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Equivariant aspects of Yang–Mills Floer theory ☆

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Abstract

We study the u-map in instanton Floer homology using Floer's exact surgery triangle. As an application we prove that the Donaldson invariants of simply-connected smooth 4-manifolds have finite type. We also construct an additive homology cobordism invariant of homology 3-spheres which is monotone with respect to definite cobordisms. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

This paper is concerned with Floer cohomology groups of SO(3) bundles $P \to Y$, where Y is a closed, oriented 3-manifold. Following [3] we only consider *admissible* bundles, which means P should be non-trivial over some surface in Y unless Y is an (integral) homology sphere. The mod 8 periodic Floer group $\mathrm{HF}^*(P;G)$ with coefficients in the abelian group G is then a topological invariant of Y, P. We will often omit the coefficient group from notation. When nothing else is specified our results hold for any coefficient group.

If P is non-trivial then "cup product" with a certain four-dimensional cohomology class (four times the μ-class of a point) defines a homomorphism $u\colon \mathrm{HF}^*(P) \to \mathrm{HF}^{*+4}(P)$. We use Floer's exact triangle to show that there is always a positive integer n such that

$$(u^2 - 64)^n = 0.$$

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When Y is a homology sphere the u-map is, in general, defined on $\mathrm{HF}^q(Y)$ only for $q \not\equiv 4,5 \bmod 8$, due to the presence of the trivial SU(2) connection over Y. However, by "factoring out" interaction with the trivial connection we construct a *reduced* Floer group $\widehat{\mathrm{HF}}^*(Y)$ on which the u-map is defined in all degrees. We expect that $\widehat{\mathrm{HF}}^*(Y;\mathbb{R})$ is isomorphic to the $\mathbb{R}[u]$ torsion submodule of Austin and Braam's equivariant Floer group [2], but this will not be proved in this paper.

For the reduced Floer group we again find that $(u^2-64)^n=0$ for some n. Combining this with the splitting theorem of Freedman and Taylor [13,24] we obtain a proof, in the simply connected case, of the finite type conjecture of Kronheimer and Mrowka:

Theorem 1. *Let X be a smooth, compact, simply-connected, oriented 4-manifold with $b_2^+(X)$ odd and $\geqslant 3$. Then the Donaldson invariants of X have finite type.*

See Section 7 for more details. Quite different proofs of this theorem have been given by Muñoz [22] (without any assumptions on the fundamental group) and by Wieczorek [26].

The nilpotency of u^2-64 also leads to the following theorem, by another application of Floer's exact triangle:

Theorem 2. *If Y is any oriented homology 3-sphere and R any associative ring in which 2 is invertible then $u\colon \mathrm{HF}^q(Y;R) \to \mathrm{HF}^{q+4}(Y;R)$ is an isomorphism for $q \not\equiv 4,5 \bmod 8$. In particular, $\mathrm{HF}^*(Y;R)$ is $\bmod 4$ periodic.*

We now focus on rational coefficients. Let Y be an oriented homology 3-sphere. Computing the reduced Floer group $\widehat{\mathrm{HF}}^*(Y;\mathbb{Q})$ from the ordinary one $\mathrm{HF}^*(Y;\mathbb{Q})$ merely requires the knowledge of a single integer $h(Y)$, which measures interaction between irreducible flat SU(2) connections and the trivial connection over Y. This invariant $h(Y)$ has the following properties:

Theorem 3. (i) $h(Y_1\#Y_2)=h(Y_1)+h(Y_2)$.

(ii) *If the homology sphere Y bounds a smooth, compact, oriented 4-manifold with negative definite intersection form then $h(Y)\geqslant 0$, with strict inequality if the intersection form is not diagonal over the integers.*

(iii) *For the Brieskorn sphere $\Sigma(2,3,5)$ one has $h=1$* (compare [15]).

Since $h(S^3)=0$, property (ii) generalizes Donaldson's theorem [6,7].

Theorem 3 implies that h is a surjective group homomorphism

$$h:\theta_3^H \to \mathbb{Z},$$

where θ_3^H is the integral homology cobordism group of oriented homology 3-spheres. The mod 2 reduction of h is not the Rochlin invariant, because $\Sigma(2,3,7)$ bounds an orientable rational ball but has Rochlin invariant one [11].

Let k be a positive integer and γ a negative knot in S^3 (i.e. a knot which admits a regular projection with only negative crossings). The manifold $S^3_{\gamma,1/k}$ resulting from $1/k$ surgery on γ bounds both positive and negative definite 4-manifolds, hence $h(S^3_{\gamma,1/k})=0$. On the other hand, if γ is non-trivial then, as proved in [4], $S^3_{\gamma,-1/k}$ bounds a 4-manifold with negative definite intersection form of the type $-E_8 \oplus n(-1)$ for some $n \geqslant 0$, hence $h(S^3_{\gamma,-1/k}) > 0$. For instance, if p, q are mutually prime integers $\geqslant 2$ then the Brieskorn sphere $\Sigma(p, q, pqk \pm 1)$ results from $\pm 1/k$ surgery on the negative (p, q) torus knot, so

$$h(\Sigma(p, q, pqk + 1)) = 0, \quad h(\Sigma(p, q, pqk - 1)) > 0.$$

Further properties of the h-invariant will be given in [14], including estimates on the behaviour under surgery on knots.

If one takes a field of characteristic $p > 2$ as coefficients for the Floer groups then one obtains a group homomorphism

$$h_p : \theta^H_3 \to \mathbb{Z}$$

(which depends only on p) with the following property: if Y bounds a negative definite, smooth, compact, oriented 4-manifold without p-torsion in its homology then $h_p(Y) \geqslant 0$, with strict inequality if the intersection form is non-standard. Unfortunately, the author is unable to prove anything about h_p that does not also hold for h. The invariants h_p will therefore not be pursued any further in this paper.

We intend to discuss related topics in Seiberg–Witten Floer theory, and in Yang–Mills Floer theory with $\mathbb{Z}/2$ coefficients, in forthcoming papers.

2. Preliminaries

This section is mostly a review of well known material. For more details see [9,12,3,10,5].

If X is an n-dimensional smooth manifold, with or without boundary, and $E \to X$ a rank 2 unitary vector bundle let $\mathscr{A}(E)$ denote the space of all connections in E which induce a fixed connection in $\Lambda^2 E$. Let $\mathscr{G}(E)$ be the group of automorphisms (or gauge transformations) of E of determinant 1 and set $\mathscr{B}(E) = \mathscr{A}(E)/\mathscr{G}(E)$.

In the case of an SO(3) bundle $P \to X$ we define $\mathscr{G}(P)$ to be the group of all automorphisms of P and $\mathscr{G}_S(P)$ to be the subgroup of *even* automorphisms, i.e. automorphisms that lift to sections of $P \times_{Ad(\mathrm{SO}(3))} \mathrm{SU}(2)$. Note that there is a natural exact sequence

$$1 \to \mathscr{G}_S(P) \to \mathscr{G}(P) \xrightarrow{\eta} H^1(X; \mathbb{Z}/2) \to 0. \tag{1}$$

If $\mathfrak{g}_E$ is the SO(3) bundle associated to the $U(2)$ bundle E then we can identify $\mathscr{B}(E)$ with the space of all connections in $\mathfrak{g}_E$ modulo even automorphisms. A connection in E is called *irreducible* if its stabilizer in $\mathscr{G}(E)$ is $\{\pm 1\}$; otherwise it is called *reducible*. We say a connection in E is *twisted reducible* if the induced connection in $\mathfrak{g}_E$ respects a splitting $\mathfrak{g}_E = \lambda \oplus L$, where λ is a non-trivial real line bundle and L a non-orientable real 2-plane bundle.

If X is compact and we consider L^p_1 connections in E modulo L^p_2 gauge transformations, where $p > n/2$, then the subspace $\mathscr{B}^*(E) \subset \mathscr{B}(E)$ of irreducible connections is a Banach manifold; if in addition p is an even integer then $\mathscr{B}^*(E)$ admits C^∞ partitions of unity.

2.1. Floer cohomology groups of homology 3-spheres

Let Y be an oriented (integral) homology 3-sphere. The Floer cohomology group $\mathrm{HF}^*(Y)$ was defined in [12] by applying Morse theoretic ideas to a suitably perturbed Chern–Simons function $\mathrm{cs}_\pi : \mathscr{B}(Y \times \mathrm{SU}(2)) \to \mathbb{R}/\mathbb{Z}$. A critical point A of cs_π is called *non-degenerate* if the Hessian is non-singular on $\ker(d_A^*) \subset \Omega_Y^1(\mathrm{su}(2))$. For the unperturbed Chern–Simons function cs the critical points are the flat connections and the Hessian is $*d_A$. In this case there is up to gauge equivalence only one reducible critical point, namely the trivial connection θ, which is non-degenerate since $H^1(Y;\mathbb{R})=0$. The trivial connection is, in fact, a critical point of cs_π for all perturbations π (this is a consequence of the gauge invariance of cs_π). We will always work with a small, generic perturbation; then we may assume that cs_π has only finitely many critical points, all of which are non-degenerate, and that θ is the only reducible critical point. We will usually not refer explicitly to the perturbation π or the corresponding perturbations of the anti-self-dual equations.

For any pair of flat SU(2) connections α,β over Y and any real number κ with $\kappa \equiv \mathrm{cs}(\alpha) - \mathrm{cs}(\beta) \bmod \mathbb{Z}$ let $M(\alpha,\beta;\kappa)$ be the moduli space of all anti-self-dual SU(2) connections A over $\mathbb{R}\times Y$ which are asymptotic to α at ∞ and to β at $-\infty$, and have second relative Chern class $(1/8\pi^2)\int_{\mathbb{R}\times Y}\mathrm{tr}(F_A^2)=\kappa$. Here F_A is the curvature of A. These moduli spaces are orientable, and orientations should be chosen compatible with gluing maps and addition of instantons over S^4, see [7]. We denote by $M(\alpha,\beta)$ the moduli space $M(\alpha,\beta;\kappa)$ whose expected dimension lies in the interval $[0,7]$, and set

$$\check{M}(\alpha,\beta)=M(\alpha,\beta)/\mathbb{R},$$

where $\mathbb{R}$ acts by translation. For an irreducible flat connection α we define the index $i(\alpha)\in\mathbb{Z}/8$ by

$$i(\alpha)\equiv \dim M(\alpha,\theta) \bmod 8.$$

The Floer cohomology group $\mathrm{HF}^*(Y)$ is the cohomology of the $\mathbb{Z}/8$ graded cochain complex (CF^*,d), where CF^i is the free abelian group generated by the gauge equivalence classes of irreducible flat SU(2) connections of index i over Y. The differential d has matrix coefficient $\#\check{M}(\alpha,\beta)$ when $i(\alpha)-i(\beta)=1$, where # means the number of points counted with sign. To show that $d^2=0$ one counts the ends of $\check{M}(\alpha,\gamma)$ when $i(\alpha)-i(\gamma)=2$.

The Floer homology group $\mathrm{HF}_*(Y)$ is the homology of the dual complex of (CF^*,d). There is then a canonical identification $\mathrm{HF}_q(Y)=\mathrm{HF}^{5-q}(\bar{Y})$.

The trivial connection over Y gives rise to a homomorphism $\delta\colon \mathrm{CF}^4\to\mathbb{Z}$ and an element $\delta'\in\mathrm{CF}^1$, defined by

$$\delta\alpha=\#\check{M}(\theta,\alpha),\qquad \delta'=\sum_\beta \#\check{M}(\beta,\theta)\,\beta,$$

where β runs through the generators of CF^1. These satisfy $\delta d=0$ and $d\delta'=0$ (for the same reason that $d^2=0$) and so define

$$\delta_0\colon \mathrm{HF}^4(Y)\to\mathbb{Z},\quad \delta_0'\in\mathrm{HF}^1(Y).$$

These will play a central role in this paper.

2.2. Floer cohomology groups of non-trivial SO(3) bundles

Now let Y be a closed, oriented 3-manifold and $P\to Y$ a non-trivial SO(3) bundle which is admissible in the sense of [3]. This means that the Stiefel–Whitney class $w_2(P)$ is not the mod 2 reduction of a torsion class in $H^2(Y;\mathbb{Z})$, or equivalently that $w_2(P)$ defines a non-zero map $H_2(Y;\mathbb{Z}) \to \mathbb{Z}/2$. In particular, the Betti number $b_1(Y)$ must be positive. In this case there are no reducible flat connections in P. The Floer group $\mathrm{HF}^*(P)$ is defined just as for homology spheres, using a small, generic perturbation of the Chern–Simons function. However, the group is now only *affinely* $\mathbb{Z}/8$ graded, i.e. only the index difference of two flat connections is well defined in $\mathbb{Z}/8$.

On the other hand, given a *spin structure* on Y one can define a mod 4 grading on $\mathrm{HF}^*(P)$ as follows. Let α be a (non-degenerate) flat connection in P. As shown in [18], Y spin bounds a simply connected, spin 4-manifold X'. Choose a cylindrical end metric on the corresponding open 4-manifold X. Since the adjoint vector bundle of P is isomorphic to $\mathbb{R} \oplus L$ for some complex line bundle $L \to Y$, and since the restriction map $H^2(X) \to H^2(Y)$ is surjective, there is an SO(3) bundle $Q \to X$ with $Q|_Y \approx P$. Now define the index $i(\alpha) \in \mathbb{Z}/4$ by

$$i(\alpha) \equiv \dim M(Q,\alpha) + 3b_2^+(X) \mod 4.$$

Here $M(Q,\alpha)$ is the moduli space of anti-self-dual connections in the bundle Q which are asymptotic to α over the end, while $b_2^+(X)$ is the maximal dimension of a positive subspace for the intersection form on $H_2(X;\mathbb{Q})$. It follows easily from the dimension formula for anti-self-dual moduli spaces over closed 4-manifolds that $i(\alpha)$ is well defined mod 4.

Recall that the group $H^1(Y;\mathbb{Z}/2)$ acts simply transitively on the set of spin structures on Y. If s_1, s_2 is a pair of spin structures then the corresponding index functions i_1, i_2 are related by

$$(i_1 - i_2)/2 \equiv ((s_1 - s_2) \cup w_2(P))[Y] \mod 2.$$

In particular, the grading mod 2 is independent of the spin structure.

To any class $\zeta \in H^1(Y;\mathbb{Z}/2)$ we can associate an involution ζ^* of $\mathrm{HF}^*(P)$, well defined up to an overall sign; this involution is induced by the twisted bundle

$$(\mathbb{R}_- \times P) \cup_g (\mathbb{R}_+ \times P)$$

over $\mathbb{R} \times Y$ where g is any automorphism of P with $\eta(g) = \zeta$ (recall the exact sequence (1)). The reason for the sign ambiguity is that one has to make a choice concerning the orientation of moduli spaces in this bundle. In any case, one does get a group homomorphism

$$H^1(Y;\mathbb{Z}/2) \to \mathrm{Aut}(\mathrm{HF}^*(P))/\{\pm 1\}.$$

It is easy to see that each ζ^* has degree 0 or 4, and if ζ has an integral lift then

$$\deg(\zeta^*)/4 \equiv (w_2(P) \cup \zeta)[Y] \mod 2.$$

(See [3] for the general formula.) In particular, there is always a degree 4 involution ζ^*.

2.3. *Invariants of* 4-*manifolds with boundary*

The purpose of this subsection is merely to review the definition of Donaldson invariants of 4-manifolds with boundary. A gluing theorem for these invariants will be stated in Section 7.

Let X be a smooth, oriented Riemannian 4-manifold with one tubular end $\mathbb{R}_+ \times Y$, and let $E \to X$ be a $U(2)$ bundle. For simplicity, suppose the SO(3) bundle $P \to Y$ associated to $E|_Y$

is non-trivial and admissible. In this case the Donaldson invariant for the bundle E is a linear map

$$D_E : \mathbb{A}(X) \to \mathrm{HF}^*(P;\mathbb{Q}), \tag{2}$$

well defined up to an overall sign, where

$$\mathbb{A}(X) = \mathrm{Sym}(H_{\mathrm{even}}(X;\mathbb{Q})) \otimes \Lambda(H_{\mathrm{odd}}(X;\mathbb{Q})).$$

(When Y is a homology sphere care must be taken to handle reducible connections in the moduli spaces; insofar as the corresponding invariants are defined in this case they will be denoted D_X^c, where $c = c_1(E)$.)

This invariant is defined in much the same way as the instanton invariants of closed 4-manifolds (see [8,21]). Let $\Sigma_1,\dots,\Sigma_m \subset X$ be a collection of smooth, compact, connected submanifolds without boundary and in general position. As in [21], Section 2(ii) choose for $j = 1,\dots,m$ a smooth, compact, codimension 0 submanifolds U_j of X containing Σ_j such that the map $H_1(U_j;\mathbb{Z}/2) \to H_1(X;\mathbb{Z}/2)$ is surjective, and such that the sets $U_1,\dots,U_m$ are disjoint. Let $\mathscr{B}^*(U_j) = \mathscr{B}^*(E|_{U_j})$ be as in Section 2 and let $\mathbb{E}_j \to \mathscr{B}^*(U_j) \times U_j$ be the universal SO(3) bundle. As in [21] choose a generic geometric representative $V_j \subset \mathscr{B}^*(U_j)$ for the cohomology class $\mu(\Sigma_j) = -\frac{1}{4}p_1(\mathbb{E}_j)/[\Sigma_j]$. Let $d = \sum_j (4 - \dim \Sigma_j)$ and for any flat connection α in P set

$$Z_\alpha = r_1^{-1}(V_1) \cap \cdots \cap r_m^{-1}(V_m) \subset M(E,\alpha).$$

Here $M(E,\alpha)$ is the moduli space of (projectively) anti-self-dual connections in E which are asymptotic to α at the end, and $r_j : M(E,\alpha) \to \mathscr{B}^*(U_j)$ is the restriction map. For the monomial $z = [\Sigma_1]\cdots[\Sigma_m] \in \mathbb{A}(X)$ define

$$D_E(z) = \left[\sum_\alpha (\#Z_\alpha)\alpha\right] \in \mathrm{HF}^*(P;\mathbb{Q}),$$

where α runs through the equivalence classes of flat connections in P for which Z_α has dimension 0.

To show that $D_E(z)$ is independent of U_j, V_j and linear one can follow the arguments in [8,21] and show that when computing $D_E(z)$ one of the classes $\mu(\Sigma_j)$ may be evaluated "abstractly" (e.g. using Čech-type (co)homology).

3. The *u*-map and the reduced Floer group

In this section one could use any coefficient group for the Floer cohomology groups, but for simplicity we will work with integral coefficients.

If P is a non-trivial admissible SO(3) bundle over a closed, oriented 3-manifold Y, then the u-map $\mathrm{HF}^*(P) \to \mathrm{HF}^{*+4}(P)$ is defined, roughly speaking, by evaluating the four-dimensional class $4\mu(x)$ over four-dimensional moduli spaces $M(\alpha,\beta)$, where α,β are flat connections in P. If Y is a homology sphere, then the construction can still be carried out on cochain level to give a homomorphism $v\colon \mathrm{CF}^*(Y) \to \mathrm{CF}^{*+4}(Y)$ (which depends on certain choices). But due to the presence of the trivial connection, this homomorphism is not quite a cochain map (see Theorem 4 below), and in general only defines a homomorphism $u\colon \mathrm{HF}^q(Y) \to \mathrm{HF}^{q+4}(Y)$

for $q \not\equiv 4,5 \bmod 8$. However, by "factoring out" interaction with the trivial connection we will construct a *reduced* Floer group $\widehat{\mathrm{HF}}^*(Y)$, in which the u-map is defined in all degrees.

3.1. The u-map

Let Y be a closed, oriented 3-manifold and $P \to Y$ an admissible SO(3) bundle. We will define a graded homomorphism $v\colon \mathrm{CF}^*(P) \to \mathrm{CF}^{*+4}(P)$. Let α, β be flat connections in P, not both reducible. Let $\mathbb{E} = \mathbb{E}(\beta,\alpha) \to M(\beta,\alpha)$ and $\mathbb{F} \to \mathscr{B}^*((-1,1)\times P)$ be the natural oriented, euclidean 3-plane bundles associated to the base-point $(0,y_0)$. Here $(-1,1)\times P$ is the obvious SO(3) bundle. There is a natural restriction map $r: M(\beta,\alpha) \to \mathscr{B}^*((-1,1)\times P)$, and we have $r^*\mathbb{F} = \mathbb{E}$. Choose sections s_1, s_2 of the complexified bundle $\mathbb{F}\otimes\mathbb{C}$ and let $\sigma_j = r^*s_j$ be the induced sections of $\mathbb{E}\otimes\mathbb{C}$. If $\dim M(\beta,\alpha) \leqslant 5$ then after perturbing the s_j's we may assume σ_1 has no zeros and that the section $\sigma = \sigma_2 \bmod \sigma_1$ of the quotient bundle $(\mathbb{E}\otimes\mathbb{C})/\mathbb{C}\sigma_1$ is transverse to the zero section. If α,β are both irreducible and $\dim M(\beta,\alpha) = 4$ then $\sigma^{-1}(0)$ is a finite set of oriented points, and we define the matrix coefficient $\langle v(\alpha),\beta\rangle$ by

$$\langle v(\alpha),\beta\rangle = \#\sigma^{-1}(0).$$

The following theorem is due to Donaldson and Furuta [5], but we include a proof for the sake of completeness and because we will need certain generalizations later.

Theorem 4 (Donaldson and Furuta).
(i) *If P is a non-trivial, admissible bundle then $dv - vd = 0$.*
(ii) *If Y is a homology 3-sphere then*

$$dv - vd + 2\delta\otimes\delta' = 0,$$

where by definition $\delta = 0$ in degrees $\not\equiv 4 \bmod 8$.

In case (i) it follows that v induces a homomorphism $u\colon \mathrm{HF}^*(P) \to \mathrm{HF}^{*+4}(P)$, while in case (ii) one gets u-maps

$$\mathrm{HF}^i(Y) \to \mathrm{HF}^{i+4}(Y), \quad i \neq 4,5,$$

$$\ker(\delta_0) \to \mathrm{HF}^0(Y); \qquad \mathrm{HF}^5(Y) \to \mathrm{HF}^1(Y)/(\mathbb{Z}\delta_0').$$

Proof. Since (i) is essentially a special case of (ii), we focus on the latter. Let α, β be irreducible, flat SU(2) connections over a homology 3-sphere Y such that $i(\beta) \equiv i(\alpha) + 5 \bmod 8$. We will show that

$$\langle (dv - vd + 2\delta\otimes\delta')\alpha,\beta\rangle = 0.$$

Our plan is to modify the section σ for connections that are close to the trivial connection over $(-1,1)\times Y$, in order to gain control over the ends of $\sigma^{-1}(0)$. Counting the number of such ends with sign (this number must be zero) will then give (ii).

The proof is divided into four parts.

(I) We first show that suitable modifications can be made to the sections s_j without affecting the definition of the chain map u. This will be used in (IV) (b) and (c) below.

If γ_1, γ_2 are flat SU(2) connections over Y and γ_1 is irreducible then by taking the holonomy of connections along the path $\mathbb{R}_- \times \{y_0\}$ one obtains a trivialization

$$f_- : M(\gamma_1,\gamma_2) \times \mathbb{C}^3 \xrightarrow{\approx} \mathbb{E}(\gamma_1,\gamma_2) \otimes \mathbb{C}$$

of the natural complex 3-plane bundle over $M(\gamma_1,\gamma_2)$. Similarly, if γ_2 is irreducible one gets a trivialization $f_+$of $\mathbb{E}(\gamma_1,\gamma_2) \otimes \mathbb{C}$ in terms of holonomy along $\mathbb{R}_+ \times \{y_0\}$.

Now fix linearly independent elements $e_1, e_2 \in \mathbb{C}^3$. If $D_1 \subset M(\theta,\alpha)$, $D_2 \subset M(\beta,\theta)$ are compact sets then by modifying the sections s_1, s_2 in a small neighbourhood of $r(D_1 \cup D_2)$ (this will not affect the chain map u) one can arrange that

$$s_j(r(A)) = \begin{cases} f_-(A,e_j) & \text{if } [A] \in D_1, \\ f_+(A,e_j) & \text{if } [A] \in D_2. \end{cases}$$

Here we are making use of the following four facts:

- If $K \subset \mathscr{B}^*((-1,1) \times Y)$ is the union of all images $r(M(\gamma_1,\gamma_2))$ where the flat connections γ_j are irreducible and $\dim M(\gamma_1,\gamma_2) \leqslant 4$ then K is compact.
- Unique continuation: If two anti-self-dual connections over $\mathbb{R} \times Y$ are gauge equivalent over $(-1,1) \times Y$ then they must be gauge equivalent over $\mathbb{R} \times Y$.
- Restriction to $(-1,1)\times Y$ defines smooth embeddings of $M(\theta,\alpha)$ and $M(\beta,\theta)$ into $\mathscr{B}^*((-1,1)\times Y)$.
- $\mathscr{B}^*((-1,1) \times Y)$ admits smooth partitions of unity (recall that we are working with L^p_1 connections with $p > 4$ an even integer).

(II) We will now state a gluing theorem which describes the elements of $M(\beta,\alpha)$ that are close to the trivial connection over $(-1,1) \times Y$.

Fix a small positive constant ε_1 and let U be the set of all elements of $M(\beta,\alpha)$ which over the band $(-1,1)\times Y$ can be represented by a connection form a with $||a||_{L^2_1} < \varepsilon_1$. As ε_1 becomes smaller, elements of U will more and more resemble broken gradient lines from α to β factoring via the trivial connection θ. Hence for sufficiently small ε_1 there is a natural map

$$U \xrightarrow{\psi} \check{M}(\theta,\alpha) \times \check{M}(\beta,\theta).$$

Let ε_2 be another small positive constant and choose a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi' \geqslant 0$, $\phi(t) = 1$ for $t \leqslant -1$, and $\phi(t) = 0$ for $t \geqslant 1$. For any flat SU(2) connection γ over Y define smooth, real functions τ_1 on $M(\gamma,\alpha)$ and τ_2 on $M(\beta,\gamma)$ implicitly by

$$\int_{\mathbb{R}\times Y} |F(A)_{(t,y)}|^2 \phi(\tau_1(A) + t)\,\mathrm{d}t\,\mathrm{d}y = \varepsilon_2,$$

$$\int_{\mathbb{R}\times Y} |F(A)_{(t,y)}|^2 \phi(\tau_2(A) - t)\,\mathrm{d}t\,\mathrm{d}y = \varepsilon_2.$$

For any real number T let U_T be the subset of U defined by the inequalities $\tau_j > T$, $j = 1,2$.

For any $[A] \in M(\beta,\alpha)$ let $\eta(A) \in \mathrm{SO}(3)$ be the holonomy of A along the path $\mathbb{R} \times \{y_0\}$ in the positive direction.

In gluing theory (see [9,5]) one proves the following theorem.

Theorem 5. *For sufficiently large $T > 0$ (depending on ε_1 and ε_2), the map*

$$(\tau_1, \tau_2, \eta, \psi) : U_T \to (T, \infty) \times (T, \infty) \times SO(3) \times \check{M}(\theta, \alpha) \times \check{M}(\beta, \theta)$$

is an orientation preserving diffeomorphism.

This theorem expresses our convention for relating the orientations of the moduli spaces $M(\theta, \alpha)$ to those of the spaces $M(\beta, \theta)$.

We now continue the proof of Theorem 4.

(III) This part of the proof is related to the computation of the coefficient of $\delta \otimes \delta'$ in the Theorem. Choose smooth maps

$$\zeta_j : \mathbb{R} \times \mathrm{SO}(3) \to \mathbb{C}^3, \quad j = 1, 2$$

satisfying

$$\zeta_j(t, g) = \begin{cases} \bar{e}_j, & t \leqslant -1, \\ g^{-1}\bar{e}_j, & t \geqslant 1, \end{cases}$$

where $\bar{e}_j \in \mathbb{C}^3$ is a vector close to e_j which will be specified later. We may arrange that ζ_1 has no zeros and that if $\underline{\mathbb{C}}^3$ denotes the trivial complex 3-plane bundle over $\mathbb{R} \times \mathrm{SO}(3)$ then the section $\zeta = \zeta_2 \bmod \zeta_1$ of the quotient bundle $\underline{\mathbb{C}}^3 / \mathbb{C}\zeta_1$ is transverse to the zero-section.

We will now compute the number of zeros of ζ, counted with sign. Let Σ be the suspension of SO(3), which is the union of two cones: $\Sigma = C_+ \cup C_-$. Let $\Sigma_0 \to \Sigma$ be the principal SO(3) bundle whose "clutching map" $C_+ \cap C_- = \mathrm{SO}(3) \to \mathrm{SO}(3)$ is the identity map. Then

$$\#\zeta^{-1}(0) = -\langle p_1(\Sigma_0), [\Sigma] \rangle = -2.$$

(IV) Fix $T > 0$ such that the conclusion of Theorem 5 holds. Choose a smooth function $w : \mathbb{R} \to \mathbb{R}$ such that $w(t) = 1$ for $t \leqslant T$ and $w(t) = 0$ for $t \geqslant 2T$. Define two real functions τ, ρ on $M(\beta, \alpha)$ by

$$\tau = (\tau_1^{-1} + \tau_2^{-1})^{-1}$$

(this will serve as a smooth approximation to $\min(\tau_1, \tau_2)$), and

$$\rho = \begin{cases} w \circ \tau & \text{on } U, \\ 1 & \text{on } M(\beta, \alpha) \setminus U. \end{cases}$$

We can ensure that ρ is smooth by choosing T so large that $\tau < T$ on ∂U. For $j = 1, 2$ define two sections ξ_j and $\tilde{\sigma}_j$ of $\mathbb{E}(\beta, \alpha) \otimes \mathbb{C}$ by

$$\xi_j(A) = f_-(A, \zeta_j(\tau_1(A) - \tau_2(A), \eta(A))),$$

$$\tilde{\sigma}_j = \rho\sigma_j + (1 - \rho)\xi_j.$$

For a generic choice of the $\bar{e}_j$ and ζ_j's the sections $\tilde{\sigma}_1, \tilde{\sigma}_2$ will satisfy the same transversality assumptions as σ_1, σ_2. Thus if $\tilde{\sigma}$ is defined as σ with $\tilde{\sigma}_j$ in place of σ_j then $Z = \tilde{\sigma}^{-1}(0)$ is an oriented, smooth, one-dimensional submanifold of $M(\beta, \alpha)$, and Z can be described as the locus in $M(\beta, \alpha)$ where $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are linearly dependent.

We will now determine the ends of Z. Suppose $[A_n]$ is a sequence in Z which has no subsequence which converges in $M(\beta,\alpha)$. After passing to a subsequence and applying suitable gauge transformations we may assume A_n converges over compact subsets of $\mathbb{R}\times Y$ to some anti-self-dual connection A. There are now four possibilities:

(a) The flat limits of A are both irreducible. Then $[A]$ must lie in a 4-dimensional moduli space $M(\gamma,\alpha)$ (where $i(\gamma)=0$) or $M(\beta,\gamma)$ (where $i(\gamma)=5$). Gluing theory tells us that the corresponding number of ends of Z is $\langle(dv-vd)\alpha,\beta\rangle$.
(b) $[A]\in M(\theta,\alpha)$. This means that $\tau_1(A_n)$ stays bounded while $\tau_2(A_n)\to\infty$ as $n\to\infty$. Hence $\xi_j(A_n)\to f_-(A,\bar{e}_j)$. Moreover, $[A]$ must satisfy $T\leqslant\tau_1(A)\leqslant 2T$. Now the inequalities $T\leqslant\tau_1\leqslant 2T$ define a compact subset $D_1\subset M(\theta,\alpha)$, so by (I) we may assume the sections s_1,s_2 are chosen such that $\sigma_j(A_n)\to f_-(A,e_j)$. Together this implies that $\tilde{\sigma}_1(A_n)$ and $\tilde{\sigma}_2(A_n)$ must be linearly independent for large n, contradicting $\tilde{\sigma}(A_n)=0$. Thus $[A]$ cannot lie in $M(\theta,\alpha)$.
(c) $A\in M(\beta,\theta)$. This is ruled out just like case (b).
(d) A is trivial. Then $\tau(A_n)\to\infty$, so $\tilde{\sigma}_j(A_n)=\xi_j(A_n)$ for large n. The corresponding number of ends of Z is $-\#\zeta^{-1}(0)\cdot\langle(\delta\alpha)\delta',\beta\rangle$, and by (III) we have $\#\zeta^{-1}(0)=-2$.

This completes the proof of Theorem 4. □

3.2. Cobordisms

We will now study how the u-map is related to maps between Floer groups induced by cobordisms. There are many different cases here that one could consider (i.e. different kinds of cobordisms, bundles, etc.), and we will focus on what is in a sense the most general case that we will encounter.

Let W be a connected Riemannian 4-manifold with two cylindrical ends, $\mathbb{R}_-\times Y_1$ and $\mathbb{R}_+\times Y_2$. Suppose Y_1 and Y_2 are homology spheres, $H_1(W;\mathbb{Z})=0$, and the intersection form of W is negative definite. If α_1,α_2 are flat SU(2) connections over Y_1,Y_2, respectively, not both trivial, let $M(W;\alpha_2,\alpha_1)$ denote the moduli space of anti-self-dual SU(2) connections over W with flat limits α_1 at $-\infty$ and α_2 at ∞, and with dimension in the range $[0,7]$. There is a degree preserving cochain homomorphism

$$W^*:\ \mathrm{CF}^*(Y_1)\to\mathrm{CF}^*(Y_2),\quad \alpha\mapsto\sum_\beta(\#M(W;\beta,\alpha))\beta,$$

where the sum is taken over all gauge equivalence classes β of flat SU(2) connections over Y_2 of the same index as α. There is also a homomorphism $\delta_W:\mathrm{CF}^5(Y_1)\to\mathbb{Z}$ and an element $\delta'_W\in\mathrm{CF}^0(Y_2)$ obtained by counting points in zero-dimensional moduli spaces over W with trivial limit over one end.

For $j=1,2$ choose a generic pair of sections of the natural complex 3-plane bundle $\mathbb{F}_j\otimes\mathbb{C}\to\mathscr{B}^*(Y_j\times(-1,1))$. As above this defines a homomorphism $v:\mathrm{CF}^*(Y_j)\to\mathrm{CF}^{*+4}(Y_j)$.

Theorem 6. *There exists a graded homomorphism* $\phi:\mathrm{CF}^*(Y_1)\to\mathrm{CF}^{*+3}(Y_2)$ *such that*

$$vW^*-W^*v+2(\delta_W\otimes\delta'+\delta\otimes\delta'_W)=d\phi+\phi d$$

as homomorphisms $\mathrm{CF}^*(Y_1) \to \mathrm{CF}^{*+4}(Y_2)$, *where* $\delta\colon \mathrm{CF}^4(Y_1) \to \mathbb{Z}$ *and* $\delta' \in \mathrm{CF}^1(Y_2)$ *are as defined in Section* 2.

Proof. The proof will be quite similar to the proof of Theorem 4, so we only indicate the new features. We will use the technique of "moving the base-point". In a sense, this will make up for the lack of translation in moduli spaces over W. To this end, choose base-points $y_j \in Y_j$, $j = 1,2$, and a smooth path $\gamma : \mathbb{R} \to W$ such that

$$\gamma(t) = \begin{cases} (t, y_1) & \text{for } t \leqslant -1, \\ (t, y_2) & \text{for } t \geqslant 1. \end{cases}$$

Set $W_0 = W \setminus ((\infty, -2] \times Y_1 \cup [2, \infty) \times Y_2)$. We may assume that $\gamma(t) \in W_0$ for $|t| \leqslant 2$. Set $\gamma_0 = \gamma|_{(-2,2)}$.

Now let α, β be flat SU(2) connections over Y_1 and Y_2, respectively, not both trivial, and set $M = M(W; \beta, \alpha)$. Let $\mathbb{U} \to M \times W$ and $\mathbb{U}_0 \to \mathscr{B}^*(W_0) \times W_0$ be the universal Euclidean 3-plane bundles (see [9]), and let

$$\mathbb{E} = (\mathrm{id}_W \times \gamma)^*(\mathbb{U}), \quad \mathbb{E}_0 = (\mathrm{id}_{W_0} \times \gamma_0)^*(\mathbb{U}_0)$$

be the pull-back bundles over $M \times \mathbb{R}$ and $\mathscr{B}^*(W_0) \times (-2, 2)$, respectively. Choose a generic pair of sections of $\mathbb{E}_0 \otimes \mathbb{C}$. We can pull back these sections and the sections of $\mathbb{F}_j \otimes \mathbb{C}$ by the restriction maps

$$r_0 : M \times (-2, 2) \to \mathscr{B}^*(W_0) \times (-2, 2), \quad ([A], t) \mapsto ([A|_{W_0}], t),$$

$$r_1 : M \times (-\infty, -1) \to \mathscr{B}^*((-1, 1) \times Y_1), \quad ([A], t) \mapsto [A|_{(t-1,t+1)\times Y_1}],$$

$$r_2 : M \times (1, \infty) \to \mathscr{B}^*((-1, 1) \times Y_2), \quad ([A], t) \mapsto [A|_{(t-1,t+1)\times Y_2}]$$

to obtain pairs of sections of $\mathbb{E} \otimes \mathbb{C}$ over $M \times (-2, 2)$, $M \times (-\infty, -1)$, and $M \times (1, \infty)$, respectively. Piecing these together using a partition of unity we obtain two sections σ_1, σ_2 of $\mathbb{E} \otimes \mathbb{C}$. If $\dim M \leqslant 4$ then we may assume σ_1 has no zeros and that the section $\sigma = \sigma_2 \bmod \sigma_1$ of the quotient bundle $(\mathbb{E} \otimes \mathbb{C})/\mathbb{C}\sigma_1$ is transverse to the zero section. If α, β are both irreducible and $\dim M = 3$ then $\sigma^{-1}(0)$ is a finite set of oriented points, and we define the matrix coefficient $\langle \phi(\alpha), \beta \rangle$ by

$$\langle \phi(\alpha), \beta \rangle = \#\sigma^{-1}(0).$$

The remainder of the proof follows the proof of Theorem 4 quite closely. Let α and β be irreducible and $i(\beta) \equiv i(\alpha) + 4 \bmod 8$, so that $\dim M = 4$. We first note that, as in (I), certain alterations may be made to the sections of $\mathbb{F}_j \otimes \mathbb{C}$ used above, without affecting the definition of the u-map on $\mathrm{CF}^*(Y_j)$. Also, there is an analogue of Theorem 5 for the present setup which we use to redefine the sections σ_1, σ_2 for elements $([A], t)$ of $M \times \mathbb{R}$ where either $t \ll 0$ and $r_1(A)$ is close to the trivial connection, or $t \gg 0$ and $r_2(A)$ is close to the trivial connection.

This being done, let $Z \subset M \times \mathbb{R}$ be the zero-set of $\tilde{\sigma}$ (the modification of σ). Thus Z is an oriented, smooth, one-dimensional submanifold of $M \times \mathbb{R}$. To describe the ends of Z, suppose $([A_n], t_n)$ is a sequence in Z which has no convergent subsequence in $M \times \mathbb{R}$. After passing to a subsequence we may assume t_n has a limit L in $[-\infty, \infty]$.

If $L\in\mathbb{R}$ then we may pass to a subsequence in which A_n converges modulo gauge transformations over compact subsets of W to some $[A]\in M(W;\gamma_1,\gamma_2)$. For dimensional reasons $[A]$ cannot be reducible. Hence for transversality reasons we must have $\dim M(W;\gamma_1,\gamma_2)=3$, so either $\gamma_1=\alpha$ or $\gamma_2=\beta$. The corresponding number of ends of Z is $-\langle(d\phi+\phi d)\alpha,\beta\rangle$.

The case $L=-\infty$ is analogous to the proof of Theorem 4, and the corresponding number of ends is $\langle(-W^*v+2\delta\otimes\delta'_W)\alpha,\beta\rangle$. Similarly, in the case $L=\infty$ the number of ends is $\langle(vW^*+2\delta_W\otimes\delta')\alpha,\beta\rangle$. □

3.3. Reduced Floer groups

Let Y be an oriented homology 3-sphere. If n is a non-negative integer then $v^n\delta'$ lies in $\mathrm{CF}^{1+4n}(Y)$, hence by Theorem 4,

$$dv^n\delta'=v^nd\delta'=0$$

and similarly $\delta v^n d=0$. Let

$$\delta'_n\in\mathrm{HF}^{1+4n}(Y)$$

be the cohomology class of $v^n\delta'$, and let

$$\delta_n:\ \mathrm{HF}^{4-4n}(Y)\to\mathbb{Z}$$

be the homomorphism induced by δv^n.

It follows from the cochain homotopy formula in Theorem 4(ii) that either δ_0 or δ'_0 must be zero. Moreover, if δ_0 is zero then δ_n vanishes for all n, and similarly if δ'_0 is zero.

In general, we do not expect that δ_n is a topological invariant of Y for $n>1$. However, the following theorem shows that δ_n is a topological invariant modulo $\delta_0,\dots,\delta_{n-1}$. Recall that one can compare Floer groups defined by different metrics and perturbations by means of the homomorphism induced by a cobordism $\mathbb{R}\times Y$ where the metric and perturbation interpolates between the given ones on Y (see [12]). Thus the question is how δ_n behaves under maps induced by cobordisms.

Lemma 1. *If W,Y_1,Y_2 are as in Theorem* 6 *then*

$$\delta W^*=\delta+\delta_W d$$

as maps $\mathrm{CF}^4(Y_1)\to\mathbb{Z}$.

Proof. Let α be an irreducible flat SU(2) connection of index 4 over Y_1. We will determine the ends of the one-dimensional SU(2) moduli space $M=M(W;\theta,\alpha)$. Let $[A_n]$ be a sequence in M. By taking a subsequence we may arrange that $[A_n]$ converges modulo gauge transformations to some instanton A over W. For index reasons A must be either irreducible or trivial.

If A is irreducible then it must have index 0, and factorization has occurred through an irreducible flat connection of index 4 over Y_1 or Y_2. The corresponding number of ends of M is $(\delta W^*-\delta_W d)\alpha$.

The number of ends of M corresponding to the case when A is trivial is $-\delta\alpha$. Here we are using the assumption that $b_2^+(W)=0$, which just means that the trivial connection over W is a regular solution of the instanton equation. □

Theorem 7. *Let W, Y_1, Y_2 be as in Theorem 6. Then $\delta_0 W^* = \delta_0$ and $W^*\delta'_0 = \delta'_0$. More generally, there are integers a_{ij}, b_{ij} such that for any non-negative integer n,*

$$\delta_n W^* = \delta_n + \sum_{i=0}^{n-1} a_{in}\delta_i, \qquad W^*\delta'_n = \delta'_n + \sum_{i=0}^{n-1} b_{in}\delta'_i.$$

Proof. The statement $\delta_0 W^* = \delta_0$ follows from Lemma 1, and the proof that $W^*\delta'_0 = \delta'_0$ is similar. The remainder of the theorem is then a simple consequence of Theorem 6. □

Of course, one can take $a_{in} = 0 = b_{in}$ when i and n have different parity.

Let $B^* \subset \mathrm{HF}^*(Y)$ be the linear span of the classes δ'_n, $n \geqslant 0$. Thus B^1 is spanned by the δ'_{2n}, and B^5 is spanned by the δ'_{2n+1}. If $q \not\equiv 1 \bmod 4$ then $B^q = 0$. Also, set

$$Z^* = \bigcap_{n \geqslant 0} \ker(\delta_n) \subset \mathrm{HF}^*(Y),$$

where by definition δ_n is zero in degrees different from $(4 - 4n) \bmod 8$.

Definition 1. The reduced Floer group $\widehat{\mathrm{HF}}^*(Y)$ is defined by

$$\widehat{\mathrm{HF}}^q(Y) = Z^q/B^q.$$

We will now define the u-map on the reduced Floer group. Let $\ker(d) \subset \mathrm{CF}^*(Y)$ be the Floer cocycles, and let $\pi\colon \ker(d) \to \mathrm{HF}^*(Y)$. Using Theorem 4 it is easy to check that v maps $\pi^{-1}(Z^*)$ into itself and $\pi^{-1}(B^*)$ into itself, hence v induces a degree 4 endomorphism u of

$$\widehat{\mathrm{HF}}^*(Y) = \pi^{-1}(Z^*)/\pi^{-1}(B^*).$$

Theorem 8. *Let W, Y_1, Y_2 be as in Theorem 6. Then $(W^*)^{-1}(Z^q) = Z^q$ and $W^*(B^q) = B^q$ for every q. In particular, W induces a homomorphism $\widehat{\mathrm{HF}}^*(Y_1) \to \widehat{\mathrm{HF}}^*(Y_2)$. Moreover, this homomorphism commutes with the u-maps.*

Proof. This follows immediately from Theorems 6 and 7. □

Just as for the ordinary Floer groups the map between the reduced Floer groups induced by W is independent of the metric on W, as long as the metric is on product form on the ends.

Corollary 1. *The $\mathbb{Z}[u]$ module $\widehat{\mathrm{HF}}^*(Y)$ is an invariant of the oriented, smooth manifold Y.*

4. Reducible connections

We will now see how one can obtain information about the homomorphisms $\delta_n\colon \mathrm{HF}^{4-4n}(Y;\mathbb{Q}) \to \mathbb{Q}$ when the oriented homology sphere Y bounds a smooth, compact, oriented 4-manifold with non-standard definite intersection form. The main ideas here are due to Donaldson [5]. It should be stressed that our results do by no means give a complete description, since we only take into account the lowest stratum of reducibles.

Let X be a smooth, oriented Riemannian 4-manifold with one cylindrical end $\mathbb{R}_+ \times Y$, where Y is an integral homology sphere. Let $E \to X$ be a $U(2)$ bundle and fix a smooth connection $A^{\det}$ in $\Lambda^2(E)$ which is trivial over the end. For any integer k let $M(E,k)$ denote the moduli space of (projectively) anti-self-dual connections A in E, with central part $A^{\det}$, which are asymptotically trivial over the end and satisfy

$$\frac{1}{8\pi^2}\int_X \operatorname{tr}(F(A)^2) = k - \frac{1}{2}c_1(E)^2.$$

Here $F(A)$ is the curvature of A. Then the expected dimension of $M(E,k)$ is

$$\dim M(E,k) = 8k - 2c_1(E)^2 - 3(1 - b_1(X) + b_2^+(X)).$$

Note that if $F_0(A)$ is the traceless part of $F(A)$ then

$$0 \leqslant \frac{1}{8\pi^2}\int_X \operatorname{tr}(F_0(A)^2) = k - \frac{1}{4}c_1(E)^2.$$

The inequality follows because $\operatorname{tr}(F_0(A)^2) = |F_0(A)|^2$ when $F_0^+(A) = 0$.

Lemma 2. *If X is connected and $b_1(X) = 0$ then the set of reducible points in the moduli space $M = M(E,k)$ is in one-to-one correspondence with the set of unordered pairs*

$$R = \{\{z_1,z_2\} \subset H^2(X;\mathbb{Z}) \,|\, (z_1 - z_2)_{\mathbb{R}} \in \mathscr{H}^-;\ z_1 + z_2 = c_1(E);\ z_1 z_2 = k\}.$$

Here $(\cdot)_{\mathbb{R}}$ denotes the real reduction of an integral cohomology class and $\mathscr{H}^-$ is the space of anti-self-dual closed L^2 2-forms on X.

Proof. We define a map $r: M^{\mathrm{red}} \to R$, where M^{red} is the set of reducible points in M. If u is any automorphism of E such that $u \notin U(1)$ and $u(A) = A$ then A preserves the eigenspace decomposition $E = L_1 \oplus L_2$. This splitting is unique up to order unless A is projectively trivial, in which case all A-invariant rank 1 sub-bundles of E are isomorphic.

Set $z_j = c_1(L_j)$ and $r(A) = \{z_1, z_2\}$. Then r is well defined. For instance, to verify $z_1 z_2 = k$ let $F(A|_{L_j}) = i\phi_j \in i\Omega^2(X;\mathbb{R})$, which represents $2\pi i c_1(L_j)$. Since the anti-self-dual closed L^2 forms ϕ_j decay exponentially on the end, we have

$$k - \frac{1}{2}(z_1 + z_2)^2 = \frac{1}{8\pi^2}\int_X \operatorname{tr}(F(A)^2) = \frac{1}{8\pi^2}\int_X (-\phi_1^2 - \phi_2^2) = -\frac{1}{2}(z_1^2 + z_2^2),$$

hence $k = z_1 z_2$ as claimed.

To see that r is a bijection recall that by Hodge theory a $U(1)$ bundle $L \to X$ admits a finite energy anti-self-dual connection precisely when the L^2-harmonic form representing $c_1(L)$ is anti-self-dual, and in that case the connection is unique up to gauge since $b_1(X) = 0$. $\square$

With X as above, let $\mathscr{T} \subset H^2(X;\mathbb{Z})$ be the torsion subgroup and set $\mathscr{L} = H^2(X;\mathbb{Z})/\mathscr{T}$. Let $\tilde{R}$ be the set of *ordered* pairs (z_1, z_2) such that $\{z_1, z_2\} \in R$. Then the torsion subgroup $\mathscr{T}$ of

$H^2(X;\mathbb{Z})$ acts freely on $\tilde{R}$ by $t\cdot(z_1,z_2)=(z_1+t,z_2-t)$ for $t\in\mathcal{T}$. By associating to a pair (z_1,z_2) the class $(z_1-z_2)_{\mathbb{R}}$ we obtain a natural identification

$$\tilde{R}/\mathcal{T}=\{z\in(c+2\mathcal{L})\cap\mathcal{H}^-(g)\,|\,z^2+4k=c^2\},\tag{3}$$

where $c=c_1(E)$. Note that if $c\notin 2H^2(X;\mathbb{Z})$ then the projection $\tilde{R}\to R$ is two to one.

By a *lattice* we shall mean a finitely generated free abelian group $\mathcal{L}$ with a non-degenerate symmetric bilinear form $b:\mathcal{L}\times\mathcal{L}\to\mathbb{Z}$. For $x,y\in\mathcal{L}$ we usually write $x\cdot y$ for the pairing $b(x,y)$, and x^2 instead of $x\cdot x$. The dual lattice $\mathrm{Hom}(\mathcal{L},\mathbb{Z})$ will be denoted $\mathcal{L}^{\#}$.

Definition 2. Let $\mathcal{L}$ be a (positive or negative) definite lattice. A vector $w\in\mathcal{L}$ is called *extremal* if $|w^2|\leqslant|z^2|$ for all $z\in w+2\mathcal{L}$. If $w\in\mathcal{L}$, $a\in\mathcal{L}^{\#}$, and m is a non-negative integer satisfying $w^2\equiv m \bmod 2$ set

$$\eta(\mathcal{L},w,a,m)=\sum_{\pm z}(-1)^{((z+w)/2)^2}(a\cdot z)^m,$$

where the sum is taken over all unordered pairs $\{z,-z\}\subset w+2\mathcal{L}$ such that $z^2=w^2$. If $m=0$ then we write $\eta(\mathcal{L},w)=\eta(\mathcal{L},w,a,m)$.

Proposition 1. *Let X be a smooth, compact, oriented 4-manifold with a homology sphere Y as boundary and with $b_1(X)=0$. Suppose the intersection form on $\mathcal{L}=H^2(X;\mathbb{Z})/\mathcal{T}$ is negative definite, where $\mathcal{T}$ is the torsion subgroup. Let $c\in H^2(X;\mathbb{Z})$, $a\in H_2(X;\mathbb{Z})$, and let m be a non-negative integer such that $c^2\equiv m \bmod 2$ and $-c^2\geqslant 2$. Set $n=-(c^2+m)/2-1$. If c reduces to a non-zero extremal vector $w\in\mathcal{L}$ then the Donaldson invariant $D_X^c(a^m)\in \mathrm{HF}^{4-4n}(Y;\mathbb{Q})$ is well defined and*

$$\delta_0u^j\cdot D_X^c(a^m)=\begin{cases}0 & \text{for } 0\leqslant j<n,\\ \pm 2^{-m}\,|\mathcal{T}|\,\eta(\mathcal{L},w,a,m) & \text{for } j=n.\end{cases}$$

Here we set $\delta_0x=0$ if $x\in\mathrm{HF}^0$.

Corollary 2 (Donaldson). *If the intersection form of X is not diagonal over the integers then $\delta_0:\mathrm{HF}^4(Y;\mathbb{Q})\to\mathbb{Q}$ is non-zero.*

Proof. Let $\bar{\mathcal{L}}\subset\mathcal{L}$ be the orthogonal complement of all vectors of square -1. The assumption is that $\bar{\mathcal{L}}\neq 0$. Let $w\in\bar{\mathcal{L}}$ be a non-zero vector of smallest length, and set $m=-w^2-2$. Choose a class $a\in H_2(X;\mathbb{Z})$ with $a\cdot w=1$. Then $\eta(\mathcal{L},w,a,m)=1$, so $\delta_0\neq 0$ by the proposition. □

Proof of Proposition. First, add a half-infinite cylinder $\mathbb{R}_+\times Y$ to X and choose a cylindrical end metric on this new manifold (also denoted X). Let $E\to X$ be the $U(2)$ bundle with $c_1(E)=c$, and set $M_k=M(E,k)$.

By Lemma 1, the moduli space M_k contains no reducibles if $k<0$. Now consider $k=0$. Let R and $\bar{R}$ be as above. It is convenient to fix an ordering of each pair $\{z_1,z_2\}\in R$. By making a small perturbation of the anti-self-dual equations near each reducible point as in [6] we can arrange that for each $\mathbf{z}=(z_1,z_2)$ the corresponding reducible point in M_0 has an open neighbourhood $C_\mathbf{z}$ in M_0 which is homeomorphic to a cone on some complex projective space $P_\mathbf{z}$. Note that reversing the order of z_1,z_2 reverses the complex structure on $P_\mathbf{z}$.

The boundary orientation that $P_\mathbf{z}$ inherits from $M_0^\# = M_0 \setminus \cup_\mathbf{z} C_\mathbf{z}$ differs from the complex orientation of $P_\mathbf{z}$ by a sign $\varepsilon(z_1,z_2)$. If $(\bar{z}_1,\bar{z}_2)$ is another element of $\tilde{R}$ then it follows from [7] that

$$\varepsilon(z_1,z_2)\varepsilon(\bar{z}_1,\bar{z}_2)=(-1)^{(z_1-\bar{z}_1)^2}.$$

Now consider the universal SO(3) bundle

$$\mathbb{E} \to P_\mathbf{z} \times X$$

and define the μ-map $H_i(X) \to H^{4-i}(P_\mathbf{z})$ as usual by $\mu(b) = -\frac{1}{4}p_1(\mathbb{E})/b$. If $e \in H^2(P_\mathbf{z})$ is the Chern class of the tautological line bundle then we have

$$\mu(1) = -\tfrac{1}{4}e^2, \quad \mu(a) = -\tfrac{1}{2}\langle z_1 - z_2, a\rangle e$$

for any $a \in H_2(X;\mathbb{Z})$ (see [9]).

Let M_0' be the oriented 1-manifold with boundary obtained by cutting down $M_0^\#$ according to the monomial $x^j a^m \in A(X)$ as in Section 2. The boundary points of M_0' lie on the links of the reducibles points, while the ends correspond to factorizations on $\mathbb{R}_+ \times Y$ through flat connections of index 4. Counted with sign, the number of boundary points in M_0' plus the number of ends must be zero. This gives

$$\delta D_X^c(x^n a^m) = \pm\, 2^{-2n-m}\, |\mathscr{T}|\, \eta(\mathscr{L}, w, a, m),$$

where the left-hand side is the number of ends. The invariant $D_X^c(x^n a^m) \in \mathrm{HF}^4(Y)$ is well defined because M_k contains no reducibles when $k<0$. The same argument shows that $\delta D_X^c(x^j a^m) = 0$ for $0 \leqslant j < n$. The Proposition now follows because

$$D_X^c(x^j a^m) = (\tfrac{1}{4}u)^j D_X^c(a^m) \quad \text{for } 0 \leqslant j \leqslant n.$$

One can prove this by "moving the base-point" along the path $[0,\infty) \times Y$ (as in Section 3). However, to run this argument one needs to know that no irreducible connections in M_k can restrict to a reducible connection over the end $\mathbb{R}_+ \times Y$. In other words, M_k must not contain any twisted reducibles (cf. Lemma 4.3.21 in [9]). Fortunately, this holds at least generically:

After perturbing the metric on X in a small ball we may assume that there are no non-flat twisted reducibles in M_k for any k (see [21, Section 2(i)]). The main point here is that if λ is a real line bundle over X and $b_i(\lambda)$ are the Betti numbers of X with coefficients in λ then as pointed out in [21],

$$b_0(\lambda) - b_1(\lambda) + b_2^+(\lambda)$$

is the same for all line bundles λ. If $b_1(X)=0$ and λ is non-trivial this gives $b_2^+(\lambda) > b_2^+(X) = 0$.

Moreover, since w is not divisible by 2 in $\mathscr{L}$, there are no flat reducibles in M_k. Any flat, twisted reducible connection in M_k must therefore be irreducible. But M_k can only contain a flat connection for $k = c_1(E)^2/4$; in that case M_k has expected dimension -3 and is generically empty (cf. [7]). Hence, we may assume there are no twisted reducibles in M_k for any k. In particular, no irreducible connection in M_k will restrict to a reducible connection over the end. □

We will now apply this proposition to obtain information about the Floer cohomology of the Poincaré sphere $S=\Sigma(2,3,5)$. It is well known that the Milnor fibre of the E_8 singularity can be smoothly embedded in a $K3$ surface X. This gives a splitting $X=X_1\cup_S X_2$, where the intersection form of X_1 is $-E_8$ and the intersection form of X_2 is $-E_8\oplus 3\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. If $e_1,\dots,e_8$ is an orthonormal basis for $\mathbb{R}^8$ then

$$E_8=\left\{\sum_{i=1}^{8}x_ie_i\,|\,2x_i\in\mathbb{Z};\ x_i-x_j\in\mathbb{Z};\ \sum_{i=1}^{8}x_i\equiv 0\,(2)\right\}.$$

For $j=1,2$, let $w_j\in H^2(X_j;\mathbb{Z})$ be the element corresponding to $e_1+e_2\in E_8$, and let $z_j\in H^2(X_j;\mathbb{Z})$ be the element corresponding to $e_1+e_2+e_3+e_4\in E_8$. Recall that the relative invariants of X_1 and X_2 take values in the Floer cohomology and homology of S, respectively. By gluing theory and knowledge of a certain Donaldson invariant of X (see [20], or [9], Proposition 9.1.3) we have

$$D_{X_1}^{w_1}(1)\cdot D_{X_2}^{z_2}(1)=D_X^{w_1+z_2}(1)=\pm 1,$$

$$D_{X_1}^{z_1}(1)\cdot D_{X_2}^{w_2}(1)=D_X^{z_1+w_2}(1)=\pm 1,$$

where the signs depend on the homology orientations of X_1 and X_2.

But S has precisely two equivalence classes of irreducible flat SU(2) connections, which are both non-degenerate, so $\alpha=D_{X_1}^{w_1}(1)$ and $\beta=D_{X_1}^{z_1}(1)$ must be generators of $\mathrm{HF}^4(S;\mathbb{Z})=\mathbb{Z}$ and $\mathrm{HF}^0(S;\mathbb{Z})=\mathbb{Z}$, respectively.

It is easy to check that e_1+e_2 and $e_1+e_2+e_3+e_4$ are extremal vectors in E_8 satisfying

$$\eta(E_8,e_1+e_2)=1,$$

$$\eta(E_8,e_1+e_2+e_3+e_4)=8.$$

Proposition 1 (with $m=0$ and $a=0$) now gives $\delta_0\alpha=\pm 1$ and $\delta_0u\beta=\pm 8$, and we deduce the following proposition.

Proposition 2. *For the Poincaré sphere* $S=\Sigma(2,3,5)$ *the following holds*:

(i) $\delta_0:\mathrm{HF}^4(S;\mathbb{Z})\to\mathbb{Z}$ *is an isomorphism.*
(ii) $u:\mathrm{HF}^0(S;\mathbb{Z})\to\mathrm{HF}^4(S;\mathbb{Z})$ *is multiplication by* ± 8.

This result was first proved by Kronheimer (unpublished) and Austin [1].

It follows from the proposition that the reduced Floer group of the Poincaré sphere is zero.

5. Floer's exact triangle

This section gives a brief description of Floer's exact triangle; for further details we refer to the exposition [3]. Let Y_0 be a closed, oriented 3-manifold and $P_0\to Y_0$ an admissible principal SO(3) bundle. Given any orientation preserving SO(3) equivariant embedding

$$\kappa_0:D^2\times S^1\times\mathrm{SO}(3)\to P_0 \tag{4}$$

we can form the surgery cobordism

$$Q_0 = \overline{D^2 \times D^2} \times \mathrm{SO}(3) \cup_\kappa [0,1] \times P_0,$$

where κ maps into $\{1\} \times P_0$. Here the bar means we take the opposite of the standard orientation. The oriented boundary of Q_0 is $P_1 \cup \bar{P}_0$, where P_1 is the result of the surgery on P_0 determined by κ_0. Let $f: S^1 \to \mathrm{SO}(3)$ be a homotopically non-trivial map and define an equivariant embedding

$$\kappa_1 : D^2 \times S^1 \times \mathrm{SO}(3) \xrightarrow{\approx} S^1 \times D^2 \times \mathrm{SO}(3) \subset P_1,$$

$$(z,w,u) \mapsto (w^{-1}, zw^{-1}, f(w)u).$$

Here we regard D^2 as the unit disc in $\mathbb{C}$, and $S^1 = \partial D^2$. Iterating this process we get a sequence (P_n, κ_n), $n = 0,1,2,\dots$. The bundle P_{n+1} is obtained from P_n by cutting out $\mathrm{im}(\kappa_n)$ and re-gluing using a certain equivariant diffeomorphism ξ of $S^1 \times S^1 \times \mathrm{SO}(3)$. A crucial point here is that ξ^3 covers the identity map on $S^1 \times S^1$, and its associated map $S^1 \times S^1 \to \mathrm{SO}(3)$ is null-homotopic. Thus, we may identify

$$(P_{n+3}, \kappa_{n+3}) = (P_n, \kappa_n).$$

From now on we assume all bundles P_n are admissible. There are then two possibilities: either (i) Y_n is not a homology sphere for any n, or (ii) for some n, the manifold Y_n is a homology sphere, while Y_{n+1} and Y_{n+2} are the result of -1 surgery and 0 surgery (respectively) on the knot in Y_n determined by κ_n. In both cases the surgery cobordism Q_n from P_n to P_{n+1} induces a homomorphism α_n: $\mathrm{HF}^*(P_n) \to \mathrm{HF}^*(P_{n+1})$. Floer's theorem now says that for every n, the composite homomorphism $\alpha_{n+2}\alpha_{n+1}\alpha_n$ shifts degrees by $-3 \bmod 8$, and

$$\mathrm{im}(\alpha_n) = \ker(\alpha_{n+1}).$$

(It is easy to compute the shift in degrees in case (ii) above; on the other hand, the shift must always be the same, as can be seen from the addition property for the index over 4-manifolds with tubular ends.) If Y_0 and Y_1 are homology spheres and we give Y_2 the spin structure that extends over W_1 then the long exact sequence takes the form

$$\cdots \to \mathrm{HF}^q(Y_0) \to \mathrm{HF}^q(Y_1) \to \mathrm{HF}^q(Y_2) \to \mathrm{HF}^{q-3}(Y_0) \to \cdots,$$

where we have written $\mathrm{HF}^q(Y_2)$ instead of $\mathrm{HF}^q(P_2)$.

6. The nilpotency of $u^2 - 64$

In this section we establish the nilpotency of $u^2 - 64$ in both the Floer group of a non-trivial, admissible SO(3) bundle and in the reduced Floer group of a homology sphere. In both cases the proof of nilpotency of $u^2 - 64$ begins by representing the 3-manifold as surgery on an algebraically split framed link in S^3. We then use Floer's exact triangle and a link reduction scheme. A central part in the reduction argument is that if F is the non-trivial SO(3)-bundle over the 2-torus then $u^2 = 8^2$ in $\mathrm{HF}^*(S^1 \times F)$. The number 8 is derived from Proposition 2(ii).

We first discuss non-trivial SO(3) bundles.

Proposition 3. *Let Y be a closed oriented 3-manifold with $H_1(Y;\mathbb{Z})$ torsion free and non-zero. Let $P \to Y$ be a non-trivial* SO(3) *bundle. Then for some $n \geqslant 1$,*

$$(u^2 - 64)^n = 0$$

as an endomorphism of $\mathrm{HF}^*(P)$.

Before proving this proposition we will deduce from it a stronger result. Let Σ be a surface of genus g and $F \to \Sigma$ the non-trivial SO(3) bundle. Consider the Floer cohomology HF^*_g of the SO(3) bundle $S^1 \times F \to S^1 \times \Sigma$. Let $\psi \in \mathrm{HF}^*_g$ be the element obtained by counting points (with signs) in zero-dimensional instanton moduli spaces in the bundle $D^2 \times F \to D^2 \times \Sigma$ (adding a tubular end to $D^2 \times \Sigma$ as usual). For $g \geqslant 0$ let N_g be the smallest non-negative integer n such that $(u^2 - 64)^n \psi = 0$. (For rational coefficients it should, in principle, be possible to compute all the constants N_g from [23]. For instance one has $N_1 = N_2 = 1$.)

Theorem 9. *Let Y be a closed, oriented 3-manifold with $b_1(Y) > 0$, and let $P \to Y$ be an* SO(3) *bundle such that $\mathbb{R} \times P$ is non-trivial over some surface $\Sigma \subset \mathbb{R} \times Y$ of genus g. Then*

$$(u^2 - 64)^{N_g} = 0$$

as an endomorphism of $\mathrm{HF}^*(P)$.

Proof of Theorem 9 (Assuming Proposition 3)**.** For any non-negative integer n let

$$K_n : \mathrm{HF}^*(P) \to \mathrm{HF}^{*+4n}(P)$$

be the homomorphism defined by cutting down $(4n)$-dimensional moduli spaces $M(\beta,\alpha)$ according to the monomial $(4x)^n$ as in Section 2, where $x \in H_0(\mathbb{R} \times Y)$ is the point class. By moving one base-point along a path $[0,\infty) \times \{y_0\}$ as in the proof of Theorem 2 one finds that $K_n = uK_{n-1}$ for $n \geqslant 1$, so by induction,

$$K_n = u^n.$$

Let $f : D^2 \times \Sigma \to \mathbb{R} \times Y$ be a smooth embedding which maps $\{0\} \times \Sigma$ onto Σ. Let $\{g_t\}_{t\geqslant 0}$ be a smooth family of metrics on $\mathbb{R} \times Y$ which stretches $\mathbb{R} \times Y$ along $f(S^1 \times \Sigma)$. More precisely, let $A \subset D^2$ be an annulus about the origin and set $U = f(A \times \Sigma)$. Then

- g_0 should be a product metric on $\mathbb{R} \times Y$,
- g_t should be independent of t outside U,
- under the identification $U \approx [0,1] \times S^1 \times \Sigma$ the restriction of g_t to U should have the form $t^2 dr^2 + ds^2$ for $t \geqslant 1$, where r is the coordinate on $[0,1]$ and ds^2 a fixed metric on $S^1 \times \Sigma$.

Now let $K_{n,t}$ be defined as K_t above but using the metric g_t, and such that the geometric representatives for $4\mu(x)$ are obtained by restricting instantons to fixed subsets of $(\mathbb{R}\times Y)\setminus U$. As is well known, $K_{n,t}$ is independent of t, because the cochain map that defines $K_{n,t}$ is independent of t up to cochain homotopy. We will now describe $K_{n,t}$ for large t. Let Q denote the restriction of $\mathbb{R} \times P$ to $f(S^1 \times \Sigma)$, and let W be the manifold $(\mathbb{R} \times Y) \setminus \Sigma$ with a metric which is on product form on the end $\mathbb{R}_- \times S^1 \times \Sigma$ and agrees with g_0 outside $U \setminus \Sigma$. Then moduli spaces in $(\mathbb{R} \times P)|_W$ cut down according to $(4x)^n$ define a homomorphism

$$L_n : \mathrm{HF}^*(Q) \to \mathrm{End}(\mathrm{HF}^*(P))$$

such that for large t one has $K_{n,t} = L_n(\psi)$.

For the final step in the proof we move one base-point along a path $\mathbb{R}_- \times \{z_0\}$ in $\mathbb{R}_- \times (S^1 \times \Sigma)$ to deduce

$$L_n(\rho) = L_{n-1}(u\rho)$$

for any $\rho \in \mathrm{HF}^*(Q)$ and $n \geqslant 1$. By induction on n,

$$L_n(\rho) = L_0(u^n \rho).$$

Putting all this together we obtain

$$(u^2 - 64)^n = L_0((u^2 - 64)^n \psi)$$

as an endomorphism of $\mathrm{HF}^*(P)$. Therefore the theorem follows from the proposition. □

The proof of Proposition 3 begins with three lemmas.

Lemma 3. *Let Y be a closed oriented 3-manifold with $H_1(Y;\mathbb{Z})$ torsion free. Then Y can be represented by a framed link in S^3 whose linking matrix is diagonal. The entries on the diagonal are either 0 or ± 1, and the number of zeros is $b_1(Y)$.*

Proof. Let L be any framed link in S^3 representing Y. Thus if X is the 4-manifold obtained by attaching 2-handles to the 4-ball according to L then $Y = \partial X$. Let $j : H_2(Y) \to H_2(X)$ be the map induced by inclusion. Then the intersection form on $H_2(X)$ descends to a unimodular form

$$q : H_2(X)/\mathrm{im}(j) \to \mathbb{Z}.$$

After adding an unknot with framing ± 1 if necessary, we may assume q is odd and indefinite. By the classification of such forms [16] $H_2(X)/\mathrm{im}(j)$ then has a basis over $\mathbb{Z}$ with respect to which q is diagonal. We can therefore modify L by a sequence of Kirby moves $\mathcal{O}_2$ (see [17]) to obtain a framed link, also representing Y, whose linking matrix is diagonal. □

Lemma 4. *Let $P \to Y$ be a non-trivial admissible $\mathrm{SO}(3)$ bundle. If $\mathbb{R} \times P$ is non-trivial over some embedded torus $T \subset \mathbb{R} \times Y$ then the cup product u on $\mathrm{HF}^*(P)$ satisfies $u^2 = 64$.*

Proof. Let $F \to T$ be the non-trivial $\mathrm{SO}(3)$ bundle over the 2-torus. Then $\mathrm{HF}^*(S^1 \times F)$ has rank 1 in two degrees differing by 4, and is zero in the remaining degrees. Let τ be the natural involution of degree 4 on $\mathrm{HF}^*(S^1 \times F)$. Since u and τ commute there is a constant c such that

$$u = c\tau.$$

Stretching $\mathbb{R} \times Y$ along $S^1 \times T$ we find as above that $u^2 = c^2$ on $\mathrm{HF}^*(P)$. To compute c, let S and S' be the result of -1 and 0 surgery on the negative $(2,3)$ torus knot in S^3, respectively. The exact triangle provides an isomorphism $\mathrm{HF}^*(S) \xrightarrow{\approx} \mathrm{HF}^*(S')$ which commutes with the cup product

$\mathrm{HF}^0 \to \mathrm{HF}^4$. But for S this cup product is multiplication by ± 8, according to Proposition 2, so $c^2 = 64$. $\square$

Before stating the next lemma we observe that if Y is a closed, oriented 3-manifold and $\gamma \subset Y$ a null-homologous knot, then any SO(3) bundle over the complement of γ has an (up to isomorphism) unique extension to an SO(3) bundle over Y.

Lemma 5. *Suppose Proposition* 3 *holds for* $P \to Y$. *Let* $\gamma \subset Y$ *be a knot which bounds a surface of genus* 1. *If* Y' *is the result of* ± 1 *surgery on* γ *then Proposition* 3 *also holds for the inherited bundle* $P' \to Y'$.

Proof. Since Proposition 3 is insensitive to the orientation of Y it suffices to consider $+1$ surgery on γ. Choose an embedding $\beta : D^2 \times S^1 \xrightarrow{\approx} N \subset Y$ onto a tubular neighbourhood of γ, and a surface $Z \subset Y$ of genus 1, such that

$$\partial Z = Z \cap N = \beta(\{w\} \times S^1)$$

for some $w \in S^1$. Also, choose a trivialization of $P|_N$ that does not extend over $N \cup Z$. This trivialization together with the embedding β determines a map κ_0 as in (4). In the notation of Section 5, the manifold Y_1 is obtained by 0 surgery on γ and $P_1 \to Y_1$ is non-trivial over the torus that one gets by closing up Z with a disc. Furthermore, Y_2 is the result of $+1$ surgery on γ. Floer's theorem now provides an exact sequence

$$\mathrm{HF}^*(P_1) \xrightarrow{\alpha_1} \mathrm{HF}^*(P_2) \xrightarrow{\alpha_2} \mathrm{HF}^*(P_0).$$

Suppose there is a positive integer n such that $(u^2 - 64)^n = 0$ on $\mathrm{HF}^*(P_0)$. For any $x \in \mathrm{HF}^*(P_2)$ the class $y = (u^2 - 64)^n x$ will then lie in the kernel of α_2. Hence $y = \alpha_1 z$ for some $z \in \mathrm{HF}^*(P_1)$. By Lemma 4 we have $(u^2 - 64)z = 0$, so

$$(u^2 - 64)^{n+1} x = \alpha_1 (u^2 - 64) z = 0. \qquad \square$$

Proof of Proposition 3. We first show that if the proposition holds when $b_1(Y) = r - 1 \geqslant 1$ then it also holds when $b_1(Y) = r$. Let Y be represented by a framed link L as in Lemma 3 and let $L_1, \ldots, L_r$ be the components with framing 0, where $r \geqslant 2$. If γ_i is a small linking circle of L_i then $[\gamma_1], \ldots, [\gamma_r]$ is a basis for $H_1(Y; \mathbb{Z})$. The dual basis for $H_2(Y; \mathbb{Z})$ can be represented by surfaces $Z_1, \ldots, Z_r$, where Z_i is obtained by capping off a Seifert surface of $\gamma_i \subset S^3$ by a disc. Here the Seifert surface should be disjoint from the other components of L; this can be arranged by the obvious tubing construction. It is clear that the bundle P is specified by the element of $(\mathbb{Z}/2)^r$ whose i'th component indicates whether $P|_{Z_i}$ is trivial or not. Without loss of generality we may assume $P|_{Z_1}$ is non-trivial.

We form two other SO(3) bundles $P' \to Y'$ and $P'' \to Y''$ as follows. The 3-manifolds Y' and Y'' are described by framed links L' and L'' in S^3; here L' is obtained from L by changing the framing of L_r from 0 to -1, while $L'' \subset L$ is the result of deleting the component L_r. Then $b_1(Y') = b_1(Y'') = r - 1$. Let the bundles P' and P'' both be specified by the element of $(\mathbb{Z}/2)^{r-1}$ which is the natural restriction of the vector specifying P. By Floer's theorem we have an exact sequence

$$\mathrm{HF}^*(P') \xrightarrow{\alpha'} \mathrm{HF}^*(P) \xrightarrow{\alpha} \mathrm{HF}^*(P'').$$

Arguing as in the proof of Lemma 5 we conclude that the proposition holds for the bundle P, since by assumption it holds for P' and P''.

It remains to prove the proposition when $b_1(Y)=1$. Again, let Y be represented by a framed link L as in Lemma 3. Choose a regular projection of L and fix some component L_i. It is well known that changing a crossing within L_i corresponds to ± 1 surgery on a knot in Y which bounds a surface of genus 1. By Lemma 5 we may therefore assume L_i is unknotted. Since Kirby calculus allows us to remove any unknotted component of L with framing ± 1 (at the expense of twisting the remainder of the link, but without changing framings or linking numbers) we are left to consider the manifold $S^1 \times S^2$ described by the unknot in S^3 with framing 0. But in this case $\mathrm{HF}^*(P)=0$, so the proposition is proved. □

We now turn to homology spheres.

Theorem 10. *For any oriented homology* 3*-sphere* Y *there is a positive integer* n *such that*

$$(u^2-64)^n = 0$$

as an endomorphism of $\widehat{\mathrm{HF}}^*(Y)$.

Proof. Let Y be an oriented homology 3-sphere and $\gamma \subset Y$ a knot. Let Y' and Y'' be the result of -1 surgery and 0 surgery on γ, respectively. There is then a long exact sequence

$$\cdots \to \mathrm{HF}^{q+3}(Y'') \xrightarrow{\alpha''} \mathrm{HF}^q(Y) \xrightarrow{\alpha} \mathrm{HF}^q(Y') \xrightarrow{\alpha'} \mathrm{HF}^q(Y'') \to \cdots . \tag{5}$$

Note that $\delta_0\alpha = \delta_0$, since the cobordism from Y to Y' is negative definite and has no integral homology in dimension 1. As explained in Section 3 this sequence induces a sequence of homomorphisms of $\mathbb{Q}[u]$ modules

$$\mathrm{HF}^{q+3}(Y'') \to \widehat{\mathrm{HF}}^q(Y) \to \widehat{\mathrm{HF}}^q(Y') \to \mathrm{HF}^q(Y''). \tag{6}$$

Moreover, it follows easily from Theorem 8 and the exactness of (5) that (6) is exact at the terms Y and Y' for every q. We can then prove the theorem by the same link reduction scheme as in the final paragraph of the proof of Proposition 3, reducing the problem to S^3, where it is trivial. □

Theorem 11. *If* Y *is any oriented homology* 3*-sphere and* R *any associative ring in which* 2 *is invertible then* $u: \mathrm{HF}^q(Y;R) \to \mathrm{HF}^{q+4}(Y;R)$ *is an isomorphism for* $q \not\equiv 4,5 \bmod 8$.

Proof. By Theorem 10 the u-map $\mathrm{HF}^q(Y;R) \to \mathrm{HF}^{q+4}(Y;R)$ is an isomorphism for $q \equiv 2,3 \bmod 4$, since in these degrees we have $\mathrm{HF}^q(Y;R) = \widehat{\mathrm{HF}}^q(Y;R)$. Thus, it only remains to show that u is an isomorphism for $q \equiv 0,1 \bmod 8$. But this now follows from the exact sequence (5) and Lemma 4, by the five-lemma and the same induction scheme as we used in the proof of Theorem 10. □

Corollary 3. *If* 2 *is invertible in* R *then* $\widehat{\mathrm{HF}}^*(Y;R)$ and $\widehat{\mathrm{HF}}^*(Y;R)$ *are both* mod 4 *periodic*.

Proof. Recall that the Floer groups are $\mathbb{Z}/8$ graded. In the case of $\widehat{\mathrm{HF}}^*(Y;R)$, it follows from Theorem 10 that u^2 is invertible, hence u is an isomorphism of $\widehat{\mathrm{HF}}^*(Y;R)$ onto itself. As for $\mathrm{HF}^*(Y;R)$, note that for every i, the u-map is an isomorphism $\mathrm{HF}^q(Y;R) \to \mathrm{HF}^{q+4}(Y;R)$ for either $q=i$ or $i-4$ (or both), by Theorem 11, hence $\mathrm{HF}^i(Y;R)$ and $\mathrm{HF}^{i+4}(Y;R)$ are isomorphic. □

7. Finite type of Donaldson invariants

We will now show that the nilpotency of u^2-64 on the reduced Floer groups leads to a proof of the finite type conjecture of Kronheimer and Mrowka in the simply connected case.

Theorem 12. *Let X be a smooth, compact, oriented* 4*-manifold such that $b_1(X)=0$ and $b_2^+(X)$ is odd. Suppose there exists a splitting of X along an embedded homology* 3*-sphere Y,*

$$X = X_1 \cup_Y X_2,$$

where $b_2^+(X_j) > 0$ for $j=1,2$. Then there exists a positive integer n such that for any homology orientation of X and any $w \in H^2(X;\mathbb{Z})$ the Donaldson invariant $D_X^w : \mathbb{A}(X) \to \mathbb{Q}$ satisfies

$$D_X^w((x^2-4)^n z) = 0$$

for every $z \in \mathbb{A}(X)$, where $x \in H_0$ is the point class. In other words, X has finite type.

Here

$$\mathbb{A}(X) = \mathrm{Sym}(H_0(X;\mathbb{Q}) \oplus H_2(X;\mathbb{Q}))$$

and D_X^w is defined in terms of U(2) bundles E over X with $c_1(E)=w$.

Proof. Because of the simplest blow-up formula [19] there is no loss of generality in assuming $w_j = w|_{X_j}$ is not divisible by 2 in $H^2(X_j;\mathbb{Z})$/torsion. To define relative invariants

$$D_j = D_{X_j}^{w_j} : \mathbb{A}(X_j) \to \mathrm{HF}^*(\partial X_j),$$

fix a metric on Y, and choose a generic metric on $X_j \cup (\mathbb{R}_+ \times \partial X_j)$ which restricts to the given product metric on the end. If $b_2^+(X_j)=1$ then D_j depends on the chamber of the metric on X_j, but this will not be reflected in our notation. Donaldson's gluing theorem now says that for any $z_j \in \mathbb{A}(X_j)$ we have

$$D(z_1 z_2) = D_1(z_1) \cdot D_2(z_2),$$

where $D = D_X^w$ and we use the natural pairing $\mathrm{HF}^*(Y) \otimes \mathrm{HF}_*(Y) \to \mathbb{Q}$ together with the identification $\mathrm{HF}^*(\bar{Y}) = \mathrm{HF}_{5-*}(Y)$.

The crucial observation now is that $\delta_n D_j = 0$ for every $n \geqslant 0$ because of the absence of reducible connections (see Section 4). Hence D_j defines an invariant

$$\hat{D}_j : \mathbb{A}(X_j) \to \widehat{\mathrm{HF}}^*(\partial X_j)$$

and we can use the natural pairing $\widehat{\mathrm{HF}}^*(Y) \otimes \widehat{\mathrm{HF}}^{5-*}(\bar{Y}) \to \mathbb{Q}$ in the gluing theorem:

$$D(z_1 z_2) = \hat{D}_1(z_1) \cdot \hat{D}_2(z_2).$$

By Theorem 10 there exists a positive integer n such that $(u^2 - 64)^n = 0$ on $\widehat{\mathrm{HF}}^*(Y)$. This gives

$$\hat{D}_1((x^2-4)^n z_1) = \left(\left(\frac{u}{4}\right)^2 - 4\right)^n \hat{D}_1(z) = 0$$

for every $z_1 \in \mathbb{A}(X_1)$. (As in the proof of Proposition 1 we can avoid twisted reducibles, since $b_1(X_1) = 0$.) But any class $z \in \mathbb{A}(X)$ can be expressed as $z = z_1 z_2$ for some $z_j \in \mathbb{A}(X_j)$, hence

$$D((x^2-4)^n z) = \hat{D}_1((x^2-4)^n z_1) \cdot \hat{D}_2(z_2) = 0. \qquad \square$$

Theorem 13. *Let X be a smooth, compact, simply connected, oriented 4-manifold with $b_2^+(X)$ odd and $\geqslant 3$. Then there exists a splitting of X as in Theorem 12, hence X has finite type.*

Proof. By the classification of indefinite forms and Donaldson's theorem we can express the intersection form of X as an orthogonal sum

$$H_2(X; \mathbb{Z})/\text{torsion} = V_1 \oplus V_2,$$

where both V_1 and V_2 contain vectors of positive square. Since X is simply connected we can invoke a theorem of Freedman and Taylor [13,24] which says that any orthogonal splitting of the intersection form of X is realized by some splitting of X along an embedded homology sphere Y:

$$X = X_1 \cup_Y X_2. \qquad \square$$

8. The *h*-invariant

In this section we consider Floer groups with rational coefficients, unless otherwise stated. We define the h-invariant and establish two basic properties: additivity under connected sums, and monotonicity with respect to negative definite cobordisms.

Definition 3. For any oriented homology 3-sphere define

$$h(Y) = \tfrac{1}{2}(\chi(\mathrm{HF}^*(Y)) - \chi(\widehat{\mathrm{HF}}^*(Y))),$$

where χ is the Euler characteristic over $\mathbb{Q}$.

We will see in a moment that $h(Y)$ is always an integer. Of course, by Taubes' theorem [25],

$$\chi(\mathrm{HF}^*(Y)) = -2\lambda(Y),$$

where λ is Casson's invariant.

Notice that we can identify $\delta_n(Y)$ with $\delta'_n(\bar{Y})$ under the canonical isomorphisms

$$\mathrm{HF}^{5-q}(\bar{Y}) = \mathrm{HF}_q(Y) = (\mathrm{HF}^q(Y))^*.$$

Now let $B_* \subset \mathrm{HF}_*(Y)$ be the linear span of the classes δ_n, $n \geqslant 0$. We usually think of HF_q as the dual space of HF^q. Since $\delta_{2n+1} = \delta_{2n}u$, and $u\colon \mathrm{HF}^0(Y) \to \mathrm{HF}^4(Y)$ is an isomorphism by Theorem 11, it follows that $\dim B_0 = \dim B_4$ and

$$h(Y) = \dim B_4(Y) - \dim B_4(\bar{Y}).$$

As observed in Section 3, either $B_4(Y) = 0$ or $B_4(\bar{Y}) = 0$.

Proposition 4. *If n is a non-negative integer then $h(Y) > n$ if and only if there exists an $x \in \mathrm{HF}^4(Y)$ such that $\delta u^{2j}x = 0$ for $0 \leqslant j < n$ but $\delta u^{2n}x \neq 0$.*

Proof. It follows from Theorem 4 that if δ_{2k} lies in the linear span of $\{\delta_{2j}\}_{0 \leqslant j < k}$ then so does δ_{2k+2}. Therefore, $h(Y) > n$ if and only if $\{\delta_{2j}\}_{0 \leqslant j \leqslant n}$ are linearly independent. The proposition now follows because

$$\delta_{2n} = \delta u^{2n} \quad \text{on} \bigcap_{j=0}^{n-1} \ker(\delta_{2j}). \qquad \square$$

Theorem 14. $h(Y_1 \# Y_2) = h(Y_1) + h(Y_2)$.

We begin the proof of the theorem with five lemmas.

Lemma 6. *If $h(Y_i) > 0$ for $i = 1, 2$ then $h(Y_1 \# Y_2) \geqslant h(Y_1) + h(Y_2)$.*

Proof. Let W be the standard homology cobordism from $\bar{Y}_1 \cup \bar{Y}_2$ to $Y_1 \# Y_2$, and let

$$W^*\colon \mathrm{HF}^p(Y_1) \otimes \mathrm{HF}^q(Y_2) \to \mathrm{HF}^{p+q}(Y_1 \# Y_2)$$

be the homomorphism defined by W. We also consider the homomorphism

$$W_u^*\colon \mathrm{HF}^p(Y_1) \otimes \mathrm{HF}^q(Y_2) \to \mathrm{HF}^{p+q-4}(Y_1 \# Y_2)$$

defined by cutting down moduli spaces over W by four times the μ-class of a point. Then the proof of Theorem 4 can be adapted to show that

$$\delta W_u^*(a_1 \otimes a_2) = \pm 2(\delta a_1)(\delta a_2).$$

On the other hand, moving the base point along a path $[0, \infty) \times \{x_0\}$ in $\mathbb{R}_+ \times (Y_1 \# Y_2)$ one finds that

$$W_u^*(a_1 \otimes a_2) = uW^*(a_1 \otimes a_2).$$

Furthermore, one has

$$W^*(ua_1 \otimes a_2) = uW^*(a_1 \otimes a_2)$$

whenever $\delta a_1 = 0$, and similarly with the roles of Y_1 and Y_2 interchanged. Now set $k_i = h(Y_i)$. By Proposition 4 and Theorem 11 there is an element $a_i \in \mathrm{HF}^0(Y_i)$ such that

$$\delta u^{2r-1} a_i = \begin{cases} 0 & \text{if } 1 \leqslant r < k_i, \\ \neq 0 & \text{if } r = k_i. \end{cases}$$

So if $1 \leqslant r_i \leqslant k_i$ one has

$$\begin{aligned}\delta u^{2(r_1+r_2)-1} W^*(a_1 \otimes a_2) &= \delta u W^*(u^{2r_1-1} a_1 \otimes u^{2r_2-1} a_2)\\ &= \pm 2(\delta u^{2r_1-1} a_1)(\delta u^{2r_2-1} a_2),\end{aligned}$$

and the lemma follows. □

Lemma 7. *Let W be a smooth, compact, oriented 4-manifold with boundary $\partial W = \bar{Y}_1 \cup Y_2$, where both Y_i are homology spheres. If the intersection form of W is negative definite and $H_1(W;\mathbb{Z}) = 0$ then*

$$h(Y_2) \geqslant h(Y_1).$$

Proof. This follows from Theorem 7. □

Lemma 8. *If $h(Y_1) = h(Y_2) > 0$ then $h(Y_1 \# \bar{Y}_2) = 0$.*

Proof. If $h(Y_1 \# \bar{Y}_2) > 0$ then by the homology cobordism invariance of h one would have

$$h(Y_1) \geqslant h(Y_2) + h(Y_1 \# \bar{Y}_2) > h(Y_2).$$

A similar argument applies if $h(Y_1 \# \bar{Y}_2) < 0$, since $h(\bar{Y}) = -h(Y)$. □

Lemma 9. *Let Y be an oriented homology 3-sphere such that $h(Y) \geqslant 0$, and let Y' be the result of -1 surgery on a knot in Y of genus 1. Then*

$$0 \leqslant h(Y') - h(Y) \leqslant 1.$$

Once we have established additivity of h it will be clear that the lemma holds without the assumption $h(Y) \geqslant 0$.

Proof. Since the surgery cobordism W from Y to Y' is negative definite and satisfies $H_1(W;\mathbb{Z}) = 0$, we have $h(Y) \leqslant h(Y')$. Now set $n = h(Y)$. To prove $h(Y') \leqslant n+1$ we use the exact sequence (5). Suppose $x \in \mathrm{HF}^4(Y')$ satisfies $\delta u^{2j} x = 0$ for $0 \leqslant j \leqslant n$. As in the proof of Lemma 5 we have

$$\alpha'((u^2 - 64)x) = (u^2 - 64)\alpha' x = 0,$$

so $(u^2 - 64)x = \alpha y$ for some $y \in \mathrm{HF}^4(Y)$. Since $\alpha u = u\alpha$ on $\ker \delta_0$ by Theorem 6 we find that for $0 \leqslant j \leqslant n$,

$$\delta u^{2j} y = \delta \alpha u^{2j} y = \delta u^{2j}(u^2 - 64)x = \delta u^{2(j+1)} x.$$

Therefore, $0 = \delta u^{2n} y = \delta u^{2(n+1)} x$. It follows that $h(Y') \leqslant n+1$. □

Let nY denote the n-fold connected sum $\#_n Y$ for $n \geqslant 0$ (if $n=0$ we agree that $nY=S^3$), and set $(-n)Y=n\bar{Y}$. Let S be the Brieskorn sphere $\Sigma(2,3,5)$.

Lemma 10. *For any integer n we have $h(nS)=n$.*

Proof. We may assume $n>0$. By Proposition 2 (i) we have $h(S)=1$. Hence $h(nS) \geqslant n$ by Lemma 6. But S is also the result of -1 surgery on the negative $(2,3)$ torus knot, which has genus 1, so n applications of Lemma 9 gives $h(nS) \leqslant n$. Thus $h(nS)=n$. □

Proof of Theorem 14. Let W be a smooth, compact, oriented, connected 4-manifold with boundary components $\bar{Z}_1, Z_2, \bar{V}_1, \ldots, \bar{V}_r$, where each component is a homology sphere. Suppose $H_j(W;\mathbb{Z})=0$ for $j=1,2$ and $h(V_i)=0$ for each i. We will show that $h(Z_1)=h(Z_2)$. If $h(Z_1)$ and $h(Z_2)$ are both zero then there is nothing to prove, so after perhaps reversing orientations we may assume $h(Z_2) \leqslant h(Z_1) > 0$.

Since $h(V_i)=0$ we can find $\rho_i \in CF^0(V_i)$ such that $d\rho_i = \delta'$. Let $\hat{W} = W \cup (\mathbb{R}_+ \times \partial W)$ have a tubular end metric. Then zero-dimensional moduli spaces over $\hat{W}$ with the chain $\theta + \rho_i$ as "flat limit" over the end $\mathbb{R}_- \times V_i$ define a degree preserving homomorphism

$$f : \mathrm{HF}^*(Z_1) \to \mathrm{HF}^*(Z_2),$$

which satisfies $fu = uf$ on $\ker \delta_0$ and $\delta_0 f = \delta_0$. This implies $h(Z_2) \geqslant h(Z_1)$, so $h(Z_1)=h(Z_2)$.

To prove the theorem, set $k_i = h(Y_i)$, $k = k_1 + k_2$, and let W have boundary components $Y_1 \# Y_2$, $k\bar{S}$, $\bar{Y}_1 \# k_1 S$, and $\bar{Y}_2 \# k_2 S$. □

Theorem 15. *Let W be a smooth, compact, oriented 4-manifold with boundary $\partial W = \bar{Y}_1 \cup Y_2$, where both Y_i are homology spheres. Suppose the intersection form of W is negative definite. Then*

$$h(Y_2) \geqslant h(Y_1)$$

with strict inequality if the intersection form is not diagonal over the integers.

Proof. Let L be the intersection form of W. Then $\tilde{Y} = Y_2 \# \bar{Y}_1 \# S$ bounds a smooth, compact, oriented 4-manifold with negative definite intersection form $L \oplus (-E_8)$, which is not diagonal over the integers. Hence $h(\tilde{Y}) \geqslant 1$ by Corollary 2. Since $h(S)=1$ and h is additive, we deduce $h(Y_2) \geqslant h(Y_1)$.

(It is possible that a more direct proof of this can be found by first surgering away the free part of $H_1(W;\mathbb{Z})$ and then analysing the abelian flat SU(2) connections over W as in [7, Section 4b].)

If L is not diagonal then $h(Y_2) - h(Y_1) = h(Y_2 \# \bar{Y}_1) > 0$ because $Y_2 \# \bar{Y}_1$ bounds a smooth, compact, oriented 4-manifold with the same intersection form as W. □

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