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LINEAR ALGEBRA AND ITS APPLICATIONS

Elementary bidiagonal factorizations

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Abstract

An elementary bidiagonal (EB) matrix has every main diagonal entry equal to 1, and exactly one off-diagonal nonzero entry that is either on the sub- or super-diagonal. If matrix A can be written as a product of EB matrices and at most one diagonal matrix, then this product is an EB factorization of A. Every matrix is shown to have an EB factorization, and this is related to LU factorization and Neville elimination. The minimum number of EB factors needed for various classes of n-by-n matrices is considered. Some exact values for low dimensions and some bounds for general n are proved; improved bounds are conjectured. Generic factorizations that correspond to different orderings of the EB factors are briefly considered. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

Let E_{ij}, $1 \leqslant i,j \leqslant n$, denote the n-by-n 0,1 matrix whose (i,j) entry, and no other, is 1, and define $L_i(t) = I + tE_{i,i-1}$ and $U_j(t) = I + tE_{j,j+1}$, $2 \leqslant i \leqslant n$,

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$1 \leqslant j \leqslant n-1$, for $0 \neq t \in F$, a given field. An n-by-n matrix B of the form $L_i(t)$ or $U_j(t)$ is called an *elementary bidiagonal* (EB) *matrix* and, more specifically, a *lower* EB (LEB) *matrix* in the case of $L_i(t)$ and an *upper* EB (UEB) *matrix* in the case of $U_j(t)$. We say that $A \in M_n(F)$ has an *elementary bidiagonal* (EB) *factorization* if A is a product of EB matrices *and* at most one diagonal matrix. If all the EB matrices are lower (upper), then A is lower (upper) triangular and is said to have a *lower* (*upper*) *bidiagonal factorization*.

Our interest here is in the existence and nature of EB factorizations, which arise in matrix analysis in various contexts. For example, 'Neville' elimination [8,5] is a variant (working up from the bottom of columns and from the left) of Gaussian elimination in which only elementary bidiagonal matrices are used. Previous work on EB factorization has been mainly concerned with the factorization of totally nonnegative matrices that corresponds to Neville elimination. Totally nonnegative matrices are those matrices with every minor nonnegative. In the theory of nonsingular totally nonnegative matrices, an important parameterization is based upon unique EB factorization into totally nonnegative matrices corresponding to Neville elimination [9,7,6], and such factorizations have been studied more abstractly in [1,4]. In [5,3], specific bidiagonal factorizations are studied and related to the nonvanishing of certain minors of the matrix to be factored. All of these prior references deal with specific orders (and numbers) of the bidiagonal factors, which necessarily restricts the class of matrices that can be factored. Initially, we place no restriction on the number or order of EB factors.

In Section 2 we show that any matrix has an EB factorization and that any nonsingular lower (upper) triangular matrix has a lower (upper) bidiagonal factorization. This allows us to relate bidiagonal factorization to LU factorization, as is done in [2] for totally nonnegative matrices. In Section 3 we begin to discuss efficiency of factorization by categorizing the number of factors needed for various classes of matrices in low dimensions. *Generically* a nonsingular n-by-n lower triangular matrix needs $\binom{n}{2}$ EB factors, but some specific matrices need more. In Section 4 we begin the process of giving bounds on the number of factors needed in an arbitrary dimension, and this leads to a number of important questions. Finally, in Section 5 we comment upon the possible order of factors in an EB factorization of a generic matrix.

We conclude Section 1 by discussing an important point of view. EB factorization may naturally (and often conveniently will) be viewed as reduction to diagonal form via the special elementary operations conveyed by EB matrices. These special elementary operations are elementary row (column) operations on two consecutive rows (columns). But here there is a slight ambiguity that should be noted. If

$$A = B_1 B_2 \cdots B_k$$

is an EB factorization of nonsingular A, in which no diagonal matrix appears, then we may equally well write

$$B_k^{-1} \cdots B_1^{-1} A = I,$$

$$B_l^{-1} \cdots B_1^{-1} A B_k^{-1} \cdots B_{l+1}^{-1} = I,$$

or

$$A B_k^{-1} \cdots B_1^{-1} = I,$$

in which each B_i^{-1} is also an EB matrix. In the first, all the EB matrices are viewed as special elementary row operations upon A, in the third as elementary column operations, and in the second as some row and some column operations. Of course, the order of operations is important in any event. If an EB matrix B is $L_i(t)$, then $B^{-1} = L_i(-t)$; similarly, $U_j^{-1}(t) = U_j(-t)$. If $L_i(-t)$ $(U_j(-t))$ is multiplied on the left of A, then it is the elementary row operation in which t times row $i-1(j+1)$ is subtracted from row $i(j)$; if on the right, it is the elementary column operation in which t times column $i(j)$ is subtracted from column $i-1(j+1)$. We say that two EB matrices B and B' are 'of the same type' if $B = L_i(t)$ and $B' = L_i(s)$ or if $B = U_j(t)$ and $B' = U_j(s)$.

2. Existence of bidiagonal factorizations

We first consider nonsingular $A \in M_n(F)$ and make some simple observations if an EB factorization exists.

Lemma 1. *Suppose that $A \in M_n(F)$ is nonsingular and has an EB factorization*
(i) $A = B_1 \cdots B_l D B_{l+1} \cdots B_k$,
for some l, $0 \leqslant l \leqslant k$, in which D is diagonal and each B_i is an EB matrix. Then for each h, $0 \leqslant h \leqslant k$, A has an EB factorization
(ii) $A = B'_1 \cdots B'_h D B'_{h+1} \cdots B'_k$
in which $B'_i = B_i$ if B'_i lies on the same side of D in (ii) *as B_i does in* (i), *and otherwise B'_i is of the same type as B_i.*

Proof. It suffices to show that D may be moved one position to the left or right, if it is not already at one end of the factorization. The two cases are similar, so consider the case $h = l - 1$, assuming $k \geqslant 1$. Then

$$A = B_1 \cdots B_l D B_{l+1} \cdots B_k = B_1 \cdots B_{l-1} D B'_l B_{l+1} \cdots B_k,$$

where $B'_l = D^{-1} B_l D$ is an EB matrix of the same type as B_l. In the case of a move to the right, $B'_{l+1} = D B_{l+1} D^{-1}$. $\square$

Since Lemma 1 shows that the diagonal factor in an EB factorization of a nonsingular matrix may be placed in any relative position, this 'rippling' of D

may be used to consolidate the diagonal factors into just one diagonal matrix in a product of nonsingular EB factorable matrices.

Lemma 2. *If $A, C \in M_n(F)$ are nonsingular and have EB factorizations, then AC has an EB factorization. Furthermore, this factorization can be written in terms of the EB factors of A and C.*

Proof. Using Lemma 1, write $A = B_1 \cdots B_k D_1$ and $C = D_2 B_{k+1} \cdots B_l$. Then $AC = B_1 \cdots B_k (D_1 D_2) B_{k+1} \cdots B_l$, which is an EB factorization. □

It follows that if $A \in M_n(F)$ is nonsingular and has a factorization into EB factorable matrices, then A itself is EB factorable. However, this need not lead to an efficient factorization of A.

The following example and lemmas lead to a general existence theorem for EB factorization.

Example 3. The reverse permutation matrix P_2 has an *EB* factorization

$$P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

This factorization contains the minimum number, namely 3, of EB factors for P_2.

We call an n-by-n permutation matrix P a *consecutive transposition* if $P = I \oplus P_2 \oplus I$, in which either identity block may be empty. By corresponding direct summation in each factor, it follows from Example 3 that any consecutive transposition has an EB factorization with 3 EB factors. Since *any* n-by-n permutation matrix is a product of (at most $\binom{n}{2}$) consecutive transpositions, it follows by repeated application of Lemma 2 that the following holds.

Lemma 4. *Any permutation matrix has an EB factorization.*

We now prove that any matrix associated with a type 3 elementary operation also has an EB factorization.

Lemma 5. *The n-by-n matrix $I + tE_{ij}$, $i \neq j$, has an EB factorization.*

Proof. There is a permutation matrix P such that $I + tE_{ij} = P^{\mathrm{T}} L_k(t) P$ for any $k \in \{2, 3, \ldots n\}$. Since P^{T} and P have EB factorizations by Lemma 4 and $L_k(t)$ is an EB matrix, it follows from Lemma 2 that $I + tE_{ij}$ has an EB factorization. □

The existence of a bidiagonal factorization for an arbitrary square matrix now follows.

Theorem 6. *Every $A \in M_n(F)$ has an EB factorization.*

Proof. If A is nonsingular, then it is well known that A may be reduced to a diagonal matrix D via permutations and type 3 elementary (row and/or column) operations. Thus, $A = E_1 \cdots E_k D F_1 \cdots F_l$ in which each $E_p,\ F_q$ is either a permutation matrix or a matrix $I + tE_{ij},\ i \neq j$. But, since each E_p and F_q has an EB factorization by Lemmas 4 and 5, and D is diagonal, it follows from Lemma 2 that A does. If A is singular, then A may be written as

$$A = S(I \oplus 0)T,$$

where S and T are nonsingular. Thus S and T have EB factorizations, which by Lemma 1 may be assumed to be of the form

$$S = B_1 \cdots B_p D_1 \quad \text{and} \quad T = D_2 B_{p+1} \cdots B_q,$$

where each B_i is EB and $D_1,\ D_2$ are (nonsingular) diagonal. Then

$$A = B_1 \cdots B_p (D_1 (I \oplus 0) D_2) B_{p+1} \cdots B_q$$

is an EB factorization of A. □

Notice that the singular case differs from the nonsingular case, in that the diagonal factor may *not* in general be put in an arbitrary relative position.

If we extend the notion of EB factorization to nonsquare matrices in the natural way (with the 'diagonal' matrix in the 'middle' and of the same order as A, and with EB matrices on the left (right) square and with the same number of rows (columns) as A), then the proof above shows that *any* matrix has an EB factorization.

The following result on EB factorization of triangular matrices leads to an equivalence (in Theorem 9) with LU factorization.

Theorem 7. *Each nonsingular lower (upper) triangular matrix $A \in M_n(F)$ has a lower (upper) bidiagonal factorization.*

Proof. By transposition it suffices to consider the lower triangular case. Without loss of generality, assume that the diagonal entries are all 1. If A is diagonal, then the result is trivially true. Otherwise, by induction on n, there is a lower EB factorization with no diagonal factor. Clearly, this is true for $n = 2$, in which case A is an EB matrix. For general n, if the first column of A has no zero entry, then multiply A on the left by $L_2(t_2) \cdots L_n(t_n)$, with suitably chosen $t_2, \ldots, t_n$, so as to zero out the first column of A (except for the $(1,1)$ entry) and apply the induction hypothesis to the remaining $(n-1)$-by-$(n-1)$ principal submatrix in the last $n-1$ rows. If there are zero entries in the first column of A, each can first be made nonzero by multiplications of the form $L_i(t_i)$, $i \geqslant 2$, as needed. Then the elimination may be applied. Since reduction of A to the identity in this manner is equivalent to EB factorization of A (see the Introduction), the proof is complete. □

Example 8. The matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

has an EB factorization, but no lower bidiagonal factorization. At least two diagonal matrices are required to produce the two zero diagonal entries if all other factors are LEB. Thus the nonsingularity restriction in Theorem 7 cannot in general be eliminated.

Recall that an n-by-n matrix A has an LU factorization if A can be written as $A = LU$, where L is lower triangular and U is upper triangular. Not every matrix has an LU factorization; for example, P_2 in Example 3 does not. But if an EB factorization of A has all LEB factors occurring to the left of all UEB factors, then A has an LU factorization. The converse is true for nonsingular A.

Theorem 9. *A nonsingular matrix $A \in M_n(F)$ has an LU factorization if and only if A has an EB factorization in which all the elementary lower bidiagonal factors occur to the left of all the upper bidiagonal factors.*

Proof. Sufficiency of the condition has already been mentioned. If A has an LU factorization, then the product of an LEB factorization of L and a UEB factorization of U (as guaranteed by Theorem 7) gives an EB factorization of A when the diagonal factors of L and U are rippled together. □

For fixed A, Lemma 1 shows that an EB factorization is not in general unique. Furthermore, since, for example, $L_i(t)L_i(s) = L_i(t+s)$, even the number of EB matrices in an EB factorization is not unique.

3. EB factorizations in low dimensions

Let $\mu(A)$ denote the minimum number of EB matrices in a bidiagonal factorization of a nondiagonal matrix $A \in M_n(F)$. For example, $\mu(P_2) = 3$ (see Example 3); $\mu(A) = n-1$ if bidiagonal $A = I_n + \sum_{i=2}^{n} a_{i,i-1}E_{i,i-1}$ with $\prod_{i=2}^{n} a_{i,i-1} \neq 0$, since $A = \prod_{i=2}^{n} L_i(a_{i,i-1})$ is its unique EB factorization; and

$\mu(A) \leqslant n(n-1)$ if A is an n-by-n nonsingular totally nonnegative matrix (see, e.g., [6], p. 110).

For $n = 2$, explicit factorization gives the following categorization.

Theorem 10. *If* $A = [a_{ij}] \in M_2(F)$, *then*

$$\mu(A) = \begin{cases} 1 & \textit{if } a_{11}a_{22} \neq 0 \textit{ and exactly one of } a_{12}, a_{21} \textit{ is } 0, \\ 2 & \textit{if } a_{12}a_{21} \neq 0 \textit{ and } a_{11} \textit{ or } a_{22} \neq 0, \\ 3 & \textit{if both } a_{11} = a_{22} = 0. \end{cases}$$

For $n = 3$ consider first the lower triangular matrices.

Theorem 11. *If* $A = [a_{ij}] \in M_3(F)$ *is nonsingular and lower triangular, then*

$$\mu(A) = \begin{cases} 1 & \textit{if } a_{31} = 0 \textit{ and exactly one of } a_{21}, a_{32} \textit{ is } 0, \\ 2 & \textit{if } a_{31} = 0 \textit{ and } a_{21}a_{32} \neq 0, \textit{ or if } 0 \neq a_{31}a_{22} = a_{21}a_{32}, \\ 3 & \textit{if } a_{31} \neq 0,\ a_{31}a_{22} \neq a_{21}a_{32}, \textit{ and } a_{21} \textit{ or } a_{32} \neq 0, \\ 4 & \textit{if } a_{31} \neq 0 \textit{ and } a_{21} = a_{32} = 0. \end{cases}$$

Proof. The existence of an LEB factorization is given by Theorem 7. By explicit factorization, the cases for $\mu(A) = 1, 2$ are straightforward. Taking $D = \operatorname{diag}(a_{11}, a_{22}, a_{33})$, the cases for $\mu(A) = 3$ are covered by the 'generic' factorizations:

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}a_{32} - a_{31}a_{22}}{a_{11}a_{32}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{32}}{a_{22}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{31}a_{22}}{a_{11}a_{32}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} D,$$

when $a_{32} \neq 0$, or

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{31}}{a_{21}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{21}a_{32} - a_{31}a_{22}}{a_{22}a_{21}} & 1 \end{bmatrix} D,$$

when $a_{21} \neq 0$, which corresponds to the Neville elimination. The final case needs four LEB matrices, namely

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{a_{31}}{a_{11}} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-a_{31}}{a_{11}} & 1 \end{bmatrix} D.$$

Viewing this lower bidiagonal factorization as resulting from a reduction of the matrix to I_3, either the $(2,1)$ or $(3,2)$ entry must first be made nonzero by an elementary row or column operation to allow a_{31} to be eliminated at the second step. $\square$

Note that for the A of Theorem 11, the nonvanishing of the minors a_{31}, a_{21} and $a_{31}a_{22} - a_{21}a_{32}$ corresponds to the consecutive column (CC) conditions of [3].

The 3-by-3 reverse permutation matrix P_3 can be factored using eight EB matrices (and one diagonal matrix); see Theorem 17. Thus $\mu(P_3) \leqslant 8$, and we conjecture that $\mu(A) \leqslant 8$ for all $A \in M_3(F)$.

For $n = 4$, we begin with a class of matrices that can be factored with at most $\binom{n}{2}$ EB matrices.

Theorem 12. *If $A = [a_{ij}] \in M_4(F)$ is lower triangular with $a_{ij} \neq 0$ for all $i \geqslant j$, then $\mu(A) \leqslant 6$.*

Proof. Without loss of generality assume all a_{ii} are equal to 1. Since all entries in the first column of A are nonzero, three elementary row operations (corresponding to EB matrices) can be used to zero out entries in the $(i, 1)$ positions for $i = 2, 3, 4$. Then by Theorem 11, A can be reduced to I_4 with at most three more elementary operations unless

$$a_{32}a_{21} = a_{31}, \quad a_{43}a_{31} = a_{41} \quad \text{and} \quad a_{31}a_{42} \neq a_{41}a_{32}$$

(see the final case of Theorem 11). However if all of the three conditions above hold, then w.l.o.g. A must have the form

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 \\ a_{31} & \frac{a_{31}}{a_{21}} & 1 & 0 \\ a_{41} & \frac{a_{41}}{a_{21}} + \delta & \frac{a_{41}}{a_{31}} & 1 \end{bmatrix},$$

where $\delta \neq 0$. The $(4, 1)$ entry is now eliminated by a column operation, followed by row operations to eliminate the $(3, 1)$ and $(2, 1)$ entries, giving

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{a_{41}}{a_{21}} + \delta & \frac{a_{41}}{a_{31}} & 1 \end{bmatrix}.$$

Three more elementary operations are required to reduce this matrix to I_4, as can be seen from Theorem 11. Thus six LEB matrices suffice in this case. □

However, as in the case of $n = 3$, some specific 4-by-4 lower triangular matrices with $a_{ij} = 0$ for some $i > j$ require more than $\binom{n}{2}$ EB factors. By consideration of all possible cases, we have the following result.

Theorem 13. *If $A = [a_{ij}] \in M_4(F)$ is nonsingular and lower triangular, then $\mu(A) \leqslant 6$, except in the following cases.*

$$\mu(A) = \begin{cases} 7 & if\ a_{41}a_{32} \neq 0,\ a_{21} = a_{31} = a_{42} = a_{43} = 0, \\ & if\ a_{41}a_{43}a_{21} \neq 0,\ a_{31} = a_{42} = a_{32} = 0, \\ & if\ a_{42} = 0, 0 \neq a_{41}a_{33} = a_{43}a_{31},\ a_{31}a_{22} = a_{32}a_{21}, \\ & if\ a_{31} = 0,\ 0 \neq a_{41}a_{22} = a_{42}a_{21},\ a_{42}a_{33} = a_{43}a_{32}, \\ 8 & if\ a_{41} \neq 0,\ a_{21} = a_{31} = a_{32} = a_{42} = a_{43} = 0. \end{cases}$$

4. Minimum number of EB factors

We begin this section by considering a particular n-by-n lower triangular matrix that requires the greatest number of EB factors for dimensions $n \leqslant 4$. Interestingly, for sufficiently large n, it requires fewer than the generic number of factors.

Example 14. For $n \geqslant 3$, let $A = I_n + a_{n1}E_{n1}$ with $a_{n1} \neq 0$. Then A has a lower bidiagonal factorization into $4(n-2)$ LEB matrices. This number can be seen most easily by considering the factorization as resulting from reduction of A to I_n. The first $n-2$ steps put nonzeros in the first column of A by consecutive row operations from the top. The next $n-1$ steps use row operations from the bottom to zero out entries $(i,1)$ for $i = n, \ldots, 2$. The next $n-3$ steps use column operations to zero out entries (n,j) for $j = 2, \ldots, n-2$. A bidiagonal matrix results, that takes $n-2$ more operations to reduce to I_n. Each operation corresponds to multiplication by an LEB matrix, thus $4(n-2)$ LEB matrices are needed. We conjecture that $\mu(A) = 4(n-2)$, and this has been established for $n = 3$ and 4 (see Section 3).

For $n = 3, 4, 5$ and 6, the number $4(n-2)$ from Example 14 is greater than $\binom{n}{2}$, which is the required number of LEB matrices for factorization of a generic lower triangular n-by-n matrix. However, for $n \geqslant 7$, $4(n-2) < \binom{n}{2}$, thus the matrix in Example 14 can be factored with fewer factors than a generic matrix. The following result for any nonsingular triangular matrix gives a bound on the number of EB factors.

Theorem 15. *If $A \in M_n(F)$ is an n-by-n nonsingular triangular matrix, then $\mu(A) \leqslant (n-1)^2$.*

Proof. W.l.o.g. assume that A is lower triangular with all $a_{ii} = 1$. For $n = 2$ and 3, the claim with equality holding has been established in Theorems 10 and 11. Assume that the result is true for $B \in M_k(F)$ with fixed k. Consider $A \in M_{k+1}(F)$. If $a_{k+1,1} \neq 0$, using at most $k-1$ elementary row operations corresponding to EB matrices, each entry in column 1 of A can be made nonzero. An additional k such operations zero out the first column of A (except for the $(1,1)$ entry). If $a_{k+1,1} = 0$, fewer such operations are required. The resulting matrix is of the form $[1] \oplus B$, which by the inductive hypothesis can be reduced to I_{k+1} with at most $(k-1)^2$ operations. Thus the total number of operations for A is at most $2k - 1 + (k-1)^2 = k^2$. Since this is equivalent to the number of LEB matrices in an EB factorization of A, the result follows by induction. □

However, we believe that for $n \geqslant 3$ this bound can be improved as in the following conjecture.

Conjecture 16. If $A = [a_{ij}]$ is an n-by-n nonsingular triangular matrix, then $\mu(A) \leqslant n - 2 + \binom{n}{2}$. If, in addition, $a_{ij} \neq 0$ for all $i > j\,(i < j)$ when A is lower (upper) triangular, then $\mu(A) \leqslant \binom{n}{2}$.

For $n = 3$ and 4, the first part of Conjecture 16 gives the same tight bounds as in Theorems 11 and 13. We believe that for $n \geqslant 4$, $n - 2$ elementary row operations (corresponding to EB matrices) can be applied to A that enable it to be reduced to I_n generically (i.e., with at most $\binom{n}{2}$ additional such operations, see Section 5). The second part of Conjecture 16 has been established as a tight bound for $n = 3$ and 4 (see Theorems 11 and 12). When $n \geqslant 7$, we know of no nonsingular triangular matrix that requires more than $\binom{n}{2}$ EB factors.

A bound on the number of EB factors for a class of permutation matrices is given in the following result.

Theorem 17. *The n-by-n reverse permutation matrix P_n can be factored using $n^2 - 1$ EB matrices (and one diagonal matrix).*

Proof. Let $S_n = \mathrm{diag}(1, -1, 1, \cdots, (-1)^{n-1})$. Consider reducing P_n to S_n by elementary operations corresponding to EB matrices. In $n - 1$ row operations P_n can be reduced to a matrix Q that has 1 in each entry above and on the reverse diagonal, i.e., $q_{ij} = 1$ for $i + j \leqslant n + 1$, and 0 otherwise. A further $2(n-1)$ operations eliminate entries in the first column and row (except for the $(1,1)$ entry). Thus P_n is reduced to $[1] \oplus (C_{n-1} - P_{n-1})$, where $C_{n-1} = [c_{ij}]$ with $c_{ij} = 1$ for $i + j = n + 1$, and 0 otherwise. We claim that $C_k - P_k$ can be reduced to $-S_k$ by $2\binom{k}{2}$ elementary operations corresponding to EB matrices. For $k = 2$, this is true from the second case of Theorem 10. To prove the claim by induction, assume that the result is true for a fixed k, and consider $C_{k+1} - P_{k+1}$. Use k row operations to fill up the first column, then a further k row operations to zero out the first column (except for the $(1,1)$ entry). The resulting matrix is $[-1] \oplus (P_k - C_k)$. Thus $C_{k+1} - P_{k+1}$ can be reduced to $-S_k$ in $2k + 2\binom{k}{2} = 2\binom{k+1}{2}$ operations, proving the claim. Matrix P_n can thus be reduced to S_n in

$$3(n-1)+2\binom{n-1}{2}=n^2-1$$

operations, with each operation corresponding to an EB matrix. □

We conjecture that $\mu(P_n)=n^2-1$.

5. Generic factorizations

By a generic factorization, we mean that any minor of the given matrix that is required to be nonzero in order for the factorization to proceed is actually nonzero. In [5] it is assumed that, if necessary, the rows of the matrix have been permuted so that this is the case. In [3], necessary and sufficient conditions are given for the existence (and uniqueness) of such a generic Neville factorization.

We assume in this section that A is an n-by-n lower triangular matrix with every main diagonal entry equal to 1. The generic Neville factorization of A has the form $A=\prod_{k=2}^{n}\prod_{j=n}^{k}L_j$, which gives $A=L_4L_3L_2L_4L_3L_4$ when $n=4$. Each $L_j=L_j(t)$ for appropriate $t\neq 0$, and in general, each of the $\binom{n}{2}$ factors L_j in this factorization has a different parameter t. In the remainder of this section, the parameter t will usually be omitted.

Many other generic EB factorizations are possible. For example, if the EB factorization is viewed as a sequence of elementary column operations that zero out entries of A from row n to row 2 (and left to right in each row), then the resultant factorization is $A=\prod_{k=2}^{n}\prod_{j=k}^{2}L_j$. Similarly, the factorization $A=\prod_{k=n}^{2}\prod_{j=k}^{n}L_j$ results from the zeroing out of entries of A 'diagonally downward' (i.e., entries $(n,1)$, $(n-1,1)$, $(n,2)$, etc.) using elementary row operations, whereas the factorization $A=\prod_{k=n}^{2}\prod_{j=2}^{k}L_j$ results from the zeroing out of entries of A 'diagonally upward' (i.e., entries $(n,1),(n,2),(n-1,1)$, etc.) using elementary column operations.

Each of these four factorizations above may be obtained from any of the others using only the two relations (see [1], p. 57)

$$L_iL_j=L_jL_i \quad \text{if } |i-j|\geqslant 2,$$

and

$$L_iL_{i\pm 1}L_i=L_{i\pm 1}L_iL_{i\pm 1}, \quad \text{generically};$$

the proofs are by induction.

For $n=3$, there are only two distinct generic LEB factorizations, namely

$$A=L_2(a_{21}-a_{31}/a_{32})L_3(a_{32})L_2(a_{31}/a_{32}),$$

and

$$A = L_3(a_{31}/a_{21})L_2(a_{21})L_3(a_{32} - a_{31}/a_{21});$$

see the proof of Theorem 11. For $n = 4$, there are 16 generic LEB factorizations (including the four factorizations listed above), and each may be obtained from any of the others by using the two relations above. For example, $A = L_4L_3L_2L_3L_4L_3$ is one such factorization, which can easily be obtained from the Neville factorization by using the second relation above.

Necessary and sufficient conditions for the existence of each of the above four factorizations for matrix A can be given in terms of the nonvanishing of certain minors of A. The Neville and diagonally downward factorizations require the CC conditions given in [3], whereas the other two factorizations given above require conditions similar to the CC conditions but reflected in the reverse diagonal.

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