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A mixed finite volume element method based on rectangular mesh for biharmonic equations

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Abstract

This paper presents a mixed finite volume element scheme based on rectangular partition for solving biharmonic equations. It also gives a kind of adaptive Uzawa iteration method for the scheme. It is rigorously proved that the scheme has first-order accuracy in H^1 semi-norm and L^2 norm according to the characteristics of the scheme. Finally, two numerical examples illustrate the effectiveness of the method.

MSC: 65N12

Keywords: Biharmonic equation; Rectangular mesh; Mixed finite volume element method; Error estimate

1. Introduction

Finite volume element method (FVEM) [4,10,14] or its generalized form, finite volume method [5,9], uses a volume integral formulation of the differential equation with a finite partitioning set of volume to discretize the equation. As far as the method is concerned, it is identical to the special case of the generalized difference method (GDM) proposed by Professor Ronghua Li [6,8,11,13,16], that is, linear or bilinear finite element space is used as trial or admissible finite element space and piecewise constant space is used as test function space. As for theoretical analysis, there are some differences in FVEM and GDM. For example, FVEM conventionally estimates the error by discrete energy norm, whereas, GDM absorbs more ideas from finite element method. Because these methods keep conservation law of mass or energy, they are widely used in computational fluid mechanics.

In this paper, we are concerned about biharmonic equations. Because of their importance, lot of methods have been developed to treat biharmonic equations, for instance, 13-point finite difference

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scheme, higher-order finite element methods and more commonly used, mixed finite element methods [2,3,7,12,15]. As for generalized difference methods, Wei Wu [11] presented a kind of Ciarlet–Raviart mixed generalized difference method by triangulation and circumcenter dual partition. Zhongying Chen [16] proposed another method by other variational principle. Still based on Ciarlet–Raviart mixed variational principle, we present a kind of mixed FVEM by rectangular partition in Section 2. In Section 3, first-order error estimate is derived strictly in accordance with the characteristics of the FVEM. We note that the method of error estimate in this paper is somewhat different from the method in [8,11]. We do not introduce the so-called Neumann projection and the method in this paper is more concise. Because the linear system of algebraic equations derived by mixed finite volume element scheme is indefinite, classical iteration methods such as Gauss–Seidel and SOR are not valid for the scheme. In Section 4, we construct a class of adaptive Uzawa iteration method. To verify the method in this paper, we compute some typical numerical examples and the results are very satisfactory. Compared with 13-point finite difference scheme, the method in this paper has higher computation accuracy and can be used to solve more general problems.

2. Mixed FVEM

Consider the following two-dimensional biharmonic equation on domain D:

$$\Delta^2\psi = f(x,y), \quad (x,y)\in D, \tag{2.1a}$$

$$\psi = \frac{\partial\psi}{\partial n} = 0, \quad (x,y)\in\partial D, \tag{2.1b}$$

where $f(x,y)$ is sufficiently smooth and n denotes the unit outward normal vector of ∂D. For convenience, assume $D=[0,1]^2$.

By introducing vorticity $\Omega = -\Delta\psi$, (2.1a) is equivalent to

$$-\Delta\Omega = f, \quad -\Delta\psi = \Omega. \tag{2.2}$$

Denote $H^m(D)$ by the standard Sobolev space of order m. Also denote by $H_0^{(m)}(D) = H_0^1(D)\cap H^m(D)$. Let $V\subset\bar{D}$ be any control volume with piecewise smooth boundary ∂V. Integrate (2.2) over control volume V, then by Green's formula, the conservative integral form of (2.2) reads, finding $(\psi,\Omega)\in H_0^{(2)}(D)\times H^2(D)$, such that

$$-\int_{\partial V}\frac{\partial\Omega}{\partial n}\,\mathrm{d}s = \int_V f\,\mathrm{d}x\,\mathrm{d}y \quad \forall V\subset D, \tag{2.3a}$$

$$-\int_{\partial V\backslash\partial D}\frac{\partial\psi}{\partial n}\,\mathrm{d}s = \int_V \Omega\,\mathrm{d}x\,\mathrm{d}y \quad \forall V\subset\bar{D}. \tag{2.3b}$$

It is easy to prove that (2.3) is equivalent to (2.1) for $(\psi,\Omega)\in(C^4(D)\cap C_0^1(\bar{D}))\times C^2(D)$ and $f\in C(D)$. In fact, from (2.3a) and by Green's formula, we have $\int_V(-\Delta\Omega - f)\,\mathrm{d}x\,\mathrm{d}y = 0 \quad \forall V\subset D$. From the continuity of $-\Delta\Omega - f$ and the arbitrariness of $V\subset D$, we can derive $-\Delta\Omega = f$ in D. Restricting

$V \subset D$ for (2.3b), we know $-\Delta\psi = \Omega$ in D, then $\int_{\partial V \cap \partial D}(\partial\psi/\partial n)\,\mathrm{d}s = 0$ for arbitrary $V \subset \bar{D}$. In a similar argument, we know $\partial\psi/\partial n|_{\partial D} = 0$. Hence, (2.1) holds.

The FVE approach of (2.3) consists of replacing $H_0^{(2)}(D)(H^2(D))$ by finite-dimensional space of piecewise smooth functions and using a finite set of volumes. In this paper, we consider rectangular partition of D and piecewise bilinear interpolations for Ω and ψ.

First, give a nonuniform rectangular partition Q_h for D and the nodes are (x_i, y_j) $(i=0,1,\ldots,N, j=0,1,\ldots,M)$. Let $\bar{D}_h=\{(x_i, y_j), 0 \leqslant i \leqslant N, 0 \leqslant j \leqslant M\}$ and denote all the interior nodes by D_h. Further let $h_i = x_i - x_{i-1}$ $(i = 1,2,\ldots,N)$, $k_j = y_j - y_{j-1}$ $(j = 1,2,\ldots,M)$, $x_{i-1/2} = x_i - h_i/2$, $x_{i+1/2} = x_i + h_{i+1}/2$, $y_{j-1/2} = y_j - k_j/2$, $y_{j+1/2} = y_j + k_{j+1}/2$, then $V_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$ is a control volume or dual element of node (x_i, y_j). For boundary nodes, their control volumes should be modified correspondingly. For instance, $V_{00} = [x_0, x_{1/2}] \times [y_0, y_{1/2}]$, $V_{i0} = [x_{i-1/2}, x_{i+1/2}] \times [y_0, y_{1/2}]$ for $i = 1,2,\ldots,N-1$. All the control volumes constitute the dual partition Q_h^* of domain D.

Second, let $H_h \subset H^1(D)$ and $H_{0h} \subset H_0^1(D)$ be both the piecewise bilinear finite element subspaces over partition Q_h, then the mixed finite volume element scheme of (2.3) reads, finding $(\psi_h, \Omega_h) \in H_{0h} \times H_h$, such that

$$-\int_{\partial V_{ij}} \frac{\partial \Omega_h}{\partial n}\,\mathrm{d}s = \int_{V_{ij}} f\,\mathrm{d}x\,\mathrm{d}y, \quad i(j) = 1,2,\ldots,N-1(M-1), \tag{2.4a}$$

$$-\int_{\partial V_{ij}\setminus\partial D} \frac{\partial \psi_h}{\partial n}\,\mathrm{d}s = \int_{V_{ij}} \Omega_h\,\mathrm{d}x\,\mathrm{d}y, \quad i(j) = 0,1,\ldots,N(M). \tag{2.4b}$$

Eq. (2.4) can be further written as difference equations. Denote by $\Omega_{ij} = \Omega_h(x_i, y_j)$, $\psi_{ij} = \psi_h(x_i, y_j)$. For a uniform partition with $M = N$ and $h_i = k_j = h$, (2.4) can be written as

$$\begin{aligned}&\tfrac{1}{4}[12\Omega_{ij} - (\Omega_{i-1,j-1} + \Omega_{i+1,j-1} + \Omega_{i+1,j+1} + \Omega_{i-1,j+1})\\ &\quad -2(\Omega_{i,j-1} + \Omega_{i+1,j} + \Omega_{i,j+1} + \Omega_{i-1,j})] = \int_{V_{ij}} f\,\mathrm{d}x\,\mathrm{d}y, \quad i,j = 1,2,\ldots,N-1.\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\tfrac{1}{4}[12\psi_{ij} - (\psi_{i-1,j-1} + \psi_{i+1,j-1} + \psi_{i+1,j+1} + \psi_{i-1,j+1})\\ &\quad -2(\psi_{i,j-1} + \psi_{i+1,j} + \psi_{i,j+1} + \psi_{i-1,j})]\\ &\quad =\frac{h^2}{64}[36\Omega_{ij} + (\Omega_{i-1,j-1} + \Omega_{i+1,j-1} + \Omega_{i+1,j+1} + \Omega_{i-1,j+1})\\ &\qquad +6(\Omega_{i,j-1} + \Omega_{i+1,j} + \Omega_{i,j+1} + \Omega_{i-1,j})], \quad i,j = 1,2,\ldots,N-1.\end{aligned} \tag{2.6}$$

Bottom boundary $(y = 0)$:

$$\frac{h^2}{16}[9\Omega_{00} + 3\Omega_{10} + 3\Omega_{01} + \Omega_{11}] = 3\psi_{00} - (\psi_{10} + \psi_{01} + \psi_{11}),$$

$$\frac{h^2}{16}[18\Omega_{i0} + 3\Omega_{i-1,0} + 3\Omega_{i+1,0} + \Omega_{i-1,1} + 6\Omega_{i1} + \Omega_{i+1,1}]$$

$$= -\psi_{i-1,0} + 6\psi_{i,0} - \psi_{i+1,0} - \psi_{i-1,1} - 2\psi_{i,1} - \psi_{i+1,1}, \quad i = 1,2,\dots,N-1,$$

$$\frac{h^2}{16}[9\Omega_{N0} + 3\Omega_{N-1,0} + 3\Omega_{N1} + \Omega_{N-1,1}] = -\psi_{N-1,0} + 3\psi_{N0} - \psi_{N-1,1} - \psi_{N1}. \tag{2.7}$$

Top boundary ($y = 1$):

$$\frac{h^2}{16}[9\Omega_{0N} + 3\Omega_{0,N-1} + 3\Omega_{1N} + \Omega_{1,N-1}] = 3\psi_{0N} - (\psi_{0,N-1} + \psi_{1N} + \psi_{1,N-1}),$$

$$\frac{h^2}{16}[18\Omega_{iN} + 3\Omega_{i-1,N} + 3\Omega_{i+1,N} + \Omega_{i-1,N-1} + 6\Omega_{i,N-1} + \Omega_{i+1,N-1}]$$

$$= -\psi_{i-1,N} + 6\psi_{iN} - \psi_{i+1,N} - \psi_{i-1,N-1} - 2\psi_{i,N-1} - \psi_{i+1,N-1}, \quad i = 1,2,\dots,N-1,$$

$$\frac{h^2}{16}[9\Omega_{NN} + 3\Omega_{N-1,N} + 3\Omega_{N,N-1} + \Omega_{N-1,N-1}]$$

$$= -\psi_{N-1,N} + 3\psi_{NN} - \psi_{N-1,N-1} - \psi_{N,N-1}. \tag{2.8}$$

Left boundary ($x = 0$):

$$\frac{h^2}{16}[18\Omega_{0j} + 3\Omega_{0,j-1} + 3\Omega_{0,j+1} + \Omega_{1,j-1} + 6\Omega_{1j} + \Omega_{1,j+1}]$$

$$= -\psi_{0,j-1} + 6\psi_{0,j} - \psi_{0,j+1} - \psi_{1,j-1} - 2\psi_{1,j} - \psi_{1,j+1}, \quad j = 1,2,\dots,N-1. \tag{2.9}$$

Right boundary ($x = 1$):

$$\frac{h^2}{16}[18\Omega_{Nj} + 3\Omega_{N,j-1} + 3\Omega_{N,j+1} + \Omega_{N-1,j-1} + 6\Omega_{N-1,j} + \Omega_{N-1,j+1}]$$

$$= -\psi_{N,j-1} + 6\psi_{N,j} - \psi_{N,j+1} - \psi_{N-1,j-1} - 2\psi_{N-1,j} - \psi_{N-1,j+1}, \quad j = 1,2,\dots,N-1. \tag{2.10}$$

Obviously, the linear system of equations (2.5)–(2.10) is simpler than that formed by mixed bilinear finite element method. It can be solved by Uzawa iteration method, see Section 4 for detail.

3. Error estimate

In Section 2, we derived a kind of finite volume element scheme. In this section, we further analyze the error of the scheme. Because the theory about the generalized difference method [8,11] has been established completely, we embed the scheme in Section 2 into the theoretical framework of GDM. Suppose $P(x_i, y_j)$ is an arbitrary node in $\bar{D}_h$. Denote $V_P = V_{ij}$ by the corresponding dual element of node P and χ_P by characteristic function over V_P. Let

$$\Pi_h^* \varphi_h = \sum_{P \in \bar{D}_h} \varphi_h(P)\chi_P \quad \forall \varphi_h \in H_h, \tag{3.1}$$

$$a(\omega,\Pi_h^*\varphi_h)=-\sum_{P\in\bar{D}_h}\varphi_h(P)\int_{\partial V_P\setminus\partial D}\frac{\partial\omega}{\partial n}\,\mathrm{d}s,\quad \omega\in H^2(D)\,(H_0^{(2)}(D)),\;\; \varphi_h\in H_h(H_{0h}),$$

$$(\omega,\Pi_h^*\,\varphi_h)=\sum_{P\in\bar{D}_h}\varphi_h(P)\int_{V_P}\omega\,\mathrm{d}x\,\mathrm{d}y,\quad \varphi_h\in H_h(H_{0h}). \tag{3.2}$$

By restricting the arbitrary control volume V to special V_P, (2.3) can be written as, finding $(\psi,\Omega)\in H_0^{(2)}(D)\times H^2(D)$, such that

$$a(\Omega,\Pi_h^*\phi_h)=(f,\Pi_h^*\phi_h)\quad\forall\phi_h\in H_{0h}, \tag{3.3a}$$

$$a(\psi,\Pi_h^*\varphi_h)=(\Omega,\Pi_h^*\varphi_h)\quad\forall\varphi_h\in H_h. \tag{3.3b}$$

Analogously, (2.4) equals to finding $(\psi_h,\Omega_h)\in H_{0h}\times H_h$, such that

$$a(\Omega_h,\Pi_h^*\phi_h)=(f,\Pi_h^*\phi_h)\quad\forall\phi_h\in H_{0h}, \tag{3.4a}$$

$$a(\psi_h,\Pi_h^*\varphi_h)=(\Omega_h,\Pi_h^*\varphi_h)\quad\forall\varphi_h\in H_h. \tag{3.4b}$$

Remark 1. For boundary node P and $\phi_h\in H_{0h}$, $\phi_h(P)=0$. Hence, (3.3a) and (3.4a) hold essentially for interior nodes and they just equal to (2.3a) ($V=V_P$) and (2.4a), respectively.

Suppose Q_h is a quasi-uniformly regular partition, i.e., there exist constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4>0$, satisfying

$$\alpha_1\max_i h_i\leqslant\min_i h_i,\quad \alpha_2\max_j k_j\leqslant\min_j k_j,\quad \alpha_3 k_j\leqslant h_i\leqslant\alpha_4 k_j. \tag{3.5}$$

Let $h=\max(\max h_i,\max k_j)$. Depicted as in Fig. 1, we convert the integral on the edge of dual partition to the related elements, then

$$a(\varphi_h,\Pi_h^*\phi_h)=-\sum_{E\in Q_h}\sum_{l=1}^{4}[\phi_h(P_l)-\phi_h(P_{l+1})]\int_{\overline{M_lQ}}\frac{\partial\varphi_h}{\partial n}\,\mathrm{d}s\quad\forall\phi_h\in H_{0h},\;\;\varphi_h\in H_h, \tag{3.6a}$$

$$a(\phi_h,\Pi_h^*\varphi_h)=-\sum_{E\in Q_h}\sum_{l=1}^{4}[\varphi_h(P_l)-\varphi_h(P_{l+1})]\int_{\overline{M_lQ}}\frac{\partial\phi_h}{\partial n}\,\mathrm{d}s\quad\forall\phi_h\in H_{0h},\;\;\varphi_h\in H_h, \tag{3.6b}$$

where $P_5=P_1$.

Remark 2. It is easy to see that (3.6a) also holds for $\varphi_h\in H^2(D)$ and (3.6b) holds for $\phi_h\in H_0^{(2)}(D)$.

Denote $\|\cdot\|_s$ and $|\cdot|_s$ by continuous norm and continuous semi-norm of order s in Sobolev space, respectively. Define discrete H^1 semi-norm and discrete L^2 norm, respectively, by

$$|\varphi_h|_{1,h}=\left\{\sum_{E\in Q_h}|\varphi_h|_{1,h,E}^2\right\}^{1/2},\quad \|\varphi_h\|_{0,h}=\left\{\sum_{E\in Q_h}\|\varphi_h\|_{0,h,E}^2\right\}^{1/2}\quad\forall\varphi_h\in H_h\,(H_{0h}), \tag{3.7}$$

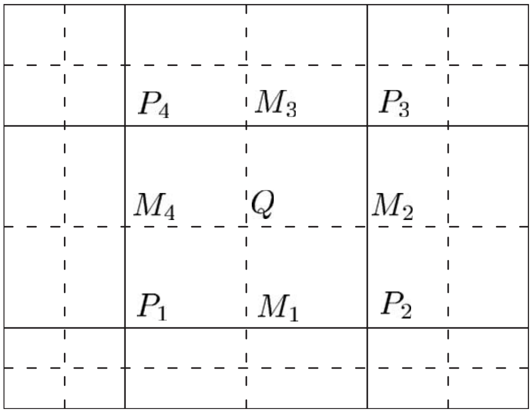

Fig. 1. Illustration for an element and its nodes.

where $E = \overline{P_1P_2P_3P_4} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, shown as in Fig. 1 and

$$|\varphi_h|_{1,h,E}^2 = \frac{k_j}{2h_i} \sum_{l=1,3} (\varphi_h(P_{l+1}) - \varphi_h(P_l))^2 + \frac{h_i}{2k_j} \sum_{l=2,4} (\varphi_h(P_{l+1}) - \varphi_h(P_l))^2,$$

$$\|\varphi_h\|_{0,h,E}^2 = \frac{h_i k_j}{4} \sum_{l=1}^{4} \varphi_h(P_l)^2.$$

Lemma 1. *For* $\forall \varphi_h \in H_h\,(H_{0h})$, $|\varphi_h|_{1,h}$ *is equivalent to* $|\varphi_h|_1$ *and* $\|\varphi_h\|_{0,h}$ *is equivalent to* $\|\varphi_h\|_0$, *that is, the following inequalities hold*:

$$\frac{\sqrt{3}}{3} |\varphi_h|_{1,h} \leqslant |\varphi_h|_1 \leqslant |\varphi_h|_{1,h}, \quad \frac{1}{3} \|\varphi_h\|_{0,h} \leqslant \|\varphi_h\|_0 \leqslant \|\varphi_h\|_{0,h}. \tag{3.8}$$

Proof. Suppose Q is the center of element E. Let $\xi = 2(x - x_Q)/h_i$, $\eta = 2(y - y_Q)/k_j$, then E is transformed to $\hat{E} = [-1, 1]^2$. Construct bilinear interpolating base functions on $\hat{E}$, which are

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta),\ N_2 = \tfrac{1}{4}(1+\xi)(1-\eta),\ N_3 = \tfrac{1}{4}(1+\xi)(1+\eta),\ N_4 = \tfrac{1}{4}(1-\xi)(1+\eta).$$

Then $\varphi_h = \sum_{l=1}^4 \varphi_h(P_l) N_l(\xi, \eta)$. According to the definition of $|\varphi_h|_{1,E}$, we have

$$\begin{aligned}
|\varphi_h|_{1,E}^2 &= \int_{\hat{E}} \left[\frac{k_j}{h_i} \left(\frac{\partial \varphi_h}{\partial \xi} \right)^2 + \frac{h_i}{k_j} \left(\frac{\partial \varphi_h}{\partial \eta} \right)^2 \right] \mathrm{d}\xi \, \mathrm{d}\eta \\
&= \frac{k_j}{3h_i} [(\varphi_h(P_2) - \varphi_h(P_1))^2 + (\varphi_h(P_3) - \varphi_h(P_4))^2 + (\varphi_h(P_2) - \varphi_h(P_1))(\varphi_h(P_3) - \varphi_h(P_4))] \\
&\quad + \frac{h_i}{3k_j} [(\varphi_h(P_4) - \varphi_h(P_1))^2 + (\varphi_h(P_3) - \varphi_h(P_2))^2 \\
&\quad + (\varphi_h(P_4) - \varphi_h(P_1))(\varphi_h(P_3) - \varphi_h(P_2))].
\end{aligned}$$

By Cauchy inequality and (3.7), the first inequality of (3.8) is proved. As for the second one, a straightforward computing shows

$$\|\varphi_h\|_{0,E}^2 = \frac{h_i k_j}{36}[\varphi_h(P_1), \varphi_h(P_2), \varphi_h(P_3), \varphi_h(P_4)] \begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} \varphi_h(P_1) \\ \varphi_h(P_2) \\ \varphi_h(P_3) \\ \varphi_h(P_4) \end{bmatrix}.$$

The eigenvalues of the matrix of the right-hand side of the above formula are $\lambda_l = 1, 3, 3, 9$, from which we can obtain

$$\tfrac{1}{9}\,\|\varphi_h\|_{0,h}^2 \leqslant \|\varphi_h\|_0^2 \leqslant \|\varphi_h\|_{0,h}^2.$$

Lemma 1 is proved. □

Lemma 2.

$$a(\varphi_h, \Pi_h^* \phi_h) = a(\phi_h, \Pi_h^* \varphi_h) \quad \forall \phi_h \in H_{0h} \quad \forall \varphi_h \in H_h, \tag{3.9}$$

$$a(\phi_h, \Pi_h^* \phi_h) \geqslant \tfrac{1}{2}|\phi_h|_{1,h}^2 \geqslant \tfrac{1}{2}|\phi_h|_1^2 \quad \forall \phi_h \in H_{0h}. \tag{3.10}$$

Proof. By (3.6), further computing the integrals, we have

$$\int_{\overline{M_1Q}} \frac{\partial \phi_h}{\partial n}\,\mathrm{d}s = \frac{k_j}{8h_i}[3(\phi_h(P_2) - \phi_h(P_1)) + (\phi_h(P_3) - \phi_h(P_4))],$$

$$\int_{\overline{M_3Q}} \frac{\partial \phi_h}{\partial n}\,\mathrm{d}s = -\frac{k_j}{8h_i}[(\phi_h(P_2) - \phi_h(P_1)) + 3(\phi_h(P_3) - \phi_h(P_4))],$$

$$\int_{\overline{M_2Q}} \frac{\partial \phi_h}{\partial n}\,\mathrm{d}s = \frac{h_i}{8k_j}[(\phi_h(P_4) - \phi_h(P_1)) + 3(\phi_h(P_3) - \phi_h(P_2))],$$

$$\int_{\overline{M_4Q}} \frac{\partial \phi_h}{\partial n}\,\mathrm{d}s = -\frac{h_i}{8k_j}[3(\phi_h(P_4) - \phi_h(P_1)) + (\phi_h(P_3) - \phi_h(P_2))].$$

The integrals about φ_h can be obtained analogously. A straightforward verification shows

$$\sum_{l=1,3}[\phi_h(P_l) - \phi_h(P_{l+1})] \int_{\overline{M_lQ}} \frac{\partial \varphi_h}{\partial n}\,\mathrm{d}s = \sum_{l=1,3}[\varphi_h(P_l) - \varphi_h(P_{l+1})] \int_{\overline{M_lQ}} \frac{\partial \phi_h}{\partial n}\,\mathrm{d}s.$$

Similar formula can also be obtained for $l = 2, 4$. Thus, (3.9) holds. As for (3.10), we have

$$a(\phi_h, \Pi_h^* \phi_h) \geqslant \frac{1}{4} \sum_{E \in Q_h} \left\{ \frac{k_j}{h_i} \sum_{l=1,3} (\phi_h(P_{l+1}) - \phi_h(P_l))^2 + \frac{h_i}{k_j} \sum_{l=2,4} (\phi_h(P_{l+1}) - \phi_h(P_l))^2 \right\}$$

$$= \tfrac{1}{2}\,|\phi_h|_{1,h}^2 \geqslant \tfrac{1}{2}\,|\phi_h|_1^2.$$

The proof is completed. □

Lemma 3.

$$(\varphi_h, \Pi_h^* \varphi_h) \geqslant \tfrac{1}{4} \|\varphi_h\|_{0,h}^2 \geqslant \tfrac{1}{4} \|\varphi_h\|_0^2 \quad \forall \varphi_h \in H_h, \tag{3.11a}$$

$$|(\omega, \Pi_h^* \varphi_h)| \leqslant \|\omega\|_0 \|\varphi_h\|_{0,h} \leqslant 3\|\omega\|_0 \|\varphi_h\|_0 \quad \forall \omega \in H^1(D) \quad \forall \varphi_h \in H_h. \tag{3.11b}$$

Proof.

$$(\varphi_h, \Pi_h^* \varphi_h) = \sum_{P \in \bar{D}_h} \varphi_h(P) \int_{V_P} \varphi_h \, \mathrm{d}x \, \mathrm{d}y = \sum_{E \in Q_h} \sum_{l=1}^{4} \varphi_h(P_l) \int_{V_{P_l} \cap E} \varphi_h \, \mathrm{d}x \, \mathrm{d}y,$$

where for example

$$\int_{V_{P_1} \cap E} \varphi_h \, \mathrm{d}x \, \mathrm{d}y = \frac{h_i k_j}{64} [9\varphi_h(P_1) + 3\varphi_h(P_2) + \varphi_h(P_3) + 3\varphi_h(P_4)].$$

Analogously, we can get the other integrals. Add these integrals, then

$$\sum_{l=1}^{4} \varphi_h(P_l) \int_{V_{P_l} \cap E} \varphi_h \, \mathrm{d}x \, \mathrm{d}y = \frac{h_i k_j}{64} [\varphi_h(P_1), \varphi_h(P_2), \varphi_h(P_3), \varphi_h(P_4)] M \begin{bmatrix} \varphi_h(P_1) \\ \varphi_h(P_2) \\ \varphi_h(P_3) \\ \varphi_h(P_4) \end{bmatrix},$$

where

$$M = \begin{bmatrix} 9 & 3 & 1 & 3 \\ 3 & 9 & 3 & 1 \\ 1 & 3 & 9 & 3 \\ 3 & 1 & 3 & 9 \end{bmatrix}.$$

A simple computation shows the eigenvalues of matrix M are $\lambda_l = 4, 8, 8, 16$, from which we can get (3.11a). As for (3.11b), denote S_{V_P} by the area of the dual element V_P, then

$$|(\omega, \Pi_h^* \varphi_h)| \leqslant \left[\sum_{P \in \bar{D}_h} \varphi_h(P)^2 S_{V_P} \right]^{1/2} \left[\sum_{P \in \bar{D}_h} \frac{1}{S_{V_P}} \left(\int_{V_P} \omega \, \mathrm{d}x \, \mathrm{d}y \right)^2 \right]^{1/2} \leqslant \|\omega\|_0 \|\varphi_h\|_{0,h}.$$

By (3.8) we know (3.11b) holds. Lemma 3 is proved. □

From Lemmas 2 and 3, we know that scheme (3.4) has a unique solution. In fact, without loss of generality, assume $f \equiv 0$ in (3.4a). Let $\phi_h = \psi_h$ in (3.4a) and $\varphi_h = \Omega_h$ in (3.4b), then by (3.9), we have $(\Omega_h, \Pi_h^* \Omega_h) = 0$. By (3.11a), we can derive $\Omega_h \equiv 0$. In particular, taking $\varphi_h = \psi_h$ in (3.4b) and by (3.10), we know $\psi_h \equiv 0$. That is, the solution of (3.4) is unique.

Let $\Pi_h \Omega : H^1(D) \to H_h$ and $\Pi_h \psi : H_0^1(D) \to H_{0h}$ be two piecewise bilinear interpolation projects. By the interpolation theory in Sobolev space, we have

$$\|\Omega - \Pi_h \Omega\|_0 \leqslant Ch^2 |\Omega|_2, \quad |\psi - \Pi_h \psi|_1 \leqslant Ch|\psi|_2. \tag{3.12}$$

Lemma 4. *Assume $\psi \in H_0^2(D) \cap H^3(D)$ and $\Omega \in H^2(D)$, then there exists a positive constant C, independent of the mesh size h, such that*

$$|a(\psi - \Pi_h\psi, \Pi_h^*\varphi_h)| \leqslant Ch^2|\psi|_3|\varphi_h|_1, \quad |a(\psi - \Pi_h\psi, \Pi_h^*\varphi_h)| \leqslant Ch|\psi|_2|\varphi_h|_1 \quad \forall \varphi_h \in H_h, \tag{3.13}$$

$$|a(\Omega - \Pi_h\Omega, \Pi_h^*\phi_h)| \leqslant Ch|\Omega|_2|\phi_h|_1 \quad \forall \phi_h \in H_{0h}. \tag{3.14}$$

Proof. From (3.6b) and Remark 2, noting Fig. 1, we have

$$a(\psi - \Pi_h\psi, \Pi_h^*\varphi_h) = -\sum_{E\in Q_h}\sum_{l=1}^{4}[\varphi_h(P_l) - \varphi_h(P_{l+1})]\int_{\overline{M_lQ}} \frac{\partial(\psi - \Pi_h\psi)}{\partial n}\,\mathrm{d}s, \tag{3.15}$$

$$\int_{\overline{M_1Q}} \frac{\partial(\psi - \Pi_h\psi)}{\partial n}\,\mathrm{d}s = \frac{k_j}{h_i}\left\{\int_{-1}^{0} \frac{\partial\psi(0,\eta)}{\partial\xi}\,\mathrm{d}\eta - \frac{1}{8}[3(\psi(P_2) - \psi(P_1)) + (\psi(P_3) - \psi(P_4))]\right\}$$

$$\triangleq \frac{k_j}{h_i} I_1(\psi).$$

As a linear functional of $\psi \in H^3(D)$, $I_1(\psi)$ satisfies $|I_1(\psi)| \leqslant C\|\psi\|_{1,\infty,\hat{E}}$. In addition, $H^3(\hat{E}) \hookrightarrow C^1(\hat{E})$. Hence, $|I_1(\psi)| \leqslant C\|\psi\|_{3,\hat{E}}$. A straightforward calculation shows $I_1(\psi) \equiv 0$ for $\psi = 1, \xi, \eta, \xi^2, \xi\eta, \eta^2$. By Bramble–Hilbert Lemma [1], we know $|I_1(\psi)| \leqslant C|\psi|_{3,\hat{E}}$. By an integral transformation, we have $|I_1(\psi)| \leqslant Ch^2|\psi|_{3,E}$. Thus,

$$\left|\int_{\overline{M_1Q}} \frac{\partial(\psi - \Pi_h\psi)}{\partial n}\,\mathrm{d}s\right| \leqslant C\frac{k_j}{h_i} h^2|\psi|_{3,E}.$$

The other integrals in (3.15) have similar estimates. Using the inequality

$$\left|\sum a_ib_i\right| \leqslant \left(\sum \rho_i a_i^2\right)^{1/2}\left(\sum \frac{1}{\rho_i} b_i^2\right)^{1/2},$$

noting (3.5), we get

$$\left|\sum_{l=1}^{4}[\varphi_h(P_l) - \varphi_h(P_{l+1})]\int_{\overline{M_lQ}} \frac{\partial(\psi - \Pi_h\psi)}{\partial n}\,\mathrm{d}s\right| \leqslant Ch^2|\psi|_{3,E}|\varphi_h|_{1,h,E}.$$

Again from (3.15), by Cauchy inequality and Lemma 1, we obtain the first inequality of (3.13). Now we prove the second one of (3.13). $I_1(\psi)$ is still defined as above. Using trace theorem, we further have

$$|I_1(\psi)| \leqslant C(\|\psi\|_{2,\hat{E}} + \|\psi\|_{0,\infty,\hat{E}}).$$

Because $H^2(\hat{E}) \hookrightarrow C^0(\hat{E})$, thus, $|I_1(\psi)| \leqslant C\|\psi\|_{2,\hat{E}}$. Again by Bramble–Hilbert Lemma, we can prove the second inequality of (3.13). As for (3.14), from (3.6a), a formula analogous to (3.15) can be derived. Thus, (3.14) holds. Lemma 4 is proved. □

Lemma 5 (Ciarlet [1]). *For $\forall \varphi_h \in H_h$, $\forall \phi_h \in H_{0h}$, we have*

$$|\varphi_h|_1 \leqslant Ch^{-1}\|\varphi_h\|_0, \quad \|\phi_h\|_0 \leqslant C|\phi_h|_1. \tag{3.16}$$

Subtracting (3.4) *from* (3.3), *we obtain the following error equations*:

$$a(\Omega-\Omega_h,\Pi_h^*\phi_h)=0 \quad \forall\phi_h\in H_{0h}, \tag{3.17a}$$

$$a(\psi-\psi_h,\Pi_h^*\varphi_h)=(\Omega-\Omega_h,\Pi_h^*\varphi_h) \quad \forall\varphi_h\in H_h. \tag{3.17b}$$

Now we state the main result of this section.

Theorem. *Assume* $(\psi,\Omega)\in(H_0^2(D)\cap H^3(D))\times H^2(D)$ *is the solution of* (2.3) *and* Q_h *is a quasi-uniformly rectangular partition of domain D, then the approximate solution* (ψ_h,Ω_h) *of mixed finite volume element scheme* (2.4) *converges to the true solution* (ψ,Ω) *with the following estimate*:

$$|\psi-\psi_h|_1+\|\Omega-\Omega_h\|_0\leqslant Ch(|\psi|_3+|\psi|_2+|\Omega|_2+h|\Omega|_2). \tag{3.18}$$

Proof. From Lemmas 2 to 5 and by ε-Cauchy inequality, we have

$$\begin{aligned}
\|\Pi_h\Omega-\Omega_h\|_0^2 &\leqslant 4(\Pi_h\Omega-\Omega_h,\Pi_h^*(\Pi_h\Omega-\Omega_h))\\
&=4[a(\psi-\Pi_h\psi,\Pi_h^*(\Pi_h\Omega-\Omega_h))+a(\Pi_h\psi-\psi_h,\Pi_h^*(\Pi_h\Omega-\Omega_h))\\
&\quad-(\Omega-\Pi_h\Omega,\Pi_h^*(\Pi_h\Omega-\Omega_h))]\\
&=4[a(\psi-\Pi_h\psi,\Pi_h^*(\Pi_h\Omega-\Omega_h))-a(\Omega-\Pi_h\Omega,\Pi_h^*(\Pi_h\psi-\psi_h))\\
&\quad-(\Omega-\Pi_h\Omega,\Pi_h^*(\Pi_h\Omega-\Omega_h))]\\
&\leqslant Ch^2|\psi|_3|\Pi_h\Omega-\Omega_h|_1+Ch|\Omega|_2|\Pi_h\psi-\psi_h|_1+Ch^2|\Omega|_2\|\Pi_h\Omega-\Omega_h\|_0\\
&\leqslant Ch|\psi|_3\|\Pi_h\Omega-\Omega_h\|_0+Ch|\Omega|_2|\Pi_h\psi-\psi_h|_1+Ch^2|\Omega|_2\|\Pi_h\Omega-\Omega_h\|_0\\
&\leqslant Ch^2|\psi|_3^2+Ch^2|\Omega|_2^2+Ch^4|\Omega|_2^2+C\varepsilon|\Pi_h\psi-\psi_h|_1^2+\tfrac12\|\Pi_h\Omega-\Omega_h\|_0^2.
\end{aligned}$$

Hence,

$$\|\Pi_h\Omega-\Omega_h\|_0^2\leqslant Ch^2|\psi|_3^2+Ch^2|\Omega|_2^2+Ch^4|\Omega|_2^2+C\varepsilon|\Pi_h\psi-\psi_h|_1^2. \tag{3.19}$$

Because $H_{0h}\subset H_h$, (3.17b) also holds for $\forall\phi\in H_{0h}$, thus

$$\begin{aligned}
|\Pi_h\psi-\psi_h|_1^2 &\leqslant 2a(\Pi_h\psi-\psi_h,\Pi_h^*(\Pi_h\psi-\psi_h))\\
&=2[(\Omega-\Pi_h\Omega,\Pi_h^*(\Pi_h\psi-\psi_h))+(\Pi_h\Omega-\Omega_h,\Pi_h^*(\Pi_h\psi-\psi_h))\\
&\quad-a(\psi-\Pi_h\psi,\Pi_h^*(\Pi_h\psi-\psi_h))]\\
&\leqslant Ch^2|\Omega|_2\|\Pi_h\psi-\psi_h\|_0+C\|\Pi_h\Omega-\Omega_h\|_0\|\Pi_h\psi-\psi_h\|_0+Ch|\psi|_2|\Pi_h\psi-\psi_h|_1.
\end{aligned}$$

By Lemma 5, we obtain

$$|\Pi_h\psi-\psi_h\|_1\leqslant C\|\Pi_h\Omega-\Omega_h\|_0+Ch^2|\Omega|_2+Ch|\psi|_2. \tag{3.20}$$

Substituting (3.20) into (3.19) and taking $C\varepsilon = \frac{1}{2}$, we have

$$\|\Pi_h\Omega - \Omega_h\|_0 \leqslant Ch(|\psi|_3 + |\psi|_2 + |\Omega|_2 + h|\Omega|_2). \tag{3.21}$$

Further substitute (3.21) into (3.20), then

$$|\Pi_h\psi - \psi_h|_1 \leqslant Ch(|\psi|_3 + |\psi|_2 + |\Omega|_2 + h|\Omega|_2). \tag{3.22}$$

Error estimate (3.18) now follows directly from interpolating error estimate (3.12). The proof is complete. □

4. Adaptive Uzawa iteration algorithm and numerical experiment

Because the linear system of equations derived by (2.4) is a typical indefinite one, we must adopt the methods which are suitable for this problem. Here we use Uzawa iteration method [1] to solve (2.4). Denote N_0 by the number of boundary nodes of Q_h and μ_i $(i=1,2,\ldots,N_0)$, the piecewise bilinear interpolating base functions corresponding to the boundary nodes. Let $M_h = \operatorname{span}\{\mu_1, \mu_2, \ldots, \mu_{N_0}\}$, then $H_h = H_{0h} \oplus M_h$. For $\forall\alpha,\ \beta \in M_h$, define their inner product by

$$(\alpha,\beta)_{M_h} = \sum_{k=1}^{N_0} \alpha_k\beta_k \int_{V_k} \mu_k \,\mathrm{d}x\,\mathrm{d}y, \text{ where } \alpha = \sum_{k=1}^{N_0} \alpha_k\mu_k, \quad \beta = \sum_{k=1}^{N_0} \beta_k\mu_k. \tag{4.1}$$

Then the adaptive Uzawa iteration algorithm can be stated as follows:

1. Given arbitrary $\lambda_h^0 \in M_h$ and $\rho > 0$. Let norm $= 10^4$.
2. Assume $\lambda_h^l \in M_h$ is known, find Ω_h^l, such that
$$\Omega_h^l - \lambda_h^l \in H_{0h}, \quad a(\Omega_h^l, \Pi_h^*\phi_h) = (f, \Pi_h^*\phi_h) \quad \forall\phi_h \in H_{0h}. \tag{4.2}$$
3. Find $\psi_h^l \in H_{0h}$, such that
$$a(\psi_h^l, \Pi_h^*\phi_h) = (\Omega_h^l, \Pi_h^*\phi_h) \quad \forall\phi_h \in H_{0h}. \tag{4.3}$$
4. Solve λ_h^{l+1}, which satisfies
$$(\lambda_h^{l+1} - \lambda_h^l, \mu_h)_{M_h} = \rho[a(\psi_h^l, \mu_h) - (\Omega_h^l, \mu_h)] \quad \forall\mu_h \in M_h. \tag{4.4}$$
5. Compute $\text{norm}_1 = (\lambda_h^{l+1} - \lambda_h^l, \lambda_h^{l+1} - \lambda_h^l)^{1/2}$. If $\text{norm}_1 \geqslant \text{norm}$, then set $\rho/2 \Rightarrow \rho$.
6. If $\text{norm}_1 \leqslant 10^{-7}$, stop; otherwise, $\text{norm}_1 \Rightarrow \text{norm}$, repeat 2–6.

From the theory of Uzawa iteration method [1,11], there exists $0 < \rho < \rho_{\max}$ such that the above iteration procedure is convergent. It is usually difficult to compute $\rho_{\max}$, so, we write the above procedure as adaptive one.

In the following we provide two numerical examples to illustrate the effectiveness of scheme (2.4).

Example 1. Let $D = [0,\pi]^2$ and $f(x,y) = 16\cos(2x)\cos(2y) - 4\cos(2x) - 4\cos(2y)$ in (2.1a), then the true solution to (2.1) is $\psi = (\sin x)^2(\sin y)^2$. Let $h = \pi/31$, compute this problem by mixed FVEM (2.4) and the results are shown in Fig. 2. We also compute the maximum absolute errors of ψ and

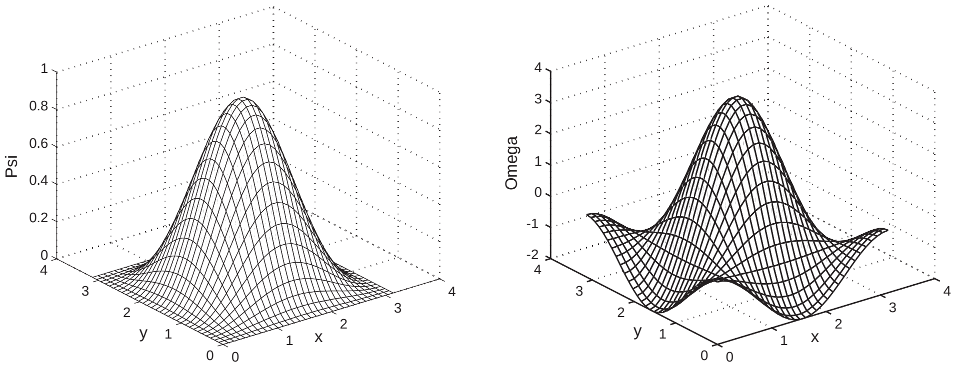

Fig. 2. The results computed by scheme (2.4) in Example 1: left: the computational surface of ψ; right: the computational surface of Ω.

Ω, respectively, which are $E_\psi = 1.7147 \times 10^{-3}$ and $E_\Omega = 1.4483 \times 10^{-2}$, where $E_\psi = \|\psi - \psi_h\|_\infty$ and $E_\Omega = \|\Omega - \Omega_h\|_\infty$. For comparison, we further compute this problem by 13-point finite difference scheme (denote by FDS13P) and the results are $E_\psi = 6.8787 \times 10^{-3}$ and $E_\Omega = 2.7485 \times 10^{-2}$, from which we know the accuracy of the scheme in this paper is obviously higher than that of 13-point finite difference scheme.

Example 2. Compute the deflections of the thin clamped unit square plate [12]. Consider two cases: case A and case B. In case A, the load is uniform and we take $f(x,y) = 1$. From [12], we know $\psi(\frac{1}{2},\frac{1}{2}) \approx 1.265 \times 10^{-3}$. In case B, we choose $f = \delta(x - \frac{1}{2}, y - \frac{1}{2})$, where δ being the Delta function. Thus, in case B, the plate is under the action of a concentrated central load. Also from [12], $\psi(\frac{1}{2},\frac{1}{2}) \approx 5.6 \times 10^{-3}$. In case A, we choose $h = 0.05$ and 0.025 and compute the problem by scheme (2.4) and 13-point finite difference scheme (FDS13P). In case B, because $\delta(x - \frac{1}{2}, y - \frac{1}{2})$ tends infinite at $(\frac{1}{2},\frac{1}{2})$, it is impossible to use FDS13P. Only scheme (2.4) is implemented. For cases A and B, the approximate solution of $\psi(\frac{1}{2},\frac{1}{2})$ is shown in Table 1. We further plot the approximate solutions of ψ and Ω, depicted as in Figs. 3 and 4 for cases A and B, respectively.

From Examples 1 and 2, we know that the scheme in this paper gets very satisfactory results for different load $f(x,y)$ and the scheme can be well applied to solve biharmonic equations.

Table 1
The approximate solution of $\psi(\frac{1}{2},\frac{1}{2})$ computed by some schemes in Example 2

	$h = 0.05$		$h = 0.025$	
	Case A	Case B	Case A	Case B
MFVEM (2.4)	1.2794×10^{-3}	5.7114×10^{-3}	1.2685×10^{-3}	5.6412×10^{-3}
FDS13P	1.2979×10^{-3}		1.2391×10^{-3}	

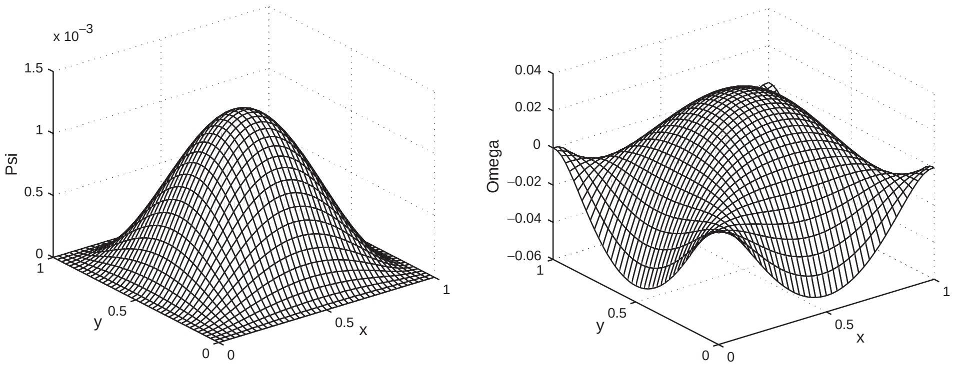

Fig. 3. The results computed by scheme (2.4) in Example 2 (case A, $h = 0.025$): left: the computational surface of ψ; right: the computational surface of Ω.

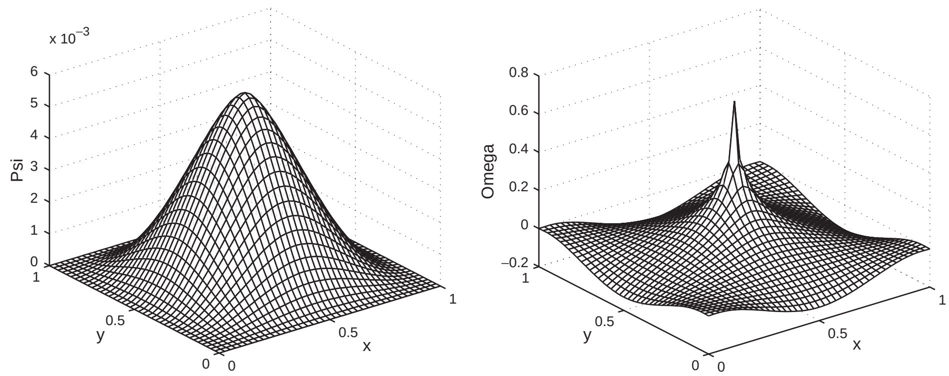

Fig. 4. The results computed by scheme (2.4) in Example 2 (case B, $h = 0.025$): left: the computational surface of ψ; right: the computational surface of Ω.

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References

[1] P.G. Ciarlet, The Finite Element Methods for Elliptic Problems, North-Holland, Amsterdam, 1978.

[2] P.G. Ciarlet, P. Raviart, A mixed finite element method for the biharmonic equations, in: C. de Boor (Ed.), Symposium on Mathematical Aspects of Finite Elements in Partial Differential Equations, Academic Press, New York, 1974, pp. 125–145.

[3] R.S. Falk, J.E. Osborn, Error estimates for mixed methods, RAIRO, Anal. Numer. 14 (1980) 249–277.

[4] Jianguo Huang, Shitong Xi, On the finite volume elements method for general self-adjoint elliptic problems, SIAM J. Numer. Anal. 35 (5) (1998) 1762–1774.

[5] W.P. Jones, K.R. Menziest, Analysis of the cell-centred finite volume method for the diffusion equation, J. Comput. Phys. 165 (2000) 45–68.
[6] Peiqi Zhu, Ronghua Li, Generalized difference methods for 2nd order elliptic partial differential equations (II), Numer. Math. J. Chinese Univ. 4 (4) (1982) 360–375.
[7] Qun Lin, Jichun Li, Aihui Zhou, A rectangle test for Ciarlet–Raviart and Herrmann–Miyoshi schemes, Proceedings of the Systems Science and Systems Engineering, Great Wall Culture Publ. Co., Hongkong, 1991, pp. 230–233.
[8] Ronghua Li, Zhongying Chen, Wei Wu, Generalized Difference Methods for Differential Equations—Numerical Analysis of Finite Volume Methods, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 226, Marcel Dekker Inc., New York, 2000.
[9] V. Selmin, The node-centred finite volume approach: bridge between finite differences and finite elements, Comput. Methods Appl. Mech. Engrg. 102 (1993) 107–138.
[10] E. Süli, Convergence of finite volume schemes for Poisson's equation on nonuniform meshes, SIAM J. Numer. Anal. 28 (5) (1991) 1419–1430.
[11] Wei Wu, The mixed generalized difference method for biharmonic equations, J. Jilin Univ. (Nat. Sci.) 1 (1986) 14–22.
[12] Xiao-liang Cheng, Weimin Han, Hong-ci Huang, Some mixed finite element methods for biharmonic equation, J. Comput. Appl. Math. 126 (2000) 91–109.
[13] Yonghai Li, Ronghua Li, Generalized difference methods on arbitrary quadrilateral networks, J. Comput. Math. 17 (6) (1999) 653–672.
[14] Zhiqiang Cai, S. McCormick, On the accuracy of the finite volume element method for diffusion equations on composite grid, SIAM J. Numer. Anal. 27 (3) (1990) 636–655.
[15] Zhiyuan Pang, Error estimates for mixed finite element methods, Math. Numer. Sinica 4 (1986) 337–344.
[16] Zhongying Chen, Variational principle and numerical analysis of generalized difference method for biharmonic equation, Numer. Math. J. Chinese Univ. 15 (2) (1993) 182–194.