Notes on a conservative nonlinear oscillator

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\textbf{ABSTRACT}

The amplitude–frequency relationship is an important mathematical property for a nonlinear oscillator. He's amplitude–frequency formulation and the Max–Min approach are used to handle the conservative nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2) x = 0$ for the amplitude–frequency relationship. The obtained result is compared with those in the open literature, revealing the effectiveness of the used methods.

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1. Introduction

Nonlinear oscillators play an important role in physics and engineering. Recently, much attention was paid to a conservative oscillator

$$\ddot{x} + (1 + \dot{x}^2) x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0,$$

characterized by a velocity-dependent stiffness coefficient and dependence on only one parameter [1–7].

There are many new techniques appeared in the open literature to solve Eq. (1); see Refs. [1–6]. Here, we will introduce two simple methods for the oscillator, He’s amplitude–frequency formulation [8–10] and the Max–Min approach [11], to get the amplitude–frequency relationship, which is a fundamental property of any nonlinear oscillators.

2. He’s amplitude–frequency formulation

He’s frequency–amplitude formulation is the development of an ancient Chinese algorithm [8–10]; it is very effective to nonlinear oscillators as shown by many authors [12–15].

According to He’s frequency–amplitude formulation [8,9], we choose a trial solution

$$x_1 = A \cos \omega_1 t.$$

(2)

Submitting Eq. (2) into Eq. (1) results in the following residual

$$R_1 = (1 + \omega_1^2 A^2 \sin^2 \omega_1 A) A \cos \omega_1 t - \omega_1^2 A \cos \omega_1 t$$

$$= \left(1 + \omega_1^2 A^2 \frac{1 - \cos 2\omega_1 t}{2}\right) A \cos \omega_1 t - \omega_1^2 A \cos \omega_1 t.$$

(3)
Introducing $\tilde{R}_1$ defined as [9]

$$\tilde{R}_1 = \frac{1}{T_1} \int_0^{T_1} R_1 \cos \omega_1 t \, dt, \quad T_1 = \frac{2\pi}{\omega_1}. \tag{4}$$

Submitting Eq. (3) into Eq. (4), we obtain

$$\tilde{R}_1 = \frac{1}{2\pi} \int_0^{2\pi} \left[ A \cos^2 t - \omega_1^2 A \cos^2 t + \frac{\omega_1^2 A^3}{4} \cos^2 t - \frac{\omega_1^3 A^3}{8} (\cos 4t + \cos 2t) \right] \, dt$$

$$= \frac{1}{2} \left( A - \omega_1^2 A + \frac{\omega_1^2 A^3}{4} \right). \tag{5}$$

Let $x_2 = A \cos \omega_2 t$ with $\omega_1 \neq \omega_2$, by the same operation as the above, we have

$$\tilde{R}_2 = \frac{1}{2} \left( A - \omega_2^2 A + \frac{\omega_2^2 A^3}{4} \right). \tag{6}$$

Using He’s frequency–amplitude formulation [9], we have

$$\omega^2 = \frac{\omega_2^2 \tilde{R}_2 - \omega_1^2 \tilde{R}_1}{\tilde{R}_2 - \tilde{R}_1} = \frac{\omega_2^2 \left( A - \omega_2^2 A + \frac{\omega_2^2 A^3}{4} \right) - \omega_1^2 \left( \frac{A}{2} - \omega_1^2 A + \frac{\omega_1^2 A^3}{4} \right)}{\frac{1}{2} \left[ (\omega_1^2 - \omega_2^2) A + \frac{A^3}{4} (\omega_2^2 - \omega_1^2) \right]} \tag{7}$$

Simplifying Eq. (7) results in

$$\omega = \frac{2}{\sqrt{4 - A^2}}. \tag{8}$$

This agrees well with those in Refs. [1–3].

### 3. Max–min approach

The max–min approach [11] was also derived from an ancient Chinese inequality, called He Chengtian’s inequality. The approach is much attractive; it can be seen in Refs. [16–19].

The approach begins with a trial solution in the form

$$x = A \cos \omega t, \tag{9}$$

where $\omega$ is the frequency to be further determined.

Rewrite Eq. (1) in an approximate form

$$\ddot{x} + \left( 1 + A^2 \omega^2 \sin^2 \omega t \right) x = 0. \tag{10}$$

It is obvious that we have the following inequality

$$1 < \omega^2 < 1 + \omega^2 A^2. \tag{11}$$

Using He Chengtian's average [11], we have

$$\omega^2 = \frac{m + n \left( 1 + \omega^2 A^2 \right)}{m + n} = \frac{k + (1 + \omega^2 A^2)}{k + 1}, \tag{12}$$

where $m$ and $n$ are positive numbers, $k = m/n$.

Solving $\omega$ from Eq. (12), we have

$$\omega = \sqrt{\frac{k + 1}{k + 1 - A^2}}. \tag{13}$$

We choose $k = m/n = 3$, this leads to Eq. (8); we can also choose $k = m/n = 7$, this results in another result:

$$\omega = \sqrt{\frac{8}{8 - A^2}}. \tag{14}$$

This agrees with that in Ref. [6].

As pointed out in Ref. [16], the solution accuracy does not strongly depend upon the choice of $k.$
4. Conclusions

We introduce two simple methods for nonlinear oscillators, He’s amplitude–frequency formulation and the Max–Min approach. Both methods were derived from ancient Chinese mathematics. The main property of the both methods is simple solution procedure and acceptable accuracy of the solution.

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