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# A degree by degree recursive construction of Hermite spline interpolants

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ABSTRACT

Based on the classical Hermite spline interpolant $H_{2n-1}$, which is the piecewise interpolation polynomial of class $C^{n-1}$ and degree $2n-1$, a piecewise interpolation polynomial $H_{2n}$ of degree $2n$ is given. The formulas for computing $H_{2n}$ by $H_{2n-1}$ and computing $H_{2n+1}$ by $H_{2n}$ are shown. Thus a simple recursive method for the construction of the piecewise interpolation polynomial set $\{H_j\}$ is presented. The piecewise interpolation polynomial $H_{2n}$ satisfies the same interpolation conditions as the interpolant $H_{2n-1}$, and is an optimal approximation of the interpolant $H_{2n+1}$. Some interesting properties are also proved.



## 1. Introduction

Given a bounded interval $[a, b]$ of $\mathbb{R}$, and a partition $\Delta : a = x_1 < x_2 < \cdots < x_m = b$ of the interval $[a, b]$, and let $n \geq 1, f_i^{(r)} = f^{(r)}(x_i), r = 0, 1, \ldots, n-1$, be the corresponding values and the first $n-1$ derivatives of the function $f(x)$. It is well known (see, e.g. [1–3]) that the classical Hermite spline interpolant $H_{2n-1} \in C^{n-1}([a, b])$ and satisfies the following interpolation conditions:

$$H_{2n-1}^{(r)}(x_i) = f_i^{(r)}, \quad i = 1, 2, \ldots, m; r = 0, 1, \ldots, n-1.$$

When $f$ is of class $C^{2n}([a, b])$, the approximation of $f$ by $H_{2n-1}$ leads to an error estimate $\|f - H_{2n-1}\|_\infty = O(h^{2n})$, where

$$\|f\|_\infty := \max_{x\in[a,b]} |f(x)|, \qquad h = \max_{1\leq k\leq m-1} h_k, \quad h_k = x_{k+1} - x_k.$$

In [4,5], it was shown that the first $n$ derivatives of $H_{2n-1}$ are good approximations to the corresponding derivatives of $f$:

$$|f^{(r)}(x) - H_{2n-1}^{(r)}(x)| \leq \frac{|(x - x_k)(x - x_{k+1})|^{n-r}}{r!(2n-2r)!} h_k^r \max_{\xi\in[x_k,x_{k+1}]} |f^{(2n)}(\xi)| \tag{1}$$

for all $x \in [x_k, x_{k+1}]$, $r = 0, 1, \ldots, n$, $k = 1, 2, \ldots, m-1$, and therefore

$$\|f^{(r)} - H_{2n-1}^{(r)}\|_\infty \leq \frac{h^{2n-r}}{2^{2n-2r} r!(2n-2r)!} \|f^{(2n)}\|_\infty. \tag{2}$$

Usually, $H_{2n-1}$ is expressed by the Hermite basis, which makes the use of formula $H_{2n-1}$ rather complicated. In order to remedy this problem, a recursive method for the construction of $H_{2n-1}$ was presented in [6]. The method allows us to compute $H_{2n+1}$ by $H_{2n-1}$ recursively. It was described in [6], that the decomposition of $H_{2n+1}$ has several advantages, and can be used for some applications in numerical approximation fields. In [7], a new method for smoothing functions and compressing Hermite data was developed. This method is based on hierarchical bases. The hierarchical bases are useful in

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several areas of mathematics. For example, they are used in [8] for compressing surfaces and in [9–11] for solving some boundary-value problems. The recursive computation of bivariate Hermite spline interpolants is given in [12].

The aim of this paper is to give piecewise interpolation polynomials $H_{2n}$ of even degree which make the recursive computation of $H_{2n+1}$ rather simple, and have interesting interpolation properties. The piecewise interpolation polynomial $H_{2n}$ satisfies the same interpolation conditions as the interpolant $H_{2n-1}$, and is an optimal approximation of the interpolant $H_{2n+1}$. When the accuracy of $H_{2n}$ is enough, we need not compute $H_{2n+1}$.

The present paper is organized as follows. In Section 2, a piecewise interpolant of even degree is given, and the recursive construction of the Hermite spline interpolants is shown. In Section 3, the estimates for the correction terms are given when $f$ is of class $C^n$ or $C^{2n}$. In Section 4, the error estimates for the piecewise interpolants of even degree are given and a numerical example is shown.

## 2. Recursive construction of Hermite interpolant

In this section, we give a recursive construction of Hermite spline interpolants, by constructing piecewise interpolation polynomials of even degree.

Let $\mathbb{F}_{2n}^{n-1}([a,b],\Delta)=\{S: S^{(r)}(x_i)=f_i^{(r)},\ 1\le i\le m,\ 0\le r\le n-1,\ S|_{[x_k,x_{k+1}]}\in\mathbb{P}_{2n},\ 1\le k\le m-1\}$, where $\mathbb{P}_{2n}$ denotes the space of polynomials of degree at most $2n$. We want to construct a piecewise interpolation polynomial set $\{H_{2n}\}$ so that $H_{2n}\in\mathbb{F}_{2n}^{n-1}([a,b],\Delta)$ and

$$\int_a^b [H_{2n+1}(x)-H_{2n}(x)]^2\,\mathrm{d}x=\min_{P\in\mathbb{F}_{2n}^{n-1}([a,b],\Delta)}\int_a^b [H_{2n+1}(x)-P(x)]^2\,\mathrm{d}x. \tag{3}$$

The condition $H_{2n}\in\mathbb{F}_{2n}^{n-1}([a,b],\Delta)$ implies that $H_{2n}$ satisfies the same interpolation conditions as the interpolant $H_{2n-1}$. For constructing $H_{2n}$, it is convenient to express $H_{2n-1}$ by the Bernstein basis functions $B_{2n-1,i}$. We can easily deduce the following lemma.

**Lemma 1.** *The interpolant $H_{2n-1}$ is given for $x\in[x_k,x_{k+1}]$ in term of $f_k^{(r)}, f_{k+1}^{(r)}$ by*

$$H_{2n-1}(x)=\sum_{i=0}^{2n-1}B_{2n-1,i}(u)p_{2n-1,i}, \tag{4}$$

*where*

$$B_{2n-1,i}(u)=\binom{2n-1}{i}(1-u)^{2n-1-i}u^i,\quad i=0,1,\ldots,2n-1,$$

$$p_{2n-1,i}=\frac{1}{(2n-1)!}\sum_{j=0}^{i}(2n-1-j)!\binom{i}{j}h_k^jf_k^{(j)},\quad i=0,1,\ldots,n-1, \tag{5}$$

$$p_{2n-1,2n-1-i}=\frac{1}{(2n-1)!}\sum_{j=0}^{i}(-1)^j(2n-1-j)!\binom{i}{j}h_k^jf_{k+1}^{(j)},\quad i=0,1,\ldots,n-1, \tag{6}$$

$h_k=x_{k+1}-x_k$, $u=(x-x_k)/h_k$, $1\le k\le m-1$.

**Lemma 2.** *For (4), we have*

$$H_{2n-1}(x)=\sum_{i=0}^{2n+1}B_{2n+1,i}(u)q_i, \tag{7}$$

*where*

$$q_n=\frac{n}{(2n+1)!}\sum_{j=0}^{n-1}(2n-1-j)!\binom{n-1}{j}h_k^j\left[(3n+1-j)f_k^{(j)}+(-1)^j(n+1)f_{k+1}^{(j)}\right], \tag{8}$$

$$q_{n+1}=\frac{n}{(2n+1)!}\sum_{j=0}^{n-1}(2n-1-j)!\binom{n-1}{j}h_k^j\left[(n+1)f_k^{(j)}+(-1)^j(3n+1-j)f_{k+1}^{(j)}\right], \tag{9}$$

$$q_i=p_{2n+1,i},\qquad q_{2n+1-i}=p_{2n+1,2n+1-i} \tag{10}$$

*for $i=0,1,\ldots,n-1$.*

**Proof.** By Lemma 1 and the degree elevation (see, e.g. [13]), we have

$$H_{2n-1}(x) = \sum_{i=0}^{2n} B_{2n,i}(u)a_i,$$

where

$$a_i = \left(1 - \frac{i}{2n}\right) p_{2n-1,i} + \frac{i}{2n} p_{2n-1,i-1}$$

for $i = 0, 1, \ldots, 2n$, and $p_{2n-1,-1} = p_{2n-1,2n} = 0$. From (5) and (6), and these, we get

$$a_n = \frac{1}{2(2n-1)!} \sum_{j=0}^{n-1} (2n-1-j)! \binom{n-1}{j} h_k^j \left[f_k^{(j)} + (-1)^j f_{k+1}^{(j)}\right],$$

$$a_i = \frac{1}{(2n)!} \sum_{j=0}^{i} (2n-j)! \binom{i}{j} h_k^j f_k^{(j)},$$

$$a_{2n-i} = \frac{1}{(2n)!} \sum_{j=0}^{i} (-1)^j (2n-j)! \binom{i}{j} h_k^j f_{k+1}^{(j)}$$

for $i = 0, 1, \ldots, n-1$.

In the same way, we have

$$H_{2n-1}(x) = \sum_{i=0}^{2n+1} B_{2n+1,i}(u)q_i,$$

where

$$q_i = \left(1 - \frac{i}{2n+1}\right) a_i + \frac{i}{2n+1} a_{i-1}$$

for $i = 0, 1, \ldots, 2n+1$, and $a_{-1} = a_{2n+1} = 0$. From these, we get (8)–(10). □

**Theorem 1.** *For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, m-1$, we have*

$$H_{2n}(x) = H_{2n-1}(x) + \lambda_{n,k}(1-u)^n u^n, \tag{11}$$

*where $u = (x - x_k)/h_k$,*

$$\lambda_{n,k} = \frac{1}{n!2} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!} h_k^j \left[f_k^{(j)} + (-1)^j f_{k+1}^{(j)}\right]. \tag{12}$$

**Proof.** Based on the condition $H_{2n} \in \mathbb{F}_{2n}^{n-1}([a, b], \Delta)$, we can deduce that $H_{2n}$ is of the following form:

$$H_{2n}(x) = H_{2n-1}(x) + \lambda_{n,k}(1-u)^n u^n,$$

for $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, m-1$. Thus, by Lemma 2, we have

$$\begin{aligned} H_{2n+1}(x) - H_{2n}(x) &= H_{2n+1}(x) - H_{2n-1}(x) - \lambda_{n,k}(1-u)^n u^n, \\ &= B_{2n+1,n}(u)(p_{2n+1,n} - q_n) + B_{2n+1,n+1}(u)(p_{2n+1,n+1} - q_{n+1}) - \lambda_{n,k}(1-u)^n u^n. \end{aligned}$$

For the condition (3), we let

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_{n,k}} \int_{x_k}^{x_{k+1}} [H_{2n+1}(x) - H_{2n}(x)]^2 \,\mathrm{d}x = 0.$$

From this we have

$$\begin{aligned} \lambda_{n,k} \int_{x_k}^{x_{k+1}} (1-u)^{2n} u^{2n} \mathrm{d}x = \binom{2n+1}{n} \int_{x_k}^{x_{k+1}} &\left[(p_{2n+1,n} - q_n)(1-u)^{2n+1} u^{2n}\right. \\ &\left. + (p_{2n+1,n+1} - q_{n+1})(1-u)^{2n} u^{2n+1}\right] \mathrm{d}x, \end{aligned}$$

and then we get

$$\lambda_{n,k} = \frac{1}{2}\binom{2n+1}{n}(p_{2n+1,n} + p_{2n+1,n+1} - q_n - q_{n+1}). \tag{13}$$

From this and (5), (6), (8) and (9), a straightforward computation gives (12). □

The expression (11) shows that we can get $H_{2n}$ recursively by $H_{2n-1}$. The piecewise interpolant $H_{2n} \in \mathbb{F}_{2n}^{n-1}([a, b], \Delta)$ is an optimal approximation of the interpolant $H_{2n+1}$. In the following, we will show that $H_{2n+1}$ can be obtained recursively by $H_{2n}$.

**Theorem 2.** *For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, m-1$, we have*

$$H_{2n+1}(x) = H_{2n}(x) + \sigma_{n,k}(1-u)^n u^n (1-2u), \tag{14}$$

*where $u = (x - x_k)/h_k$,*

$$\sigma_{n,k} = \frac{1}{n!2}\sum_{j=0}^{n}\frac{(2n-j)!}{(j)!(n-j)!}h_k^j\left[f_k^{(j)} - (-1)^j f_{k+1}^{(j)}\right]. \tag{15}$$

**Proof.** For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, m-1$, by (13) we have

$$\begin{aligned} H_{2n+1}(x) - H_{2n}(x) &= \left\{\binom{2n+1}{n}\left[(1-u)(p_{2n+1,n} - q_n) + u(p_{2n+1,n+1} - q_{n+1})\right] - \lambda_{n,k}\right\}(1-u)^n u^n \\ &= \frac{1}{2}\binom{2n+1}{n}(p_{2n+1,n} - p_{2n+1,n+1} - q_n + q_{n+1})(1-u)^n u^n (1-2u). \end{aligned}$$

From (5), (6), (8) and (9), we have

$$p_{2n+1,n} - p_{2n+1,n+1} = \frac{n!}{(2n+1)!}\sum_{j=0}^{n}\frac{(2n-j+1)!}{(j)!(n-j)!}h_k^j\left[f_k^{(j)} - (-1)^j f_{k+1}^{(j)}\right]$$

$$q_{n+1} - q_n = -\frac{n!}{(2n+1)!}\sum_{j=0}^{n-1}\frac{(2n-j)!}{(j)!(n-j-1)!}h_k^j\left[f_k^{(j)} - (-1)^j f_{k+1}^{(j)}\right].$$

Thus, we obtain

$$H_{2n+1}(x) - H_{2n}(x) = \sigma_{n,k}(1-u)^n u^n (1-2u),$$

where $\sigma_{n,k}$ is given by (15). □

The piecewise interpolation polynomial set $\{H_{2n}\}$ brings a perfect piecewise interpolation polynomial set $\{H_j\}$. Based on Theorems 1 and 2, we can provide a recursive construction of the piecewise interpolation polynomial set $\{H_j\}$. Let $H_1 \in \mathbb{F}_1^0([a, b], \Delta)$ be the piecewise linear interpolant to $f$ at the knots $x_i$, $i = 1, 2, \ldots, m$. For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \ldots, m-1$, let $u = (x - x_k)/h_k$, $b_n(x) = (1-u)^n u^n$ and

$$w_{2n}(x) = \lambda_{n,k} b_n(x), \qquad w_{2n+1}(x) = \sigma_{n,k} b_n(x)(1-2u). \tag{16}$$

Then $b_0(x) = 1$, $b_n(x) = (1-u)u b_{n-1}(x)$, $H_j(x) = H_{j-1}(x) + w_j(x)$. By repeating this decomposition, we finally obtain

$$H_j = H_1 + w_2 + w_3 + \cdots + w_j. \tag{17}$$

It is clear that the decomposition (17) allows us to determine $H_{2n+1}$ step by step, without computing all the corresponding classical Hermite basis functions. The quantities $w_j$, $j = 2, 3, \ldots, 2n+1$, added to $H_1$, are expressed in terms of $b_i(x)$ or $b_i(x)(1-2u)$ which have simpler expressions than the $\phi_{i,k}^i$ or $\bar{\phi}_{i,k}^i$ given in [6] since

$$\phi_{i,k}^i = \frac{h_k^i}{i!}(1-u)^{i+1}u^i, \qquad \bar{\phi}_{i,k}^i = \frac{h_k^i}{i!}(1-u)^i u^{i+1}.$$

Moreover, for $\lambda_{i,k}$ and $\sigma_{i,k}$, the absolute values of the coefficients of $f_k^{(j)}$ and $f_{k+1}^{(j)}$ are the same. For the $\delta_k^i$ or $\bar{\delta}_k^i$ given in [6], the coefficients of $f_k^{(j)}$ and $f_{k+1}^{(j)}$ are different. Therefore, the $\lambda_{i,k}$ or $\sigma_{i,k}$ has a simpler expression than the $\delta_k^i$ or $\bar{\delta}_k^i$ given in [6]. The above arguments mean that the proposed method for computing $H_{2n+1}$ is very simple and easy.

The following theorem shows the derivative property of $H_{2n}(x)$.

**Theorem 3.** *For $k = 1, 2, \ldots, m-1$, we have*

$$\int_{x_k}^{x_{k+1}} \left[H_{2n-1}^{(n)}(x)\right]^2 \mathrm{d}x \le \int_{x_k}^{x_{k+1}} \left[H_{2n}^{(n)}(x)\right]^2 \mathrm{d}x \le \int_{x_k}^{x_{k+1}} \left[H_{2n+1}^{(n)}(x)\right]^2 \mathrm{d}x. \tag{18}$$

**Proof.** For $i = 0, 1, \ldots, 2n-1$, integration by parts yields

$$\begin{aligned}\int_{x_k}^{x_{k+1}} \left[(1-u)^{2n-1-i}u^i\right]^{(n)} \left[(1-u)^n u^n\right]^{(n)} \mathrm{d}x &= -\int_{x_k}^{x_{k+1}} \left[(1-u)^{2n-1-i}u^i\right]^{(n+1)} \left[(1-u)^n u^n\right]^{(n-1)} \mathrm{d}x \\ &= \cdots \\ &= (-1)^{n-1} \int_{x_k}^{x_{k+1}} \left[(1-u)^{2n-1-i}u^i\right]^{(2n-1)} \left[(1-u)^n u^n\right]^{(1)} \mathrm{d}x = 0.\end{aligned}$$

Therefore, from (11) we have

$$\begin{aligned}\int_{x_k}^{x_{k+1}} \left[H_{2n}^{(n)}(x)\right]^2 \mathrm{d}x &= \int_{x_k}^{x_{k+1}} \left[H_{2n-1}^{(n)}(x)\right]^2 \mathrm{d}x + 2\lambda_{n,k} \int_{x_k}^{x_{k+1}} H_{2n-1}^{(n)}(x) \left[(1-u)^n u^n\right]^{(n)} \mathrm{d}x \\ &\quad + \lambda_{n,k}^2 \int_{x_k}^{x_{k+1}} \left\{\left[(1-u)^n u^n\right]^{(n)}\right\}^2 \mathrm{d}x \\ &= \int_{x_k}^{x_{k+1}} \left[H_{2n-1}^{(n)}(x)\right]^2 \mathrm{d}x + \lambda_{n,k}^2 \int_{x_k}^{x_{k+1}} \left\{\left[(1-u)^n u^n\right]^{(n)}\right\}^2 \mathrm{d}x.\end{aligned}$$

Thus, we get the first inequality of (18).

For $i = 0, 1, \ldots, 2n-1$, since

$$\int_{x_k}^{x_{k+1}} \left[(1-u)^{2n-1-i}u^i\right]^{(n)} \left[(1-u)^n u^n (1-2u)\right]^{(n)} \mathrm{d}x = 0,$$

$$\int_{x_k}^{x_{k+1}} \left[(1-u)^n u^n\right]^{(n)} \left[(1-u)^n u^n (1-2u)\right]^{(n)} \mathrm{d}x = 0,$$

we have

$$\int_{x_k}^{x_{k+1}} H_{2n}^{(n)}(x) \left[(1-u)^n u^n (1-2u)\right]^{(n)} \mathrm{d}x = \int_{x_k}^{x_{k+1}} \left[H_{2n-1}(x) + \lambda_{n,k}(1-u)^n u^n\right]^{(n)} \left[(1-u)^n u^n (1-2u)\right]^{(n)} \mathrm{d}x = 0.$$

Therefore, from (14) we get

$$\int_{x_k}^{x_{k+1}} \left[H_{2n+1}^{(n)}(x)\right]^2 \mathrm{d}x = \int_{x_k}^{x_{k+1}} \left[H_{2n}^{(n)}(x)\right]^2 \mathrm{d}x + \sigma_n^2 \int_{x_k}^{x_{k+1}} \left\{\left[(1-u)^n u^n (1-2u)\right]^{(n)}\right\}^2 \mathrm{d}x.$$

This means that the second inequality of (18) holds. □

## 3. The estimates of $\|w_j\|_\infty$

According to the structure of the decomposition of $H_j$, the piecewise linear interpolant $H_1$ can be considered as a coarse approximation of a function $f$, and $w_i$ $(2 \le i \le j)$ are the correction terms which we add to $H_1$ in order to improve the approximation. In this section, we give the estimates of $\|w_j\|_\infty$. To do this, we need the following lemmas.

**Lemma 3** (*See [14]*). *Let $l, i, r$ be nonnegative integers, $s = \min\{i, r\}$, then*

$$\sum_{j=0}^{s} (-1)^j \frac{(l+r-j)!}{j!(i-j)!(r-j)!} = \frac{l!}{i!r!} \prod_{j=1}^{r} (l+j-i).$$

**Lemma 4.** *For $u \in [0, 1]$, let*

$$\delta_r(u) = [(1-u)^n u^n (1-2u)]^{(r)},$$

*and the $r$th derivatives in $\delta_r(u)$ be with respect to $u$. Then*

$$|\delta_r(u)| \le \begin{cases} \left[\dfrac{n}{2(2n+1)}\right]^n \dfrac{\sqrt{2n+1}}{2n+1}, & r = 0, \\ \dfrac{(2n+1)!}{(2n+1-r)!2^{2n-r}}, & 1 \le r \le 2n+1. \end{cases}$$

**Proof.** A straightforward computation gives that $\delta_0(u)$ has a maximum value at $(1-u)u = n/[2(2n+1)]$ and then

$$|\delta_0(u)| \le \left[\frac{n}{2(2n+1)}\right]^n \frac{\sqrt{2n+1}}{2n+1} < \frac{1}{2^{2n+1}}.$$

Since $\delta_0(u)$ has $2n+1$ zeros in the interval $[0, 1]$, we can deduce that $\delta_r(u)$ has $2n+1-r$ zeros in the interval $[0, 1]$ for $r = 0, 1, \ldots, 2n+1$, and

$$\delta_r(u) = (-1)^{n+1} \frac{2(2n+1)!}{(2n+1-r)!} \prod_{i=1}^{2n+1-r} (u - \xi_{r,i})$$

for some $\xi_{r,i} \in [0, 1]$.

Let $0 \le \xi_{r,1} \le \xi_{r,2} \le \cdots \le \xi_{r,2n+1-r} \le 1$. Since $\delta_r(u) = (-1)^{r+1}\delta_r(v)$ for $v = 1-u$, we can set $\xi_{r,i} = 1 - \xi_{r,2n+2-r-i}$, and then

$$|(x - \xi_{ri})(x - \xi_{r,2n+2-r-i})| \le \frac{1}{4}$$

for $i = 1, 2, \ldots, 2n+1-r$. Therefore

$$|\delta_r(u)| \le \frac{(2n+1)!}{(2n+1-r)!2^{2n-r}},$$

for $r = 1, 2, \ldots, 2n+1$. □

**Theorem 4.** *If $f \in C^{2n}([a, b])$, then*

$$\lambda_{n,k} = \frac{(-1)^n}{(2n)!} h_k^{2n} f^{(2n)}(\xi_k), \tag{19}$$

*for some $\xi_k \in (x_k, x_{k+1})$ and*

$$\|w_{2n}\|_\infty \le \frac{h^{2n}}{(2n)!4^n} \|f^{(2n)}\|_\infty. \tag{20}$$

**Proof.** From (12), we have

$$\begin{aligned}
\lambda_{n,k} &= \frac{1}{n!2} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!} h_k^j \left\{ f_k^{(j)} + (-1)^j \left[ \sum_{i=j}^{2n-1} \frac{h_k^{i-j}}{(i-j)!} f_k^{(i)} + \frac{1}{(2n-j-1)!} \int_{x_k}^{x_{k+1}} (x_{k+1} - t)^{2n-j-1} f^{(2n)}(t)\mathrm{d}t \right] \right\} \\
&= \frac{1}{n!2} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!} h_k^j f_k^{(j)} + \frac{1}{n!2} \sum_{i=1}^{2n-1} \sum_{j=1}^{s} \frac{(-1)^j (2n-j-1)!}{(j-1)!(n-j)!(i-j)!} h_k^i f_k^{(i)} \\
&\quad + \frac{1}{n!2} \sum_{j=1}^{n} \frac{(-1)^j h_k^j}{(j-1)!(n-j)!} \int_{x_k}^{x_{k+1}} (x_{k+1} - t)^{2n-j-1} f^{(2n)}(t)\mathrm{d}t,
\end{aligned}$$

where $s = \min\{i, n\}$. By Lemma 3, we have

$$\begin{aligned}
\sum_{i=1}^{2n-1} \sum_{j=1}^{s} \frac{(-1)^j (2n-j-1)!}{(j-1)!(n-j)!(i-j)!} &= \sum_{i=1}^{2n-1} \sum_{j=0}^{s-1} \frac{(-1)^{j+1}(2n-j-2)!}{j!(n-j-1)!(i-j-1)!} \\
&= -\sum_{i=1}^{2n-1} \frac{1}{(i-1)!} \prod_{j=1}^{n-1} (n-i+j) = -\sum_{i=1}^{n} \frac{(2n-i-1)!}{(i-1)!(n-i)!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\lambda_{n,k} &= \frac{1}{n!2} \sum_{j=0}^{n-1} \frac{(-1)^{j+1} h_k^{j+1}}{j!(n-j-1)!} \int_{x_k}^{x_{k+1}} (x_{k+1} - t)^{2n-j-2} f^{(2n)}(t)\mathrm{d}t \\
&= -\frac{h_k}{n!(n-1)!2} \int_{x_k}^{x_{k+1}} (x_k - t)^{n-1} (x_{k+1} - t)^{n-1} f^{(2n)}(t)\mathrm{d}t \\
&= \frac{(-1)^n}{(2n)!} h_k^{2n} f^{(2n)}(\xi_k)
\end{aligned}$$

for some $\xi_k \in (x_k, x_{k+1})$. From this we obtain (20) immediately. □

**Theorem 5.** *If* $f \in C^{2n+1}([a,b])$, *then*

$$\sigma_{n,k} = \frac{(-1)^{n+1}}{(2n+1)!2} h_k^{2n+1} f^{(2n+1)}(\eta_k), \tag{21}$$

*for some* $\eta_k \in (x_k, x_{k+1})$ *and*

$$\|w_{2n+1}\|_\infty \le \frac{h^{2n+1}}{(2n+1)!4^{n+1}} \|f^{(2n+1)}\|_\infty. \tag{22}$$

**Proof.** From (15), we have

$$\begin{aligned}
\sigma_{n,k} &= \frac{1}{n!2} \sum_{j=0}^{n} \frac{(2n-j)!}{j!(n-j)!} h_k^j \left\{ f_k^{(j)} - (-1)^j \left[ \sum_{i=j}^{2n} \frac{h_k^{i-j}}{(i-j)!} f_k^{(i)} + \frac{1}{(2n-j)!} \int_{x_k}^{x_{k+1}} (x_{k+1}-t)^{2n-j} f^{(2n+1)}(t)\mathrm{d}t \right] \right\} \\
&= \frac{1}{n!2} \sum_{j=0}^{n} \frac{(2n-j)!}{j!(n-j)!} h_k^j f_k^{(j)} - \frac{1}{n!2} \sum_{i=0}^{2n} \sum_{j=0}^{s} \frac{(-1)^j (2n-j)!}{j!(n-j)!(i-j)!} h_k^i f_k^{(i)} \\
&\quad - \frac{1}{n!2} \sum_{j=0}^{n} \frac{(-1)^j h_k^j}{j!(n-j)!} \int_{x_k}^{x_{k+1}} (x_{k+1}-t)^{2n-j} f^{(2n+1)}(t)\mathrm{d}t,
\end{aligned}$$

where $s = \min\{i, n\}$. By Lemma 3, we have

$$\begin{aligned}
\sum_{i=0}^{2n} \sum_{j=0}^{s} \frac{(-1)^j (2n-j)!}{j!(n-j)!(i-j)!} &= \sum_{i=0}^{2n} \frac{1}{i!} \prod_{j=1}^{n} (n-i+j) \\
&= \sum_{i=0}^{n} \frac{(2n-i)!}{i!(n-i)!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\sigma_{n,k} &= -\frac{1}{n!2} \sum_{j=0}^{n} \frac{(-1)^j h_k^j}{j!(n-j)!} \int_{x_k}^{x_{k+1}} (x_{k+1}-t)^{2n-j} f^{(2n+1)}(t)\mathrm{d}t \\
&= -\frac{1}{n!n!2} \int_{x_k}^{x_{k+1}} (x_k-t)^n (x_{k+1}-t)^n f^{(2n+1)}(t)\mathrm{d}t \\
&= \frac{(-1)^{n+1}}{(2n+1)!2} h_k^{2n+1} f^{(2n+1)}(\eta_k)
\end{aligned}$$

for some $\eta_k \in (x_k, x_{k+1})$.

From Lemma 4 we have

$$|(1-u)^n u^n (1-2u)| \le \left[\frac{n}{2(2n+1)}\right]^n \frac{\sqrt{2n+1}}{2n+1} < \frac{1}{2^{2n+1}}.$$

From this we obtain (22) immediately. □

In [6], the bound of $\|H_{2n+1} - H_{2n-1}\|_\infty$ was estimated. Here we give the exact expression of $H_{2n+1} - H_{2n-1}$ as follows.

**Theorem 6.** *If* $f \in C^{2n}([a,b])$, *then for* $x \in [x_k, x_{k+1}]$,

$$H_{2n+1}(x) - H_{2n-1}(x) = \frac{(-1)^n}{(2n)!} (1-u)^n u^n h_k^{2n} f^{(2n)}(\zeta_k), \tag{23}$$

*where* $u = (x - x_k)/h_k$, $\zeta_k \in (x_k, x_{k+1})$, *and*

$$\|H_{2n+1} - H_{2n-1}\|_\infty \le \frac{h^{2n}}{(2n)!4^n} \|f^{(2n)}\|_\infty. \tag{24}$$

**Proof.** From (11) and (14), we have

$$\begin{aligned}
H_{2n+1}(x) - H_{2n-1}(x) &= w_{2n}(x) + w_{2n+1}(x) \\
&= (1-u)^n u^n \left[(1-u)(\lambda_{n,k} + \sigma_{n,k}) + u(\lambda_{n,k} - \sigma_{n,k})\right].
\end{aligned} \tag{25}$$

In the same way as the proof in Theorems 4 and 5, by Lemma 3, we have

$$\begin{aligned}\lambda_{n,k}+\sigma_{n,k} &= \frac{1}{n!}\sum_{j=0}^{n}\frac{(2n-j-1)!}{j!(n-j)!}h_k^j\left[nf_k^{(j)}-(-1)^j(n-j)f_{k+1}^{(j)}\right]\\ &= \frac{(-1)^n}{n!(n-1)!}\int_{x_k}^{x_{k+1}}(t-x_k)^{n-1}(x_{k+1}-t)^nf^{(2n)}(t)\mathrm{d}t,\\ \lambda_{n,k}-\sigma_{n,k} &= \frac{1}{n!}\sum_{j=0}^{n}\frac{(2n-j-1)!}{j!(n-j)!}h_k^j\left[(j-n)f_k^{(j)}+(-1)^jnf_{k+1}^{(j)}\right]\\ &= \frac{(-1)^n}{n!(n-1)!}\int_{x_k}^{x_{k+1}}(t-x_k)^{n}(x_{k+1}-t)^{n-1}f^{(2n)}(t)\mathrm{d}t.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}H_{2n+1}(x)-H_{2n-1}(x) &= \frac{(-1)^n}{n!(n-1)!}(1-u)^nu^nf^{(2n)}(\zeta_k)\\ &\quad\times\int_{x_k}^{x_{k+1}}(t-x_k)^{n-1}(x_{k+1}-t)^{n-1}\left[(1-u)(x_{k+1}-t)+u(t-x_k)\right]\mathrm{d}t\\ &= \frac{(-1)^n}{(2n)!}(1-u)^nu^nh_k^{2n}f^{(2n)}(\zeta_k)\end{aligned}$$

for some $\zeta_k\in(x_k,x_{k+1})$. From this we obtain (24) immediately. □

If we weaken the condition on $f$, then the following results hold.

**Theorem 7.** *If $f\in C^n([a,b])$, then*

$$|\lambda_{n,k}|\le\frac{c_{n-1}}{n!2}h_k^n\omega(f^{(n)},h_k),\tag{26}$$

*where $c_{n-1}=\sum_{j=0}^{n-1}\binom{n-1}{j}\binom{2n-j-2}{n-1}$, $\omega(f^{(n)},\cdot)$ is the modulus of continuity of $f^{(n)}$, and*

$$\|w_{2n}\|_\infty\le\frac{c_{n-1}}{n!4^n2}h^n\omega(f^{(n)},h).\tag{27}$$

**Proof.** In a similar way as the proof in Theorem 4, we have

$$\begin{aligned}\lambda_{n,k} &= \frac{1}{n!2}\sum_{j=1}^{n-1}\frac{(2n-j-1)!}{(j-1)!(n-j)!}h_k^j\left\{f_k^{(j)}+(-1)^j\left[\sum_{i=j}^{n-1}\frac{h_k^{i-j}}{(i-j)!}f_k^{(i)}+\frac{1}{(n-j-1)!}\int_{x_k}^{x_{k+1}}(x_{k+1}-t)^{n-j-1}f^{(n)}(t)\mathrm{d}t\right]\right\}\\ &\quad+\frac{h_k^n}{n!2}\left[f_k^{(n)}+(-1)^nf_{k+1}^{(n)}\right]\\ &= \frac{1}{n!2}\sum_{j=1}^{n-1}\frac{(-1)^j(2n-j-1)!}{(j-1)!(n-j)!(n-j-1)!}h_k^j\int_{x_k}^{x_{k+1}}(x_{k+1}-t)^{n-j-1}f^{(n)}(t)\mathrm{d}t+\frac{h_k^n}{n!2}\left[f_k^{(n)}+(-1)^nf_{k+1}^{(n)}\right]\\ &= \frac{1}{n!2}\sum_{j=1}^{n-1}\frac{(-1)^j(2n-j-1)!}{(j-1)!(n-j)!(n-j)!}h_k^nf^{(n)}(\theta_{kj})+\frac{h_k^n}{n!2}\left[f_k^{(n)}+(-1)^nf_{k+1}^{(n)}\right],\end{aligned}$$

for some $\theta_{kj}\in(x_k,x_{k+1})$. By Lemma 3,

$$\sum_{j=1}^{n}\frac{(-1)^j(2n-j-1)!}{(j-1)!(n-j)!(n-j)!}=\sum_{j=0}^{n-1}\frac{(-1)^{j+1}(2n-j-2)!}{j!(n-j-1)!(n-j-1)!}=-1.$$

Therefore, we obtain

$$\lambda_{n,k}=\frac{h_k^n}{n!2}\sum_{j=1}^{n}\frac{(-1)^j(2n-j-1)!}{(j-1)!(n-j)!(n-j)!}\left[f^{(n)}(\theta_{kj})-f_k^{(n)}\right],$$

where $\theta_{kn}=x_{k+1}$. Thus, we obtain (26) and (27) immediately. □

**Theorem 8.** *If $f \in C^n([a,b])$, then*

$$|\sigma_{n,k}| \le \frac{c_n}{n!2} h_k^n \omega(f^{(n)}, h_k), \tag{28}$$

*where $c_n = \sum_{j=0}^{n} \binom{n}{j}\binom{2n-j}{n}$, and*

$$\|w_{2n+1}\|_\infty \le \frac{c_n}{n!4^{n+1}} h^n \omega(f^{(n)}, h). \tag{29}$$

**Proof.** In a similar way as the proof in Theorem 5, we have

$$\begin{aligned}
\sigma_{n,k} &= \frac{1}{n!2}\sum_{j=0}^{n-1}\frac{(2n-j)!}{j!(n-j)!}h_k^j\left\{f_k^{(j)} - (-1)^j\left[\sum_{i=j}^{n-1}\frac{h_k^{i-j}}{(i-j)!}f_k^{(i)} + \frac{1}{(n-j-1)!}\int_{x_k}^{x_{k+1}}(x_{k+1}-t)^{n-j-1}f^{(n)}(t)\mathrm{d}t\right]\right\} \\
&\quad + \frac{h_k^n}{n!2}\left[f_k^{(n)} - (-1)^n f_{k+1}^{(n)}\right] \\
&= -\frac{1}{n!2}\sum_{j=0}^{n-1}\frac{(-1)^j(2n-j)!}{j!(n-j)!(n-j-1)!}h_k^j\int_{x_k}^{x_{k+1}}(x_{k+1}-t)^{n-j-1}f^{(n)}(t)\mathrm{d}t + \frac{h_k^n}{n!2}\left[f_k^{(n)} - (-1)^n f_{k+1}^{(n)}\right] \\
&= -\frac{1}{n!2}\sum_{j=0}^{n-1}\frac{(-1)^j(2n-j)!}{(j-1)!(n-j)!(n-j)!}h_k^n f^{(n)}(\theta_{kj}) + \frac{h_k^n}{n!2}\left[f_k^{(n)} - (-1)^n f_{k+1}^{(n)}\right],
\end{aligned}$$

for some $\theta_{kj} \in (x_k, x_{k+1})$. By Lemma 3,

$$\sum_{j=0}^{n}\frac{(-1)^j(2n-j)!}{j!(n-j)!(n-j)!} = 1.$$

Therefore, we obtain

$$\sigma_{n,k} = \frac{h_k^n}{n!2}\sum_{j=0}^{n}\frac{(-1)^j(2n-j)!}{j!(n-j)!(n-j)!}\left[f_k^{(n)} - f^{(n)}(\theta_{kj})\right],$$

where $\theta_{kn} = x_{k+1}$. Thus, we obtain (28) and (29) immediately. □

It is to be expected that the constant $c_n/(n!4^n)$ tends to zero when $n$ is large. In [6], it is illustrated numerically that a corresponding constant tends to zero when $n$ is large. Now we show that $c_n/(n!4^n)$ tends to zero when $n$ is large as follows. Since

$$\binom{2n-j}{n} \le \frac{1}{2}\binom{2n-j+1}{n} \le \frac{1}{2^2}\binom{2n-j+2}{n} \le \cdots \le \frac{1}{2^j}\binom{2n}{n},$$

we have

$$c_n = \sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n} \le \binom{2n}{n}\sum_{j=0}^{n}\frac{1}{2^j}\binom{n}{j} = \frac{3^n}{2^n}\binom{2n}{n}.$$

For $i = 1, 2, \ldots, n$, we have $(n+i)/[i(n-i+1)] \le 2$. Thus we have

$$\begin{aligned}
\frac{c_n}{n!4^n} &\le \frac{3^n}{4^n 2^n n!}\binom{2n}{n} = \frac{3^n}{4^n 2^n}\prod_{i=1}^{n}\frac{n+i}{i(n-i+1)} \\
&\le \frac{3^n}{4^n} \to 0 \quad (n \to \infty).
\end{aligned}$$

## 4. The error estimates of $H_{2n}$ and a numerical example

$H_{2n}(x)$ has the same continuity as the $H_{2n-1}(x)$. The following theorem shows that $H_{2n}(x)$ has accuracy $O(h^{2n+1})$ and so has better accuracy than $H_{2n-1}(x)$.

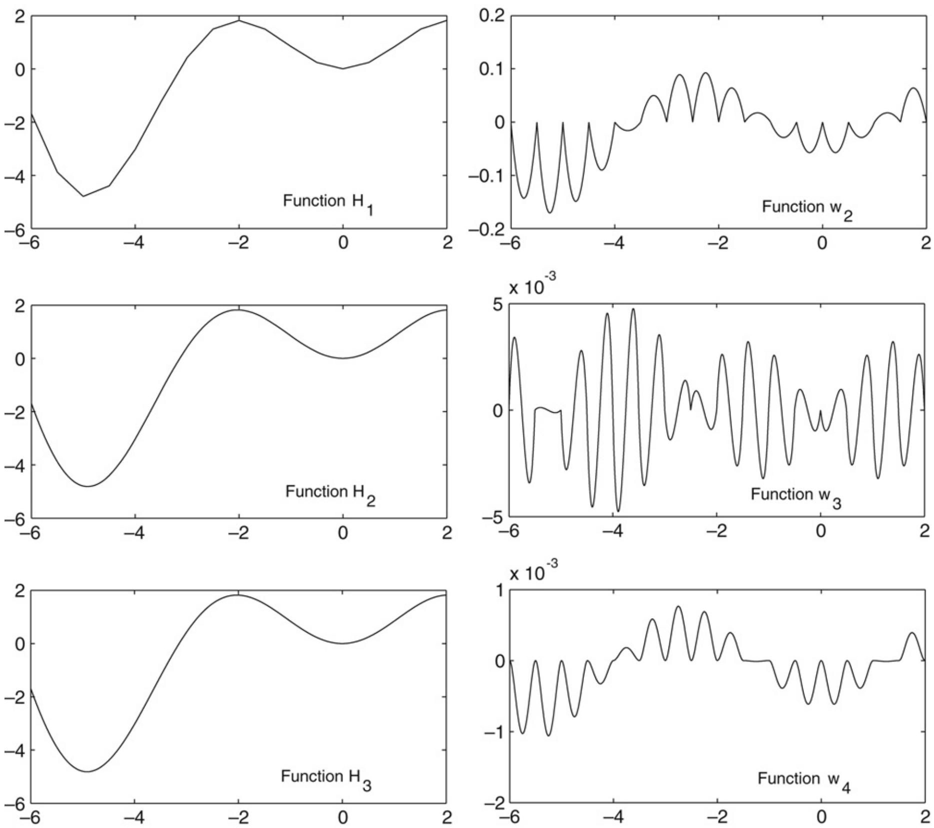


**Fig. 1.** The piecewise interpolants and their correction terms.

**Theorem 9.** *If* $\in C^{2n+1}([a, b])$, *then*

$$\|f - H_{2n}\|_\infty \le \|f - H_{2n+1}\|_\infty + \frac{h^{2n+1}}{(2n+1)!4^{n+1}}\|f^{(2n+1)}\|_\infty, \tag{30}$$

$$\|f^{(r)} - H_{2n}^{(r)}\|_\infty \le \|f^{(r)} - H_{2n+1}^{(r)}\|_\infty + \frac{h^{2n-r+1}}{(2n-r+1)!2^{2n-r+1}}\|f^{(2n+1)}\|_\infty \tag{31}$$

*with* $r = 1, 2, \dots, 2n+1$.

**Proof.** For $x \in [x_k, x_{k+1}]$, $k = 1, 2, \dots, m-1$, $r = 0, 1, \dots, 2n+1$, $u = (x - x_k)/h_k$, we have

$$\begin{aligned} f^{(r)}(x) - H_{2n}^{(r)}(x) &= f^{(r)}(x) - H_{2n+1}^{(r)}(x) + H_{2n+1}^{(r)}(x) - H_{2n}^{(r)}(x) \\ &= f^{(r)}(x) - H_{2n+1}^{(r)}(x) + \frac{\sigma_{n,k}}{h_k^r}[(1-u)^n u^n (1-2u)]^{(r)}. \end{aligned}$$

From this, Lemma 4 and Theorem 5, we obtain (30) and (31) immediately. □

According to (2) and (30), $H_{2n}$ has accuracy $O(h^{2n+1})$ and reproduces all polynomials of degree $\le 2n$. When the accuracy of $H_{2n}$ is enough, we may not compute $H_{2n+1}$. From (14) we can see that

$$H_{2n}(x) = H_{2n+1}(x)$$

for $x = x_k, (x_k + x_{k+1})/h_k, x_{k+1}$, and

$$\int_{x_k}^{x_{k+1}} H_{2n}(x)\mathrm{d}x = \int_{x_k}^{x_{k+1}} H_{2n+1}(x)\mathrm{d}x,$$

$k = 1, 2, \dots, m-1$. Therefore, $H_{2n}$ approximate $H_{2n+1}$ very well. This point can also be seen from (22).

In order to illustrate the theoretical results, we end this section by a numerical example. We describe a decomposition of the interpolants $H_j, j = 1, 2, \dots, 6$, which interpolates the values of the function $f(x) = x\sin(x)$, and its derivatives at the

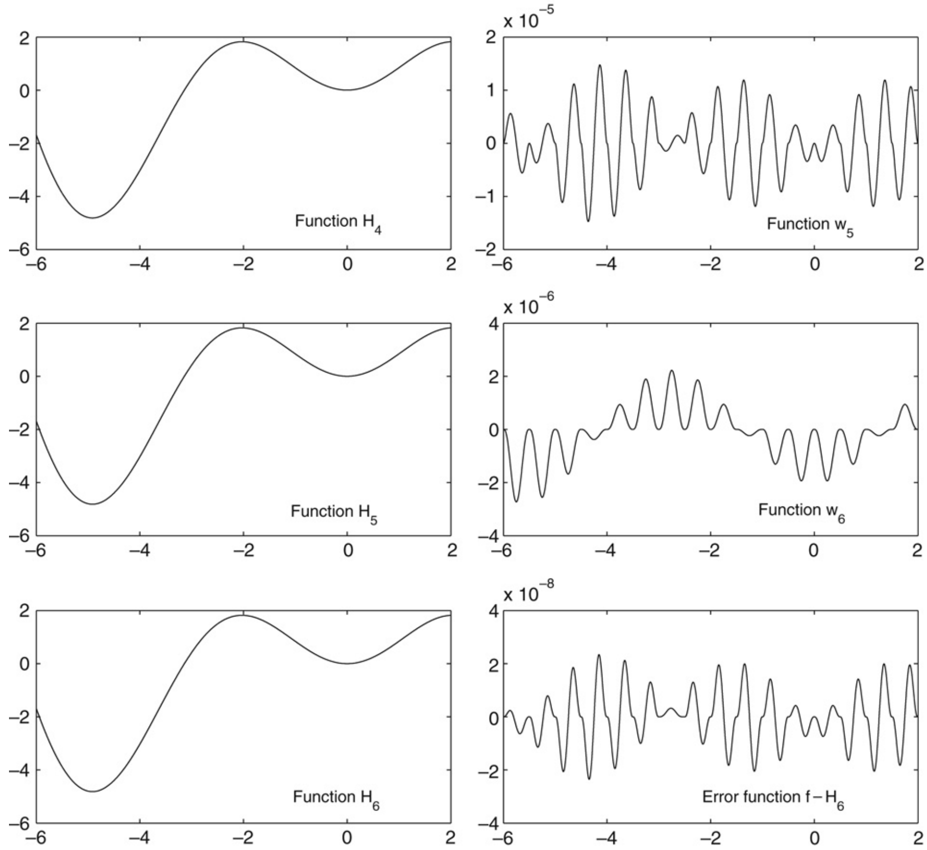


**Fig. 1.** (*continued*)

knots $x_k = -6.5 + 0.5k$, $k = 1, 2, \ldots, 17$. The function is the same as the function given in [6], and the number of the knots is less than the number of the knots given in [6]. Fig. 1 shows the graphs of the functions $H_j$ $(1 \le j \le 6)$, $w_j$ $(2 \le j \le 6)$, and the error function $f - H_6$. The corresponding functions of the graphs are marked in Fig. 1. From the graphs we can see that $w_{2i}$ is near $w_{2i-1}$, and then correction terms $w_{2i}$ are effective. With the graphs in [6], we can see that the error bounds of $f - H_6$ and $f - H_7$ are close since the both error bounds are about $10^{-8}$.

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