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Two-Dimensional Euler Equations in a Time Dependent Domain

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We consider the following Euler equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u = -\nabla p + f, & x \in \Omega_t, \quad t > 0, \\ \operatorname{div} u = 0, & x \in \Omega_t, \quad t > 0, \\ u \cdot v = 0, & x \in \partial\Omega_t, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega_0. \end{cases} \tag{1.1}$$

Here $u = u(x, t) = (u_1, u_2)$ and $p = p(x, t)$ denote the unknown velocity vector field and the pressure of the ideal fluid at point $(x, t) \in \Omega_t \times \{t\}$, respectively, while $u_0(x)$ is a given initial velocity vector field, and $f = f(x, t)$ is the external force vector field. v is the unit outward normal vector of $\partial\Omega_t$. Let Ω_0 denote the region filled a fluid at $t = 0$, while Ω_t represents the region where the fluid is located at time $t > 0$. Let $\Omega_t (t \geqslant 0)$ be a given bounded, simply connected domain in R^2 with smooth boundary.

From a physical point of view, a real fluid is evolutional, so the region filled with a moving fluid usually move along the trajectories of the incompressible fluid motion. Thus, the space-time domain is not a cylindrical one as often treated. So we treat with the case of a noncylindrical space-time

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domains in this paper. In fact, we are interested in the following questions on the Euler Eqs. (1.1):

(1) Under what conditions on Ω_t, does there exists a weak solution to (1.1) for the given initial data (u_0, f)?

(2) How about the uniqueness and regularity of the weak solution, if it exists?

(3) If the movement of Ω_t is periodic and f is periodic, then does there exist a periodic solution to (1.1)?

(4) If the domain Ω_t move along the trajectories of the fluid motion, how about the questions (1)–(3) above?

In this paper we study above questions and give an affirmative but partial answer to questions (1)–(4). Similar questions were studied for incompressible Navier–Stokes equations in [1, 8, 13].

Firstly, we study the existence, periodicity, uniqueness and regularity of weak solutions to the Euler equations in a time dependent domain, which is diffeomorphic to a cylindrical one. It should be noticed that (1.1) is not a free boundary problem in this case. In a analogous time dependent domain, a weak solution of Navier–Stokes equations was constructed in [1, 8, 13], and in [10], the existence and uniqueness of a global classical solution have been obtained for two-dimensional Euler equations by Schauder fixed point theorem. While in a cylindrical space-time one, the existence, uniqueness and regularity of weak solutions to the Euler equations have already been studied, and these problems are much better understood, see Kato (1967), Marchioro and Pulvirenti (1994) and Lions (1996). We first establish the existence of a weak solution of the Euler equations in a noncylindrical space-time domain which can be reduced to a cylindrical space-time one by a diffeomorphism, with the initial data such that the initial velocity $u_0 \in L^2(\Omega)$ and the initial vorticity $\omega_0 \in L^r(\Omega)$ for $1 < r \leqslant \infty$. For this purpose, we combine the method discussing the Navier–Stokes equations in a time dependent domain with the technique treating the Euler equations in a cylindrical space-time domain, and construct the approximate solutions by making use of Navier–Stokes equations in the same time dependent domain with modified Dirichlet boundary conditions. Moreover, if the movement of Ω_t and the diffeomorphism are periodic with period $T > 0$, then the weak solution is also periodic with some initial velocity $u_0(x) = u(x, T)$ in $H^r(\text{curl}, \Omega_0) = \{v \mid v \in L^2(\Omega_0), \text{curl}\, v \in L^r(\Omega_0)\}$ for $1 < r \leqslant \infty$. Further, we obtain the existence of a weak solution of the Euler equations in $\Omega_t = \Psi_t(\Omega_0)$, in which $\Psi_t(x)$ denotes the path line starting from $x \in \Omega_0$, that is, the solution of the ordinary differential equation

$$\begin{cases} \dfrac{d}{dt}\Psi_t(x) = u(\Psi_t(x), t), \\ \Psi_0(x) = x \end{cases} \tag{1.2}$$

for any $x \in \Omega_0$. Applying DiPerna–Lions theory for ordinary differential equations in [5], the generalized flow can be defined for weak solutions of the Euler equations, as made in Desjardins [3, 4]. In fact, our analysis show that, if $u_0 \in L^2(\Omega_0)$, $\omega_0 = \text{curl}\, u_0 \in L^\infty(\Omega_0)$ and $f \in L^1(0, \infty; W^{1,\infty}(R^2))$, then there exists a unique domain Ω_t such that each Ω_t is diffeomorphic to Ω_0 for all $t \geqslant 0$. Next we discuss the uniqueness of weak solutions of the Euler equations. The first result was established by Yudovitch [16] when the initial vorticity is bounded. And same results are obtained for Euler equations in two-dimensional bounded domain and whole space, see Lions (1996) and its literature. In a analogous time dependent domain, Kozono (1985) discussed the uniqueness of a global classical solution. In this paper, motivated by the approach used in [12] to discuss the uniqueness of weak solutions, we establish the local dependence on initial data provided $\|\omega_0^1 - \omega_0^2\|_{r,\Omega}$ $(1 < r < 2)$ small, $\omega^i \in L^\infty(\Omega_0)$ and $f^i = 0$. This implies the uniqueness of weak solutions in this case. Furthermore we study the regularity when $u_0 \in W^{k,r}(\Omega_0)$ and $f \in L^1(0, \infty; W^{k,r}(R^2))$ with $k \geqslant 1$ or $u_0 \in C^{k,\alpha}(\bar{\Omega}_0)$ and $f \in L^1(0, \infty; BC^{k,\alpha}(R^2))$ with $k \geqslant 1$ and $\alpha \in (0, 1)$. For the case of bounded domain and whole space R^2, the corresponding results have been established and are summed up in Lions (1996).

2. TRANSFORMATION OF THE EQUATIONS

In order to show the existence and periodicity of weak solutions, we need to reduce the Eqs. (1.1) to the one in a cylindrical space-time domain, as made by Inoue and Wakimoto (1977), Miyakawa and Teramoto (1982). Hence we assume that there exists a diffeomorphism which reduce the given time dependent domain to a cylindrical one. Let $Q_\infty = \bigcup_{t \in R} \Omega_t \times \{t\}$ be a noncylindrical space-time domain. Then we make the following assumptions on the domain Ω_t.

Assumption 1. There exist a cylindrical domain $\tilde{Q}_\infty = \tilde{\Omega} \times R$ and a level-preserving C^∞ diffeomorphism $\Phi : (y, s) = \Phi(x, t) = (\Phi^1(x, t), \Phi^2(x, t), t)$ from $\bar{Q}_\infty \to \bar{\tilde{Q}}_\infty$, which satisfies

$$\det\left(\frac{\partial \Phi^i(x, t)}{\partial x_j}\right) \equiv J^{-1}(t) > 0 \tag{2.1}$$

for all $(x, t) \in \bar{Q}_\infty$.

Assumption 2. The derivatives $\partial\Phi^i/\partial x_j$ and $\partial\Phi^i/\partial t (1 \leqslant i, j \leqslant 2)$ are bounded functions on $\bar{Q}_\infty$. $J(t)$ is bounded in R^+.

According to the discussion in Section 4 in Miyakawa and Teramoto (1982), the assumption (2.1) for Jacobian is of no restriction. Meanwhile, it is worth to point out, for classical solution, that $J(t) \equiv 1$ for the case that $\Omega_t = \Psi_t(\Omega_0)$, which will be discussed in Section 5.

Let

$$\tilde{u}^i(y, s) = \frac{\partial y_i}{\partial x_1} u^1(\Phi^{-1}(y, s)) + \frac{\partial y_i}{\partial x_2} u^2(\Phi^{-1}(y, s)),$$

$$\tilde{u}_0^i(y) = \frac{\partial y_i}{\partial x_1} u_0^1(\Phi^{-1}(y, 0)) + \frac{\partial y_i}{\partial x_2} u_0^2(\Phi^{-1}(y, 0)),$$

$$\tilde{f}^i(y, s) = \frac{\partial y_i}{\partial x_1} f^1(\Phi^{-1}(y, s)) + \frac{\partial y_i}{\partial x_2} f^2(\Phi^{-1}(y, s)),$$

$$\tilde{p}(y, s) = p(\Phi^{-1}(y, s))$$

for $i = 1, 2$. Then problem (1.1) can be reduced to the following problem in a cylindrical domain for $\tilde{u} = (\tilde{u}^1, \tilde{u}^2)$ and $\tilde{p}$:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial s} + M\tilde{u} + N\tilde{u} = -\nabla_g \tilde{p} + \tilde{f}, & y \in \tilde{\Omega}, \quad s > 0, \\ \operatorname{div} \tilde{u} = 0, & y \in \tilde{\Omega}, \quad s > 0, \\ \tilde{u} \cdot \tilde{v} = 0, & y \in \partial\tilde{\Omega}, \quad s > 0, \\ \tilde{u}(y, 0) = \tilde{u}_0(y), & y \in \tilde{\Omega}. \end{cases} \tag{2.2}$$

Where $\tilde{f} = (\tilde{f}^1, \tilde{f}^2)$ and $\tilde{u}_0 = (\tilde{u}_0^1, \tilde{u}_0^2)$. $\tilde{v}$ denotes the unit exterior normal along $\partial\tilde{\Omega}$. And $(M\tilde{u})^i = (\partial y_j/\partial t)\, \nabla_j \tilde{u}^i + (\partial y_i/\partial x_k)(\partial^2 x_k/\partial s \partial y_j)\, \tilde{u}^j$, $(N\tilde{u})^i = \tilde{u}^j\, \nabla_j \tilde{u}^i$, $(\nabla_g \tilde{p})^i = g^{ij} \partial\tilde{p}/\partial y_j$, $\nabla_j \tilde{u}^i = \partial\tilde{u}^i/\partial y_j + \tilde{u}^k (\partial y_i/\partial x_l)(\partial^2 x_l/\partial y_j \partial y_k)$, $g^{ij} = (\partial y_i/\partial x_k)(\partial y_j/\partial x_k)$, $g_{ij} = (\partial x_k/\partial y_i)(\partial x_k/\partial y_j)$. For more details, see Inoue and Wakimoto (1977). From now on, we use the summation convention, i.e. take sum over repeated indices. Moreover, we let $\tilde{v}$ denote the vector field on $\tilde{Q}$ obtained by transformation $\tilde{v}^j(y, s) = \partial y_j/\partial x_k \cdot v^k(\Phi^{-1}(y, s))$ for each vector field v on Q_∞. Conversely, v is the vector field obtained by inverse transformation for $\tilde{v}$. As pointed out in Inoue and Wakimoto (1977), the divergence operator is left invariant under the coordinate transformation. By the assumption 1, it is not difficult to see that $(g^{ij})^{-1} = (g_{ij})$ and $\det(g_{ij}) = J^2(t)$. Also it is worth to note that $\partial\tilde{u}/\partial s + M\tilde{u}$ and $N\tilde{u}$ correspond respectively to $\partial u/\partial t$ and $(u \cdot \nabla)\, u$ under the transformation Φ.

LEMMA 1. (1) *The matrixes* (g_{ij}) *and* (g^{ij}) *are positive definite and bounded.*

(2) *The derivatives* $\partial x_i/\partial y_j (1 \leqslant i, j \leqslant 2)$ *are bounded functions on* $\tilde{Q}_\infty$.

Proof. Due to the assumption 1 and 2, (1) is obvious. By the fact $(\partial x_i/\partial y_k)\cdot(\partial y_k/\partial x_j)=\delta_{ij}$ and the assumption 2, (2) follows. ∎

Before giving the definition of weak solutions, we first introduce some notations. Let $L^p(\Omega_t)$, $1\leqslant p \leqslant +\infty$, represent the usual Lesbegue space of scalar functions as well as that of vector functions, with norm $\|\cdot\|_{p,\Omega_t}$. Let $C^\infty_{0,\sigma}(\Omega_t)$ denote the set of all C^∞ real vector functions $\phi=(\phi_1,\phi_2)$ with compact support in Ω_t, such that $\operatorname{div}\phi=0$. $W^{k,p}(\Omega_t)$ is the usual Sobolev space of order (k, p). $W^{k,p}_0(\Omega_t)=$ completion of $C^\infty_0(\Omega_t)$ in $W^{k,p}(\Omega_t)$, while $W^{-k,p}(\Omega_t)$ is the dual space of $W^{k,p'}_0(\Omega_t)(1/p+1/p'=1)$. $C^{k,\alpha}(\Omega_t)$, $k\geqslant 1$ and $\alpha\in(0,1)$, is the usual Hölder continuous functions space. $BC^{k,\alpha}(\Omega_t)$ denote the space of functions whose derivatives up to k are continuous and bounded in Ω_t. We may analogously define the function spaces of functions defined on $\tilde{\Omega}$. Let H_t is the closure of $C^\infty_{0,\sigma}(\Omega_t)$ with respect to $\|\cdot\|_{2,\Omega_t}$. Finally, given a Banach space X with norm $\|\cdot\|_X$, we denote by $L^p(0, T; X)$, $1\leqslant p\leqslant +\infty$, the set of functions $f(t)$ defined on $(0, T)$ with values in X such that $\int_0^T \|f(t)\|_X^p\,dt<+\infty$ for $1\leqslant p<\infty$; $\sup_{[0,T]}\|f(t)\|_X<\infty$ for $p=\infty$. In general, $g\in L^p(0,T; L^r(\Omega_t))$ means $\int_0^T \|g(t)\|^p_{r,\Omega_t}\,dt<\infty$, and we may define the corresponding space if $L^r(\Omega_t)$ is replaced by Sobolev spaces or others. In the end, by symbol C, we represent a generic constant.

Now we may define a weak solution to the Euler Eqs. (1.1).

DEFINITION. A velocity field $u\in L^\infty(0, T; L^2(\Omega_t))$ for any $T>0$ is called a weak solution of the Euler equations with initial data $u_0(x)$ and external force field $f(x,t)$ provided that

(i) for all test function $\tilde{\phi}\in C^\infty_{0,\sigma}(\tilde{\Omega}\times[0,\infty))$

$$\int_0^\infty \int_{\tilde{\Omega}} g_{ij}(y,s)\left(\tilde{u}^i(y,s)\frac{\partial\tilde{\phi}^j}{\partial s}+\tilde{u}^i M\tilde{\phi}^j - N\tilde{u}^i\cdot\tilde{\phi}^j(y,s)\right.$$

$$\left.+\tilde{f}^i(y,s)\,\tilde{\phi}^j(y,s)\right) J(s)\,dy\,ds$$

$$=-\int_{\tilde{\Omega}} g_{ij}(y,0)\,\tilde{u}^i_0(y)\,\tilde{\phi}^j(y,0)\,J(0)\,dy.$$

(ii) the velocity $\tilde{u}$ is incompressible in the weak sense, i.e. for all scalar function $\tilde{\phi} \in C_0^\infty(\tilde{\Omega} \times R^+)$

$$\int_{R^+} \int_{\tilde{\Omega}} \nabla\tilde{\phi} \cdot \tilde{u} \, dy \, ds = 0,$$

(iii) $\tilde{u} \cdot \tilde{v} = 0$ on $\partial\tilde{\Omega} \times (0, \infty)$.

3. EXISTENCE OF WEAK SOLUTIONS

In this section, we obtain the first result on the existence of weak solutions to (1.1).

THEOREM 1. *Let* $u_0 \in H_0$, $\omega_0 = \text{curl}\, u_0 \in L^r(\Omega_0)$ *and* $f \in L^1(0, \infty; W^{1,r}(\Omega_t))$ *for* $1 < r \leqslant \infty$, *then there exists a weak solution* $u \in L^\infty(0, \infty; L^2(\Omega_t))$, *which satisfies that*

$$\nabla u, \qquad \omega \in L^\infty(0, +\infty; L^r(\Omega_t)) \tag{3.1}$$

if $1 < r < \infty$; *and*

$$\nabla u \in L^\infty(0, \infty; L^p(\Omega_t)), \qquad \omega \in L^\infty(0, \infty; L^\infty(\Omega_t)) \tag{3.2}$$

for any $1 < p < \infty$, *if* $r = \infty$. *Moreover*,

$$\frac{\partial u}{\partial t} \in \begin{cases} L^\infty(0, \infty; L^{2r/(4-r)}(\Omega_t)) & \text{if } 4/3 \leqslant r < 2, \\ L^\infty(0, \infty; L^p(\Omega_t)) \ \ \text{for} \ \ p \in [1, 2), & \text{if } r = 2, \\ L^\infty(0, \infty; L^r(\Omega_t)) & \text{if } r > 2, \\ L^\infty(0, \infty; L^p(\Omega_t)) \ \ \text{for} \ \ p \in [1, \infty), & \text{if } r = \infty. \end{cases} \tag{3.3}$$

Remark 1. 1. For the case that Q_∞ is a cylindrical space-time domain, analogous results have already been obtained. cf. Kato (1967), Lions (1996).

2. In a analogous time dependent domain, Kozono (1985) constructed a global classical solution.

3. When $r = 2$, the conditions of $u_0 \in H_0$ and $\omega_0 \in L^2(\Omega_0)$ are equivalent to $u_0 \in W^{1,2}(\Omega_0)$, (see Theorem 6.1, Chap. 7, Duvaut and Lions (1972)).

4. It is easy to see, from the procedure of proof given later, that the deducement of estimates (3.10) and (3.13) is independent of assumption 2. In order to obtain the strong convergence, we take transformation $\Phi^{-1}(y, s)$, and need the estimates (3.17) and (3.18), for which we have to

use the assumption 2. So the assumption about $\partial \Phi^i/\partial x_j$, $\partial \Phi^i/\partial t$ can be weakened to that $\partial \Phi^i/\partial x_j, \partial \Phi^i/\partial t \in L^\infty(0, T; L^p(\Omega_t))$ for $1 \leqslant i, j \leqslant 2$ and $p > \max\{1, r/2(r-1)\}$, if $1 < r < \infty$; some $p > 1$ if $r = \infty$. In fact, instead of (3.16), we deduce that $\|\tilde{\omega}_\varepsilon\|_{r', \tilde{\Omega}} \leqslant C(\|\omega_0\|_{r, \Omega_0} + \int_0^\infty \|\text{curl } f(\tau)\|_{r, \Omega_t} d\tau$ for $1/r' = 1/r + 1/2p$ and $1 < r' < r$. Through a similar procedure, we can show the strong convergence of u_ε in $L^2(0, T; L^2(\Omega_t))$ for any $T > 0$.

Proof of Theorem 1. We first construct the approximate solutions of Eqs. (1.1), utilizing the solutions to the Navier–Stokes equations in a corresponding time dependent domain with the modified boundary conditions, as made in Lions(1996). Namely, we introduce the equations

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon \Delta u_\varepsilon + (u_\varepsilon \cdot \nabla) u_\varepsilon = -\nabla p_\varepsilon + f, & x \in \Omega_t, \quad t > 0, \\ \text{div } u_\varepsilon = 0, & x \in \Omega_t, \quad t > 0, \\ u_\varepsilon \cdot \nu = 0, \quad \omega_\varepsilon = \text{curl } u_\varepsilon = 0, & x \in \partial\Omega_t, \quad t > 0, \\ u_\varepsilon(x, 0) = u_0(x), & x \in \Omega_0. \end{cases} \tag{3.4}$$

Equations (3.4) can be solved exactly as the usual Navier–Stokes equations with the usual Dirichlet boundary conditions in a time dependent domain, through reducing to the problem in a corresponding cylindrical domain. cf. [1, 8, 13]. Thus we obtain a unique weak solution $u_\varepsilon \in L^\infty(0, T; L^2(\Omega_t)) \cap L^2(0, T; W^{1,2}(\Omega_t))$ for all $T > 0$ and all $\varepsilon > 0$. Here we only give the modified energy inequality which implies the existence of a weak solution.

We multiply both sides of (3.4) by u_ε and integrate over Ω_t, to get

$$\int_{\Omega_t} \frac{\partial u_\varepsilon}{\partial t} \cdot u_\varepsilon \, dx + \varepsilon \|\nabla u_\varepsilon\|_{2, \Omega_t}^2 = \varepsilon \int_{\partial\Omega_t} u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial \nu} \, dS + \int_{\Omega_t} u_\varepsilon \cdot f \, dx. \tag{3.5}$$

Take the transformation $\Phi^{-1}(y, s)$. The first term at the right hand of (3.5) is

$$\begin{aligned} \int_{\Omega_t} \frac{\partial u_\varepsilon}{\partial t} \cdot u_\varepsilon \, dx &= \int_{\tilde{\Omega}} g_{ij}(y, s) \left(\frac{\partial \tilde{u}_\varepsilon^i}{\partial s} + (M\tilde{u})_\varepsilon^i \right) u_\varepsilon^j J(s) \, dy \\ &= \frac{1}{2} \frac{d}{ds} \int_{\tilde{\Omega}} g_{ij}(y, s) \, \tilde{u}_\varepsilon^i \tilde{u}_\varepsilon^j(y, s) \, J(s) \, dy. \end{aligned}$$

Here we have used the Lemma 2.7 in [13]. Returning to Ω_t, we obtain that

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon\|_{2, \Omega_t}^2 + \varepsilon \|\nabla u_\varepsilon\|_{2, \Omega_t}^2 = \varepsilon \int_{\partial\Omega_t} u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial \nu} \, dS + \int_{\Omega_t} u_\varepsilon \cdot f \, dx. \tag{3.6}$$

Since $u_\varepsilon \cdot \nu = 0$ on $\partial\Omega_t$, it is easy to deduce that

$$u_\varepsilon \cdot \nabla(u_\varepsilon \cdot \nu) = 0 \qquad \text{on } \partial\Omega_t, \qquad t > 0.$$

By the fact that $\omega_\varepsilon = 0$ on $\partial\Omega_t$, we may deduce that

$$u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial \nu} = -\kappa\, |u_\varepsilon|^2 \qquad \text{on } \partial\Omega_t, \qquad t > 0$$

where κ is the curvatures of $\partial\Omega_t$. By using the above fact and Cauchy inequality, the right hand side of (3.6) can be estimated as

$$\begin{aligned}
&\varepsilon \int_{\partial\Omega_t} u_\varepsilon \cdot \frac{\partial u_\varepsilon}{\partial \nu}\, dS + \int_{\Omega_t} u_\varepsilon \cdot f\, dx \\
&\quad \leqslant C\varepsilon \int_{\partial\Omega_t} |u_\varepsilon|^2\, dS + \|u_\varepsilon\|_{2, \Omega_t} \|f\|_{2, \Omega_t} \\
&\quad \leqslant C\varepsilon\, \|u_\varepsilon\|_{2, \Omega_t} \|\nabla u_\varepsilon\|_{2, \Omega_t} + \|u_\varepsilon\|_{2, \Omega_t} \|f\|_{2, \Omega_t} \\
&\quad \leqslant \frac{1}{2}\varepsilon\, \|\nabla u_\varepsilon\|^2_{2, \Omega_t} + \|u_\varepsilon\|_{2, \Omega_t} [C\varepsilon\, \|u_\varepsilon\|_{2, \Omega_t} + \|f\|_{2, \Omega_t}].
\end{aligned}$$

Here we have used the classical trace inequality and the Young inequality. Thus

$$\frac{d}{dt} \|u_\varepsilon\|^2_{2, \Omega_t} + \varepsilon\, \|\nabla u_\varepsilon\|^2_{2, \Omega_t} \leqslant \|u_\varepsilon\|_{2, \Omega_t} [C\varepsilon\, \|u_\varepsilon\|_{2, \Omega_t} + \|f\|_{2, \Omega_t}]$$

which implies that

$$\|u_\varepsilon\|_{2, \Omega_t} \leqslant e^{C\varepsilon t} \|u_0\|_{2, \Omega_0} + \int_0^t e^{C\varepsilon(t-\tau)} \|f(\tau)\|_{2, \Omega_\tau}\, d\tau \tag{3.7}$$

for all $t \geqslant 0$.

In the following, we consider the vorticity equation of u_ε

$$\begin{cases}
\dfrac{\partial \omega_\varepsilon}{\partial t} - \varepsilon \Delta \omega_\varepsilon + (u_\varepsilon \cdot \nabla)\, \omega_\varepsilon = \operatorname{curl} f, & x \in \Omega_t, \quad t > 0, \\
\omega_\varepsilon = 0, & x \in \partial\Omega_t, \quad t > 0, \\
\omega_\varepsilon(x, t=0) = \omega_0(x) & x \in \Omega_0.
\end{cases} \tag{3.8}$$

Let $1 < r < \infty$. Set $T_R(\omega_\varepsilon) = \max(\min(\omega_\varepsilon, R), -R)$ for any $R > 0$. We multiply both sides of (3.8) by $|T_R(\omega_\varepsilon)|^{r-2}\, T_R(\omega_\varepsilon)$ and integrate over Ω_t to get that

$$\int_{\Omega_t} \frac{\partial \omega_\varepsilon}{\partial t}\, |T_R(\omega_\varepsilon)|^{r-2}\, T_R(\omega_\varepsilon)\, dx + (r-1)\, \varepsilon \int_{\Omega_t} |T_R(\omega_\varepsilon)|^{r-2}\, |\nabla T_R(\omega_\varepsilon)|^2\, dx$$
$$= \int_{\Omega_t} \operatorname{curl} f \cdot |T_R(\omega_\varepsilon)|^{r-2}\, T_R(\omega_\varepsilon)\, dx. \tag{3.9}$$

But

$$\begin{aligned}
\int_{\Omega_t} \frac{\partial \omega_\varepsilon}{\partial t}\, |T_R(\omega_\varepsilon)|^{r-2}\, T_R(\omega_\varepsilon)\, dx
&= \frac{1}{r} \int_{\Omega_t} \frac{\partial}{\partial t}\, |T_R(\omega_\varepsilon)|^r\, dx \\
&= \frac{1}{r} \int_{\tilde{\Omega}} \left(\frac{\partial}{\partial s} + \frac{\partial y_i}{\partial s} \cdot \frac{\partial}{\partial y_i} \right) |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J(s)\, dy \\
&= \frac{1}{r} \frac{d}{ds} \int_{\tilde{\Omega}} |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J(s)\, dy \\
&\quad + \frac{1}{r} \int_{\tilde{\Omega}} \frac{\partial y_i}{\partial s} \cdot \frac{\partial}{\partial y_i}\, |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J(s)\, dy \\
&\quad - \frac{1}{r} \int_{\tilde{\Omega}} |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J'(s)\, dy.
\end{aligned}$$

By the fact (cf. Miyakawa and Teramoto (1982))

$$J'(t) = -J(t)\, \nabla_j \left(\frac{\partial y_j}{\partial t} \right) = -J(t)\, \frac{\partial}{\partial y_i} \left(\frac{\partial y_i}{\partial t} \right),$$

the third term at the right hand side of the last equality is

$$\begin{aligned}
&\frac{1}{r} \int_{\tilde{\Omega}} |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J(s)\, \frac{\partial}{\partial y_i} \left(\frac{\partial y_i}{\partial s} \right) dy \\
&\qquad = -\frac{1}{r} \int_{\tilde{\Omega}} \frac{\partial y_i}{\partial s} \cdot \frac{\partial}{\partial y_i}\, |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r\, J(s)\, dy.
\end{aligned}$$

Using the Green formula, it follows

$$\int_{\Omega_t} \frac{\partial \omega_\varepsilon}{\partial t} |T_R(\omega_\varepsilon)|^{r-2} T_R(\omega_\varepsilon)\, dx = \frac{1}{r} \frac{d}{ds} \int_{\tilde{\Omega}} |T_R(\omega_\varepsilon(\Phi^{-1}(y, s)))|^r J(s)\, dy$$

$$= \frac{1}{r} \frac{d}{dt} \|T_R(\omega_\varepsilon)\|^r_{r, \Omega_t}.$$

Therefore, by Hölder inequality and the last equality, we deduce from (3.9) that

$$\|T_R(\omega_\varepsilon)(t)\|_{r, \Omega_t} \leqslant \|T_R(\omega_\varepsilon)(0)\|_{r, \Omega_0} + \int_0^t \|\text{curl}\, f(\tau)\|_{r, \Omega_\tau}\, d\tau$$

for any $t > 0$. Let $R \to \infty$, we obtain that

$$\|\omega_\varepsilon(t)\|_{r, \Omega_t} \leqslant \|\omega_0\|_{r, \Omega_0} + \int_0^t \|\text{curl}\, f(\tau)\|_{r, \Omega_\tau}\, d\tau \tag{3.10}$$

for $t > 0$ and $1 < r < \infty$. It is obvious that (3.10) is valid for $r = \infty$.

Since $\text{div}\, u_\varepsilon = 0$, there exists a stream function ψ_ε such that

$$u_\varepsilon(x) = \nabla^\perp \psi_\varepsilon(x) = \left(\frac{\partial \psi_\varepsilon}{\partial x_2}, -\frac{\partial \psi_\varepsilon}{\partial x_1} \right)$$

and ψ_ε satisfies that

$$\begin{cases} -\Delta \psi_\varepsilon = \omega_\varepsilon & \text{in } \Omega_t \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega_t \end{cases}$$

Let G_{Ω_t} be the Green function with homogeneous boundary condition. Then

$$\psi_\varepsilon(x) = \int_{\Omega_t} G_{\Omega_t}(x, y)\, \omega_\varepsilon(y)\, dy$$

and

$$u_\varepsilon = \int_{\Omega_t} \nabla^\perp G_{\Omega_t}(x, y)\, \omega_\varepsilon(y)\, dy. \tag{3.11}$$

For Green function G_{Ω_t} and its derivatives, the following estimates hold

$$\begin{cases} |G_{\Omega_t}(x, y)| \leqslant C(\ln|x-y|+1), \\ |\nabla^\perp G_{\Omega_t}(x, y)| \leqslant C\,|x-y|^{-1}, \\ \left|\sum_{i=1}^{2} \frac{\partial}{\partial x_i} \nabla^\perp G_{\Omega_t}(x, y)\right| \leqslant C\,|x-y|^{-2}. \end{cases} \tag{3.12}$$

By the Calderón–Zygmund theorem on singular integral (cf. Stein (1970)), we deduce, from (3.10), (3.11), and (3.12), that

$$\begin{aligned} \|\nabla u_\varepsilon\|_{r, \Omega_t} &\leqslant C\,\|\omega_\varepsilon\|_{r, \Omega_t} \\ &\leqslant C\left(\|\omega_0\|_{r, \Omega_0} + \int_0^t \|\text{curl}\, f(\tau)\|_{r, \Omega_\tau}\, d\tau\right), \end{aligned} \tag{3.13}$$

for $1 < r < \infty$ and $t \geqslant 0$. Moreover, if $r > 1$,

$$C \leqslant C_1 \frac{r^2}{r-1} \tag{3.14}$$

where C_1 is a constant independent of r. This fact follows from the constant which is calculated carefully in Marcinkiewicz interpolation inequality (cf. [7]).

In two-dimensional case, we have

$$\Delta u_\varepsilon = \nabla\, \text{div}\, u_\varepsilon + \nabla^\perp\, \text{curl}\, u_\varepsilon.$$

This, combined to the fact that $\text{div}\, u_\varepsilon = 0$, implies

$$\Delta u_\varepsilon \cdot \nu = (\nu \cdot \nabla^\perp)\, \text{curl}\, u_\varepsilon \qquad \text{on } \partial\Omega_t, \quad t > 0.$$

Since vector $\nabla^\perp$ is along the tangent vector direction, we know

$$\Delta u_\varepsilon \cdot \nu = 0 \qquad \text{on } \partial\Omega_t, \quad t > 0.$$

Let P denote the orthogonal projection from $L^2(\Omega_t)$ to H_t. Applying the projection operator P to both sides of (3.4), we get formally

$$\frac{\partial u_\varepsilon}{\partial t} + P(u_\varepsilon \cdot \nabla)\, u_\varepsilon - \varepsilon\, \Delta u_\varepsilon = Pf.$$

Applying the estimate (3.13), it is easy to deduce that

$$\left\| \frac{\partial u_\varepsilon}{\partial t} \right\|_{W^{-1,q}(\Omega_t)} \leqslant C \tag{3.15}$$

for $0 < t < T$ and some $q \in (1, 2)$.

Since

$$\tilde{\omega}_\varepsilon = \operatorname{curl} \tilde{u}_\varepsilon = \left. \frac{\partial x_i}{\partial y_1} \frac{\partial y_2}{\partial x_j} \frac{\partial u_\varepsilon^j}{\partial x_i} \right|_{x=\Phi^{-1}(y,s)} - \left. \frac{\partial x_i}{\partial y_2} \frac{\partial y_1}{\partial x_j} \frac{\partial u_\varepsilon^j}{\partial x_i} \right|_{x=\Phi^{-1}(y,s)},$$

we deduce, with the help of the assumption 2, Lemma 1 and estimate (3.13), that

$$\|\tilde{\omega}_\varepsilon\|_{r,\tilde{\Omega}} \leqslant C \|\nabla u_\varepsilon\|_{r,\Omega_t} \leqslant C \left(\|\omega_0\|_{r,\Omega_0} + \int_0^t \|\operatorname{curl} f(\tau)\|_{r,\Omega_\tau} \, d\tau \right). \tag{3.16}$$

Due to the fact that $\operatorname{div} \tilde{u}_\varepsilon = 0$, it can be seen that there exists a stream function $\tilde{\psi}_\varepsilon$ such that

$$\tilde{u}_\varepsilon(x) = \nabla^\perp \tilde{\psi}_\varepsilon(x) = \left(\frac{\partial \tilde{\psi}_\varepsilon}{\partial y_2}, -\frac{\partial \tilde{\psi}_\varepsilon}{\partial y_1} \right)$$

and $\tilde{\psi}_\varepsilon$ satisfies

$$\begin{cases} -\Delta \tilde{\psi}_\varepsilon = \tilde{\omega}_\varepsilon & \text{in } \tilde{\Omega} \\ \tilde{\psi}_\varepsilon = 0 & \text{on } \partial \tilde{\Omega} \end{cases}$$

Similarly,

$$\tilde{u}_\varepsilon = \int_{\tilde{\Omega}} \nabla^\perp G_{\tilde{\Omega}}(x, y) \, \tilde{\omega}_\varepsilon(y) \, dy,$$

where $G_{\tilde{\Omega}}$ is the Green function with homogeneous boundary condition. The estimates similar to (3.12) are valid for $G_{\tilde{\Omega}}$. Hence

$$\begin{aligned} \|\nabla \tilde{u}_\varepsilon\|_{r,\tilde{\Omega}} &\leqslant C \|\tilde{\omega}_\varepsilon\|_{r,\tilde{\Omega}} \\ &\leqslant C \left(\|\omega_0\|_{r,\Omega_0} + \int_0^S \|\operatorname{curl} f(\tau)\|_{r,\Omega_\tau} \, d\tau \right), \end{aligned} \tag{3.17}$$

for $1 < r < \infty$ and $s \geqslant 0$. In view of the fact

$$\frac{\partial \tilde{u}_\varepsilon^j}{\partial s} = \left. \frac{\partial y_j}{\partial y_k} \frac{\partial u_\varepsilon^k}{\partial t} \right|_{y=\Phi(x,t)} - \left. \frac{\partial y_j}{\partial x_k} \frac{\partial y_i}{\partial t} \frac{\partial u_\varepsilon^k}{\partial y_i} \right|_{y=\Phi(x,t)},$$

we deduce, with the help of the assumption 2 and estimates (3.15) and (3.17), that

$$\left\| \frac{\partial \tilde{u}_\varepsilon}{\partial s} \right\|_{W^{-1,q}(\tilde{\Omega})} \leqslant C \tag{3.18}$$

for $0 < s < S$ and some $q \in (1, 2)$.

By the Kondrakov compact imbedding theorem, $W^{1,r}(\tilde{\Omega})$ can be compactly imbedded into $L^q(\tilde{\Omega})$ for $1 < q < 2r/(2-r)$ if $r < 2$; into $L^q(\tilde{\Omega})$ for $1 < q < \infty$ if $r \geqslant 2$. By using the compactness theorem (cf. Temam (1977) p. 271), together with the estimates (3.7), (3.10), (3.13), (3.17), and (3.18), we deduce that there exists a subsequence of $\{u_\varepsilon\}$ or $\{\tilde{u}_\varepsilon\}$ (still denote by itself) and $u \in L^2(0, \infty; L^2(\Omega_t))$ or $\tilde{u} \in L^\infty(0, \infty; L^2(\tilde{\Omega}))$ such that

$$\begin{cases} \nabla \tilde{u}_\varepsilon \text{ converges to } \nabla \tilde{u} \text{ in } L^\infty(0, \infty; L^r(\tilde{\Omega})) \text{ weak-star}, \\ \tilde{u}_\varepsilon \text{ converges to } \tilde{u} \text{ in } L^2(0, S; L^2(\tilde{\Omega})) \text{ strongly}, \\ \omega_\varepsilon \text{ converges to } \omega \text{ in } L^\infty(0, \infty; L^r(\Omega_t)) \text{ weak-star}, \\ \nabla u_\varepsilon \text{ converges to } \nabla u \text{ in } L^\infty(0, \infty; L^r(\Omega_t)) \text{ weak-star}. \end{cases} \tag{3.19}$$

This shows, by a routine argument, that $\tilde{u}$ is the weak solution of (1.2), which is the limit equations of the corresponding one reduced from Navier–Stokes Eqs. (3.4) by diffeomorphism Φ^{-1}. Thus u is a weak solution of (1.1). Since norm is weak lower-continuous, (3.7) (3.10), and (3.13) give us the estimates

$$\|u(t)\|_{2, \Omega_t} \leqslant \|u_0\|_{2, \Omega_0} + \int_0^t \|f(\tau)\|_{2, \Omega_\tau} \, d\tau,$$

$$\|\nabla u(t)\|_{r, \Omega_t} \leqslant C \left(\|\omega_0\|_{r, \Omega_0} + \int_0^t \|\text{curl}\, f(\tau)\|_{r, \Omega_\tau} \, d\tau \right),$$

$$\|\omega(t)\|_{r, \Omega_t} \leqslant \|\omega_0\|_{r, \Omega_0} + \int_0^t \|\text{curl}\, f(\tau)\|_{r, \Omega_\tau} \, d\tau$$

for $t > 0$ and $1 < r < \infty$. Using these estimates and

$$\frac{\partial u}{\partial t} = -P(u \cdot \nabla)\, u + Pf, \tag{3.20}$$

we obtain the estimates about $\partial u / \partial t$ which yield (3.3) for $1 < r < \infty$. Thus we complete the proof of Theorem 1 when $1 < r < \infty$.

Let $r=\infty$, then $\omega_0 \in L^q(\Omega_0)$ for $1<q<\infty$. From (3.10), it is known that $\omega_\varepsilon \in L^q(\Omega_t)$. Similar to the case of $1<r<\infty$, we obtain a weak solution $u \in L^\infty(0, \infty; L^2(\Omega_t))$ of (1.1), such that (3.2) hold. Moreover,

$$\|\nabla u(t)\|_{q, \Omega_t} \leqslant C\left(\|\omega_0\|_{q, \Omega_0} + \int_0^t \|\operatorname{curl} f(\tau)\|_{q, \Omega_\tau} \, d\tau \right). \tag{3.21}$$

Utilizing estimates (3.21) and (3.20), it is easy to deduce the estimate about $\partial u/\partial t$ which implies (3.3) for $r=\infty$. ∎

4. EXISTENCE OF PERIODIC WEAK SOLUTIONS

For simplicity, we discuss the case $f=0$ in this section. Let

$$H^r(\text{curl}, \Omega_t) = \{v \in L^2(\Omega_t), \operatorname{curl} v \in L^r(\Omega_t)\}$$

with norm

$$\|v\|_{H^r(\text{curl}, \Omega_t)} = \|v\|_{2, \Omega_t} + \|\operatorname{curl} v\|_{r, \Omega_t}$$

for $1<r\leqslant\infty$.

THEOREM 2. *Assume the movement of Ω_t and the diffeomorphism $\Phi(x, t)$ are periodic with period $T>0$. Then there exists a weak solution u of* (1.1) *in $L^\infty(0, \infty; W^{1, r}(\Omega_t))$ satisfying the definition of weak solution with some $u_0(x) = u(x, T)$ in $H^r(\text{curl}, \Omega_0)$ for $1<r\leqslant\infty$; in $L^\infty(0, \infty; W^{1, q}(\Omega_t))$ with some $u_0(x)=u(x, T)$ in $H^\infty(\text{curl}, \Omega_0)$ for $1<q<\infty$.*

Proof. We adopt the method discussing the periodic weak solution of Navier–Stokes equations in Miyakawa and Teramoto (1982). Let $\{w_j(x, t)\}_{j=1}^\infty$ be the basis of $C_{0, \sigma}^\infty(\Omega_t)$ and $\{w_j(x, t)\}_{j=1}^\infty$ be a Schmidt orthogonalization with respect to the L^2-inner product. For each $m \geqslant 1$, we arbitrarily take a function u_0^m from the subspace spanned by $\{w_j(x, 0)\}_{1 \leqslant j \leqslant m}$. Then we define the approximate solutions of (3.4)

$$u_\varepsilon^m = \sum_{i=1}^m h_{im}(t) \, w_i(x, t),$$

where $h_{im}(t)$ is defined by

$$\left(\frac{\partial u_\varepsilon^m}{\partial t}, w_j \right) + \varepsilon(\nabla u_\varepsilon^m, \nabla w_j) + ((u_\varepsilon^m \cdot \nabla) \, u_\varepsilon^m, w_j) = 0, \qquad 1 \leqslant j \leqslant m. \tag{4.1}$$

Where $(,)$ denote the L^2-inner product. Taking the transformation $x = \Phi^{-1}(y, s)$ in (4.1), we can determine u_ε^m with initial data u_0^m because of the estimate

$$\|u_\varepsilon^m(t)\|_{2, \Omega_t} \leqslant e^{C\varepsilon t} \|u_0^m\|_{2, \Omega_0}$$

which is obtained through the deducement similar to (3.7).

Let P_m be the projection from $C_{0, \sigma}^\infty(\Omega_t)$ to $\operatorname{span}\{w_1(x, t), ..., w_m(x, t)\}$, then

$$\frac{\partial u_\varepsilon^m}{\partial t} - \varepsilon \Delta u_\varepsilon^m + P_m(u_\varepsilon^m \cdot \nabla) u_\varepsilon^m = -P_m \nabla p.$$

Thus $\omega_\varepsilon^m = \operatorname{curl} u_\varepsilon^m$ satisfies

$$\frac{\partial \omega_\varepsilon^m}{\partial t} - \varepsilon \Delta \omega_\varepsilon^m + P_m(u_\varepsilon^m \cdot \nabla) \omega_\varepsilon^m = 0. \tag{4.2}$$

Similar to (3.10), we deduce that

$$\|\omega_\varepsilon^m(t)\|_{r, \Omega_t} \leqslant \|\omega_0^m\|_{r, \Omega_0}, \qquad 1 < r \leqslant \infty.$$

By the same procedure as before, we obtain a weak solution $u^m(t)$ in $L^\infty(0, \infty; W^{1, r}(\Omega_t))$ which satisfies

$$\frac{\partial u^m}{\partial t} + P_m(u^m \cdot \nabla) u^m = -P_m \nabla p,$$

and

$$\|u^m(t)\|_{2, \Omega_t} \leqslant \|u_0^m\|_{2, \Omega_0}, \tag{4.3}$$

$$\|\omega^m(t)\|_{r, \Omega_t} \leqslant \|\omega_0^m\|_{r, \Omega_0}. \tag{4.4}$$

And it is obvious that $u^m(t) \in \operatorname{span}\{w_1(x, t), ..., w_m(x, t)\}$. Hence (4.3) and (4.4) show that $\|u^m(T)\|_{H^r(\operatorname{curl}, \Omega_T)} \leqslant M$ if $\|u_0^m\|_{H^r(\operatorname{curl}, \Omega_0)} \leqslant M$. On the other hand, it is easy to show that the map: $u_0^m \to u^m(T)$ is continuous. By the assumption of periodicity, $u^m(T)$ and u_0^m are in the finite dimensional linear $\operatorname{span}\{w_1(x, 0), ..., w_m(x, 0)\} = \operatorname{span}\{w_1(x, T), ..., w_m(x, T)\}$. Thus, by the Brouwer fixed point theorem, there exists a u_0^m such that $u_0^m = u^m(T)$ and $\|u_0^m\|_{H^r(\operatorname{curl}, \Omega_0)} = \|u^m(T)\|_{H^r(\operatorname{curl}, \Omega_T)} \leqslant M$. Since M is independent of m, we may fix M, and take $m \to \infty$. Through the same argument as before, we may show that there exists a weak solution u of (1.1) in $L^\infty(0, \infty; W^{1, r}(\Omega_t))$ with some $u_0(x) = u(x, T)$ in $H^r(\operatorname{curl}, \Omega_0)$ for $1 < r < \infty$; in

$L^\infty(0, \infty; W^{1,q}(\Omega_t))$ with some $u_0(x) = u(x, T)$ in $H^\infty(\mathrm{curl}, \Omega_0)$ for arbitrary $1 < q < \infty$. ▮

5. FURTHER RESULT ON THE EXISTENCE OF WEAK SOLUTIONS

In this section we show the existence of weak solution of the Euler equations in $\Omega_t = \Psi_t(\Omega_0)$ with Ψ_t defined by (1.2). First we make a few remarks related to the integral curves of the velocity field, i.e., the trajectories of the incompressible fluid motion, in the sense of DiPerna–Lions theory for ordinary differential equations. In order to solve ordinary differential Eqs. (1.2), we use the notion of renormalized solutions introduced by DiPerna and Lions for equations related to incompressible fluid and for parabolic equation, see Lions (1996) and its literature. We refer to [3, 4] for other same settlement.

PROPOSITION. *Assume the conditions of Theorem 1 satisfied. Then the pathline Ψ_t: $[0, \infty) \times \Omega_0 \to \Omega_t$ defined almost everywhere such that*

(i) *Ψ_t is the unique renormalized solution of the following linear transport equation*

$$\begin{cases} \dfrac{\partial}{\partial t} \beta(\Psi_t) + u \cdot \nabla \beta(\Psi_t) = 0, \\ \beta(\Psi_t)|_{t=0} = \beta(x) \end{cases}$$

in the sense of distribution for $\beta \in C_0^\infty(R^2)$. And $\beta(\Psi_t) \in C([0, +\infty); L^p(\Omega_0))$ for any $1 \leqslant p < \infty$.

(ii) *For almost every $x \in \Omega_0$*

$$\begin{cases} \Psi_\cdot(x) \in C([0, +\infty)) \\ \Psi_t(x) = x + \displaystyle\int_0^t u(s, \Psi_s(x))\, ds. \end{cases}$$

The Proposition can be proved similar to that in [3, 4] and the Appendix E in Lions (1996).

In the following, we will show that

THEOREM 3. *Let $u_0 \in H_0$ and $\omega_0 = \mathrm{curl}\, u_0 \in L^\infty(\Omega_0)$, $f \in L^1(0, \infty; W^{1,\infty}(R^2))$, then there exists a weak solution $u \in L^\infty(0, \infty; L^2(\Omega_t))$ with $\Omega_t = \Psi_t(\Omega_0)$, which satisfies the definition of weak solution and*

$$
\begin{aligned}
&\omega = \operatorname{curl} u \in L^\infty(0, \infty; L^\infty(\Omega_t)), \\
&\nabla u \in L^\infty(0, \infty; L^q(\Omega_t)), \\
&\frac{\partial u}{\partial t} \in L^\infty(0, \infty; L^q(\Omega_t))
\end{aligned}
\tag{5.1}
$$

for any $1 < q < \infty$. *Moreover.*

$$
\begin{cases}
\|u(t)\|_{2, \Omega_t} \leqslant \|u_0\|_{2, \Omega_0} + |\Omega_0|^{1/2} \int_0^t \|f(\tau)\|_{\infty, R^2}\, d\tau, \\
\|\nabla u(t)\|_{r, \Omega_t} \leqslant C\left(\|\omega_0\|_{r, \Omega_0} + |\Omega_0|^{1/r} \int_0^t \|\operatorname{curl} f(\tau)\|_{\infty, R^2}\, d\tau \right), \qquad 1 < r < \infty, \\
\|\omega(t)\|_{r, \Omega_t} \leqslant \|\omega_0\|_{r, \Omega_0} + |\Omega_0|^{1/r} \int_0^t \|\operatorname{curl} f(\tau)\|_{\infty, R^2}\, d\tau, \qquad 1 < r \leqslant \infty.
\end{cases}
\tag{5.2}
$$

Remark 2. Our analysis show that if (u_0, f) satisfies the conditions in Theorem 3, then there exists a domain Ω_t such that each Ω_t is diffeomorphic to Ω_0 for all $t \geqslant 0$. Furthermore, by the result of Theorem 5, the domain Ω_t is unique for each $t \geqslant 0$.

Proof. We first define the sequences $\{u^k\}_{k \geqslant 0}$ of approximate solutions as follows:

$$
\begin{cases}
\dfrac{\partial u^0}{\partial t} + (u^0 \cdot \nabla)\, u^0 = -\nabla p^0 + f, & x \in \Omega_0, \quad t > 0, \\
\operatorname{div} u^0 = 0, & x \in \Omega_0, \quad t > 0, \\
u^0 \cdot \nu = 0, & x \in \partial\Omega_0, \quad t > 0, \\
u^0(x, 0) = u_0(x), & x \in \Omega_0.
\end{cases}
$$

and

$$
\begin{cases}
\dfrac{\partial u^k}{\partial t} + (u^k \cdot \nabla)\, u^k = -\nabla p^k + f, & x \in \Omega_k, \quad t > 0, \\
\operatorname{div} u^k = 0, & x \in \Omega_k, \quad t > 0, \\
u^k \cdot \nu = 0, & x \in \partial\Omega_k, \quad t > 0, \\
u^k(x, 0) = u_0(x), & x \in \Omega_0.
\end{cases}
\tag{5.3}
$$

for $k \geqslant 1$. Here $\Omega_k = \Psi_t^{k-1}(\Omega_{k-1})$ for $k \geqslant 1$, in which Ψ_t^{k-1} is the path lines from $x \in \Omega_0$, i.e. the solution of ordinary differential equation

$$\begin{cases} \dfrac{d}{dt} \Psi_t^{k-1}(x) = u^{k-1}(\Psi_t^{k-1}(x), t), \\ \Psi_0^{k-1}(x) = x \end{cases} \tag{5.4}$$

for any $x \in \Omega_0$. Note the facts that Jacobian of Ψ_t^k is one and $|\Omega_k| = |\Omega_0|$ if $\operatorname{div} u^k = 0$ and $u^k \in C^1(\bigcup_{t \in R^+} \Omega_k \times \{t\})$, cf. [2]. Without loss of generality, we assume $u_0 \in C_{0,\sigma}^\infty(\Omega_0)$ and $f \in L^1(0, +\infty;\ C_0^\infty(R^2))$. Otherwise, we may use vectors in $C_{0,\sigma}^\infty(\Omega_0)$ or $L^1(0, +\infty; C_0^\infty(R^2))$ to approximate u_0 and f. When $k=0$, it can be shown, by the classical existence result in a bounded domain (cf. [11]), that there exists a solution $u^0 \in C([0, +\infty; C^m(\Omega_0)) \cap L^\infty(0, \infty; W^{1,q}(\Omega_0))$ for any $m \geqslant 1$ and $1 < q < \infty$. Moreover, similar to the deducement of (3.10), (3.13), and (3.14), we obtain

$$\begin{cases} \|u^0(t)\|_{2,\Omega_0} \leqslant \|u_0\|_{2,\Omega_0} + \displaystyle\int_0^t \|f(\tau)\|_{2,\Omega_0}\, d\tau, \\ \|\omega(t)\|_{\infty,\Omega_0} \leqslant \|\omega_0\|_{\infty,\Omega_0} + \displaystyle\int_0^t \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau, \\ \|\nabla u^0(t)\|_{q,\Omega_0} \leqslant C \left\{ \|\omega_0\|_{q,\Omega_0} + |\Omega_0|^{1/q} \displaystyle\int_0^t \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau \right\} \\ \qquad \leqslant C \left\{ \|\omega_0\|_{\infty,\Omega_0} + \displaystyle\int_0^t \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau \right\} \end{cases} \tag{5.5}$$

for $0 < t < \infty$ and $1 < q < \infty$. Furthermore if $q > 1$, C can be estimated as

$$C \leqslant C_2 \left(\frac{q^2}{q-1} \right) \tag{5.6}$$

where C_2 is independent of q.

Taking $k=1$ in (5.4), we deduce that $\Psi_t^0 \in C^1([0, +\infty; C^m(\Omega_0))$ and

$$|\nabla \Psi_t^0(x)| \leqslant 1 + \int_0^t |\nabla u| \Big|_{\Psi_s^0(x)} |\nabla \Psi_s^0(x)|\, ds.$$

By Gronwall inequality,

$$|\nabla \psi_t^0(x)| \leqslant \exp\left\{ \int_0^t |\nabla u| \Big|_{\Psi_s^0(x)} ds \right\}.$$

Let $4 \leqslant p < \infty$, we deduce, by estimates (5.5) and (5.6), that

$$\|\nabla \Psi_t^0\|_{p,\Omega_0} \leqslant \left\| \exp\left\{ \int_0^t |\nabla u| \Big|_{\Psi_s^0(x)} ds \right\} \right\|_{p,\Omega_0}$$

$$\leqslant |\Omega_0|^{1/p} + \sum_{m=1}^{\infty} \frac{1}{m!} \left\| \int_0^t |\nabla u|\, ds \right\|_{mp,\Omega_0}^m$$

$$\leqslant |\Omega_0| + \sum_{m=1}^{\infty} \frac{1}{m!}$$

$$\times \left\{ C_2 mp \left(\|\omega_0\|_{\infty,\Omega_0} + \int_0^\infty \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau \right) t \right\}^m.$$

Set $T = 1/(2C_2 p e^C(\|\omega_0\|_{\infty,\Omega_0} + \int_0^\infty \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau))$, we deduce, since $m^m = m! \exp\{Cm\}$, that

$$\|\nabla \Psi_t^0\|_{p,\Omega_0} \leqslant C \tag{5.7}$$

for $0 \leqslant t \leqslant T$ and $p \geqslant 4$.

Continuing the procedure above, we can show

$$\|\nabla \Psi_t^0\|_{p,\Omega_0} \leqslant |\Omega_0| + \sum_{m=1}^{\infty} \frac{1}{m!}$$

$$\times \left\{ C_2 mp \left(\|\omega(T)\|_{\infty,\Omega_0} + \int_T^\infty \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau \right) t \right\}^m$$

$$\leqslant |\Omega_0| + \sum_{m=1}^{\infty} \frac{1}{m!}$$

$$\times \left\{ C_2 mp \left(\|\omega_0\|_{\infty,\Omega_0} + \int_0^\infty \|\operatorname{curl} f(\tau)\|_{\infty,R^2}\, d\tau \right) t \right\}^m.$$

So (5.7) is valid for $T \leqslant t \leqslant 2T$. Therefore, estimate (5.7) is valid for any $T > 0$. By remark (4) of Theorem 1, we have obtained a weak solution u^1 which satisfies (5.5) and (5.6). Employing the regularity result (see Theorem 8 in Section 7), we have $u^1 \in C([0, +\infty; C^m(\Omega_1)) \cap L^\infty(0, +\infty; W^{1,q}(\Omega_1))$ for any $m \geqslant 1$ and $1 < q < +\infty$. Continuing this procedure, we can define u^k for all $k \geqslant 1$, such that $u^k \in C([0, +\infty); C^m(\Omega_k)) \cap L^\infty(0, +\infty; W^{1,q}(\Omega_k))$ for any $m \geqslant 1$ and $1 < q < \infty$. Meanwhile, we have the estimates

$$\begin{cases} \|u^k(t)\|_{2,\Omega_k} \leqslant \|u_0\|_{2,\Omega_0} + |\Omega_0|^{1/2} \int_0^t \|f(\tau)\|_{\infty, R^2}, \\ \qquad t>0, \\ \|\omega^k(t)\|_{q,\Omega_k} \leqslant \|\omega_0\|_{q,\Omega_0} + |\Omega_0|^{1/q} \int_0^t \|\text{curl } f(\tau)\|_{\infty, R^2}, \\ \qquad t>0, \quad 1<q\leqslant\infty, \\ \|\nabla u^k(t)\|_{q,\Omega_k} \leqslant C\left(\|\omega_0\|_{\infty,\Omega_0} + \int_0^t \|\text{curl } f(\tau)\|_{\infty, R^2}\right), \\ \qquad t>0, \quad 1<q<\infty \end{cases} \tag{5.8}$$

holding uniformly for $k \geqslant 0$.

Let $p \geqslant 4$. Similar to the deducement of (5.7), we have

$$\|\nabla \Psi_t^k\|_{p,\Omega_k} \leqslant C(p,T), \qquad 0 \leqslant t \leqslant T \tag{5.9}$$

uniformly for $k \geqslant 1$.

Let $1/r = 1/(2p) + 1/q$ and $p \geqslant 4$. Instead of (3.16), we have

$$\begin{aligned} \|\tilde{\omega}^k\|_{r,\Omega_0} &\leqslant \|\nabla x\|_{p,\Omega_0} \|\nabla y\|_{p,\Omega_k} \|\nabla u^k\|_{q,\Omega_k} \\ &\leqslant C\left(\|\omega_0\|_{\infty,\Omega_0} + \int_0^\infty \|\text{curl } f\|_{\infty; R^2}\right) \end{aligned} \tag{5.10}$$

for $0 \leqslant t < T$. Here $x = \Psi_{-t}^{k-1}(y)$ and $y = \Psi_t^{k-1}(x)$ for $y \in \Omega_k$ and $x \in \Omega_0$. Thus, through a analogous procedure as before, we deduce

$$\|\nabla \tilde{u}^k\|_{r,\Omega_0} \leqslant C\left(\|\omega_0\|_{\infty,\Omega_0} + \int_0^t \|\text{curl } f(\tau)\|_{\infty, R^2}\right). \tag{5.11}$$

Since $p(\geqslant 4)$ and q in (5.8) and (5.9) are arbitrary, so $r(>1)$ in (5.11) is arbitrary. By the estimate (3.3), it follows similarly

$$\left\|\frac{\partial u^k}{\partial t}\right\|_{r,\Omega_k} \leqslant C$$

for $t>0$ and $r>1$. Similar to (3.18), we deduce that

$$\left\|\frac{\partial \tilde{u}^k}{\partial s}\right\|_{r',\Omega_0} \leqslant C, \qquad 0 \leqslant s \leqslant S \tag{5.12}$$

uniformly for $k \geqslant 0$ and $1/r' = 1/(2p) + 1/r$, $p \geqslant 4$.

Employing (5.11) and (5.12), we deduce that there exists a u in $L^\infty(0, \infty; W^{1,r}(\Omega_t))$ such that

$$\begin{cases} u^k \rightharpoonup u & \text{in } L^\infty(0, \infty; W^{1,r}(\Omega_t)) \text{ weak-star} \\ \tilde{u}^k \to \tilde{u} & \text{in } L^\infty(0, S; L^\infty(\Omega_0)) \text{ strongly for any } \quad S > 0. \end{cases} \tag{5.13}$$

Now it is routine to show that $\tilde{u}$ satisfies the definition of weak solution with Ω_0 instead of $\tilde{\Omega}$. Thus u is the weak solution in $L^\infty(0, \infty; W^{1,q}(\Omega_t))$ which satisfies (5.1) and (5.2). ▮

6. UNIQUENESS OF WEAK SOLUTION

In this section, we discuss the uniqueness of weak solutions to the Euler equations in a time dependent domain. The first result was established by Yudovitch with bounded initial vorticity. Later on, this result was extended to bounded domain in R^2 and whole space R^2, See Lions (1996) and its literature. In a time dependent domain, the uniqueness of the classical solution was obtained by Kozono (1985). Here we extend the result of Yudovitch to the case of a time dependent domains. For simplicity, we only consider the case $f = 0$ in this section.

THEOREM 4. *Let $u_0^i \in H_0$ and $\omega_0^i \in L^\infty(\Omega_0)$ for $i = 1, 2$. Let u^i be the weak solutions to* (1.1) *corresponding to the initial data (u_0^i, ω_0^i). If $A = \|\omega_0^1 - \omega_0^2\|_{r, \Omega_0} \leqslant 1$ for some $r \in (1, 2)$, then*

$$\|u^1(t) - u^2(t)\|_{1, \Omega_t} \leqslant C(A + A^{1 - C_3 Bt} \ln A^{-1}) \tag{6.1}$$

for any t, $0 < t < \min\{1, 1/(C_3 B)\}$. Especially, if $A \leqslant e^{-1/\alpha} (0 < \alpha < 2/3)$, then

$$\|u^1(t) - u^2(t)\|_{1, \Omega_t} \leqslant C A^{2/3 - \alpha} \tag{6.2}$$

for any t, $0 < t < T = \min\{1, 1/(3C_1 B)\}$. Here $B = \|\omega_0^1\|_{\infty, \Omega_0}$.

Proof. Let $\Psi_t(x)$ denote the path lines starting from $x \in \Omega_0$ defined by (1.2). Let $u^i (i = 1, 2)$ be the weak solutions corresponding to the initial vorticity ω_0^i. The corresponding path line is represented by Ψ_t^i $(i = 1, 2)$. Thus

$$\begin{aligned} |\Psi_t^1(x) - \Psi_t^2(x)| \leqslant & \left| \int_0^t (u^1(\Psi_s^1(x)) - u^1(\Psi_s^2(x)))\, ds \right| \\ & + \left| \int_0^t (u^1(\Psi_s^2(x)) - u^2(\Psi_s^2(x)))\, ds \right|. \end{aligned}$$

Let $\omega^i = \operatorname{curl} u^i$, then ω^i satisfies that

$$\frac{\partial}{\partial t}\omega^i(x,t) + (u^i \cdot \nabla)\,\omega^i = 0,$$

which implies that

$$\omega^i(x,t) = \omega_0^i(\Psi^i{}_{-t}(x)). \tag{6.3}$$

For the weak solution to (1.1), we may deduce the same expression as (3.11), i.e.

$$u(x,t) = \int_{\Omega_t} \nabla^{\perp} G_{\Omega_t}(x,y)\cdot \omega(y)\,dy. \tag{6.4}$$

By the estimate

$$\int_{\Omega_t} |\nabla^{\perp}(G_{\Omega_t}(x,y) - G_{\Omega_t}(x',y))|\,dy \leqslant C(1+|\Omega_t|)\,\Phi(|x-x'|) \tag{6.5}$$

where (cf. [12] p. 67)

$$\Phi(r) = \begin{cases} r(1-\ln r) & \text{if } \ r<1,\\ 1 & \text{if } \ r\geqslant 1,\end{cases}$$

we have

$$\begin{aligned}
&|u^1(\Psi_s^1(x)) - u^1(\Psi_s^2(x))|\\
&\quad\leqslant \int_{\Omega_s} |\nabla^{\perp}(G_{\Omega_s}(\Psi_s^1(x),y) - G_{\Omega_s}(\Psi_s^2(x),y))|\,|\omega^1(y)|\,dy\\
&\quad\leqslant CB(1+|\Omega_0|)\,\Phi(|\Psi_s^1(x) - \Psi_s^2(x)|)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_s} |u^1(x,s) - u^2(x,s)|\,dx\\
&\quad\leqslant \int_{\Omega_s}\left|\int_{\Omega_s} \nabla^{\perp} G_{\Omega_s}(x,y)(\omega_0^1(\Psi^1{}_{-s}(y)) - \omega_0^2(\Psi^2{}_{-s}(y)))\,dy\right| dx\\
&\quad\leqslant \int_{\Omega_s}\left|\int_{\Omega_s} \nabla^{\perp} G_{\Omega_s}(x,y)(\omega_0^1(\Psi^1{}_{-s}(y)) - \omega_0^1(\Psi^2{}_{-s}(y)))\,dy\right| dx\\
&\qquad+ \int_{\Omega_s}\left|\int_{\Omega_s} \nabla^{\perp} G_{\Omega_s}(x,y)(\omega_0^1(\Psi^2{}_{-s}(y)) - \omega_0^2(\Psi^2{}_{-s}(y)))\,dy\right| dx
\end{aligned}$$

$$\leqslant \int_{\Omega_s} \left| \int_{\Omega_0} \nabla^{\perp}(G_{\Omega_s}(x, \Psi_s^1(y)) - G_{\Omega_s}(x, \Psi_s^2(y)))\, \omega_0^1(y)\, dy \right| dx$$

$$+ C\, |\Omega_0|^{1-(1/r)} \left\| \int_{\Omega_s} \frac{1}{|x-y|} (\omega_0^1(\Psi^2_{-s}(y)) - \omega_0^2(\Psi^2_{-s}(y)))\, dy \right\|_{r, \Omega}$$

$$\leqslant CB \int_{\Omega_0} \int_{\Omega_s} \left| \frac{1}{|x - \Psi_s^1(y)|} - \frac{1}{|x - \Psi_s^2(y)|} \right| dx\, dy + CA$$

$$\leqslant CB \int_{\Omega_0} \Phi(|\Psi_s^1(y) - \Psi_s^2(y)|)\, dy + CA,$$

here we have used estimates (5.1), (6.3) and (6.5), the Hardy–Littlewood–Sobolev inequality (cf. Stein (1970)). Let

$$\delta(t) = \frac{1}{|\Omega_0|} \int_{\Omega_0} |\Psi_t^1(x) - \Psi_t^2(x)|\, dx.$$

Hence

$$\delta(t) \leqslant \frac{1}{|\Omega_0|}\, CB \int_0^t \int_{\Omega_0} \Phi(|\Psi_s^1(x) - \Psi_s^2(x)|)\, dx\, ds + CA$$

for $0 \leqslant t \leqslant 1$, where we have used the fact that

$$\int_{\Omega_0} (u^1(\Psi_s^2(x), s) - u^2(\Psi_s^2(x)))\, dx = \int_{\Omega_s} (u^1(y, s) - u^2(y, s))\, dy,$$

since $x \in \Omega_0 \Leftrightarrow \Psi_s^2(x) \in \Omega_s$. By Jensen inequality, it holds

$$\frac{1}{|\Omega_0|} \int_{\Omega_0} \Phi(f)\, dx \leqslant \Phi\left(\frac{1}{|\Omega_0|} \int_{\Omega_0} f(x)\, dx \right)$$

for any convex function Φ. Hence

$$\delta(t) \leqslant CB \int_0^t \Phi(\delta(s))\, ds + CAt.$$

Since $\Phi(r) \leqslant -(\ln \varepsilon)\, r + \varepsilon$ for arbitrary $0 < \varepsilon < 1$, it follows

$$\delta(t) \leqslant CB(-\ln \varepsilon) \int_0^t \delta(s)\, ds + C(A + \varepsilon B),$$

for $0 < t \leqslant 1$. Applying the Gronwall inequality, we obtain

$$\delta(t) \leqslant C(A + \varepsilon B)\, e^{-(\ln \varepsilon)\, CBt}$$

for $0<t\leqslant 1$. Let $\varepsilon=A\leqslant 1$, we deduce that

$$\delta(t)\leqslant CA^{1-C_3Bt}. \tag{6.6}$$

Moreover, from the procedure of deducement above, we get that

$$\begin{aligned}\int_{\Omega_t}|u^1(x,t)-u^2(x,t)|\,dx &\leqslant CB\int_{\Omega_0}\Phi(|\Psi_t^1(y)-\Psi_t^2(y)|)\,dy+CA\\ &\leqslant CB\{\varepsilon+\delta(\ln\varepsilon^{-1})\}+CA\\ &\leqslant CB(\varepsilon-(\ln\varepsilon)\,A^{1-C_3Bt})+CA.\end{aligned}$$

Let $\varepsilon=A$, we obtain (6.1). Noticing the inequality $\delta^{\alpha}\ln\delta^{-1}\leqslant 1$ if $0<\delta\leqslant e^{-1/\alpha}$ and taking $T=\min\{1, 1/(3C_3B)\}$, we have estimate (6.2). ∎

If $\omega^1=\omega^2$, then $u^1=u^2$ for $x\in\Omega_t$, $0\leqslant t\leqslant T$, from Theorem 4. Since T only depends on $\|\omega_0^1\|_{\infty,\Omega_0}$ and $|\Omega_0|$, we deduce the following theorem, by iterating the procedure of deducement of (6.1) and using the fact that $\|\omega^1(t)\|_{\infty,\Omega}\leqslant\|\omega_0^1\|_{\infty,\Omega_0}$ for any $t>0$.

THEOREM 5. *If $\omega_0\in L^{\infty}(\Omega_0)$, then the solution to* (1.1) *is unique.*

7. REGULARITY OF WEAK SOLUTIONS

THEOREM 6. *If $u_0\in W^{2,r}(\Omega_0)$ and $f\in L^1(0,\infty;W^{2,r}(R^2))$, then for every $T>0$, $u\in L^{\infty}(0,T;W^{2,r}(\Omega_t))$, if $r>2$; if additionally $\omega_0\in L^{\infty}(\Omega_0)$, then $u\in L^{\infty}(0,T;W^{2,r'}(\Omega))$ for $r'\in(1,r)$, if $1<r\leqslant 2$.*

Proof. Employing equality (6.3), we deduce that

$$|\nabla\omega(x,t)|\leqslant|\nabla\omega_0|\,|\nabla\Psi_{-t}(x)|+\int_0^t|\nabla f|\cdot|\nabla\Psi_{s-t}|\,ds. \tag{7.1}$$

In order to obtain the estimate about $|\nabla\omega|$, we need the estimate on $\|\nabla\Psi_{-t}\|_{r,\Omega_t}$ for any large r. Let $r>2$, then $\nabla u_0\in L^{\infty}(\Omega_0)$ and $\nabla f\in L^1(0,\infty;L^{\infty}_{loc}(R^2))$. Hence, through the same procedure as the deducement of (5.7), we have

$$\nabla\Psi_t\in L^{\infty}(0,T;L^p(\Omega)),\qquad\forall p\geqslant 4. \tag{7.2}$$

From (7.1),

$$\|\nabla\omega\|_{r',\,\Omega_t} \leqslant \|\nabla\omega_0\|_{r,\,\Omega_t}\,\|\nabla\Psi_{-t}\|_{p,\,\Omega_t} + \int_0^t \|\operatorname{curl} f(\tau)\|_{r,\,R^2}\,\|\nabla\Psi_{s-t}\|_{p,\,\Omega_t}\,ds$$

for $1/r' = 1/r + 1/p$, $2 < r' < r$.

Similarly, there is a stream function ψ corresponding to the solution of Euler equations which satisfies

$$\begin{cases} -\Delta\psi = \omega, \\ \psi\,|_{\partial\Omega} = 0, \end{cases} \tag{7.3}$$

and

$$u = \nabla^{\perp}\psi. \tag{7.4}$$

By the elliptic estimate, we may deduce that

$$\sum_{i,\,j=1}^{2} \left\| \frac{\partial^2 u}{\partial x_i\,\partial x_j} \right\|_{r',\,\Omega_t} \leqslant C, \qquad \forall 0 < t \leqslant T.$$

By Sobolev imbedding theorem, this shows that $\|\nabla u\|_{\infty,\,\Omega_t} \leqslant C$ for any $0 < t \leqslant T$. From (1.2), it is easy to deduce that $\|\nabla\Psi_t\|_{\infty,\,\Omega_t} \leqslant e^{\|\nabla u\|_{\infty,\,\Omega_t} t}$ for any $0 < t \leqslant T$. This, combined with (7.1), yields that $\|\nabla\omega\|_{r,\,\Omega_t} \leqslant C$. Utilizing this estimate, (7.3) and (7.4), we deduce that $u \in L^\infty(0, T; W^{2,r}(\Omega_t))$ for arbitrary $T > 0$. This and (3.20) imply that $\partial u/\partial t \in L^\infty(0, T; W^{1,r}(\Omega_t))$. When $1 < r \leqslant 2$, the proof is similar. ∎

If we continue the above procedure, we can show $u \in L^\infty(0, T; W^{k,r}(\Omega))$ when $u_0 \in W^{k,r}(\Omega)$ and $f \in L^1(0, \infty; W^{k,r}(R^2))$ for $k \geqslant 3$.

THEOREM 7. *Let $\partial\Omega_0 \in C^{k+1}$, $u_0 \in W^{k,r}(\Omega_0)$ and $f \in L^1(0, \infty; W^{k,r}(R^2))$ for $k \geqslant 3$, then*

$$u \in L^\infty(0, T; W^{k,r}(\Omega_t)), \qquad \frac{\partial u}{\partial t} \in L^\infty(0, T; W^{k-1,r}(\Omega_t))$$

for any $T > 0$.

THEOREM 8. *Let $\partial\Omega_0 \in C^{k,\alpha}$. If $u_0 \in C^{k,\alpha}(\bar{\Omega}_0)$ and $f \in L^1(0, \infty; BC^{k,\alpha}(R^2))$ for $k \geqslant 1$ and $0 < \alpha < 1$, then*

$$u \in L^\infty(0, T; C^{k,\alpha}(\Omega_t)), \qquad \frac{\partial u}{\partial t} \in L^\infty(0, T; C^{k-1,\alpha}(\Omega_t))$$

for any $T > 0$.

Remark 3. 1. For the case of a bounded domain or whole space, analogous results as in Theorem 5–8 had been established, see Lions (1996) and its literature.

2. Kozono showed that $(u, p) \in C^1(0, T; C^1(\Omega_t)) \times C(0, T; C^1(\Omega_t))$ for any $T > 0$ as long as $u_0 \in C^{1,\theta}(\bar{\Omega}_0)$ for $0 < \theta < 1$.

Proof. Here we only give the proof for the case of $k = 1$, since the procedure can be iterated as long as possible, up to the maximum order of derivatives of the initial data u_0.

Let $u_0 \in C^{1,\alpha}(\bar{\Omega}_0)$, then $\omega_0 = \operatorname{curl} u_0 \in C^{0,\alpha}(\bar{\Omega}_0)$, thus $\omega_0 \in L^\infty(\Omega_0)$. By (7.2), $\nabla \Psi_t \in L^\infty(0, T; L^p(\Omega_0))$ for arbitrary $p \geqslant 4$. By Sobolev imbedding theorem, $\Psi \in L^\infty(0, T; C^{0,\beta}(\Omega_0))$ with $\beta = 1 - 2/p$. Since

$$\begin{aligned} \frac{|\omega(x,t)-\omega(y,t)|}{|x-y|^{\alpha'}} &\leqslant \frac{|\omega_0(\Psi_{-t}(x))-\omega_0(\Psi_{-t}(y))|}{|\Psi_{-t}(x)-\Psi_{-t}(y)|^{\alpha}} \left(\frac{|\Psi_{-t}(x)-\Psi_{-t}(y)|}{|x-y|^{\alpha'/\alpha}}\right)^{\alpha} \\ &\quad + \int_0^t \frac{|\operatorname{curl} f(\Psi_{s-t}(x)) - \operatorname{curl} f(\Psi_{s-t}(y))|}{|\Psi_{s-t}(x)-\Psi_{s-t}(y)|^{\alpha}} \\ &\quad \times \left(\frac{|\Psi_{s-t}(x)-\Psi_{s-t}(y)|}{|x-y|^{\alpha'/\alpha}}\right)^{\alpha} ds \end{aligned} \tag{7.5}$$

for $0 < \alpha' < \alpha$, it follows

$$\omega \in L^\infty(0, T; C^{0,\alpha'}(\Omega_t)), \qquad \forall \alpha' \in (0, \alpha).$$

By the standard Schauder theory on the elliptic equation, we deduce, from (7.3), that $\psi \in L^\infty(0, T; C^{2,\alpha'}(\Omega_t))$. Hence, $u = \nabla^\perp \psi \in L^\infty(0, T; C^{1,\alpha'}(\Omega_t))$. From (1.2), it is not difficult to deduce that $\nabla \Psi_t \in L^\infty(0, T; L^\infty(\Omega_t))$, thus, $\omega \in C^{0,\alpha}(\Omega_t)$. Similarly, we deduce that

$$u \in L^\infty(0, T; C^{1,\alpha}(\Omega_t)).$$

By (3.20),

$$\frac{\partial u}{\partial t} \in L^\infty(0, T; C^{0,\alpha}(\Omega_t)). \quad \blacksquare$$

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