Topology and its Applications 158 (2011) 2520–2527

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Products and h-homogeneity

Andrea Medini

Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

ARTICLE INFO

Keywords:
h-Homogeneous
Clopen set
Pseudocompact
Zero-dimensional
First-countable
Homogeneous
Infinite power

ABSTRACT

Building on work of Terada, we prove that h-homogeneity is productive in the class of zero-dimensional spaces. Then, by generalizing a result of Motorov, we show that for every non-empty zero-dimensional space X there exists a non-empty zero-dimensional space Y such that $X \times Y$ is h-homogeneous. Also, we simultaneously generalize results of Motorov and Terada by showing that if X is a space such that the isolated points are dense then X^κ is h-homogeneous for every infinite cardinal κ. Finally, we show that a question of Terada (whether X^ω is h-homogeneous for every zero-dimensional first-countable X) is equivalent to a question of Motorov (whether such an infinite power is always divisible by 2) and give some partial answers.

All spaces in this paper are assumed to be Tychonoff. It is easy to see that every zero-dimensional space is Tychonoff. For all undefined topological notions, see [7]. For all undefined Boolean algebraic notions, see [9]. Recall that a subset of a space is *clopen* if it is closed and open.

Definition 1. A space X is *h-homogeneous* (or *strongly homogeneous*) if all non-empty clopen subsets of X are homeomorphic to each other.

The Cantor set, the rationals and the irrationals are examples of h-homogeneous spaces. Every connected space is h-homogeneous. A finite space is h-homogeneous if and only if it has size at most 1. The concept of h-homogeneity has been studied (mostly in the zero-dimensional case) by several authors: see [10] for an extensive list of references.

We will denote by $\mathrm{Clop}(X)$ the Boolean algebra of the clopen subsets of X. Recall that a Boolean algebra $\mathcal{A}$ is *homogeneous* if $\mathcal{A} \upharpoonright a$ is isomorphic to $\mathcal{A}$ for every non-zero $a \in \mathcal{A}$, where $\mathcal{A} \upharpoonright a$ denotes the *relative algebra* $\{x \in \mathcal{A}\colon\ x \leqslant a\}$. If X is h-homogeneous then $\mathrm{Clop}(X)$ is homogeneous; the converse holds if X is compact and zero-dimensional (see the remarks following Definition 9.12 in [9]).

1. The productivity of h-homogeneity

In [20], the productivity of h-homogeneity is stated as an open problem (see also [10] and [11]), and it is shown that the product of zero-dimensional h-homogeneous spaces is h-homogeneous provided it is compact or non-pseudocompact (see Theorem 3.3 in [20]). The following theorem, proved by Terada under the additional assumption that X is zero-dimensional (see Theorem 2.4 in [20]), is the key ingredient in the proof. Recall that a collection $\mathcal{B}$ consisting of non-empty open subsets of a space X is a π*-base* if for every non-empty open subset U of X there exists $V \in \mathcal{B}$ such that $V \subseteq U$.

E-mail address: medini@math.wisc.edu.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.08.011

Theorem 2 *(Terada). Assume that X is non-pseudocompact. If X has a π-base consisting of clopen sets that are homeomorphic to X then X is h-homogeneous.*

The proof of Theorem 2 uses the fact that a zero-dimensional non-pseudocompact space can be written as the disjoint union of infinitely many of its non-empty clopen subsets (the converse is also true, trivially). However, that is the only consequence of zero-dimensionality that is actually used (see Appendix A). Therefore such assumption is redundant by the following lemma, whose proof we leave to the reader.

Lemma 3. *Assume that X is non-pseudocompact. If X has a π-base consisting of clopen sets then X can be written as the disjoint union of infinitely many of its non-empty clopen subsets.*

Using Theorem 2 one can easily prove the following.

Theorem 4 *(Terada). Assume that $X = \prod_{i\in I} X_i$ is non-pseudocompact. If X_i is h-homogeneous and it has a π-base consisting of clopen sets for every $i \in I$ then X is h-homogeneous.*

For proofs of the following basic facts about βX, see Section 11 in [12]. Given any open subset U of X, define $\mathrm{Ex}(U) = \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus U)$. It is easy to see that $\mathrm{Ex}(U)$ is the biggest open subset of βX such that its intersection with X is U. If C is a clopen subset of X then $\mathrm{Ex}(C) = \mathrm{cl}_{\beta X}(C)$, hence $\mathrm{Ex}(C)$ is clopen in βX. Furthermore, the collection $\{\mathrm{Ex}(U)\colon U \text{ is open in } X\}$ is a base for βX.

Remark. It is not true that βX is zero-dimensional whenever X is zero-dimensional (see Example 6.2.20 in [7] or Example 3.39 in [22]). If βX is zero-dimensional then X is called *strongly zero-dimensional*.

We will need the following theorem (see Theorem 8.25 in [22]); see also Exercise 3.12.20(d) in [7]. Recall that a subspace Y of X is *C^*-embedded* in X if every bounded continuous function $f : Y \to \mathbb{R}$ admits a continuous extension to X.

Theorem 5 *(Glicksberg). Assume that $X = \prod_{i\in I} X_i$ is pseudocompact. Then X is C^*-embedded in $\prod_{i\in I} \beta X_i$.*

Remark. The reverse implication is also true, under the additional assumption that $\prod_{j\neq i} X_j$ is infinite for every $i \in I$. Such assumption is clearly not needed in the above statement (see Proposition 8.2 in [22]).

Proposition 6. *Assume that $X \times Y$ is pseudocompact. If C is a clopen subset of $X \times Y$ then C can be written as the union of finitely many open rectangles.*

Proof. It follows from Theorem 5 that $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$. By the universal property of the Čech–Stone compactification (see Corollary 3.6.3 in [7]), there exists a homeomorphism $h : \beta X \times \beta Y \to \beta(X \times Y)$ such that $h(x, y) = (x, y)$ whenever $(x, y) \in X \times Y$.

Let C be a clopen subset of $X \times Y$. Since $\{\mathrm{Ex}(U)\colon U \text{ is open in } X\}$ is a base for βX and $\{\mathrm{Ex}(V)\colon V \text{ is open in } Y\}$ is a base for βY, the collection

$$\mathcal{B} = \big\{\mathrm{Ex}(U) \times \mathrm{Ex}(V)\colon U \text{ is open in } X \text{ and } V \text{ is open in } Y\big\}$$

is a base for $\beta X \times \beta Y$. Therefore $\{h[B]\colon B \in \mathcal{B}\}$ is a base for $\beta(X \times Y)$, hence we can write $\mathrm{Ex}(C) = h[B_1] \cup \cdots \cup h[B_n]$ for some $B_1, \ldots, B_n \in \mathcal{B}$ by compactness.

Finally, if we let $B_i = \mathrm{Ex}(U_i) \times \mathrm{Ex}(V_i)$ for each i, we get

$$\begin{aligned}
C &= \mathrm{Ex}(C) \cap (X \times Y) \\
&= \big(h[B_1] \cup \cdots \cup h[B_n]\big) \cap h[X \times Y] \\
&= h\big[B_1 \cap (X \times Y)\big] \cup \cdots \cup h\big[B_n \cap (X \times Y)\big] \\
&= \big(B_1 \cap (X \times Y)\big) \cup \cdots \cup \big(B_n \cap (X \times Y)\big) \\
&= (U_1 \times V_1) \cup \cdots \cup (U_n \times V_n),
\end{aligned}$$

that concludes the proof. □

Lemma 7. *Assume that C is a clopen subset of $X \times Y$ that can be written as the union of finitely many rectangles. Then C can be written as the union of finitely many pairwise disjoint clopen rectangles.*

Proof. For every $x \in X$, let $C_x = \{y \in Y\colon\ (x, y) \in C\}$ be the corresponding vertical cross-section. For every $y \in Y$, let $C^y = \{x \in X\colon\ (x, y) \in C\}$ be the corresponding horizontal cross-section. Since C is clopen, each cross-section is clopen.

Let $\mathcal{A}$ be the Boolean subalgebra of $\mathrm{Clop}(X)$ generated by $\{C^y\colon\ y \in Y\}$. Since $\mathcal{A}$ is finite, it must be atomic. Let $P_1, \ldots, P_m$ be the atoms of $\mathcal{A}$. Similarly, let $\mathcal{B}$ be the Boolean subalgebra of $\mathrm{Clop}(Y)$ generated by $\{C_x\colon\ x \in X\}$, and let $Q_1, \ldots, Q_n$ be the atoms of $\mathcal{B}$.

Observe that the rectangles $P_i \times Q_j$ are clopen and pairwise disjoint. Furthermore, given any i, j, either $P_i \times Q_j \subseteq C$ or $(P_i \times Q_j) \cap C = \emptyset$. Hence C is the union of a (finite) collection of such rectangles. □

Proposition 8. *Assume that $X \times Y$ is pseudocompact. If X is h-homogeneous and Y is h-homogeneous then $X \times Y$ is h-homogeneous.*

Proof. Assume that X and Y are h-homogeneous. If X and Y are both connected then $X \times Y$ is connected. So assume without loss of generality that X is not connected. Since X is also h-homogeneous, it follows that $X \cong n \times X$ whenever $1 \leqslant n < \omega$. Therefore $X \times Y \cong n \times X \times Y$ whenever $1 \leqslant n < \omega$.

Now let C be a non-empty clopen subset of $X \times Y$. By Proposition 6 and Lemma 7, we can write C as the disjoint union of finitely many, say n, non-empty clopen rectangles. By the h-homogeneity of X and Y, each such rectangle is homeomorphic to $X \times Y$. Therefore $C \cong n \times X \times Y \cong X \times Y$. □

Corollary 9. *Assume that $X = X_1 \times \cdots \times X_n$ is pseudocompact. If each X_i is h-homogeneous then X is h-homogeneous.*

An obvious modification of the proof of Proposition 6 yields the following.

Proposition 10. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If C is a clopen subset of X then C can be written as the union of finitely many open rectangles.*

Corollary 11. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If C is a clopen subset of X then C depends on finitely many coordinates.*

Remark. The zero-dimensional case of Corollary 11 is a trivial consequence of a result by Broverman (see Theorem 2.6 in [2]).

Theorem 12. *Assume that $X = \prod_{i \in I} X_i$ is pseudocompact. If X_i is h-homogeneous for every $i \in I$ then X is h-homogeneous.*

Proof. Assume that each X_i is h-homogeneous. Let C be a non-empty clopen subset of X. By Corollary 11, there exists a finite subset F of I such that C is homeomorphic to $D \times \prod_{i \in I \setminus F} X_i$, where D is a non-empty clopen subset of $\prod_{i \in F} X_i$. But $\prod_{i \in F} X_i$ is h-homogeneous by Corollary 9, so $D \cong \prod_{i \in F} X_i$. Hence $C \cong X$. □

Theorem 13. *If X_i is h-homogeneous and it has a π-base consisting of clopen sets for every $i \in I$ then $X = \prod_{i \in I} X_i$ is h-homogeneous.*

Proof. If X is pseudocompact, apply Theorem 12; if X is non-pseudocompact, apply Theorem 4. □

Corollary 14. *If X_i is h-homogeneous and zero-dimensional for every $i \in I$ then $\prod_{i \in I} X_i$ is h-homogeneous.*

Question. Can the zero-dimensionality requirement be dropped in Corollary 14?

2. Some applications

The compact case of the following result was essentially proved by Motorov (see Theorem 0.2(9) in [16] and Theorem 2 in [15]).

Theorem 15. *Assume that X has a π-base $\mathcal{B}$ consisting of clopen sets. Then $Y = (X \times 2 \times \prod \mathcal{B})^\kappa$ is h-homogeneous for every infinite cardinal κ.*

Proof. One can easily check that Y has a π-base consisting of clopen sets that are homeomorphic to Y. Therefore, if Y is non-pseudocompact, the result follows from Theorem 2.

On the other hand, an analysis of Motorov's proof shows that the only consequence of the compactness of Y that is used is the fact that clopen sets in Y depend on finitely many coordinates. Therefore the same proof works if Y is pseudocompact by Corollary 11. We reproduce such proof for the convenience of the reader.

Assume that Y is pseudocompact and let C be a non-empty clopen subset of Y. The fact that C depends on finitely many coordinates implies that $C \cong Y \times C$. So it will be enough to show that $Y \times C \cong Y$.

Let B be a clopen subset of C that is homeomorphic to Y. Let $D = C \setminus B$ and $E = (Y \setminus C) \oplus B$. Observe that $Y \cong Y^2 \cong (Y \times D) \oplus (Y \times E)$ and that $Y \oplus Y \cong 2 \times Y \cong Y$. So

$$\begin{aligned} Y \times C &\cong (Y \times D) \oplus (Y \times B) \\ &\cong (Y \times D) \oplus Y^2 \\ &\cong (Y \times D) \oplus \big((Y \times D) \oplus (Y \times E)\big) \\ &\cong \big((Y \oplus Y) \times D\big) \oplus (Y \times E) \\ &\cong (Y \times D) \oplus (Y \times E) \\ &\cong Y, \end{aligned}$$

that concludes the proof. □

Remark. In [15] and [16], Motorov asked whether the 2 can be dropped in the definition of Y. This is certainly true if Y is non-pseudocompact, but we do not know the answer in general. Observe that if the answer were 'yes' then Theorem 18 would become an immediate corollary of Theorem 15.

Corollary 16. *For every non-empty zero-dimensional space X there exists a non-empty zero-dimensional space Y such that $X \times Y$ is h-homogeneous. Furthermore, if X is compact, then Y can be chosen to be compact.*

Question. Is it true that for every non-empty space X there exists a non-empty space Y such that $X \times Y$ is h-homogeneous? If X is compact, can Y be chosen to be compact?

Remark. In [21], using a very brief and elegant argument, Uspenskiĭ proved that for every non-empty space X there exists a non-empty space Y such that $X \times Y$ is homogeneous (in the sense of Definition 19). However, it is not true that Y can be chosen to be compact whenever X is compact: Motorov proved that the closure in the plane of $\{(x, \sin(1/x)) \colon x \in (0, 1]\}$ is not the retract of any compact homogeneous space (see Section 3 in [1] for a proof).

The following was proved by Matveev (see Proposition 3 in [10]) under the additional assumption that X is zero-dimensional, even though such assumption is not actually used in the proof (see Appendix A). Recall that a sequence $\langle A_n \colon n \in \omega\rangle$ of subsets of a space X *converges to a point* x if for every neighborhood U of x there exists $N \in \omega$ such that $A_n \subseteq U$ for each $n \geqslant N$.

Proposition 17 *(Matveev). Assume that X has a π-base consisting of clopen sets that are homeomorphic to X. If there exists a sequence $\langle U_n \colon n \in \omega\rangle$ of non-empty open subsets of X that converges to a point then X is h-homogeneous.*

The case $\kappa = \omega$ of the following result is an easy consequence of Proposition 17. Motorov first proved it under the additional assumption that X is a zero-dimensional first-countable compact space (see Theorem 0.2(2) in [16] and Theorem 1 in [15]). Terada proved it for an arbitrary infinite κ, under the additional assumption that X is zero-dimensional and non-pseudocompact (see Corollary 3.2 in [20]).

Theorem 18. *Assume that X is a space such that the isolated points are dense. Then X^κ is h-homogeneous for every infinite cardinal κ.*

Proof. We will show that X^ω is h-homogeneous and it has a π-base consisting of clopen sets. Since $X^\kappa \cong (X^\omega)^\kappa$ for every infinite cardinal κ, an application of Theorem 13 will conclude the proof.

Let D be the set of isolated points of X and let $\mathrm{Fn}(\omega, D)$ be the set of finite partial functions from ω to D. Given $s \in \mathrm{Fn}(\omega, D)$, define $U_s = \{f \in X^\omega \colon f \supseteq s\}$. Now fix $d \in D$ and let $g \in X^\omega$ be the constant function with value d. It is easy to see that $\langle U_{g \restriction n} \colon n \in \omega\rangle$ is a sequence of open sets in X^ω that converges to g. Furthermore $\mathcal{B} = \{U_s \colon s \in \mathrm{Fn}(\omega, D)\}$ is a π-base for X^ω consisting of clopen sets that are homeomorphic to X^ω. So X^ω is h-homogeneous by Proposition 17. □

3. Infinite powers of zero-dimensional first-countable spaces

Definition 19. A space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

It is well-known (and easy to prove) that every zero-dimensional first-countable h-homogeneous space is homogeneous. As announced by Motorov (see Theorem 0.1 in [16]), the converse holds for zero-dimensional first-countable compact spaces of uncountable cellularity (see Theorem 2.5 in [8] for a proof). In [3], Van Douwen constructed a zero-dimensional first-countable compact homogeneous space X that is not h-homogeneous (actually, X has no proper subspaces that are homeomorphic to X). In [17], using similar techniques, Motorov constructed a zero-dimensional first-countable compact homogeneous space that is not divisible by 2 (in the sense of Definition 21); see also Theorem 7.7 in [14].

In [20], Terada asked whether X^ω is h-homogeneous for every zero-dimensional first-countable X. In [4], the following remarkable theorem is proved.

Theorem 20 *(Dow and Pearl). If X is a zero-dimensional first-countable space then X^ω is homogeneous.*

However, Terada's question remains open. In [15] and [16], Motorov asks whether such an infinite power is always divisible by 2. Using Theorem 20, we will show that the two questions are equivalent: actually even weaker conditions suffice (see Proposition 24).

Definition 21. A space F is a *factor* of X (or X is *divisible* by F) if there exists Y such that $F \times Y \cong X$. If $F \times X \cong X$ then F is a *strong factor* of X (or X is *strongly divisible* by F).

We will use the following lemma freely in the rest of this section.

Lemma 22. *The following are equivalent.*

(1) *F is a factor of X^ω.*
(2) *F is a strong factor of X^ω.*
(3) *F^ω is a strong factor of X^ω.*

Proof. The implications (2) → (1) and (3) → (1) are clear.

Assume (1). Then there exists Y such that $F \times Y \cong X^\omega$, hence

$$X^\omega \cong \left(X^\omega\right)^\omega \cong (F \times Y)^\omega \cong F^\omega \times Y^\omega.$$

Since multiplication by F or by F^ω does not change the right-hand side, it follows that (2) and (3) hold. □

Whenever we will write $X \oplus Y$, we will assume without loss of generality that X and Y are disjoint.

Lemma 23. *Assume that Y is a non-empty zero-dimensional first-countable space. Then $X = (Y \oplus 1)^\omega$ is h-homogeneous and $X \cong Y^\omega \times (Y \oplus 1)^\omega \cong 2^\omega \times Y^\omega$.*

Proof. Recall that $1 = \{0\}$ and let $g \in X$ be the constant function with value 0. For each $n \in \omega$, define

$$U_n = \left\{f \in X\colon\ f(i) = 0 \text{ for all } i < n\right\}.$$

Observe that $\mathcal{B} = \{U_n\colon\ n \in \omega\}$ is a local base for X at g consisting of clopen sets that are homeomorphic to X. But X is homogeneous by Theorem 20, therefore it has such a local base at every point. In conclusion X has a base (hence a π-base) consisting of clopen sets that are homeomorphic to X. It follows from Proposition 17 that X is h-homogeneous.

To prove the second statement, observe that

$$X \cong (Y \oplus 1) \times X \cong (Y \times X) \oplus X,$$

hence $X \cong Y \times X$ by h-homogeneity. It follows that $X \cong Y^\omega \times (Y \oplus 1)^\omega$. Finally,

$$Y^\omega \times (Y \oplus 1)^\omega \cong \left(Y^\omega \times (Y \oplus 1)\right)^\omega \cong \left(Y^\omega \oplus Y^\omega\right)^\omega \cong 2^\omega \times Y^\omega,$$

that concludes the proof. □

Proposition 24. *Assume that X is a zero-dimensional first-countable space containing at least two points. Then the following are equivalent.*

(1) *$X^\omega \cong (X \oplus 1)^\omega$.*
(2) *$X^\omega \cong Y^\omega$ for some space Y with at least one isolated point.*
(3) *X^ω is h-homogeneous.*
(4) *X^ω has a non-empty clopen subset that is strongly divisible by* 2.
(5) *X^ω has a proper clopen subset that is homeomorphic to X^ω.*
(6) *X^ω has a proper clopen subset that is a factor of X^ω.*

Proof. The implication (1) → (2) is trivial; the implication (2) → (3) follows from Lemma 23; the implications (3) → (4) → (5) → (6) are trivial.

Assume that (6) holds. Let C be a proper clopen subset of X^ω that is a factor of X^ω and let $D = X^\omega \setminus C$. Then

$$\begin{aligned} X^\omega &\cong (C \oplus D) \times X^\omega \\ &\cong \left(C \times X^\omega\right) \oplus \left(D \times X^\omega\right) \\ &\cong X^\omega \oplus \left(D \times X^\omega\right) \\ &\cong (1 \oplus D) \times X^\omega, \end{aligned}$$

hence $X^\omega \cong (1 \oplus D)^\omega \times X^\omega$. Since $(1 \oplus D)^\omega \cong 2^\omega \times D^\omega$ by Lemma 23, it follows that 2^ω is a factor of X^ω. So 2^ω is a strong factor of X^ω. Therefore (1) holds by Lemma 23. □

The next two propositions show that in the pseudocompact case we can say something more.

Proposition 25. *Assume that X is a zero-dimensional first-countable space such that X^ω is pseudocompact. Then $C^\omega \cong (X \oplus 1)^\omega$ for every non-empty proper clopen subset C of X^ω.*

Proof. Let C be a non-empty proper clopen subset of X^ω. It follows from Corollary 11 that $C \cong C \times X^\omega$, hence $C^\omega \cong C^\omega \times X^\omega$. Since $C^\omega \times X^\omega$ clearly has a proper clopen subset that is homeomorphic to $C^\omega \times X^\omega$, Proposition 24 implies that C^ω is h-homogeneous, hence strongly divisible by 2. So $C^\omega \cong 2^\omega \times C^\omega \cong 2^\omega \times C^\omega \times X^\omega$. Since $2^\omega \times X^\omega \cong (X \oplus 1)^\omega$ by Lemma 23, it follows that $C^\omega \cong C^\omega \times (X \oplus 1)^\omega$.

On the other hand, $(X \oplus 1)^\omega \cong X^\omega \times (X \oplus 1)^\omega$ by Lemma 23. Hence $(X \oplus 1)^\omega$ has a clopen subset homeomorphic to $C \times (X \oplus 1)^\omega$. But Lemma 23 shows that $(X \oplus 1)^\omega$ is h-homogeneous, so $C \times (X \oplus 1)^\omega \cong (X \oplus 1)^\omega$. Therefore $C^\omega \times (X \oplus 1)^\omega \cong (X \oplus 1)^\omega$, that concludes the proof. □

Proposition 26. *In addition to the hypotheses of Proposition* 24*, assume that X^ω is pseudocompact. Then the following can be added to the list of equivalent conditions.*

(7) *X^ω has a non-empty proper clopen subset that is homeomorphic to Y^ω for some space Y.*

Proof. The implication (5) → (7) is trivial.

Assume that (7) holds. Let C be a non-empty proper clopen subset of X^ω that is homeomorphic to Y^ω for some space Y. Then clearly $C^\omega \cong C$. Therefore $C \cong (X \oplus 1)^\omega$ by Proposition 25. Hence C is strongly divisible by 2 by Lemma 23, showing that (4) holds. □

Finally, we point out that Proposition 24 can be used to give a positive answer to Terada's question for a certain class of spaces. We will need the following definition.

Definition 27. A space X is *ultraparacompact* if every open cover of X has a refinement consisting of pairwise disjoint clopen sets.

It is easy to see that every ultraparacompact space is zero-dimensional. As noted by Nyikos in [18], a space is ultraparacompact if and only if it is paracompact and strongly zero-dimensional (this is proved like Proposition 1.2 in [5]). A metric space X is ultraparacompact if and only if $\dim X = 0$ (see Theorem 7.2.4 in [7]); see also Theorem 7.3.3 in [7]. For such a metric space X, Van Engelen proved that X^ω is h-homogeneous if X is of the first category in itself or X has a completely metrizable dense subset (see Theorem 4.2 and Theorem 4.4 in [6]). It follows that X^ω is h-homogeneous if X is analytic (see Corollary 31). For related results, see also Theorem 8 and Theorem 9 in [19].

Proposition 28. *Assume that X is a* (*zero-dimensional*) *first-countable space. If X^ω is ultraparacompact and non-Lindelöf then X^ω is h-homogeneous.*

Proof. Let $\mathcal{U}$ be an open cover of X^ω with no countable subcovers. By ultraparacompactness, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ consisting of pairwise disjoint non-empty clopen sets. Let $\mathcal{V} = \{C_\alpha \colon \alpha \in \kappa\}$ be an enumeration without repetitions, where κ is an uncountable cardinal.

Now fix $x \in X^\omega$ and a local base $\{U_n \colon n \in \omega\}$ at x consisting of clopen sets. Since X^ω is homogeneous by Theorem 20, for each $\alpha < \kappa$ we can find $n(\alpha) \in \omega$ such that a homeomorphic clopen copy D_α of $U_{n(\alpha)}$ is contained in C_α. Since κ is uncountable, there exists an infinite $S \subseteq \kappa$ such that $n(\alpha) = n(\beta)$ for every $\alpha, \beta \in S$. It is easy to check that $\bigcup_{\alpha \in S} D_\alpha$ is a non-empty clopen subset of X^ω that is strongly divisible by 2. Therefore X^ω is h-homogeneous by Proposition 24. □

An application of Corollary 4.1.16, Theorem 7.3.2 and Theorem 7.3.16 in [7] immediately yields the following.

Corollary 29. *Assume that X is a metric space such that* $\dim X = 0$. *If X is non-separable then X^ω is h-homogeneous.*

Acknowledgements

The author thanks his advisor Ken Kunen and the anonymous referee for valuable comments on earlier versions of this paper.

Appendix A. Proofs of the results by Terada and Matveev

In this section we will present a somewhat unified approach to the proofs of Theorem 2 and Proposition 17. Notice that zero-dimensionality is never needed.

Proof of Theorem 2. Assume that X has a π-base consisting of clopen sets that are homeomorphic to X. By Lemma 3, we can fix a collection $\{X_n\colon n \in \omega\}$ consisting of pairwise disjoint non-empty clopen subsets of X such that $X = \bigcup_{n\in\omega} X_n$. Let C be a non-empty clopen subset of X. Since C contains a clopen subset that is homeomorphic to X, we can fix a collection $\{C_n\colon n \in \omega\}$ consisting of pairwise disjoint non-empty clopen subsets of C such that $C = \bigcup_{n\in\omega} C_n$.

We will recursively construct clopen sets $Y_n \subseteq X_n$ and $D_n \subseteq C_n$, together with partial homeomorphisms h_n and k_n for every $n \in \omega$. In the end, setting $h = \bigcup_{n\in\omega}(h_n \cup k_n)$ will yield the desired homeomorphism. Start by setting $Y_0 = \emptyset$ and $h_0 = \emptyset$. Then, let $D_0 \subseteq C_0$ be a clopen set that is homeomorphic to $X_0 \setminus Y_0$ and fix a homeomorphism $k_0 : X_0 \setminus Y_0 \to D_0$. Now assume that clopen sets $D_n \subseteq C_n$ and $Y_n \subseteq X_n$ have been defined. Let $Y_{n+1} \subseteq X_{n+1}$ be a clopen set that is homeomorphic to $C_n \setminus D_n$ and fix a homeomorphism $h_{n+1} : Y_{n+1} \to C_n \setminus D_n$. Then, let $D_{n+1} \subseteq C_{n+1}$ be a clopen set that is homeomorphic to $X_{n+1} \setminus Y_{n+1}$ and fix a homeomorphism $k_{n+1} : X_{n+1} \setminus Y_{n+1} \to D_{n+1}$. □

Proof of Proposition 17. Let $\langle U_n\colon n \in \omega\rangle$ be a sequence of non-empty open subsets of X that converges to a point x. One can easily obtain a sequence $\langle X_n\colon n \in \omega\rangle$ of pairwise disjoint non-empty clopen sets that converges to x, such that $x \notin X_n$ for each $n \in \omega$. Let C be a non-empty clopen subset of X. Let B be a clopen subset of C that is homeomorphic to X. Fix a homeomorphism $f : X \to B$ and let $C_n = f[X_n]$ for each $n \in \omega$.

Now define clopen sets $Y_n \subseteq X_n$ and $D_n \subseteq C_n$ for each $n \in \omega$ and a (partial) homeomorphism h as in the proof of Theorem 2, but start by choosing Y_0 homeomorphic to $C \setminus B$ and fixing a homeomorphism $h_0 : Y_0 \to C \setminus B$. Finally, extend h by setting $h(x) = f(x)$ for every $x \in X \setminus \bigcup_{n\in\omega} X_n$. It is easy to check that this yields the desired homeomorphism. □

Appendix B. Some descriptive set theory

The following results seem to be folklore, but we could not find satisfactory references. For the definitions of *analytic* and *property of Baire*, see [13].

Theorem 30. *Let X be an analytic metric space. Then either X has a completely metrizable dense subset or X has a non-empty open subset of the first category.*

Proof. Let $\widetilde{X}$ be the completion of X. By Theorem A.13.13 in [13], X has the property of Baire in $\widetilde{X}$. Therefore, by Proposition A.13.10 in [13], we can write $X = G \cup M$, where G is a G_δ subset of $\widetilde{X}$ and M is of the first category in $\widetilde{X}$.

Since G is a G_δ subset of the complete metric space $\widetilde{X}$, it is completely metrizable (see Theorem A.6.3 in [13]). Since X is dense in $\widetilde{X}$, the set M is of the first category in X as well (see Exercise A.13.7 in [13]). In conclusion, if G is dense in X then the first alternative in the statement of the theorem will hold, otherwise the second alternative will hold. □

Corollary 31. *Let X be an analytic metric space. Then either X^ω has a completely metrizable dense subset or X^ω is of the first category in itself.*

Proof. If X has a completely metrizable dense subset D then D^ω is a completely metrizable dense subset of X^ω (see Lemma A.6.2 in [13]).

So assume that X has a non-empty open subset U of the first category. Observe that $M_n = \{f \in X^\omega\colon f(n) \in U\}$ is of the first category in X^ω for every $n \in \omega$. Also, it is clear that $(X \setminus U)^\omega$ is closed nowhere dense in X^ω. It follows that X^ω is of the first category in itself. □

References

[1] A.V. Arkhangel'skiĭ, Topological homogeneity. Topological groups and their continuous images, Uspekhi Mat. Nauk 42 (2) (1987) 69–105 (in Russian); English translation in: Russian Math. Surveys 42 (2) (1987) 83–131.

[2] S. Broverman, The structure of continuous $\{0,1\}$-valued functions on a topological product, Canad. J. Math. 28 (3) (1976) 553–559.
[3] E.K. van Douwen, A compact space with a measure that knows which sets are homeomorphic, Adv. in Math. 52 (1) (1984) 1–33.
[4] A. Dow, E. Pearl, Homogeneity in powers of zero-dimensional first-countable spaces, Proc. Amer. Math. Soc. 125 (1997) 2503–2510.
[5] R.L. Ellis, Extending continuous functions on zero-dimensional spaces, Math. Ann. 186 (1970) 114–122.
[6] F. van Engelen, On the homogeneity of infinite products, Topology Proc. 17 (1992) 303–315.
[7] R. Engelking, General Topology, revised and completed edition, Sigma Ser. Pure Math., vol. 6, Heldermann Verlag, Berlin, 1989.
[8] S. Shelah, S. Geschke, Some notes concerning the homogeneity of Boolean algebras and Boolean spaces, Topology Appl. 133 (3) (2003) 241–253.
[9] S. Koppelberg, General theory of Boolean algebras, in: J.D. Monk, R. Bonnet (Eds.), Handbook of Boolean Algebras, vol. 1, North-Holland, Amsterdam, 1989.
[10] M.V. Matveev, Basic homogeneity in the class of zero-dimensional spaces, Filomat 12 (2) (1998) 13–20.
[11] M.V. Matveev, S.A. Shkarin, Some questions on topological homogeneity, Questions Answers Gen. Topology 18 (1) (2000) 1–16.
[12] J. van Mill, An invitation to $\beta\omega$, available at: http://www.few.vu.nl/~vanmill/teaching/caput/caput.pdf.
[13] J. van Mill, The Infinite-Dimensional Topology of Function Spaces, North-Holland Math. Library, vol. 64, North-Holland Publishing Co., Amsterdam, 2001.
[14] J. van Mill, Sierpiński's technique and subsets of $\mathbb{R}$, Topology Appl. 44 (1–3) (1992) 241–261.
[15] D.B. Motorov, Homogeneity and π-networks, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 44 (4) (1989) 31–34 (in Russian); English translation in: Moscow Univ. Math. Bull. 44 (4) (1989) 45–50.
[16] D.B. Motorov, Zero-dimensional and linearly ordered bicompacta: properties of homogeneity type, Uspekhi Mat. Nauk 44 (6) (1989) 159–160 (in Russian); English translation in: Russian Math. Surveys 44 (6) (1989) 190–191.
[17] D.B. Motorov, A homogeneous zero-dimensional bicompactum which is not divisible by 2, in: V.V. Fedorchuck, et al. (Eds.), General Topology. Spaces, Mappings and Functors, Collection of Scientific Works, Izdatel'stvo Moskovskogo Universiteta, Moskva, 1992, pp. 65–73 (in Russian).
[18] P. Nyikos, Strongly zero-dimensional spaces, in: General Topology and Its Relations to Modern Analysis and Algebra, III, Proc. Third Prague Topological Sympos., 1971, Academia, Prague, 1972, pp. 341–344.
[19] A.V. Ostrovskiĭ, On a question of L.V. Keldysh on the structure of Borel sets, Mat. Sb. (N.S.) 131(173) (3) (1986) 323–346, 414 (in Russian); English translation in: Math. USSR-Sb. 59 (2) (1988) 317–337.
[20] T. Terada, Spaces whose all nonempty clopen subsets are homeomorphic, Yokohama Math. J. 40 (1993) 87–93.
[21] V.V. Uspenskiĭ, For any X, the product $X \times Y$ is homogeneous for some Y, Proc. Amer. Math. Soc. 87 (1) (1983) 187–188.
[22] R.C. Walker, The Stone–Čech Compactification, Springer-Verlag, New York, 1974.