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A new approach to nonlinear L^2-stability of double diffusive convection in porous media: Necessary and sufficient conditions for global stability via a linearization principle

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Abstract

A new approach to nonlinear L^2-stability for double diffusive convection in porous media is given. An auxiliary system Σ of PDEs and two functionals V, W are introduced. Denoting by L and N the linear and nonlinear operators involved in Σ, it is shown that Σ-solutions are linearly linked to the dynamic perturbations, and that V and W depend directly on L-eigenvalues, while (along Σ) $\frac{dV}{dt}$ and $\frac{dW}{dt}$ not only depend directly on L-eigenvalues but also are independent of N. The nonlinear L^2-stability (instability) of the rest state is reduced to the stability (instability) of the zero solution of a linear system of ODEs. Necessary and sufficient conditions for general, global L^2-stability (i.e. absence of regions of subcritical instabilities for any Rayleigh number) are obtained, and these are extended to cover the presence of a uniform rotation about the vertical axis.

Keywords: Porous media; Global nonlinear stability; Peculiar Lyapunov functionals; Linearization principle

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1. Introduction

The equations governing the motion of a binary fluid mixture bounded by the horizontal planes $z = 0$, $z = d > 0$, in the Darcy–Oberbeck–Boussinesq scheme, are [1–4]

$$\begin{cases} \nabla p = -\dfrac{\mu}{k}\mathbf{v} + \rho_f \mathbf{g}, \\ \nabla \cdot \mathbf{v} = 0, \\ \tilde{A} T_{,t} + \mathbf{v} \cdot \nabla T = k_T \Delta T, \\ \varepsilon C_{,t} + \mathbf{v} \cdot \nabla C = k_C \Delta C, \end{cases} \tag{1}$$

where

$$\rho_f = \rho_0\big[1 - \gamma_T(T - T_0) + \gamma_C(C - C_0)\big] \tag{2}$$

and where the following notation is used:

$\gamma_T =$ thermal expansion coefficient,
$\gamma_C =$ solute expansion coefficient,
$\varepsilon =$ porosity,
$\mathbf{v} =$ seepage velocity field,
$C =$ concentration field,
$p =$ pressure field,
$T =$ temperature field,
$\mu =$ viscosity,
$T_0 =$ reference temperature,
$C_0 =$ reference concentration,
$k_T =$ thermal diffusivity,
$k_C =$ solute diffusivity,
$c =$ specific heat of the solid,
$\tilde{A} = \dfrac{(\rho_0 c)_m}{(\rho_0 c_p)_f}$,
$\rho_0 =$ fluid density at T_0, C_0.
$k =$ permeability coefficient,
$c_p =$ specific heat of fluid at constant pressure.

The subscripts m and f refer to the porous medium and the fluid, respectively.

To (1) we append the boundary conditions

$$\begin{cases} T_L = T_0 + \dfrac{1}{2}(T_1 - T_2), \qquad C_L = C_0 + \dfrac{1}{2}(C_1 - C_2) \quad \text{on } z = 0, \\ T_U = T_0 - \dfrac{1}{2}(T_1 - T_2), \qquad C_U = C_0 - \dfrac{1}{2}(C_1 - C_2) \quad \text{on } z = d, \end{cases} \tag{3}$$

with $T_1 > T_2$ and $C_1 > C_2$. By introducing the scaling

$$\mathbf{x} = d\mathbf{x}^*, \qquad t = \frac{\tilde{A} d^2}{k_T} t^*, \qquad \mathbf{v} = \frac{k_T}{d}\mathbf{v}^*,$$
$$p^* = \frac{k(p + \rho_0 g z)}{\mu k_T}, \qquad T^* = \frac{T - T_0}{T_1 - T_2}, \qquad C^* = \frac{C - C_0}{C_1 - C_2}$$

the dimensionless versions of (1) and (3)—omitting the stars—are respectively

$$\begin{cases} \nabla p = -\mathbf{v} + (RT - \mathcal{C}C)\mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ T_{,t} + \mathbf{v} \cdot \nabla T = \Delta T, \\ \epsilon\, Le\, C_{,t} + Le\, \mathbf{v} \cdot \nabla C = \Delta C, \end{cases} \tag{4}$$

$$\begin{cases} T_L = \dfrac{1}{2}, \qquad C_L = \dfrac{1}{2} \qquad \text{on } z = 0, \\ T_U = -\dfrac{1}{2}, \qquad C_U = -\dfrac{1}{2} \quad \text{on } z = 1, \end{cases} \tag{5}$$

with

$$\begin{cases} R = \dfrac{\gamma_T g(T_1 - T_2)kd}{\nu k_T} & \text{(thermal Rayleigh number)},\\ \mathcal{C} = \dfrac{\gamma_C g(C_1 - C_2)kd}{\nu k_T} & \text{(solutal Rayleigh number)},\\ Le = \dfrac{k_T}{k_C} & \text{(Levis number)},\\ \nu = \dfrac{\mu}{\rho_0} & \text{(kinematic viscosity)},\\ \epsilon = \dfrac{\varepsilon}{\tilde{A}} & \text{(normalized porosity)}. \end{cases}$$

Equations (4)–(5) admit the steady solution (motionless state)

$$\begin{cases} \mathbf{v}_S = 0, \qquad \nabla p_s(z) = -(R + \mathcal{C})\left(z - \dfrac{1}{2}\right)\mathbf{k},\\ T_s(z) = -\left(z - \dfrac{1}{2}\right), \qquad C_s(z) = -\left(z - \dfrac{1}{2}\right). \end{cases} \tag{6}$$

The stability of (6) has been considered by several authors (also when rotation about the vertical axis is incorporated) [4–18]. Precisely, denoting by

- $\mathbf{u} = (u, v, w)$ the velocity perturbation field,
- θ the temperature perturbation field,
- Γ the concentration perturbation field,
- $R_C^{(L)}$ the critical Rayleigh number of linear stability,
- $R_C^{(E)}$ the critical Rayleigh number of nonlinear energy stability

and assuming that the perturbations $(u, v, w, \theta, \Gamma)$

(i) are periodic in the x and y directions respectively of periods $\frac{2\pi}{a_x}$ and $\frac{2\pi}{a_y}$,
(ii) on the periodicity cell $\Omega = [0, \frac{2\pi}{a_x}] \times [0, \frac{2\pi}{a_y}] \times [0, 1]$ (in order to guarantee uniqueness) u and v have zero mean value,
(iii) belong to $L^2(\Omega)$, $\forall t \in \mathbb{R}^+$,

the results on nonlinear energy stability [4–18], as far as we know, can be summarized as follows: *there exists a bounded positive number $R^* \ll \infty$ such that*

$$R \leqslant R^* \;\Rightarrow\; \begin{cases} (1)\ R_C^{(E)} = R_C^{(L)} \text{ (i.e. absence of regions of subcritical instabilities for } R \leqslant R_C^{(E)}\text{)},\\ (2)\ R \leqslant R_C^{(E)} \text{ implies global nonlinear } L^2\text{-stability}, \end{cases}$$

$$R > R^* \;\Rightarrow\; \begin{cases} (1)\ R_C^{(E)} < R_C^{(L)} \text{ (i.e. existence of potential regions of subcritical instabilities)},\\ (2)\ R < R_C^{(E)} \text{ implies } \textit{local} \text{ nonlinear } L^2\text{-stability}. \end{cases}$$

In the present paper, I reconsider the problem with the aim of showing that *in the case at hand*

(I) $R^* = \infty$ (i.e. *absence of regions of subcritical instabilities* $\forall R > 0$),
(II) the nonlinear L^2-stability is always *global*.

Denoting by $L_2^*(\Omega)$ the class of perturbations $(\mathbf{u}, \theta, \Gamma)$ satisfying (i)–(iii) and such that all their first derivatives and second spatial derivatives can be expanded in a Fourier series absolutely and uniformly convergent in Ω, the aim is to derive the following (*main*) *theorem*:

Theorem 1. *Motion* (6) *is globally asymptotically exponentially L^2-stable if and only if the zero solution of the linear binary system of ODEs*

$$\begin{cases} \dfrac{d\xi}{dt} = a_1\xi + a_2\eta, \\ \dfrac{d\eta}{dt} = a_3\xi + a_4\eta \end{cases} \tag{7}$$

with

$$\begin{cases} a_1 = \dfrac{a^2 R}{a^2+\pi^2} - \left(a^2+\pi^2\right), \qquad a_2 = -\dfrac{a^2}{a^2+\pi^2} Le\, C, \\ a_3 = \dfrac{a^2 R}{a^2+\pi^2} \varepsilon\, Le, \qquad a_4 = -\dfrac{1}{\varepsilon}\left(\dfrac{a^2 C}{a^2+\pi^2} + \dfrac{a^2+\pi^2}{Le}\right) \end{cases} \tag{8}$$

is stable for any value of the positive parameter a^2.

As far as we know, this is the first time that for double diffusive convection, coincidence of linear stability and *global* nonlinear L^2-stability of the rest state is established $\forall R$. In fact, for $\{R_B = 4\pi^2,\ C^* = \frac{R_B}{\varepsilon\, Le(\varepsilon Le - 1)}\}$, the L^2-global stability conditions implied by (8) are

$$\begin{cases} \epsilon\, Le \leqslant 1, \\ R < R_B + Le\,\mathcal{C}; \end{cases} \qquad \begin{cases} \epsilon\, Le \geqslant 1, \qquad \mathcal{C} \geqslant \mathcal{C}^*, \\ R < \dfrac{\mathcal{C}}{\epsilon} + \left(1 + \dfrac{1}{\epsilon\, Le}\right) R_B \end{cases} \tag{9}$$

and coincide with those of linear stability $\forall R$. *Therefore—in the case at hand—the absence of regions of subcritical instabilities is established.*

The plan of the paper is as follows. In Section 2—dedicated to preliminaries—it is shown that $\mathbf{u}$ is linearly linked to (θ, Γ). In Section 3 an auxiliary system Σ of PDEs and a quadratic functional V (different from the $L^2(\Omega)$-norm of (θ, Γ)) are introduced. Denoting by L and N, respectively, the linear and nonlinear operators involved in Σ, it is shown that

(a) the solutions of Σ are linearly linked to the perturbations $(\mathbf{u}, \theta, \Gamma)$ (Theorem 2);
(b) V depends in a simple direct way on the L eigenvalues;
(c) along Σ, $\frac{dV}{dt}$ not only depends in a simple direct way on the L eigenvalues, but also does not depend on N (Theorem 3).

Section 4 is devoted to the global nonlinear stability. By virtue of (c) conditions sufficient for guaranteeing global nonlinear L^2-stability are found (Theorems 4–5). Instability is considered in Section 5. By the introduction of a functional W having the properties (b)–(c) it is shown that the conditions found in Section 4 are necessary for the stability of the rest state (Theorems 6–8). Sections 6–7 are devoted to proof of the main theorem and its generalization to a rotating layer. The paper ends with an Appendices A.1–A.3 in which—for the sake of completeness—some results, used in the paper, are discussed.

2. Preliminaries

By virtue of (4)–(5), the equations governing the perturbations $(\mathbf{u}, \theta, \varGamma)$ are

$$\begin{cases} \nabla\tilde{\pi} = -\mathbf{u} + (R\theta - \mathcal{C}\varGamma)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \\ \dfrac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \Delta\theta + w, \\ \epsilon\, Le\,\dfrac{\partial\varGamma}{\partial t} + Le\,\mathbf{u}\cdot\nabla\varGamma = \Delta\varGamma + w \end{cases} \tag{10}$$

under the boundary conditions

$$w = \theta = \varGamma = 0 \quad \text{on } z = 0, 1, \tag{11}$$

$\tilde{\pi}$ being the perturbation to the pressure field.

Since the sequence $\{\sin n\pi z\}$ $(n = 1, 2, \dots)$ is a complete orthogonal system for $L_2([0, 1])$, by virtue of the periodicity, it turns out that for $\mathcal{L} \in \{w, \theta, \varGamma\}$ there exists a sequence $\{\tilde{\mathcal{L}}_n(x, y, t)\}$ such that

$$\mathcal{L} = \sum_{n=1}^{\infty} \tilde{\mathcal{L}}_n(x, y, t)\sin n\pi z, \quad \forall t \geqslant 0, \tag{12}$$

with

$$\Delta_1\tilde{\mathcal{L}}_n = -a^2\tilde{\mathcal{L}}_n, \quad \Delta_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},\ a^2 = a_x^2 + a_y^2. \tag{13}$$

Setting

$$\zeta = (\nabla\times\mathbf{u})\cdot\mathbf{k}, \tag{14}$$

in view of $(10)_2$, one obtains

$$\begin{cases} \Delta_1 u = -\dfrac{\partial^2 w}{\partial x\partial z} - \dfrac{\partial\zeta}{\partial y}, \\ \Delta_1 v = -\dfrac{\partial^2 w}{\partial y\partial z} + \dfrac{\partial\zeta}{\partial y}. \end{cases} \tag{15}$$

On the other hand $(10)_1$ implies $\zeta = 0$, hence

$$\Delta_1 u = -\frac{\partial^2 w}{\partial x\partial z}, \qquad \Delta_1 v = -\frac{\partial^2 w}{\partial y\partial z},$$

and therefore one obtains

$$\begin{cases} u = \displaystyle\sum_{n=1}^{\infty}\tilde{u}_n(x, y, t)\frac{d}{dz}(\sin n\pi z), \\ v = \displaystyle\sum_{n=1}^{\infty}\tilde{v}_n(x, y, t)\frac{d}{dz}(\sin n\pi z), \\ \Delta_1\tilde{u}_n = -a^2\tilde{u}_n, \qquad \Delta_1\tilde{v}_n = -a^2\tilde{v}_n, \\ \tilde{u}_n = \dfrac{1}{a^2}\dfrac{\partial\tilde{w}_n}{\partial x}, \qquad \tilde{v}_n = \dfrac{1}{a^2}\dfrac{\partial\tilde{w}_n}{\partial y}. \end{cases} \tag{16}$$

Then w, θ, Γ are the effective perturbation fields. These fields are not independent, however. In fact, from $(10)_1$ it follows that

$$\nabla \times (\nabla \times \mathbf{u}) \cdot \mathbf{k} = \nabla \times \left[\nabla \times (R\theta - Le\,\mathcal{C}\Gamma)\mathbf{k}\right] \cdot \mathbf{k}$$

i.e.

$$\Delta w = \Delta_1 (R\theta - Le\,\mathcal{C}\Gamma),$$

and (10) becomes

$$\begin{cases} \Delta w = \Delta_1 (R\theta - Le\,\mathcal{C}\Gamma), \\ \nabla \cdot \mathbf{u} = 0, \\ \dfrac{\partial \theta}{\partial t} = \Delta\theta + w - \mathbf{u} \cdot \nabla\theta, \\ \dfrac{\partial \Gamma}{\partial t} = \dfrac{1}{\epsilon\, Le}(\Delta\Gamma + w) - \dfrac{1}{\epsilon}\mathbf{u} \cdot \nabla\Gamma. \end{cases} \tag{17}$$

According to (12)

$$\begin{cases} w_n = \tilde{w}_n(x, y, t) \sin n\pi z, \\ \theta_n = \tilde{\theta}_n(x, y, t) \sin n\pi z, \\ \Gamma_n = \tilde{\Gamma}_n(x, y, t) \sin n\pi z \end{cases} \tag{18}$$

and, in view of (12)–(16), one obtains

$$\begin{cases} \Delta w_n = -\left(n^2\pi^2 + a^2\right) w_n, \\ \mathbf{u}_n = (u_n, v_n, w_n), \\ u_n = \dfrac{1}{a^2} \dfrac{\partial^2 w_n}{\partial x \partial z}, \qquad v_n = \dfrac{1}{a^2} \dfrac{\partial^2 w_n}{\partial y \partial z}, \\ \Delta_1 \theta_n = -a^2 \theta_n, \qquad \Delta_1 \Gamma_n = -a^2 \Gamma_n. \end{cases} \tag{19}$$

Therefore, setting

$$\gamma_n = \frac{a^2}{\xi_n}, \quad \xi_n = a^2 + n^2\pi^2, \tag{20}$$

it follows that

$$\begin{cases} w_n = \gamma_n (R\theta_n - Le\,\mathcal{C}\Gamma_n), \\ \mathbf{u}_n = \left(\dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial x \partial z}, \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial y \partial z}, w_n\right) \end{cases} \tag{21}$$

satisfy, $\forall (a^2, n) \in \mathbb{R}^+ \times N^+$, the boundary conditions

$$w_n = \theta_n = \Gamma_n = 0 \quad \text{on } z = 0, 1,\ \forall n \in \mathbb{N}^+, \tag{22}$$

(i)–(ii) and $(17)_1$–$(17)_2$. Then—by virtue of linearity—the general solutions of $(17)_1$, $(17)_2$ are

$$w = \sum_{n=1}^{\infty} w_n, \qquad \mathbf{u} = \sum_{n=1}^{\infty} \mathbf{u}_n \tag{23}$$

with $w_n, \mathbf{u}_n$ given by (21).

3. Linearization principle via an auxiliary system

Let $Z_p = \tilde{Z}_p \sin(p\pi z)$ with $\tilde{Z}_p \in \{\tilde{\theta}_p, \tilde{\Gamma}_p\}$, $p \in \mathbb{N}^+$, and let $\langle\cdot,\cdot\rangle$, $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ denote, respectively, the scalar product in $L^2(\Omega)$ and $L^2[0,1]$. The following lemmas hold.

Lemma 1. *Let $p, q, n \in \mathbb{N}^+$. Then $\max(p,q) \leqslant n$ implies*

$$\langle\!\langle \sin(p\pi z), \sin(q\pi z)\rangle\!\rangle = \begin{cases} 1/2 & \textit{for } p = q, \\ 0 & \textit{for } p \neq q, \end{cases} \tag{24}$$

$$\langle\!\langle \sin(q\pi z)\sin(n\pi z), \cos(p\pi z)\rangle\!\rangle = \begin{cases} 0 & \textit{for } p+q \neq n, \\ 1/4 & \textit{for } p+q = n. \end{cases} \tag{25}$$

Proof. (24) immediately follows from

$$\langle\!\langle \sin(p\pi z), \sin(q\pi z)\rangle\!\rangle = \left\langle\!\!\left\langle \frac{1}{2}, \cos\bigl[(p-q)\pi z\bigr] - \cos\bigl[(p+q)\pi z\bigr] \right\rangle\!\!\right\rangle.$$

Concerning (25), we observe that, by virtue of

$$\begin{cases} \sin(q\pi z)\sin(n\pi z) = \dfrac{1}{2}\bigl\{\cos\bigl[(n-q)\pi z\bigr] - \cos\bigl[(n+q)\pi z\bigr]\bigr\}, \\ \cos\bigl[(n-q)\pi z\bigr]\cos(p\pi z) = \dfrac{1}{2}\bigl\{\cos\bigl[(n-q-p)\pi z\bigr] + \cos\bigl[(n+p-q)\pi z\bigr]\bigr\}, \\ \cos\bigl[(n+q)\pi z\bigr]\cos(p\pi z) = \dfrac{1}{2}\bigl\{\cos\bigl[(n+q+p)\pi z\bigr] + \cos\bigl[(n+q-p)\pi z\bigr]\bigr\} \end{cases}$$

it follows that

$$\begin{aligned} &\langle\!\langle \sin(q\pi z)\sin(n\pi z), \cos(p\pi z)\rangle\!\rangle \\ &\quad = \left\langle\!\!\left\langle \frac{1}{4}, \cos\bigl[(n+p-q)\pi z\bigr] + \cos\bigl[(p+q-n)\pi z\bigr] - \cos\bigl[(n+q-p)\pi z\bigr] \right\rangle\!\!\right\rangle \end{aligned} \tag{26}$$

and hence (25) immediately follows. □

Lemma 2. *Let $Z = \sum_{p=1}^{\infty} Z_p$. Then it follows that*

$$\left\langle\!\!\left\langle \sum_{p=1}^{\infty} Z_p, \sin(n\pi z) \right\rangle\!\!\right\rangle = \frac{1}{2}, \tag{27}$$

$$\langle\!\langle \mathbf{u}_p \cdot \nabla Z_q, \sin(n\pi z)\rangle\!\rangle = \begin{cases} 0 & \textit{for } p+q \neq n, \\ \frac{\pi}{4}\bigl(\frac{p}{a^2}\nabla\tilde{w}_p \cdot \nabla\tilde{Z}_q + q\tilde{w}_p\tilde{Z}_q\bigr) & \textit{for } p+q = n. \end{cases} \tag{28}$$

Proof. Lemma 2 is immediately implied by Lemma 1. □

Let us set

$$\begin{cases} b_{1n} = \gamma_n R - \xi_n, & b_{2n} = -\gamma_n \, Le\, C, \\ b_{3n} = \dfrac{\gamma_n R}{\varepsilon\, Le}, & b_{4n} = -\dfrac{\gamma_n}{\varepsilon\, Le}\left(Le\, C + \dfrac{\xi_n}{\gamma_n}\right), \end{cases} \tag{29}$$

$$S_m^{(\theta)} = \sum_{n=1}^{m} \theta_n, \qquad S_m^{(\Gamma)} = \sum_{n=1}^{m} \Gamma_n, \qquad \mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n \tag{30}$$

with w_n and $\mathbf{u}_n$ given by (19) and $m, n \in \mathbb{N}^+$. Then $\{S_m^{(\theta)}, S_m^{(\Gamma)}, \mathbf{U}_m\}$ is a dynamical perturbation iff

$$\begin{cases} \dfrac{\partial}{\partial t} S_m^{(\theta)} = \displaystyle\sum_{n=1}^{m} (b_{1n}\theta_n + b_{2n}\Gamma_n) - \mathbf{U}_m \cdot \nabla S_m^{(\theta)}, \\ \dfrac{\partial}{\partial t} S_m^{(\Gamma)} = \displaystyle\sum_{n=1}^{m} (b_{3n}\theta_n + b_{4n}\Gamma_n) - \dfrac{1}{\varepsilon}\mathbf{U}_m \cdot \nabla S_m^{(\Gamma)}, \\ \nabla \pi_n = -\mathbf{u}_n + (R\theta_n - Le\, C\Gamma_n)\mathbf{k}, \\ \nabla \cdot \mathbf{u}_n = 0, \\ w_n = \gamma_n (R\theta_n - Le\, C\Gamma_n) \end{cases} \tag{31}$$

under the boundary data (22) and $\mathbf{u}_n$ given by (20)–(21).

With (31) we associate the *auxiliary system*

$$\begin{cases} \dfrac{\partial \theta_1^*}{\partial t} = b_{11}\theta_1^* + b_{21}\Gamma_1^* - \mathbf{U}_m^* \cdot \nabla \theta_1^*, \\ \dfrac{\partial \Gamma_1^*}{\partial t} = b_{31}\theta_1^* + b_{41}\Gamma_1^* - \dfrac{1}{\varepsilon}\mathbf{U}_m^* \cdot \nabla \Gamma_1^*, \\ \dots \\ \dfrac{\partial \theta_m^*}{\partial t} = b_{1m}\theta_m^* + b_{2m}\Gamma_m^* - \mathbf{U}_m^* \cdot \nabla \theta_m^*, \\ \dfrac{\partial \Gamma_m^*}{\partial t} = b_{3m}\theta_m^* + b_{4m}\Gamma_m^* - \dfrac{1}{\varepsilon}\mathbf{U}_m^* \cdot \nabla \Gamma_m^*, \end{cases} \tag{32}$$

where

$$\begin{cases} \mathbf{U}_m^* = \displaystyle\sum_{1}^{m} \mathbf{u}_n^*, \quad \mathbf{u}_n^* = \left(\dfrac{1}{a^2}\dfrac{\partial^2 w_n^*}{\partial x \partial z}, \dfrac{1}{a^2}\dfrac{\partial^2 w_n^*}{\partial y \partial z}, w_n^*\right), \\ w_n^* = \tilde{w}_n^* \sin(n\pi z), \quad \theta_n^* = \tilde{\theta}_n^*(x, y, t)\sin(n\pi z), \\ \Gamma_n^* = \tilde{\Gamma}_n^*(x, y, t)\sin(n\pi z), \qquad \tilde{w}_n^* = \gamma_n\big(R\tilde{\theta}_n^* - LeC\tilde{\Gamma}_n^*\big) \end{cases} \tag{33}$$

with θ_n^*, Γ_n^* periodic in the x, y directions with Ω as cell of periodicity, under the boundary data analogous to (22)

$$\theta_n^* = \Gamma_n^* = 0 \quad \text{for } z = 0, 1. \tag{34}$$

Theorem 2. *Let* $\{\theta_1^*, \dots, \theta_m^*; \Gamma_1^*, \dots, \Gamma_m^*; \mathbf{U}_m^*\}$ *with* $\{\mathbf{U}_m^*$ *given by* (33) *and* $(S_m^{(\theta^*)} = \sum_1^m \theta_i^*$, $S_m^{(\Gamma^*)} = \sum_1^m \Gamma_i^*) \in [L_2^*(\Omega)]^2$, $m \in \mathbb{N}\}$ *be the solution of* (32)–(34). *Then* (30) *with* $\{\theta_n = \theta_n^*$, $\Gamma_n = \Gamma_n^*$ $(n = 1, \dots, m)$; $\mathbf{U}_m = \mathbf{U}_m^*\}$ *is the solution of* (31) *under* (20)–(22). *Vice versa, if* (30) *with* $\{S_m^{(\theta)}, S_m^{(\Gamma)}\} \in [L_2^*(\Omega)]^2$, $m \in \mathbb{N}$, *is the solution of* (31) *under* (20)–(22), *then* $\{\theta_1^*, \dots, \theta_n^*; \Gamma_1^*, \dots, \Gamma_m^*; \mathbf{U}_m^*\}$ *with* $\{\theta_n = \theta_n^*, \Gamma_n = \Gamma_n^*$ $(n = 1, \dots, m)$; $\mathbf{U}_m = \mathbf{U}_m^*\}$ *is the solution of* (32)–(34).

Proof. Let $(\theta_1^*, \dots, \theta_m^*; \Gamma_1^*, \dots, \Gamma_m^*; \mathbf{U}_m^* = \sum_1^m \mathbf{u}_n^*)$ with $\{\mathbf{u}_n^*$ given by (33) and $(S_m^{(\theta^*)} = \sum_1^m \theta_n^*$, $S_m^{(\Gamma^*)} = \sum_1^m \Gamma_n^*) \in [L_2^*(\Omega)]^2$, $m \in \mathbb{N}\}$ be the solution of (32)–(34). Then by adding (32) it immediately follows that $(S_m^{(\theta^*)} = \sum_1^m \theta_n^*, S_m^{(\Gamma^*)} = \sum_1^m \Gamma_n^*, \mathbf{U}_m^*)$ is the solution of (31) under

(20)–(22). Vice versa, let $(S_m^{(\theta)}, S_m^{(\Gamma)}, \mathbf{U}_m)$ with $\{S_m^{(\theta)}, S_m^{(\Gamma)}\} \in [L_2^*(\Omega)]^2$, $m \in \mathbb{N}$, be solution of (31) under (20)–(22) with

$$S_m^{(\theta)}(0) = \sum_1^m \theta_n(0), \qquad S_m^{(\Gamma)} = \sum_1^m \theta_n(0). \tag{35}$$

Denoting by $(\theta_1^*, \ldots, \theta_m^*; \Gamma_1^*, \ldots, \Gamma_m^*)$ the solution of (32)–(34) associated with the initial data

$$\theta_n^*(0) = \theta_n(0), \quad \Gamma_n^*(0) = \Gamma_n(0), \quad \mathbf{u}_n^*(0) = \mathbf{u}_n(0), \quad \forall n \in \{1, ..., m\}, \tag{36}$$

it follows that $(S_m^{(\theta^*)}, S_m^{(\Gamma^*)}, \mathbf{U}_m^*)$ is the solution of (31) under (20)–(22). In view of the uniqueness theorem for (31) under (20)–(22) (see Appendix A.1), it turns out that

$$\theta_n^* = \theta_n, \quad \Gamma_n^* = \Gamma_n, \quad \mathbf{u}_n^* = \mathbf{u}_n, \quad \forall n \in \{1, \ldots, m\}. \qquad \square \tag{37}$$

Remark 1. In view of Theorem 2, we can determine the stability of (6) by substituting (32)–(34) in (31) under (20)–(22).

Setting

$$I_n = b_{1n} + b_{4n}, \qquad A_n = b_{1n}b_{4n} - b_{2n}b_{3n} \tag{38}$$

the following linearization principle holds.

Theorem 3. *The time derivative of*

$$V_n = \frac{1}{2}\left[A_n\left(\|\theta_n\|^2 + \|\Gamma_n\|^2\right) + \|b_{1n}\Gamma_n - b_{3n}\theta_n\|^2 + \|b_{2n}\Gamma_n - b_{4n}\theta_n\|^2\right] \tag{39}$$

along the solutions of (31) *is given by*

$$\frac{dV_n}{dt} = A_n I_n\left(\|\theta_n\|^2 + \|\Gamma_n\|^2\right). \tag{40}$$

Proof. By virtue of Theorem 2, we may evaluate the time derivative of V_n along the solution of (32)–(34) and hence along the solution of

$$\begin{cases} \dfrac{\partial \theta_n}{\partial t} = b_{1n}\theta_n + b_{2n}\Gamma_n - \mathbf{U}_m \cdot \nabla\theta_n, \\ \dfrac{\partial \Gamma_n}{\partial t} = b_{3n}\theta_n + b_{4n}\Gamma_n - \dfrac{1}{\varepsilon}\mathbf{U}_m \cdot \nabla\Gamma_n, \end{cases} \qquad n = 1, \ldots, m. \tag{41}$$

It turns out that {cf. Appendix A.3}

$$\frac{dV_n}{dt} = A_n I_n\left(\|\theta_n\|^2 + \|\Gamma_n\|^2\right) + \Psi_n \tag{42}$$

where Ψ_n, the contribution of the nonlinear terms appearing in (41), is given by

$$\begin{cases} \Psi_n = -\langle \alpha_{1n}\theta_n - \alpha_{3n}\Gamma_n, \mathbf{U}_m \cdot \nabla\theta_n\rangle - \dfrac{1}{\varepsilon}\langle \alpha_{2n}\Gamma_n - \alpha_{3n}\theta_n, \mathbf{U}_m \cdot \nabla\Gamma_n\rangle, \\ \alpha_{1n} = A_n + b_{3n}^2 + b_{4n}^2, \quad \alpha_{2n} = A_n + b_{1n}^2 + b_{2n}^2, \quad \alpha_{3n} = b_{1n}b_{3n} + b_{2n}b_{4n}. \end{cases} \tag{43}$$

By virtue of (28), it turns out that

$$\langle\!\langle \mathbf{U}_m \cdot \nabla\theta_n, \Gamma_n \rangle\!\rangle = \langle\!\langle \mathbf{U}_m \cdot \nabla\Gamma_n, \theta_n \rangle\!\rangle = 0, \quad n = 1, \ldots, m. \tag{44}$$

Further, in view of $\nabla \cdot \mathbf{U}_m = 0$ and the boundary data, it follows that

$$\begin{cases} \langle \mathbf{U}_m \cdot \nabla \theta_n, \theta_n \rangle = \frac{1}{2} \langle \mathbf{U}_m, \nabla \theta_n^2 \rangle = 0, \\ \langle \mathbf{U}_m \cdot \nabla \Gamma_n, \Gamma_n \rangle = \frac{1}{2} \langle \mathbf{U}_m, \nabla \Gamma_n^2 \rangle = 0. \end{cases} \tag{45}$$

Then (44)–(45) imply

$$\Psi_n = 0 \quad \forall n \in \{1, 2, \ldots, m\}. \qquad \square \tag{46}$$

Remark 2. Let us observe that:

(i) denoting by $(\lambda_{1n}, \lambda_{2n})$ the eigenvalues of

$$\begin{cases} \frac{d\theta_n}{dt} = b_{1n}\theta_n + b_{2n}\Gamma_n, \\ \frac{d\Gamma_n}{dt} = b_{3n}\theta_n + b_{4n}\Gamma_n \end{cases} \tag{47}$$

it follows that

$$\begin{cases} A_n = \lambda_{1n} \cdot \lambda_{2n}, \\ I_n = \lambda_{1n} + \lambda_{2n}; \end{cases} \tag{48}$$

(ii) V_n and $\dot{V}_n$ are linked in a direct simple way to the eigenvalues of the linear operator involved in (32) and, moreover, $\dot{V}_n$ does not depend on the nonlinear operator involved in (32);

(iii) the time derivative of

$$E_n = \frac{1}{2}\left(\|\theta_n\|^2 + \|\Gamma_n\|^2\right) \tag{49}$$

along the solutions of (41) is given by

$$\frac{dE_n}{dt} = b_{1n}\|\theta_n\|^2 + (b_{2n} + b_{3n})\langle \theta_n, \Gamma_n \rangle + b_{4n}\|\Gamma_n\|^2 \tag{50}$$

and is also independent of the nonlinear terms. However, the eigenvalues of the quadratic form appearing in the right-hand side of (50)—in view of $b_{2n} \neq b_{3n}$, $\forall n \in \mathbb{N}^+$—are not, in general, those determined by $\left(\begin{smallmatrix} b_{1n} & b_{2n} \\ b_{3n} & b_{4n} \end{smallmatrix}\right)$.

4. Global stability

Lemma 3. *Setting*

$$\begin{cases} R_B = 4\pi^2, \qquad R_C^{(1)} = R_B + Le\,\mathcal{C}, \\ R_C^{(2)} = \frac{\mathcal{C}}{\epsilon} + \left(1 + \frac{1}{\epsilon\, Le}\right) R_B, \qquad R_C = \inf\left(R_C^{(1)}, R_C^{(2)}\right) \end{cases} \tag{51}$$

it follows that

$$\begin{cases} R_B = \inf \frac{\xi_n}{\gamma_n}, \\ \epsilon\, Le \leqslant 1 \quad \Rightarrow \quad R_C = R_C^{(1)} < R_C^{(2)}, \\ \left\{ \epsilon\, Le > 1, \quad \mathcal{C} > \mathcal{C}^* = \frac{R_B}{(\epsilon\, Le - 1)\, Le} \right\} \quad \Rightarrow \quad R_C = R_C^{(2)} < R_C^{(1)}. \end{cases} \tag{52}$$

Proof. By virtue of

$$\frac{\xi_n}{\gamma_n} = \frac{(a^2+n^2\pi^2)^2}{a^2} \geqslant \frac{(a^2+\pi^2)^2}{a^2},$$

$(52)_1$ immediately follows. In view of

$$R_C^{(2)} - R_C^{(1)} = \frac{1}{\epsilon}\left[(1-\epsilon\, Le)\mathcal{C} + \frac{R_B}{Le}\right], \tag{53}$$

$(52)_2$ becomes obvious. Passing to $(52)_3$, from $\mathcal{C} > \mathcal{C}^*$, it turns out that

$$R_C^{(2)} - R_C^{(1)} < \frac{1}{\epsilon}\left(-\frac{R_B}{Le} + \frac{R_B}{Le}\right) = 0. \qquad \square \tag{54}$$

Lemma 4. *Let*

$$R < R_C. \tag{55}$$

Then $\forall n \in \mathbb{N}^+$, $\forall a > 0$,

$$\begin{cases} A_n \geqslant \dfrac{\pi^4}{\epsilon\, Le}(1-\eta_1) > 0, \\ I_n \leqslant -\pi^2\left(1 + \dfrac{1}{\epsilon\, Le}\right)(1-\eta_2) < 0, \quad A_n I_n \leqslant -\delta \end{cases} \tag{56}$$

with

$$\begin{cases} \eta_1 = \dfrac{1}{R_B}(R - Le\, C), \qquad \eta_2 = \dfrac{R - C/\varepsilon}{R_B(1+1/(\varepsilon\, Le))}, \\ \delta = \dfrac{\pi^6}{\epsilon\, Le}\left(1+\dfrac{1}{\epsilon\, Le}\right)(1-\eta_1)(1-\eta_2). \end{cases} \tag{57}$$

Proof. (55) implies $0 < \eta_i < 1$ $(i = 1, 2)$. Further, by virtue of (51)–(54), it turns out that

$$\begin{cases} A_n = \dfrac{\xi_n\gamma_n}{\epsilon\, Le}\left(\dfrac{\xi_n}{\gamma_n} + Le\,\mathcal{C} - R\right) > \dfrac{\xi_n^2}{\epsilon\, Le}\left(1 - \eta_1\dfrac{R_B}{\xi_n/\gamma_n}\right) \\ \qquad \geqslant \dfrac{(a^2+n^2\pi^2)^2}{\epsilon\, Le}(1-\eta_1) > \dfrac{\pi^4}{\epsilon\, Le}(1-\eta_1), \\ -I_n = \gamma_n\left[\left(1+\dfrac{1}{\epsilon\, Le}\right)\dfrac{\xi_n}{\gamma_n} + \dfrac{\mathcal{C}}{\epsilon} - R\right] > \gamma_n\left(1+\dfrac{1}{\epsilon\, Le}\right)\left(\dfrac{\xi_n}{\gamma_n} - \eta_2 R_B\right) \\ \qquad = \left(1+\dfrac{1}{\epsilon\, Le}\right)\xi_n\left(1 - \dfrac{\eta_2 R_B}{\xi_n/\gamma_n}\right) > \left(1+\dfrac{1}{\epsilon\, Le}\right)\pi^2(1-\eta_2). \qquad \square \end{cases} \tag{58}$$

Lemma 5. *Let* (55) *hold. Setting*

$$\begin{cases} B_n = 2\max\left(b_{1n}^2, b_{2n}^2, b_{3n}^2 + b_{4n}^2\right), \\ d = \dfrac{2\pi^2(1+\epsilon\, Le)}{(1+\mu)\epsilon\, Le}(1-\eta_2), \\ \mu = \dfrac{2}{R_B^2}(1-\eta_1)\max\left\{2\epsilon\, Le\left(R^2+R_B^2\right), \dfrac{(Le\,\mathcal{C}+R_B)^2}{\epsilon\, Le}, \dfrac{R^2}{\epsilon\, Le}, \epsilon\, Le^3\,\mathcal{C}^2\right\} \end{cases} \tag{59}$$

it turns out that ($\forall n \in N^+$, $\forall a > 0$)

$$\begin{cases} \dfrac{B_n}{A_n} \leqslant \mu, \\ d_n = \dfrac{2|I_n|A_n}{A_n + B_n} \geqslant d. \end{cases} \tag{60}$$

Proof. In view of

$$\begin{cases} A_n > \dfrac{\gamma_n^2}{\epsilon\, Le}\left(\dfrac{\xi_n}{\gamma_n}\right)^2(1-\eta_1), \\ \dfrac{b_{1n}^2}{A_n} = \dfrac{\gamma_n^2(R-\xi_n/\gamma_n)^2}{A_n} \leqslant \dfrac{2\gamma_n^2(\frac{\xi_n}{\gamma_n})^2(1+\frac{R^2}{(\xi_n/\gamma_n)^2})}{A_n} \leqslant \dfrac{2\epsilon\, Le(R^2+R_B^2)}{(1-\eta_1)R_B^2}, \\ \dfrac{b_{2n}^2}{A_n} \leqslant \dfrac{\gamma_n^2\, Le^2\, \mathcal{C}^2 \epsilon\, Le}{\gamma_n^2(\frac{\xi_n}{\gamma_n})^2(1-\eta_1)} \leqslant \dfrac{\epsilon\, Le^3\, \mathcal{C}^2}{(1-\eta_1)R_B^2}, \\ \dfrac{b_{3n}^2}{A_n} \leqslant \dfrac{\gamma_n^2 \epsilon\, Le\, R^2}{\epsilon^2\, Le^2\, \gamma_n^2(\frac{\xi_n}{\gamma_n})^2(1-\eta_1)} \leqslant \dfrac{R^2}{\epsilon\, Le\, R_B^2(1-\eta_1)}, \\ \dfrac{b_{4n}^2}{A_n} \leqslant \dfrac{1}{\epsilon^2 A_n}\xi_n^2\left(\dfrac{\mathcal{C}}{\xi_n/\gamma_n} + \dfrac{1}{Le}\right)^2 \leqslant \dfrac{1}{\epsilon^2}\xi_n^2\left(\dfrac{\mathcal{C}}{R_B} + \dfrac{1}{Le}\right)^2 \cdot \dfrac{\epsilon\, Le}{\xi_n^2(1-\eta_1)} \\ \qquad = \dfrac{(Le\,\mathcal{C} + R_B)^2}{\epsilon\, Le\, R_B^2(1-\eta_1)}, \end{cases} \tag{61}$$

$(60)_2$ easily follows. On the other hand, by virtue of

$$d_n = \frac{2|I_n|}{1+\frac{B_n}{A_n}} \geqslant \frac{2|I_n|}{1+\mu}, \tag{62}$$

$(60)_3$ is implied by $(56)_2$. □

Lemma 6. *Let $A_n > 0$. Then V_n is positive definite and it turns out that* ($\forall n \in N^+$)

$$E_n < \frac{V_n}{A_n} < (1+\mu)E_n. \tag{63}$$

Proof. From Lemma 5, (63) immediately follows. □

Theorem 4. *Let either*

$$\begin{cases} \epsilon\, Le \leqslant 1, \\ R < R_B + Le\,\mathcal{C} \end{cases} \tag{64}$$

or

$$\begin{cases} \epsilon\, Le \geqslant 1, \qquad \mathcal{C} \geqslant \mathcal{C}^*, \\ R < \dfrac{\mathcal{C}}{\epsilon} + \left(1 + \dfrac{1}{\epsilon\, Le}\right)R_B \end{cases} \tag{65}$$

hold. Then the nonlinear global asymptotic exponential L^2-stability of (6)*, with respect to the perturbations* $\{S_m^{(\theta)}, S_m^{(\Gamma)}, \mathbf{U}_m\}$, $\forall m \in \mathbb{N}^+$, *is guaranteed.*

Proof. (64)–(65) imply (55). Then, by virtue of (40) and Lemmas 4–6 it turns out that

$$\frac{dV_n}{dt} = -2|I_n|A_n E_n \leqslant -\frac{2|I_n|A_n}{A_n + B_n} V_n \leqslant -dV_n, \tag{66}$$

i.e.

$$V_n(t) \leqslant V_n(0)e^{-dt} \quad \forall t \geqslant 0,\ \forall n \leqslant m, \tag{67}$$

and, in view of (63), one obtains

$$E_n(t) \leqslant (1+\mu)E_n(0)e^{-dt}. \tag{68}$$

Setting

$$V_m^* = \sum_1^m V_n, \qquad \mathcal{E}_m = \sum_1^m E_n \tag{69}$$

(66)–(67) imply

$$\begin{cases} V_m^* \leqslant V_m^*(0)e^{-dt}, \\ \mathcal{E}_m \leqslant (1+\mu)\mathcal{E}_m(0)e^{-dt}. \end{cases} \tag{70}$$ □

Theorem 5. *Let either* (64) *or* (65) *hold. Then* (6) *is nonlinearly globally exponentially* L^2*-stable with respect to any perturbation* $\{\theta, \Gamma, \mathbf{u}\}$ *according to*

$$\begin{cases} E(t) \leqslant (1+\mu)E(0)e^{-dt}, \\ V \leqslant V(0)e^{-dt} \end{cases} \tag{71}$$

with

$$E = \mathcal{E}_\infty = \sum_1^\infty E_n, \qquad V = V_\infty^* = \sum_1^\infty V_n. \tag{72}$$

Proof. In view of (70), letting $m \to \infty$, (71) immediately follow. □

5. Instability

Theorem 6. *Suppose there exists an* $\bar{a}^2 \in \mathbb{R}^+$ *such that*

$$I_1(\bar{a}^2) > 0 \tag{73}$$

or

$$A_1(\bar{a}^2) \leqslant 0. \tag{74}$$

Then (6) *is* L^2*-unstable.*

Proof. In the case (73) with $A_1 > 0$, $\forall a^2 \in \mathbb{R}^+$, in view of (72) and (63), it turns out that

$$E \geqslant E_1 \geqslant \frac{V_1}{(1+\mu)A_1} \tag{75}$$

with

$$\frac{dV_1}{dt} = 2I_1 A_1 E_1 \geqslant \frac{2I_1}{1+\mu} V_1, \tag{76}$$

i.e.

$$V_1 \geqslant V_1(0)\exp\left(\frac{2I_1}{1+\mu}t\right). \tag{77}$$

In the case (74), in view of $\{b_{31} > 0,\ \forall a^2\}$, we introduce the functional

$$W = \frac{1}{2}\left(\|X\|^2 + \|Y\|^2\right) \tag{78}$$

with

$$\begin{cases} X = (b_1-\lambda_1)\Gamma^* - b_3\theta^*, \\ Y = (b_1-\lambda_2)\Gamma^* - b_3\theta^*, \\ \lambda_1+\lambda_2 = [I_1]_{a=\bar{a}}, \qquad \lambda_1\lambda_2 = [A_1]_{a=\bar{a}}, \\ b_1 = [b_{11}]_{a=\bar{a}}, \quad b_3 = [b_{31}]_{a=\bar{a}}, \quad \theta^* = [\theta_1]_{a=\bar{a}}, \quad \Gamma^* = [\Gamma_1]_{a=\bar{a}}. \end{cases} \tag{79}$$

By straightforward calculations (cf. Appendix A.2), it follows that

$$\begin{cases} \theta^* = \dfrac{1}{b_3(\lambda_2-\lambda_1)}\left[(b_1-\lambda_2)X - (b_1-\lambda_1)Y\right], \\ \Gamma^* = \dfrac{1}{\lambda_2-\lambda_1}(X-Y) \end{cases} \tag{80}$$

and, in view of (41)—for $n=1$ and $a=\bar{a}$—we obtain

$$\begin{cases} \dfrac{\partial X}{\partial t} = \lambda_1 X + F, \\ \dfrac{\partial Y}{\partial t} = \lambda_2 Y + G \end{cases} \tag{81}$$

with

$$\begin{cases} F = b_3\mathbf{U}_m\cdot\nabla\theta^* - \dfrac{1}{\varepsilon}(b_1-\lambda_1)\bar{\mathbf{U}}_m\cdot\nabla\Gamma^*, \\ G = b_3\mathbf{U}_m\cdot\nabla\theta^* - \dfrac{1}{\varepsilon}(b_1-\lambda_2)\bar{\mathbf{U}}_m\cdot\nabla\Gamma^*, \\ \bar{\mathbf{U}}_m = [\mathbf{U}_m]_{a=\bar{a}}. \end{cases} \tag{82}$$

By virtue of (74), the eigenvalues λ_i are real, nonnegative numbers, hence (81) implies

$$\frac{dW}{dt} = \lambda_1\|X\|^2 + \langle X, F\rangle + \lambda_2\|Y\|^2 + \langle Y, G\rangle. \tag{83}$$

On the other hand $\forall a^2$, (28) implies

$$\langle X, F\rangle = \langle Y, G\rangle = 0, \tag{84}$$

hence the instability follows from

$$\frac{dW}{dt} > 0, \quad \left(W(0), t\right) \in \mathbb{R}^+ \times \mathbb{R}^+. \qquad \square \tag{85}$$

Theorem 7. *Let*

$$\epsilon\, Le > 1, \qquad R > R_C. \tag{86}$$

Then (6) *is nonlinearly L^2-unstable.*

Proof. In view of (52), it follows that

$$\epsilon\, Le > 1 \quad \Rightarrow \quad R_C = R_C^{(2)} = \frac{\mathcal{C}}{\epsilon} + \left(1 + \frac{1}{\epsilon\, Le}\right) R_B$$

and hence $(86)_2$ implies

$$R = \frac{\mathcal{C}}{\epsilon} + \left(1 + \frac{1}{\epsilon\, Le}\right)(R_B + k), \tag{87}$$

k being a positive constant. If

$$R_C^{(2)} < R < R_C^{(1)} \tag{88}$$

then (87) implies

$$\begin{cases} I_1(a^2) = \gamma_1 \left[R - \dfrac{\mathcal{C}}{\epsilon} - \dfrac{\xi_1}{\gamma_1}\left(1 + \dfrac{1}{\epsilon\, Le}\right)\right] = \gamma_1 \left(1 + \dfrac{1}{\epsilon\, Le}\right)\left(R_B + k - \dfrac{\xi_1}{\gamma_1}\right), \\ A_1(a^2) > \dfrac{\xi_1 \gamma_1}{\epsilon\, Le}\left(\dfrac{\xi_1}{\gamma_1} + Le\,\mathcal{C} - R_C^{(1)}\right) = \dfrac{\xi_1 \gamma_1}{\epsilon\, Le}\left(\dfrac{\xi_1}{\gamma_1} - R_B\right). \end{cases} \tag{89}$$

Let $0 < \varepsilon_1 < 1$ and consider the equation

$$\frac{\xi_1(a^2)}{\gamma_1(a^2)} = R_B + \varepsilon_1 k \tag{90}$$

having the positive roots

$$\bar{a}^2 = \frac{4\pi^2 + \varepsilon_1 k \pm \sqrt{(4\pi^2 + \varepsilon_1 k)^2 - 4\pi^4}}{2}. \tag{91}$$

It turns out that $\{I_1(\bar{a}) > 0,\ A_1(\bar{a}) > 0\}$ and the instability comes from Theorem 6. In the case

$$R > R_C^{(1)} > R_C^{(2)} \tag{92}$$

there exist two positive constants k, k_1 such that (87) and

$$R = R_C^{(1)} + k_1 \tag{93}$$

hold. It follows that

$$\begin{cases} I_1 = \gamma_1 \left(1 + \dfrac{1}{\epsilon\, Le}\right)\left(R_B + k - \dfrac{\xi_1}{\gamma_1}\right), \\ A_1 = \dfrac{\xi_1 \gamma_1}{\epsilon\, Le}\left(\dfrac{\xi_1}{\gamma_1} - R_B - k_1\right). \end{cases} \tag{94}$$

If $k > k_1$, then for any $\bar{a}$ such that

$$R_B + k_1 < \frac{\xi_1(\bar{a})}{\gamma_1(\bar{a})} < R_B + k \tag{95}$$

one obtains $\{I_1(\bar{a}) > 0,\ A_1(\bar{a}) > 0\}$ and instability follows. If $k \leqslant k_1$, then for any $\bar{a}$ such that

$$\frac{\xi_1(\bar{a})}{\gamma_1(\bar{a})} < R_B + k \tag{96}$$

it follows that (74) is established. □

Theorem 8. *Let*

$$\begin{cases} \varepsilon\, Le \leqslant 1, & \mathcal{C} \leqslant \mathcal{C}^*, \\ R > R_C. \end{cases} \tag{97}$$

Then (6) *is* L^2*-unstable.*

Proof. In view of (52), it follows that

$$\{\epsilon\, Le \leqslant 1, \quad \mathcal{C} \leqslant \mathcal{C}^*\} \quad \Rightarrow \quad R_C = R_C^{(1)} = R_B + Le\,\mathcal{C},$$

and hence (97) implies (93)–$(94)_2$ and (74) for any $\bar{a}$ such that

$$R_B < \frac{\xi_1(\bar{a})}{\gamma_1(\bar{a})} < R_B + k_1. \qquad \square \tag{98}$$

Remark 3. In the case $\{I_1 = 0,\ A_1 > 0\}$ it follows that $\{I_n < 0,\ A_n > 0\}$, $\forall n > 1$. By virtue of (66), it turns out that

(i) (6) is (simply) a L^2-stable *center*;
(ii) all the harmonics tend to zero, except the principal one ($n = 1$).

6. Proof of the main theorem

Collecting the L^2-stability (instability) results obtained, we have to show that they can be incapsulated in Theorem 1. By virtue of

$$\begin{cases} I_n = \gamma_n\left[R - \dfrac{C}{\varepsilon} - \left(1 + \dfrac{1}{\varepsilon\, Le}\right)\dfrac{\xi_n}{\gamma_n}\right], \\ A_n = \dfrac{\xi_n\gamma_n}{\varepsilon\, Le}\left(\dfrac{\xi_n}{\gamma_n} + Le\, C - R\right), \\ \gamma_n > 0, \quad \xi_n > 0, \quad \dfrac{\partial}{\partial n}\left(\dfrac{\xi_n}{\gamma_n}\right) > 0, \quad \forall a^2, \end{cases} \tag{99}$$

it follows that $[\forall(n^2, a^2) \in \mathbb{N}^+ \times \mathbb{R}^+]$

$$I_1 < 0 \quad \Rightarrow \quad I_n < 0; \qquad A_1 > 0 \quad \Rightarrow \quad A_n > 0$$

and that $A_{\bar{n}}(\bar{a}^2) \leqslant 0$ only if $A_1(\bar{a}^2) \leqslant 0$. Taking into account $(56)_3$, Theorems 2–8 and

$$\begin{cases} I_1 < 0, \\ A_1 > 0, \end{cases} \quad \forall a^2 \quad \Rightarrow \quad R < R_C,$$

the proof of Theorem 1, by virtue of ($a_1 = b_{11}$, $a_2 = b_{21}$, $a_3 = b_{31}$, $a_4 = b_{41}$), immediately follows.

7. Rotating layer

When the layer rotates with constant angular velocity $\underline{\omega}_* = \omega_* \mathbf{k}$ about the vertical z axis, (1) becomes

$$\begin{cases} \nabla p = -\dfrac{\mu}{k}\mathbf{v} + \rho_f \mathbf{g} - 2\dfrac{\rho_0}{\varepsilon}\underline{\omega}_* \times \mathbf{v}, \\ \nabla \cdot \mathbf{v} = 0, \\ \tilde{A} T_{,t} + \mathbf{v} \cdot \nabla T = k_T \Delta T, \\ \varepsilon C_{,t} + \mathbf{v} \cdot \nabla C = k_C \Delta C \end{cases} \tag{100}$$

with

$$p = p_1 - \frac{1}{2}\rho_0 [\underline{\omega}_* \times \mathbf{k}]^2 \tag{101}$$

under the boundary conditions (3). By using the same scalings as in Section 1, the dimensionless version of Eqs. (100) is

$$\begin{cases} \nabla p = -\mathbf{v} + (RT - \mathcal{C}C)\mathbf{k} + \mathcal{T}\mathbf{v} \times \mathbf{k}, \\ \nabla \cdot \mathbf{v} = 0, \\ T_{,t} + \mathbf{v} \cdot \nabla T = \Delta T, \\ \varepsilon\, Le\, C_{,t} + \mathbf{v} \cdot \nabla C = \Delta C, \end{cases} \tag{102}$$

where $\mathcal{T} = \frac{2k\omega_*}{\varepsilon\nu}$ is the Taylor–Darcy number. Under the boundary data (5), (6) continues to be the only equilibrium state admissible. The equations governing the perturbations $[\mathbf{u} = (u, v, w), \theta, \Gamma]$ are easily found to be

$$\begin{cases} \nabla \tilde{\pi} = -\mathbf{u} + (R\theta - \mathcal{C}\Gamma)\mathbf{k} + \mathcal{T}\mathbf{u} \times \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_{,t} = w + \Delta T - \mathbf{u} \cdot \nabla \theta, \\ \varepsilon\, Le\, C_{,t} = w + \Delta \Gamma - Le\, \mathbf{u} \cdot \nabla \Gamma \end{cases} \tag{103}$$

under the boundary data (11). Following the procedure of Section 2, it turns out that

$$\begin{cases} \Delta w + \mathcal{T}^2 w_{zz} = \Delta_1 (R\theta - Le\, \mathcal{C}\Gamma), \\ \nabla \cdot \mathbf{u} = 0, \\ \dfrac{\partial \theta}{\partial t} = \Delta\theta + w - \mathbf{u} \cdot \nabla \theta, \\ \dfrac{\partial \Gamma}{\partial t} = \dfrac{1}{\epsilon\, Le}(\Delta\Gamma + w) - \dfrac{1}{\epsilon}\mathbf{u} \cdot \nabla \Gamma \end{cases} \tag{104}$$

under the boundary data (11). It easily follows that the general solution of $(104)_1$–$(104)_2$ is given by

$$w = \sum_1^\infty w_n, \qquad \mathbf{u} = \sum_1^\infty \mathbf{u}_n \tag{105}$$

with

$$\begin{cases} w_n = \gamma_n^*(R\theta_n - \mathcal{C}\Gamma_n), \\ \gamma_n^* = \dfrac{a^2}{\xi_n + \mathcal{T}^2 n^2\pi^2}, \quad \xi_n = a^2 + n^2\pi^2, \\ \mathbf{u}_n = \left(\dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial x\partial z}, \dfrac{1}{a^2}\dfrac{\partial^2 w_n}{\partial y\partial z}, w_n\right). \end{cases} \tag{106}$$

Then—following step by step—the procedures of Sections 3–6, and setting

$$R_B^* = \pi^2\bigl(1+\sqrt{1+\mathcal{T}^2}\bigr)^2 = \inf\frac{\xi_n}{\gamma_n^*}, \tag{107}$$

one finds that, on replacing R_B by R_B^*, each result of Sections 4–6 continues to hold. *In particular, the main Theorem* 1 *continues to hold with* $a^2 + (1+\mathcal{T})\pi^2$ *in place of* $a^2+\pi^2$.

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Appendix A.1. Uniqueness theorem

Let $(\mathbf{u}, \theta, \Gamma, \tilde{\pi})$, $(\mathbf{u}^*, \theta^*, \Gamma^*, \tilde{\pi}^*)$ be two perturbations to the rest state (6) having the same initial data. Then—by virtue of the uniqueness theorem for (1)—it turns out that

$$\begin{cases} \mathbf{u} = \mathbf{u}^*, \\ \theta = \theta^*, \qquad \forall t \geqslant 0. \\ \Gamma = \Gamma^*, \end{cases} \tag{108}$$

Therefore, in view of

$$\theta = \sum_{n=1}^m \tilde{\theta}_n \sin(n\pi z), \tag{109}$$

$$\theta^* = \sum_{n=1}^m \tilde{\theta}_n^* \sin(n\pi z) \tag{110}$$

it turns out that

$$\sum_{n=1}^m \bigl(\tilde{\theta}_n - \tilde{\theta}_n^*\bigr)\sin(n\pi z) = 0 \quad \forall t \geqslant 0, \tag{111}$$

and hence

$$\bigl\|\tilde{\theta}_n - \tilde{\theta}_n^*\bigr\| = 0 \quad \forall n \leqslant m \in \mathbb{N}^+,\ t \in \mathbb{R}^+. \tag{112}$$

Analogously

$$\begin{cases} \Gamma = \sum_1^m \tilde{\Gamma}_n \sin(n\pi z), \\ \Gamma^* = \sum_1^m \tilde{\Gamma}_n^* \sin(n\pi z) \end{cases} \quad \Rightarrow \quad \|\tilde{\Gamma}_n - \tilde{\Gamma}_n^*\| = 0 \quad \forall n \leqslant m \in \mathbb{N}^+,\ t \in \mathbb{R}^+, \tag{113}$$

and hence

$$\begin{cases} \mathbf{u} = \sum_1^m \tilde{\mathbf{u}}_n \sin(n\pi z), \\ \mathbf{u}^* = \sum_1^m \tilde{\mathbf{u}}_n^* \sin(n\pi z) \end{cases} \quad \Rightarrow \quad \|\tilde{\mathbf{u}}_n - \tilde{\mathbf{u}}_n^*\| = 0 \quad \forall n \leqslant m \in \mathbb{N}^+,\ t \in \mathbb{R}^+. \tag{114}$$

In conclusion, one obtains that each harmonic $(\mathbf{u}_n, \theta_n, \Gamma_n)$ of the perturbation field $(\mathbf{u} = \sum_1^m \mathbf{u}_n, \theta = \sum_1^m \theta_m, \Gamma = \sum_1^m \Gamma_m)$ is uniquely determined by its initial value.

Appendix A.2. Time derivative of W along (41)

For the sake of generality we consider

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha^*, \\ \dfrac{\partial v}{\partial t} = \beta^* \end{cases} \tag{115}$$

with

$$\begin{cases} \alpha^* = a_{11}u + a_{12}v + \psi, \\ \beta^* = a_{21}u + a_{22}v + \psi^*, \end{cases} \tag{116}$$

a_{ij} $(i, j = 1, 2)$ being constants such that $a_{11}a_{22} - a_{12}a_{21} < 0$ and $\psi = \psi(u, v)$, $\psi^* = \psi^*(u, v)$. By virtue of

$$\begin{cases} u = \dfrac{1}{a_{21}(\lambda_2 - \lambda_1)}\big[(a_{11} - \lambda_2)X - (a_{11} - \lambda_1)Y\big], \\ v = \dfrac{1}{\lambda_2 - \lambda_1}(X - Y) \end{cases} \tag{117}$$

it turns out that

$$\begin{cases} (a_{11} - \lambda_2)\dfrac{\partial X}{\partial t} - (a_{11} - \lambda_1)\dfrac{\partial Y}{\partial t} = a_{21}(\lambda_2 - \lambda_1)\alpha^*, \\ \dfrac{\partial X}{\partial t} - \dfrac{\partial Y}{\partial t} = (\lambda_2 - \lambda_1)\beta^* \end{cases} \tag{118}$$

and hence

$$\begin{cases} \dfrac{\partial X}{\partial t} = -a_{21}\alpha^* + (a_{11} - \lambda_1)\beta^*, \\ \dfrac{\partial Y}{\partial t} = -a_{21}\alpha^* + (a_{11} - \lambda_2)\beta^*. \end{cases} \tag{119}$$

Taking into account that

$$\begin{cases} a_{11}-\lambda_2=\lambda_1-a_{22}, \quad a_{11}-\lambda_1=\lambda_2-a_{22}, \\ a_{11}(a_{11}-\lambda_2)+a_{12}a_{21}=\lambda_1(a_{11}-\lambda_2), \\ a_{11}(a_{11}-\lambda_1)+a_{12}a_{21}=\lambda_2(a_{11}-\lambda_1) \end{cases} \tag{120}$$

one obtains

$$\begin{cases} \alpha^*=\dfrac{1}{a_{21}(\lambda_2-\lambda_1)}\big[\lambda_1(a_{11}-\lambda_2)X-\lambda_2(a_{11}-\lambda_1)Y\big]+\psi, \\ \beta^*=\dfrac{1}{\lambda_2-\lambda_1}(\lambda_1X-\lambda_2Y)+\psi^*, \\ a_{21}\alpha^*=\dfrac{1}{\lambda_2-\lambda_1}\big\{\big[a_{11}(a_{11}-\lambda_2)+a_{21}a_{12}\big]X \\ \qquad -\big[a_{11}(a_{11}-\lambda_1)+a_{12}a_{21}\big]Y\big\}+a_{21}\psi \\ \qquad =\dfrac{1}{\lambda_2-\lambda_1}\big[\lambda_1(a_{11}-\lambda_2)X-\lambda_2(a_{11}-\lambda_1)Y\big]+a_{21}\psi, \\ (a_{11}-\lambda_1)\beta^*=\dfrac{a_{11}-\lambda_1}{\lambda_2-\lambda_1}[\lambda_1X-\lambda_2Y]+(a_{11}-\lambda_1)\psi^*, \\ (a_{11}-\lambda_2)\beta^*=\dfrac{a_{11}-\lambda_2}{\lambda_2-\lambda_1}[\lambda_1X-\lambda_2Y]+(a_{11}-\lambda_2)\psi^*, \end{cases} \tag{121}$$

and hence by virtue of

$$\begin{cases} \big[(a_{11}-\lambda_1)\lambda_1-\lambda_1(a_{11}-\lambda_2)\big]X+\big[(a_{11}-\lambda_1)\lambda_2-\lambda_2(a_{11}-\lambda_1)\big]Y \\ \quad =\lambda_1(\lambda_2-\lambda_1)X, \\ \big[(a_{11}-\lambda_2)\lambda_1-\lambda_1(a_{11}-\lambda_2)\big]X+\big[(a_{11}-\lambda_1)\lambda_2-\lambda_2(a_{11}-\lambda_2)\big]Y \\ \quad =\lambda_2(\lambda_2-\lambda_1)Y \end{cases} \tag{122}$$

it turns out that

$$\begin{cases} -a_{21}\alpha^*+(a_{11}-\lambda_1)\beta^*=\lambda_1X+F^*, \\ -a_{21}\alpha+(a_{11}-\lambda_2)\beta=\lambda_2+G^*, \\ F^*=-a_{21}\psi+(a_{11}-\lambda_1)\psi^*, \\ G^*=-a_{21}\psi+(a_{11}-\lambda_2)\psi^*. \end{cases} \tag{123}$$

Therefore in view of (119) and (123) one obtains

$$\begin{cases} \dfrac{\partial X}{\partial t}=\lambda_1X+F^*, \\ \dfrac{\partial Y}{\partial t}=\lambda_2Y+G^* \end{cases} \tag{124}$$

and hence

$$W=\frac{1}{2}\big[\|X\|^2+\|Y\|^2\big]$$

implies

$$\frac{dW}{dt}=\lambda_1\|X\|^2+\langle X,F^*\rangle+\lambda_2\|Y\|^2+\langle Y,G^*\rangle. \tag{125}$$

Appendix A.3. Time derivative of V_n along (41)

For the sake of generality we consider (115)–(116). Setting [18]

$$A = a_{11}a_{22} - a_{12}a_{21}, \qquad I = a_{11} + a_{22} \tag{126}$$

and introducing the functional

$$V = \frac{1}{2}\left[A\left(\|u\|^2 + \|v\|^2\right) + \|a_{11}v - a_{12}u\|^2 + \|a_{12}v - a_{22}u\|^2\right] \tag{127}$$

it follows that

$$\begin{aligned} \frac{dV}{dt} &= \left(A + a_{21}^2 + a_{22}^2\right)\langle u, u_t\rangle + \left(A + a_{11}^2 + a_{12}^2\right)\langle v, v_t\rangle \\ &\quad - (a_{11}a_{21} + a_{12}a_{22})\langle v, u_t\rangle - (a_{11}a_{31} + a_{12}a_{22})\langle u, v_t\rangle. \end{aligned} \tag{128}$$

Since, along (115)–(116), it turns out that

$$\begin{cases} \langle u, u_t\rangle = a_{11}\|u\|^2 + a_{12}\langle u, v\rangle + \langle u, \psi\rangle, \\ \langle v, v_t\rangle = a_{21}\langle u, v\rangle + a_{22}\|v\|^2 + \langle v, \psi^*\rangle, \\ \langle v, u_t\rangle = a_{11}\langle u, v\rangle + a_{12}\|v\|^2 + \langle v, \psi\rangle, \\ \langle u, v_t\rangle = a_{21}\|u\|^2 + a_{22}\langle u, v\rangle + \langle u, \psi^*\rangle, \end{cases} \tag{129}$$

by straightforward calculations it follows that

$$\frac{dV}{dt} = AI\left(\|u\|^2 + \|v\|^2\right) + \Psi^{**} \tag{130}$$

with

$$\begin{cases} \Psi^{**} = \langle \alpha_1 u - \alpha_3 v, \psi\rangle + \langle \alpha_2 v - \alpha_3 u, \psi^*\rangle, \\ \alpha_1 = A + a_{21}^2 + a_{22}^2, \quad \alpha_2 = A + a_{11}^2 + a_{12}^2, \quad \alpha_3 = a_{11}a_{21} + a_{12}a_{22}. \end{cases} \tag{131}$$

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