partial_x \overline{w_2} w_3)\big\|_{Z_1} \lesssim \|Iw_1\|_{Y_1} \|Iw_2\|_{Y_1} \|Iw_3\|_{Y_1}, \tag{7.3}$$

and

$$\big\|I(w_1 \overline{w_2} w_3 \overline{w_4} w_5)\big\|_{Z_1} \lesssim \|Iw_1\|_{Y_1} \cdots \|Iw_5\|_{Y_1}. \tag{7.4}$$

By Lemma 12.1 in [12], it suffices to prove the multilinear estimates,

$$\|w_1 \partial_x \overline{w_2} w_3\|_{Z_s} \lesssim \|w_1\|_{Y_s} \|w_2\|_{Y_s} \|w_3\|_{Y_s}, \tag{7.5}$$

and

$$\|w_1 \overline{w_2} w_3 \overline{w_4} w_5\|_{Z_s} \lesssim \|w_1\|_{Y_s} \cdots \|w_5\|_{Y_s}. \tag{7.6}$$

These were proved in [27]. □

7.2. Rescaling

We rescale the solution of (3.1) by writing

$$w_\mu(x,t) = \mu^{-\frac{1}{2}} w\big(x/\mu, t/\mu^2\big); \qquad w_{0,\mu}(x) = \mu^{-\frac{1}{2}} w_0(x/\mu).$$

Then $w_\mu(x,t)$ is still the solution of (3.1) with the initial data $w(x,0) = w_{0,\mu}(x)$. Meanwhile, $w(x,t)$ exists on $[0,T]$ if and only if $w_\mu(x,t)$ exists on $[0, \mu^2 T]$.

By $m(\xi) \leqslant 1$ and (3.4), we know that

$$\big\|Iw_\mu(t)\big\|_{L_x^2} \leqslant \big\|w_\mu(t)\big\|_{L_x^2} = \|w_{0,\mu}\|_{L_x^2} = \|w_0\|_{L_x^2} < \sqrt{2\pi}.$$

This together with (3.3) yields

$$\big\|\partial_x I u_\mu(t)\big\|_{L_x^2}^2 \sim E_I^1\big(w_\mu(t)\big), \qquad \big\|Iw_\mu(t)\big\|_{H_x^1}^2 \lesssim E_I^1\big(w_\mu(t)\big) + 1. \tag{7.7}$$

Moreover, by (2.6), we get that

$$\|\partial_x I w_{0,\mu}\|_{L^2} \lesssim N^{1-s}/\mu^s \cdot \|w_0\|_{H^s}.$$

Hence, if we choose $\mu \sim N^{\frac{1-s}{s}}$ suitably, we have $\|I w_{0,\mu}\|_{H^1} \leqslant 5$. Thus we may take $\delta \sim 1$ by Proposition 7.1.

By standard limiting argument, the global well-posedness of w in $H^s(\mathbb{R})$ follows if for any $T > 0$, we have

$$\sup_{0\leqslant t\leqslant T} \big\| w(t)\big\|_{H^s} \lesssim C\big(\|w_0\|_{H^s}, T\big).$$

Moreover, in light of (2.6) and (7.7), it suffices to show

$$\sup_{0\leqslant t\leqslant \mu^2 T} E_I^1\big(w_\mu(t)\big) \lesssim C(T) \tag{7.8}$$

for some N. In the following subsection, we shall prove it by almost conservation law and iteration.

7.3. *Almost conservation law and iteration*

By (3.17), we have

$$E_I^3\big(w_\mu(t)\big) = E_I^3(w_{0,\mu}) + \int_0^t \bigg(\varLambda_6\big(M_6 \cdot \chi_{\Gamma_6\setminus\Omega}; w(s)\big)\, ds + \int_0^t \varLambda_8\big(M_8 + \widetilde{M}_8; w(s)\big) + \varLambda_{10}\big(M_{10}; w(s)\big)\bigg)\, ds.$$

By Proposition 5.1–Proposition 5.3 and (7.2), we have for any $t \in [0, 1]$,

$$\begin{aligned} E_I^3\big(w_\mu(t)\big) &\leqslant E_I^3(w_{0,\mu}) + C_1 N^{-\frac{5}{2}+}\big(\|I w_\mu\|_{Y_1}^6 + \|I w_\mu\|_{Y_1}^8 + \|I w_\mu\|_{Y_1}^{10}\big) \\ &\leqslant E_I^3(w_{0,\mu}) + C_2 N^{-\frac{5}{2}+}. \end{aligned}$$

Thus,

$$\begin{aligned} E_I^1\big(w_\mu(t)\big) \leqslant E_I^1(w_{0,\mu}) &+ \big(E_I^1\big(w_\mu(t)\big) - E_I^3\big(w_\mu(t)\big)\big) \\ &+ \big(E_I^3(w_{0,\mu}) - E_I^1(w_{0,\mu})\big) + C_2 N^{-\frac{5}{2}+}. \end{aligned}$$

Using (6.1), choosing N suitable large and applying the bootstrap argument, we obtain that for any $t \in [0, 1]$,

$$E_I^1\big(w_\mu(t)\big) \leqslant 10.$$

Repeating this process M times, we obtain for any $t \in [0, M]$,

$$E_I^1\big(w_\mu(t)\big) \leqslant E_I^1(w_{0,\mu}) + \big(E_I^1\big(w_\mu(t)\big) - E_I^3(w_\mu)\big) + \big(E_I^3(w_{0,\mu}) - E_I^1(w_{0,\mu})\big) + C_2 M N^{-\frac{5}{2}+}.$$

Therefore, by (6.1) again, we have $E_I^1(w_\mu(t)) \leqslant 10$ provided $M \lesssim N^{\frac{5}{2}-}$, which implies that the solution w_μ exists on $[0, M\delta] \sim [0, N^{\frac{5}{2}-}]$. Hence, w exists on $[0, \mu^2 T]$ with the relation

$$N^{\frac{5}{2}-} \gtrsim \mu^2 T \sim N^{\frac{2(1-s)}{s}} T.$$

Thus we may take $T \sim N^{\frac{9s-4}{2s}-}$. When $s \geqslant \frac{1}{2}$, we have $\frac{9s-4}{2s} > 0$. This implies (7.8) by choosing sufficient large N, and thus completes the proof of Theorem 1.1.

Acknowledgments

The authors thank the referees and the associated editor for their invaluable comments and suggestions which helped to improve the paper greatly. The authors were supported by the NSF of China (No. 10725102, No. 10801015). The second author was also partly supported by Beijing International Center for Mathematical Research.

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