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A Necessary Condition for a Power Series to Be a Formal Solution of a Singular Linear Differential Equation of Order k

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We obtain a necessary condition on the coefficients of a formal power series, which is a formal solution of a nontrivial singular linear differential equation of order k, with analytic coefficients and prove a "uniqueness" theorem for the differential equation.

INTRODUCTION

Given a linear differential equation of order k with analytic coefficients,

$$\sum_{j=0}^{j=k} a_j(z)\, Y^{(k-j)} = c(z), \tag{1}$$

where

$$a_j(z) = \sum_0^\infty a_{j,n} z^n, \quad j = 0, 1,\dots, k, \qquad c(z) = \sum_0^\infty c_n z^n$$

$a_j(z)$, $c(z)$ are absolutely convergent series in $|z| < r$, $r > 0$, and $a_0(z) \not\equiv 0$.

It is well-known (see 1, p. 22 with a slight modification) that if $z = 0$ is a singular regular point of (1) and $\sum_0^\infty A_n z^n$ is a formal solution, then these series are absolutely convergent for $|z| \leqslant \sigma$ for some $\sigma > 0$. This means by Cauchy inequality (2, p. 84) that

$$A_n = \mathcal{O}(\sigma^n). \tag{*}$$

The question of what conditions must A_n fulfill if $z = 0$ is an irregular singular point of (1) and $\sum_0^\infty A_n Z^n$ is its formal solution will be answered now by the main result of this paper, which is Theorem 2. It states that

$$A_n = \mathcal{O}(n!)^k\, \sigma^n \qquad \text{for some } \sigma > 0. \tag{**}$$

Define Eq. (1) for $k = 0$ to be an algebraic equation $a_0(z)\, Y = c'(z)$.

Consider now the set $\mathscr{F}$ of equations of type (1), with $k \geqslant 0$, where $\sum_0^\infty A_n z^n$ is a formal solution of every member of $\mathscr{F}$. It is easily verified that there exists a member of $\mathscr{F}$ with a *minimal* k. In view of the last statement we may consider (*) as a particular case of (**), if in (**) we always choose k to be the minimal one. There is a "uniqueness property" associated with the members of $\mathscr{F}$ with minimal k, which will be proved in Theorem 1.

THEOREM 1.

(i) *Let* $Y = \sum_0^\infty A_n z^n$ *be a formal solution of two nontrivial differential equations of type* (1):

$$\sum_{j=0}^{j=k} a_j(z)\, Y^{(k-j)} = c(z) \tag{2}$$

$$\sum_{j=0}^{j=k} \bar{a}_j(z)\, Y^{(k-j)} = \bar{c}(z), \tag{3}$$

where k is minimal, then

$$\frac{\bar{a}_j(z)}{a_j(z)} = f(z) \qquad j = 0, 1, 2, \ldots, n,$$

where $f(z)$ is a meromorphic function in $|\, z \,| < r$.

Proof. Assume $Y = \sum_0^\infty A_n z^n$ to satisfy Eqs. (2) and (3) where $a_0(z) \cdot \bar{a}_0(z) \not\equiv 0$. Multiply (2) by $\bar{a}_0(z)$ and (3) by $a_0(z)$, and subtract the multiplied equations to obtain

$$\sum_{j=1}^{j=k} [a_j(z)\, \bar{a}_0(z) - \bar{a}_j(z)\, a_0(z)]\, Y^{(k-j)} = c(z)\, \bar{a}_0(z) - \bar{c}(z)\, a_0(z). \tag{4}$$

Then Y satisfies (4), which is of order $k - 1$ at most; this implies

$$a_j(z)\, \bar{a}_0(z) - a_j(z)\, a_0(z) = 0, \qquad j = 1, \ldots, n,$$

and the result follows.

THEOREM 2. *Let* $Y = \sum_0^\infty A_n z^n$ *be a formal solution of* (1). *Then* $A_n = \mathcal{O}[(n!)^k \sigma^n]$, *and the result is "sharp."*

Proof. We break the proof into a few lemmas.

LEMMA 1. *The identity* (1) *can be written as a system of infinite equations of type*

$$\sum_{j=0}^{j=n+j_0} \beta_{n,j} A_j = c_n\,, \tag{5}$$

so that

(I) $\exists_{j_0}$, $\exists_{n_0}$, *such that for* $n > n_0$, $\beta_{n,n+j_0} \neq 0$, *but* $\forall n$ *and* $k \geqslant j > j_0$, $\beta_{n,n+j} = 0$.

(II) *Moreover*, $\exists_{j_1}$, $\exists_{n_1}$, *such that for* $n > n_1$

$$\frac{1}{|\beta_{n,n+j_0}|} < \frac{1}{(n+j_0)\cdots(n+2-j_1)} < \frac{1}{2}. \tag{6}$$

Proof. By observing that (1) is composed of $(k+1)$ terms on the left side, we collect from each term the coefficient of z^n and equate it to c_n. We write this explicitly:

$$\begin{aligned}
&\sum_{j=0}^{j=n} a_{k,j}A_{n-j}\\
&+\sum_{j=0}^{j=n} a_{k-1,j}(n+1-j)\,A_{n+1-j}\\
&\quad\vdots\\
&+\sum_{j=0}^{j=n} a_{k-\nu,j}(n+\nu-j)\,(n+\nu-j-1)\cdots(n+1-j)\,A_{n+\nu-j}\\
&\quad\vdots\\
&+\sum_{j=0}^{j=n} a_{0,j}(n+k-j)\,(n+k-j-1)\cdots(n+1-j)\,A_{n+k-j} = c_n.
\end{aligned} \tag{7}$$

Assume without loss of generality that (i) is normalized such that $\sum_{j=0}^{j=k} |a_{j,0}| > 0$. Consider the quantities α_p defined by

$$\alpha_p = \sum_{j=0}^{j=p} |a_{p-j,j}|. \tag{8}$$

By (1) we have that there exists a minimal index p_0 such that $\alpha_{p_0} \neq 0$ and $p_0 \leqslant k$. From (7) we have that it can be written in the form (5) by defining $\beta_{n,j}$ to be the coefficient of A_j in (7).

By our definition $\beta_{n,n+j} = 0$ for all $k \geqslant j > k - p_0$. Note that if $a_{0,0} \neq 0$, there is no j, $k \geqslant j > k - p_0$, such that $\beta_{n,n+j} = 0$, since $\alpha_0 = |\alpha_0|$. We define $k - p_0 = j_0$. To complete the first part of the lemma, we have to show the existence of n_0 so that $\beta_{n,n+j_0} \neq 0$ for $n > n_0$. By definition,

$$\begin{aligned}
\beta_{n,n+j_0} = {} & a_{k-j_0,0}(n+j_0)\,(n+j_0-1)\cdots(n+1)+\cdots\\
&+ a_{k-j_0-\nu,\nu}(n+j_0)\,(n+j_0-1)\cdots(n+1)\cdots(n+1-\nu)\\
&+ a_{0,k-j_0}(n+j_0)\,(n+j_0-1)\cdots(n+1)\cdots(n+1-\nu)\cdots\\
&\times(n+1-k+j_0).
\end{aligned} \tag{9}$$

We observe that $p_0 = k - j_0 \geqslant \nu \geqslant 0$. $\beta_{n,n+j_0}$ is composed of $(p_0 + 1)$ terms. Every term is a product of $(j_0 + \nu)$ factors, where $(j_0 + \nu - 1)$ of them are linear functions of n.

Since p_0 is minimal and $\sum_{j=0}^{j=p_0} | a_{p_0-j,j} | > 0$, there exists a maximal j that we will denote by j_1 such that $| a_{p_0-j_1,j_1} | > 0$, where as $| a_{p_0-j,j} | = 0$ for $p_0 \geqslant j > j_1$. Assume now (on the contrary) that such n_0 does not exist; then there exists an infinite sequence $\{n_m\}$, $n_m \to \infty$ for $m \to \infty$ such that

$$\beta_{n_m,n_m+j_0} = 0.$$

Divide both sides of (9) by

$$(n_m + j_0)(n_m + j_0 - 1) \cdots (n_m + 1 - j_1), \tag{10}$$

which is the exact coefficient of $a_{p_0-j_1,j_1}$ in (9), and let $n_m \to \infty$ and get by reconsidering the previous remarks that $a_{p_0-j_1,j_1} = 0$, which is a contradiction. To prove the second statement of the lemma denote

$$K = \max_{0 \leqslant j \leqslant j_1} | a_{p_0-j,j} | .$$

By applying the triangle inequality to (9), we obtain

$$| \beta_{n,n+j_0} | \geqslant | a_{p_0-j_1,j_1} | (n + j_0) \cdots (n + 1 - j_1) - j_1 K(n + j_0) \cdots (n + 2 - j_1)$$

$$\geqslant (n + j_0) \cdots (n + 2 - j_1) [| a_{p_0-j_1,j_1} | (n + 1 - j_0) - j_1 K]. \tag{11}$$

From this it is easily seen that $\exists n_1$ such that for $n > n_1$

$$| a_{p_0-j_1,j_1} | [(n + 1 - j_0) - j_1 K] > 2.$$

LEMMA 2. *Let M be a positive constant such that*

$$| a_{j,n} | \leqslant M, \qquad | c_n | \leqslant M \qquad \forall n, \forall_j; \tag{12}$$

then there exists $\sigma > 0$ and $N > 0$ such that for every m

$$| A_m | \leqslant N \cdot (m!)^k \sigma^m. \tag{13}$$

Proof. Consider $A_0, A_1 \cdots A_{n_1+j_0}$, where n_1 was the number found in the previous lemma.

Define $N = \max_j\{| A_j |, 1\}$, $0 \leqslant j \leqslant n_1 + j_0$. Choose $\sigma > 1$ such that $M(k + 1)/(\sigma - 1) < 1$.

For $m \leqslant n_1 + j_0$ (13) is obviously true. Assume the statement (13) to be true for $m > n_1 + j_0$ and proceed to prove it by induction for $m + 1$. By Lemma 1

$$\sum_{j=0}^{j=m+1} \beta_{m+1-j_0,j} A_j = c_{m+1-j_0} .$$

We transfer terms in this formula and apply the triangle inequality to obtain

$$| \beta_{m+1-j_0,m+1} | \, | A_{m+1} | \leqslant | c_{m+1-j_0} | + \sum_{j=0}^{j=m} | \beta_{m+1-j_0,j} | \, | A_j | . \tag{14}$$

Examining (5) and (7), We conclude that $\beta_{n,j}$ is composed of $(k + 1)$ terms *at most*. Taking (12) into consideration also, we obtain

$$| \beta_{n,j} | \leqslant (k + 1) M \cdot j^k. \tag{15}$$

Now insert the induction hypothesis into (14), and obtain after dividing by $| \beta_{m+1+j_0,m+1} |$ and using (6) and (15)

$$\begin{aligned} | A_{m+1} | &\leqslant M \cdot \tfrac{1}{2} + N \cdot \tfrac{1}{2} \sum_{j=0}^{j=m} (k + 1) M \cdot j^k \cdot (j!)^k \sigma^j \\ &\leqslant M \cdot \frac{1}{2} + \frac{M}{2} \cdot N(k + 1) \cdot m^k \cdot (m!)^k \frac{\sigma^{m+1} - 1}{\sigma - 1} . \end{aligned} \tag{16}$$

Without loss of generality, assume also $M < N[(m + 1)!]^k \sigma^{m+1}$, and use $\sigma > 1$, $M(k + 1)/(\sigma - 1) < 1$; then

$$| A_{m+1} | < \frac{N[(m + 1)!]^k \sigma^{m+1}}{2} + \frac{N \cdot m^k \cdot (m!)^k \sigma^{m+1}}{2} < N[(m + 1)!]^k \sigma^{m+1}.$$

Q.E.D.

LEMMA 3. *Let $\sum_0^\infty A_n z^n$ be a formal solution of* (1); *then there exists a series $\sum_0^\infty B_n \xi^n \equiv \sum_0^\infty A_n \rho^n \xi^n$ with $\rho > 0$, which is a formal solution of*

$$\sum_{j=0}^{j=k} b_j(\xi) \left(\sum_0^\infty B_n \xi^n \right)^{(k-j)} = d(\xi), \tag{17}$$

where $b_j(\xi)$, $j = 0,..., n$, $d(\xi)$ are absolutely convergent series in a circle $| \xi | < r$, $r > 0$, and $| b_{jn} | \leqslant M$, $| d_n | \leqslant M$ for some M, $M > 0$.

Proof. Since $a_j(z)$, $c(z)$ converge in $| z | \leqslant \rho$, $\rho > 0$, we have by Cauchy theorem $\exists M$, $M > 0$, such that

$$| a_{jn} | \leqslant \frac{M}{\rho^n}, \qquad | c_n | \leqslant \frac{M}{\rho^n} . \tag{18}$$

It is easily verified that the series $B = \sum_0^\infty B_n \xi^n$, where $B_n = A_n \rho^n$ is a formal solution of the equation

$$\sum_{j=0}^{j=k} \rho^j a_j(\rho\xi)\, B^{(k-j)} = c(\rho\xi)\, \rho^k. \tag{19}$$

Define $b_j(\xi) = \rho^j a_j(\rho\xi)$, $0 \leqslant j \leqslant k$ and $d(\xi) = \rho^k c(\rho\xi)$. We obtain by this

$$b_{jn} = a_{jn}\rho^n \rho^j \qquad \text{and} \qquad d_n = c_n \rho^k \rho^n. \tag{20}$$

Define $M_1 = M \max_j \rho^j$, $0 \leqslant j \leqslant k$. From (18) it is easily obtained that $| b_{jn} | \leqslant M_1$, $| c_n | \leqslant M_1$ for every j, $0 \leqslant j \leqslant k$. Applying Lemma 2 to series B, we obtain $A_n \rho^n = \mathcal{O}[(n!)^k \sigma^n]$, and the result follows.

To point out how "sharp" is the theorem, we define the following operators:

$$\phi_1 A = [zA]', \qquad \phi_{k+1} A = [z\phi_k A]'. \tag{21}$$

By (21) it follows that

$$[(n+1)!]^k z^{n+1} = z\phi_k[(n!)^k z^n]. \tag{22}$$

Since the operator is linear, we obtain

$$z\phi_k \left(\sum_0^\infty (-1)^n (n!)^k z^n \right) = (-1) \sum_0^\infty (-1)^{n+1} [(n+1)!^k]\, z^{n+1}. \tag{23}$$

And if we define

$$E_{k,1} = \sum_0^\infty (-1)^n [n!]^k z^n,$$

it follows from (23) that

$$z\phi_k E_{k,1} + E_{k,1} = 1. \tag{24}$$

Eq. (24) is readily observed to be a linear differential equation of order k. This means (by Theorem 2) also that k is the *lowest order* possible for a linear differential equation, having $E_{k,1}$ as a formal solution.

COROLLARY 1. *There exist formal series* $\sum_0^\infty A_n z^n$, *which cannot be formal solutions of* (1) *of any finite order.*

Proof. For example, choose $A_n = (n!)^n$, which never can be $(n!)^n = \mathcal{O}[(n!)^k \sigma^n]$.

COROLLARY 2. *Let*

$$z^m Y' + P(z)\, Y = c(z) \tag{25}$$

be a vectorial system, where

(i) *m is an integer,*

(ii) *Y is a column vector of n functions,*

(iii) *$P(z)$ is a matrix of $k \times k$ holomorphic functions in a circle $|z| < r$, $r > 0$, and*

(iv) *$c(z)$ is a column vector of n holomorfic functions in $|z| < r$. Then if $Y = \sum_0^\infty A_n z^n$ is a formal solution of the aforementioned system, where A_n are k-dimensional column vectors, we have in the supremum norm*

$$\| A_n \| = \mathcal{O}[(n!)^k \sigma^n].$$

Proof. Differentiate (25) $(k - 1)$ times. By elimination and substitution, it is easily verified that every one of the components of

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix}$$

satisfies an equation of type (1) of order *k at most*. Apply to every Y_i Theorem 2, and the result follows.

COROLLARY 3. *Let $f(z)$ be a solution of* (1) *in some sector having vertex at the origin. Assume $f(z)$ to have asymptotic expansion of all orders in powers of z; then if $f(z) \sim \sum_0^\infty A_n z^n$, we have an estimation of A_n without computation!*

Remark 1. At the expense of complicating the proof of Theorem 2, we could derive more delicate bounds for N and σ appearing in (13).

Remark 2. We conjecture that if $Y = \sum_0 A_n z^n$ is a formal solution of $z^m Y' = F(z, Y)$, where Y is a k-dimensional vector and $F(z, Y)$ is a k-dimensional vector function, analytic in $|z| < r$, $\|Y\| < \rho$, $m > 0$, then

$$\| A_n \| = \mathcal{O}[(n!)^k \sigma^n] \qquad \text{for some } \sigma > 0.$$

REFERENCES

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