On the Theorem of Hartogs
in Non-Archimedean Valued Fields

M. S. Stawski

Barnaul Pedagogical Institute, Barnaul, USSR

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Let $U = U_0 \times U_1 \times \cdots \times U_n$ be an open polyring in a non-Archimedean valued, locally non-compact field. Let the function $f$ be defined in the polyring $U$ and satisfy the following conditions: (1) $f$ is holomorphic for every $x \in U_0$ separately in each of the rest variables $y_i \in U_i, i = 1, 2, \ldots, n$; (2) $f$ is holomorphic in $y \in \mathcal{V}_0 \times \cdots \times \mathcal{V}_n$, where $\mathcal{V}_i$ is a certain disk from the ring $U_i$. Then, if the valuation is dense, the function $f$ is holomorphic in the polyring $U$. If the valuation is discrete, then the function $f$ is holomorphic in a domain close to the polyring $U$.

**Introduction**

Let $K$ be a non-Archimedean valued field, not locally compact. In the article [1] it was proved that an analogue of Hartogs' theorem holds in these fields. This result, in short, states that a function of several variables which can be represented by a power series in each variable, can be represented by a multiple power series. Ph. Robba [2] proved the analogous theorem for Laurent series, but under the following additional conditions: (1) the function is bounded; (2) the field $K$ is maximally complete and algebraically closed.

It turns out that these additional conditions can be omitted. Moreover, the conditions concerning the development in Laurent series in each variable can be weakened. This is the subject of the present paper.

**Notations and Terminology**

$K$ denotes a complete non-archimedean valued field, not locally compact; $||$ is the valuation and $G$ is its range, i.e., $G = \{||x|| \mid x \in K\}$.

$\langle R, S \rangle$ denotes an interval, $-\infty \leq R \leq S \leq +\infty$. A closed interval is denoted by $[R, S]$, an open one by $(R, S)$, a half-open one by $[R, S)$.
$U_{(R,S)}$ denotes the ring $\{x \in K \mid |x| \in \langle R, S \rangle\}$. In particular, $U_{[0,s]}$ and $U_{[0,s]}$ are disks, the first one closed, the second, open. $\Gamma_R$ is the circle $\{x \in K \mid |x| = R\}$.

$\hat{K}$ denotes the completion of the algebraic closure of $K$. $\hat{U}_{(R,S)}$ denotes the set $\{x \in \hat{K} \mid |x| \in \langle R, S \rangle\}$ and is called the extension of the ring $U_{(R,S)}$.

The function $f(x_1, \ldots, x_n)$ is called holomorphic in the Cartesian product $\prod_{i=1}^n U_{(R_i,S_i)}$, if it can be represented by the Laurent series

$$f(x_1, \ldots, x_n) = \sum_{(k_1, \ldots, k_n) \in \mathbb{Z}^n} a_{k_1, \ldots, k_n} x_1^{k_1} \cdots x_n^{k_n} \quad (1)$$

and if

$$\lim_{|k_1| + \cdots + |k_n| \to \infty} |a_{k_1, \ldots, k_n}| \cdot r_1^{k_1} \cdots r_n^{k_n} = 0$$

for each $r = (r_1, \ldots, r_n)$, $r_i \in \langle R_i, S_i \rangle$, $i = 1, \ldots, n$. In particular, if any of the rings is a disk, then series (1) is a power series in the corresponding variable.

If $f(x) = \sum_{k=-\infty}^{\infty} a_k x^k$, $x \in U_{(R,S)}$, then for $t \in \langle R, S \rangle$ we define

$$\mu(f, t) = \max_{k \in \mathbb{Z}} |a_k| t^k,$$

$$\nu(f, t) = \max\{k \in \mathbb{Z} \mid |a_k| t^k = \mu(f, t)\}.$$

**Auxiliary Assertions**

First we note an evident property of Laurent series analogous to Abel’s theorem for power series:

**Lemma 1.** In order for series (1) to converge in the domain $U_{(R_1,S_1)} \times U_{(R_2,S_2)} \times \cdots \times U_{(R_n,S_n)}$, it is sufficient that for each $\eta > 1$ there exists $N$ such that the inequalities

$$|a_{k_1, \ldots, k_n}| R_1^{k_1} \cdots R_n^{k_n} < \eta^{k_1 + \cdots + k_n},$$

$$|a_{k_1, \ldots, k_n}| S_1^{k_1} \cdots S_n^{k_n} < \eta^{k_1 + \cdots + k_n}$$

are true for $|k_1| + |k_2| + \cdots + |k_n| > N$.

The following lemma embraces the essential part of the proofs of the next theorems.

**Lemma 2.** Let the series

$$\sum_{k \in \mathbb{Z}^n} a_k(x) y^k, \quad (2)$$


where

\[ k = (k_1, \ldots, k_n), \quad y^k = y_1^{k_1} \cdots y_n^{k_n}, \]

converge with respect to \( y \) in the domain \( U_{(R_1,S_1)} \times U_{(R_2,S_2)} \times \cdots \times U_{(R_n,S_n)} \) for each \( x \in U_{(R,S)}, \) \( R, S \in G. \) Let

\[ a_k(x) = \sum_{i \in \mathbb{Z}} c_{k,i} x^i, \quad x \in U_{(R,S)} \]

and

\[ |v(a_k, t)| \leq A_t |k|, \quad |k| = |k_1| + \cdots + |k_n|, \quad t \in U_{(R,S)}. \]

Then the series

\[ \sum_{k \in \mathbb{Z}^n} \sum_{i \in \mathbb{Z}} c_{k,i} x^i y^k \]

converges in the domain

\[ U_{(R,S)} \times U_{(R_1,S_1)} \times \cdots \times U_{(R_n,S_n)}. \]

Proof. 1. Let \( t \in (R, S). \) If we show that for every \( \eta > 1 \) there exists \( N > 0 \) such that

\[ |k| > N \Rightarrow \mu(a_k, t) r^k \leq \eta^{|k|}, \]

then we obtain the assertion of the lemma at once. In fact, from (3) it follows that

\[ |c_{k,i}| t^i \leq \mu(a_k, t). \]

Putting this into (7), we obtain

\[ |k| > N \Rightarrow |c_{k,i}| t^i r^k \leq \eta^{|k|}. \]

Since \( t \) and \( r \) are arbitrary, from here in view of Lemma 1 it follows that series (5) converges in the domain (6), which had to be proved.

Thus, it is sufficient to prove (7). Assume the contrary. Then we can find \( \eta > 1 \) and a sequence

\[ k^{(1)}, k^{(2)}, \ldots, k^{(i)}, \ldots, \quad |k^{(i)}| < |k^{(i+1)}|, \]

such that

\[ \mu(a_{k^{(i)}}, t) > \eta^{|k^{(i)}|}. \]

The rest of the proof is different for dense and discrete valuations.
2. Let the valuation of $K$ be dense. We are given an arbitrary $\varepsilon > 0$. By (4) we can choose $t_1 \in G$, that is arbitrarily close to $t$ and satisfies the condition

$$|v(a_k(t_1), t_1 + 0) - v(a_k(t_1), t_1 - 0)| < \varepsilon |k^{(l)}|$$

for an infinite set of $l$'s and we may assume that it holds for the whole sequence (8). Let

$$E_m = \{x \in \Gamma_{t_1} \mid |a_k(x)| r^k < 1 \text{ for } k \geq m\}.$$  

We have

$$E_m \subset E_{m+1}, \quad \bigcup_{m=1}^{\infty} E_m = \Gamma_{t_1}.$$  

Let $x_{i_1}, x_{i_2}, \ldots, x_{i_{t_1}}$ be all zeros of $a_k(t_1)(x)$ in $\Gamma_{t_1}$. Then, considering also (10), we obtain

$$|a_k^{(l)}(x)| = \mu(a_k(t_1), t_1) \prod_{j=1}^{s_j} \left| \frac{x - x_{ij}}{s_j} \right|, \quad s_j \leq \varepsilon |k^{(l)}|.$$  

We denote by $\operatorname{mes} V$ the greatest lower bound of the sum of radii of disks from $K$ covering the set $V$. (For the properties of $\operatorname{mes} V$ in detail see [1]). By Lemma II [1] we can find a set $V_i \subset \Gamma_{t_1}$ such that

$$\operatorname{mes} V_i = \frac{1}{2} t_1, \quad \prod_{j=1}^{s_j} \left| x - x_{ij} \right| > \left( \frac{t_1}{2e} \right)^{s_j}, \quad x \notin V_i.$$  

Comparing this with (13), we obtain

$$|a_k^{(l)}(x)| > \mu(a_k(t_1), t_1) \left( \frac{1}{2e} \right)^{s_j} > \mu(a_k(t_1), t_1) \left( \frac{1}{2e} \right)^{s_j}.$$  

From here, by (9), it follows:

$$|a_k^{(l)}(x)| t^{k^{(l)}} > \left( \left( \frac{1}{2e} \right)^{s_j} \right)^{s_j}.$$  

Choosing $\varepsilon$ so that $(1/2e)^{s_j} > \eta^{-1}$, we get

$$|a_k^{(l)}(x)| t^{k^{(l)}} > 1, \quad x \in \Gamma_{t_1} \setminus V_i.$$  

1 This lemma is analogous to a well-known theorem of H. Cartan, Lemma II[1], which states: for each points $a_1, \ldots, a_s \in \Gamma_{t_1}$ and any $r < R$ there can be found a finite system of open disks from $\Gamma_{t_1}$ with a sum of radii equal to $r$, that outside of them the following inequality is true: $|x - a_1| \cdots |x - a_s| > (r/e)$.
Comparing this with (11), we see that

$$|k^{(i)}| \geq m \Rightarrow E_m \subset V_i.$$ 

Hence \(\text{mes } E_m \leq \text{mes } V_i = \frac{1}{2} t_1\) for all \(m\). From here by Theorem 2 \([1]^{2}\) it follows that

$$\text{mes } \left( \bigcup_{m=1}^{\infty} E_m \right) \leq \frac{1}{2} t_1 < t_1.$$ 

But, on the other hand, by (12)

$$\text{mes } \left( \bigcup_{m=1}^{\infty} E_m \right) = \text{mes } \Gamma_1 = t_1.$$ 

This contradiction proves the lemma in the case of a dense valuation.

3. Suppose the valuation on \(K\) is discrete. Then in view of its local incompactness the residue field is infinite. This means that the circle \(I\), includes an infinite set of open disks with the radius \(t\). For any \(\varepsilon > 0\) choose \(N > 2A_\varepsilon/\varepsilon\) of such disks, where \(A_\varepsilon\) is defined in (4). For every \(k^{(i)}\) the number of zeros of \(a_{k^{(i)}}(x)\) in the extension of at least one from these disks is less than \(\varepsilon |k^{(i)}|\). Hence, there can be found an open disk \(V \subset \Gamma_t\) with the radius \(t\), in which this holds for an infinite set of \(k^{(i)}\). We can assume that for the whole sequence (8), we have

$$\nu(a_{k^{(i)}}, V) < \varepsilon |k^{(i)}|,$$

where \(\nu(a_{k^{(i)}}, V)\) means the number of zeros of \(a_{k^{(i)}}(x)\) in \(V\).

We denote

$$E_m = \{x \in \Gamma_t \mid |a_k(x)| r^k < 1, |k| \geq m\}. \quad (14)$$

Let \(x_1, x_2, \ldots, x_{s_i}\) be all the zeros of \(a_{k^{(i)}}(x)\) in \(V\). Then

$$|a_{k^{(i)}}(x)| = \mu(a_{k^{(i)}}, t) \prod_{j=1}^{s_i} \frac{|x - x_{ij}|}{t}, \quad s_i < \varepsilon |k^{(i)}|, \quad x \in V. \quad (15)$$

Let \(a\) be such that \(at \in G, at < \text{mes } V\). By Lemma II \([1]\) there exists a set \(V_t \subset V\) such that \(\text{mes } V_t \leq at\) and

$$\prod_{j=1}^{s_i} |x - x_{ij}| > \left( \frac{at}{\varepsilon} \right)^{s_i} \quad \text{for } x \in V \setminus V_t.$$ 

$${}^{2}$$ Theorem 2 \([1]\) states in particular that if \(\{E_m\}\) is an increasing sequence of sets from \(K\), then \(\text{mes } (\bigcup_{m=1}^{\infty} E_m) = \lim_{m \to \infty} (\text{mes } E_m)\).
Substituting this in (15), we obtain

\[ |a_{k(i)}(x)| > \mu(a_{k(i)}(x), \mu(a_{k(i)}, t) \left( \frac{a}{e} \right)^{e(k(i))}, \quad x \in V \setminus V_i. \]

If we choose \( \varepsilon \) such that \( (a/e)^{k(i)} > \eta^{-i} \), then

\[ |a_{k(i)}(x)| > \mu(a_{k(i)}, t) \eta^{-k(i)}. \]

From this inequality and (9) it follows:

\[ |a_{k(i)}(x)| r^{k(i)} > 1, \quad x \in V \setminus V_i. \]

Comparing this with (14), we see that

\[ (E_m \cap V) \subset V_i \quad \text{for} \quad |k(i)| \geq m. \]

Hence

\[ \operatorname{mes}(E_m \cap V) \leq \operatorname{mes} V_i \leq at < V \]

and at the same time by Theorem 2

\[ \lim_{m \to \infty} (\operatorname{mes}(E_m \cap V)) = \operatorname{mes} \bigcup_m (E_m \cap V) = \operatorname{mes}(\Gamma_i \cap V) = \operatorname{mes} V. \]

The lemma is proved.

**Main Assertions**

Now we turn to the theorem of Hartogs. For dense and discrete valuations the formulations will be slightly different. First we show an intermediate result.

**Theorem 1.** Let the function \( f \), defined in the domain

\[ U_{(R,S)} \times U_{(10,T)} \times \cdots \times U_{(10,T^n)}, \tag{16} \]

satisfy the following conditions:

1. for each \( x \in U_{(R,S)} \), \( f \) is holomorphic separately in every of the other variables;
2. for each \( y = (y_1, \ldots, y_n) \in V_1 \times \cdots \times V_n \), where \( V_i \) is a certain disk from \( U_{(10,T)} \), the function \( f \) is holomorphic in \( x \).
Then:

1. if the valuation on $K$ is dense the function $f$ is holomorphic in the domain (16);

2. if the valuation on $K$ is discrete and $(R, S)$ includes at least four elements from $G$ then $f$ is holomorphic in the domain

$$U \cup U_1 \cup \cdots \cup U_n \cup W,$$

where

$$U = U_{(R', S', T)} \times U_{[0, T_1]} \times \cdots \times U_{[0, T_n]},$$

$$U_i = U_{(R, S)} \times U_{[0, T_1]} \times \cdots \times U_{[0, T_{i-1}]},$$

$$\times U_{[0, T_1]} \times U_{[0, T_{i-1}]} \times \cdots \times U_{[0, T_n]},$$

$$W = U_{(R', S')} \times U_{[0, T_1]} \times \cdots \times U_{[0, T_n]}.$$  

(17)

$R', S', T_i$ are the nearest absolute values to $R, S, T_i$, respectively, and such that $[R', S'] \subset (R, S), T_i \subset (0, T_i)$.

Proof. 1. Let $n = 1$. We write $y$ instead $y_1$. The domain (16) takes the form

$$U_{(R, S)} \times U_{[0, T]}.$$  

(18)

Suppose the valuation on $K$ is dense. We may assume that the disk in condition (2) has the form $U_{[0, r]}$. We show that for each closed interval $[R', S'] \subset (R, S)$ there exists such $\rho > 0$ that the function $f$ is bounded in $U_{[R', S']} \times U_{[0, \rho]}$. This is done in the usual way (see, for instance [1]). Suppose

$$|f(x, y)| \leq M \quad \text{for} \quad (x, y) \in U_{[R', S']} \times U_{[0, \rho]}.$$  

(19)

From condition (1) it follows that in domain (18) the function $f$ expands in a series:

$$f(x, y) = \sum_{k=0}^{\infty} a_k(x) y^k.$$  

As in [1] we prove that all $a_k(x)$ are holomorphic. Let

$$a_k(x) = \sum_{i=-\infty}^{\infty} c_{ki} x^i, \quad x \in U_{(R, S)}.$$
Now we estimate the \( \nu(a_k, t) \) and consider the case when \( \nu(a_k, t) \geq 0 \). Then we have

\[
\mu(a_k, t) = |c_k, \nu(a_k, t)| t^{\nu(a_k, t)},
\]

\[
\mu(a_k, t_1) \geq |c_k, \nu(a_k, t_1)| t_1^{\nu(a_k, t_1)}, \quad t_1 > t,
\]

from where

\[
\nu(a_k, t) \leq (\log \mu(a_k, t_1) - \log \mu(a_k, t))/(t_1 - t).
\]  \( \text{(20)} \)

From (19) it follows that \( |a_k(x)| \geq M \rho^{-k} \); hence

\[
\log \mu(a_k, t) \leq \log M - k \log \rho.
\]

This together with (20) gives

\[
\nu(a_k, t) \leq A \cdot k, \quad t \in (R, S).
\]

In a similar way we argue the case when \( \nu(a_k, t) < 0 \). Thus we obtain

\[
|\nu(a_k, t)| \leq A \cdot |k|, \quad t \in (R, S).
\]

Now we can apply Lemma 2, and the proof is finished for this case.

2. Let the valuation on \( K \) be discrete. We describe briefly the proof, which is eventually analogous to the previous one. First we establish the boundedness of \( f \) in the domain \( \mathcal{U}_{(R', S')} \times \mathcal{U}_{(0, \rho)} \), \( R', S' \in G \). Then we estimate the \( \nu(a_k, t) \) for \( t \in (R', S') \). Using Lemma 2 we show that \( f(x, y) \) is holomorphic in the domain \( \mathcal{U}_{(R'', S'')} \times \mathcal{U}_{(0, \tau)} \), where \( R'', S'' \in G \), \( [R'', S''] \subset (R', S') \).

The function \( f \) can be represented now in the form

\[
f(x, y) = \sum_{i=-\infty}^{\infty} g_i(y)x^i, \quad g_i(y) = \sum_{k=0}^{\infty} c_{ki} y^k.
\]

As above, estimating the \( \nu(g_i, t) \), we obtain in view of condition (2) that \( f(x, y) \) is holomorphic in the domain \( \mathcal{U}_{(R, S)} \times \mathcal{U}_{(0, \tau)} \), i.e., in the second of the domains indicated in the statement of the theorem. Hence, it can be prolonged onto the extension \( \hat{\mathcal{U}}_{(R, S)} \times \hat{\mathcal{U}}_{(0, \tau)} \), and since the valuation on \( \hat{K} \) is dense, we obtain that the function is bounded in the domain \( \hat{\mathcal{U}}_{(R'', S'')} \times \hat{\mathcal{U}}_0 \), where \( [R', S'] \subset (R'', S'') \). Now one can estimate in a proper way the \( (a_k, t) \) for \( t \in [R', S'] \), and by Lemma 2 we obtain the holomorphy of \( f \) in the domain \( \mathcal{U}_{(R', S')} \times \mathcal{U}_{(0, \tau)} \). Finally by virtue of the logarithmic convexity of
the domain of convergence of a power series, \( f \) is holomorphic in the domain \( U_{(R', S')} \times U_{[0, T']} \). The theorem is proved for \( n = 1 \).

3. By induction it is easy to extend the theorem to \( n \) variables \( y_i \). This completes the proof.

Now we go to the main theorem.

**Theorem 2.** Let the function \( f \), defined in the domain

\[
U_{(R_0, S_0)} \times U_{(R_1, S_1)} \times \cdots \times U_{(R_n, S_n)},
\]

(21)
satisfy the following conditions:

1. for every \( x \in U_{(R_0, S_0)} \), \( f \) is holomorphic separately in each of the remaining variables \( y_i \in U_{(R_i, S_i)} \), \( i = 1, 2, \ldots, n \);

2. for every \( y = (y_1, \ldots, y_n) \times V_1 \times \cdots \times V_n \), where \( V_i \) is a certain disk from the ring \( U_{(R_i, S_i)} \), the function \( f \) is holomorphic in \( x \in U_{(R_0, S_0)} \).

Then

1. if the valuation on \( K \) is dense, the function \( f \) is holomorphic in domain (21);

2. if the valuation on \( K \) is discrete and each of the intervals \( (R_i, S_i) \) includes at least four elements from \( G \) then \( f \) is holomorphic in the domain

\[
U_0 \cup U_1 \cup \cdots \cup U_n \cup U_{n+1},
\]

(22)

where

\[
U_i = U_{(R_0, S_0)} \times \cdots \times U_{(R_{i-1}, S_{i-1})} \\
\times U_{(R_i, S_i)} \times U_{(R_{i+1}, S_{i+1})} \times \cdots \times U_{(R_n, S_n)},
\]

\[
U_{n+1} = U_{(R_0, S_0)} \times U_{(R_1, S_1)} \times \cdots \times U_{(R_n, S_n)},
\]

\( R'_i, S'_i \) denotes the nearest absolute values respectively to \( R_i, S_i \) such that \( |R'_i, S'_i| \subset (R_i, S_i) \).

*Proof.* Suppose the valuation on \( K \) is dense. First we prove the theorem for \( n = 1 \). Let \( y_1 = y \). By Theorem 1 the function \( f \) is holomorphic in the domain \( U_{(R_0, S_0)} \times V_1 \). Hence its partial derivatives are holomorphic in this domain too. Let

\[
f(x, y) = \sum_{k = -\infty}^{\infty} a_k(x) y^k
\]

\[
= a_0(x) + \sum_{k = 1}^{\infty} (a_k(x) y^k + a_{-k}(x) y^{-k}).
\]

(23)
In view of condition (1) the series converges for each \( x \in U_{(R_a,S_0)} \). Now we show that the \( a_k(x) \) are holomorphic. The method from Theorem 1 is not valid here, since the series contains, in general, an infinite set of negative powers. We introduce the operator \( D \) by

\[
Df = y \frac{\partial}{\partial y} \left( y \frac{\partial f}{\partial y} \right).
\]

This operator, by the above remark, is defined for our function \( f \) and it is linear and continuous in the topology of the uniform convergence. Moreover,

\[
D(a_k(x)y^k + a_{-k}(x)y^{-k}) = k^2(a_k(x)y^k + a_{-k}(x)y^{-k}).
\]

Hence, from (23) it follows:

\[
Df(x, y) = \sum_{k=1}^{\infty} k^2(a_k(x)y^k + a_{-k}(x)y^{-k}).
\]

Let

\[
f_1(x, y) = f(x, y) - a_0(x),
\]

\[
f_{m+1}(x, y) = f_m(x, y) - \frac{1}{m^2} Df_m(x, y), \quad m \geq 1, \tag{24}
\]

and

\[
D_mf = \frac{1}{m^2} Df_m. \tag{25}
\]

It is easy to show that

\[
D_mf(x, y) = \sum_{k=m}^{\infty} \frac{k^2}{m^2} \left( 1 - \frac{k^2}{(m-1)^2} \right) \left( a_k(x)y^k + a_{-k}(x)y^{-k} \right). \tag{26}
\]

The coefficients in this series are equal to \( \binom{k}{m} \binom{k+1}{m-1} \); hence they are integers, and therefore series (26) is majorised by the corresponding remainder of series (23). From (24) and (25) we get

\[
a_0(x) = f(x, y) - D_1f(x, y) - \cdots - D_mf(x, y) - \cdots. \tag{27}
\]

This series by condition (1) converges for each \( x \in U_{(R_a,S_0)} \).
Now we consider series (27) from a different point of view. By Theorem 1 the function \( f \) is holomorphic in the domain

\[
W = U_{[R_0, s_0]} \times V,
\]

(28)

where \([R_0, S_0] \subseteq (R_0, S_0), \quad V = \{y \mid |y - y_0| \leq r_1 \} \subseteq V_1\). Therefore it expands in this domain into a uniformly convergent series

\[
f(x, y) = \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} c_{kj} x^j (y - y_0)^k.
\]

(29)

Let

\[
\mu_{kj} = \sup \{|c_{kj}| r^j r_1^k \mid R_0' \leq r \leq S_0'\}.
\]

Then

\[
\lim_{k+|j| \to \infty} \mu_{kj} = 0.
\]

(30)

From (29) we get

\[
D_m f(x, y) = \sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} c_{kj} x^j D_m ((y - y_0)^k).
\]

But from (26) it follows that \(D_m ((y - y_0)^k) = 0\) for \(m > k\). Hence

\[
D_m f(x, y) = \sum_{k=m}^{\infty} \sum_{j=-\infty}^{\infty} c_{kj} x^j D_m ((y - y_0)^k).
\]

Observing that \(|D_m ((y - y_0)^k)| \leq r_1^k\) for \(y \in V\), we obtain that in the domain \(W\) defined by (28) we have

\[
\sup_{(x,y) \in W} |D_m f(x, y)| \leq \max_{k \geq m} |\mu_{kj}|.
\]

From here and (30) it follows that

\[
\lim_{m \to \infty} \sup_{(x,y) \in W} |D_m f(x, y)| = 0.
\]

Hence, series (27) converges uniformly in domain (28), in particular, uniformly in \(x\) for \(y = y_0\). But from (29) it follows that the \(D_m f(x, y_0)\) are holomorphic with respect to \(x\). Therefore \(a_n(x)\) is holomorphic in \(U_{[R_0, s_0]}\) and consequently in \(U_{(R_0, S_0)}\) too. Now let \(g(x, y) = y^{-n} f(x, y)\). Then

\[
a_n(x) = g(x, y) - D_1 g(x, y) - \cdots - D_m g(x, y) - \cdots,
\]
and it follows as above, that the \( a_n(x) \) are holomorphic in \( U_{(R_0, S_0)} \). Let \( M = \sup_{x \in U_{(R_0, S_0)}} f(x, y_0) \). Then in view of the Cauchy inequality we get

\[
|a_t(x)| \leq M |y_0|^{-s}.
\]

Estimating now \( v(a_t, t) \) as in Theorem 1, by means of Lemma 2 we find that \( f \) is holomorphic in domain (21) for \( n = 1 \). The case when the value is discrete is considered analogously. Thus the theorem is proved for \( n = 1 \), i.e., for two variables.

Now we suppose that the theorem holds for \( n - 1 \) variables \( y_i \). Let \( x \) be fixed. Then by virtue of condition \( (1) \) the function \( f \) is holomorphic in each of the variables \( y_i \) and therefore by the induction hypothesis it is holomorphic in all of them. Hence we can write it in the form

\[
f(x, y) = \sum_{k \in \mathbb{Z}^n} a_k(x) y^k,
\]

where \( k = (k_1, \ldots, k_n) \), \( y^k = y_1^{k_1} \cdots y_n^{k_n} \). Fixing \( y_2, \ldots, y_n \) we obtain by Theorem 1 that the \( a_k(x) \) are holomorphic in \( U_{(R_0, S_0)} \). Further, as in Theorem 1, one proves that \( f \) is bounded in the domain \( U_{[R_0, S_0]} \times V'_1 \times \cdots \times V'_n \), where \([R'_0, S'_0] \subset (R_0, S_0)\) and \( V'_i \) is a closed disk in \( V_i \). Let in this domain \( |f(x, y)| \leq M \). Without loss of generality we can think that \( 0 \in V'_i \), \( i = 1, \ldots, n \). Then by (31) in view of the Cauchy inequality we obtain

\[
|a_k(x)| \leq M r_1^{-k_1} r_2^{-k_2} \cdots r_n^{-k_n}, \quad \text{where } r_i \text{ is the radius of } V'_i.
\]

Then

\[
\log \mu(a_k, t) \leq \log M - k_1 \log r_1 - \cdots - k_n \log r_n \leq A |k|,
\]

where \( |k| = |k_1| + \cdots + |k_n| \), \( t \in (R'_0, S'_0) \). From here as in Theorem 1 we obtain the inequality \( \nu(a_k, t) \leq A \ |k| \), and using Lemma 2 we prove that \( f \) is holomorphic in domain (21). In a similar way we consider the case of discrete valuation. Theorem 2 is proved completely.

**References**