Some characterizations of convex functions

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Abstract

The main result in this paper is to establish some new characterizations of convex functions, in which we also simplify the proof of the characterizations given by Bessenyei and Páles.

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1. Introduction

In general, local properties may or may not imply global properties. For instance, continuous at every point implies continuous on the whole space, while local integrable does not imply integrable. Recently, Bessenyei and Páles [1] discussed characterizations of convexity via Hermite–Hadamard’s inequality by using two-dimensional Chebyshev systems. Inspired by the above-mentioned results, the main purpose of this paper is to characterize convex functions. A characterization of a locally convex set is also given.

We organize this paper as follows. In Section 2, we introduce some properties of a locally convex function and a locally convex set (see Definition 2.1). We show that the local convexity of a real-valued function on a convex subset of a normed linear space has to be convex. Since every nonempty open subset of a normed linear space is locally convex, a locally convex set may not be convex. We show that every connected, closed, and locally convex set in a normed linear space is convex. In Section 3, we prove some characterizations of convex function in which we simplify the proof of Bessenyei and Páles’ result concerning the characterizations of convexity via Hermite–Hadamard’s inequality. Some of them are new.

For related results, we refer to [2–13].

2. Characterizations of local convexity and convexity

In this section, we shall study the convexity of a locally convex set and a locally convex function in a normed linear space.

Definition 2.1. Let X be a normed linear space over \( \mathbb{R} \) and let \( \Omega \) be a nonempty subset of X.

(a) A function \( f : \Omega \to \mathbb{R} \) is said to be convex on \( \Omega \) if it satisfies the inequality

\[
    f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad \text{for all } x, y \in \Omega \text{ and } 0 \leq t \leq 1.
\]


Lemma 2.2. Let $I$ be an interval of $\mathbb{R}$ and let $a, b \in I$ exist, with $a < b$. If $f : I \to \mathbb{R}$ is convex on $I \cap [a, \infty)$ and $I \cap (-\infty, b]$, respectively, then $f$ is convex on $I$.

Proof. If $a$ or $b$ is an endpoint of $I$, then we are done; so we assume that both $a$ and $b$ are not endpoints of $I$. Let $x, y \in I$ exist, with $x < y$, and let $t \in (0, 1)$ be arbitrary. Write $z := tx + (1 - t)y$. We show

$$f(z) \leq tf(x) + (1 - t)f(y).$$

Since every convex function on an open interval of $\mathbb{R}$ is continuous, $f$ is continuous on the interior points of $I$. Therefore we may assume that $x < a, b < y$.

Case 1. If $a < z < b$, then there exist $r, s \in (0, 1)$ such that

$$a = rx + (1 - r)z \quad \text{and} \quad z = sa + (1 - s)y \quad \text{if} \quad z \in (a, b).$$

This implies

$$z = sa + (1 - s)y = s[rx + (1 - r)z] + (1 - s)y$$

and so $z = tx + (1 - t)y$, where $t = \frac{sr}{1 - s(1 - t)}$. Since $f$ is continuous on $[a, b]$ and convex on $I \cap [a, \infty)$ and $I \cap (-\infty, b]$, respectively, we have

$$f(z) \leq sf(a) + (1 - s)f(y) \leq s[tf(x) + (1 - r)f(z)] + (1 - s)f(y).$$

Therefore,

$$[1 - s(1 - r)]f(z) \leq stf(x) + (1 - s)f(y),$$

which implies

$$f(z) \leq \frac{sr}{1 - s(1 - t)}f(x) + \frac{s(1 - s)}{1 - s(1 - t)}f(y) = tf(x) + (1 - t)f(y).$$

Case 2. If $b < z < y$, then there are $r, s, t \in (0, 1)$ such that $z = rb + (1 - r)y, b = sa + (1 - s)y, a = tx + (1 - t)b$. Hence, $[1 - s(1 - t)]b = stx + (1 - s)y$ and so $z = rb + (1 - r)y = \frac{rst}{1 - s(1 - t)}x + [(1 - r) + \frac{r(1 - s)}{1 - s(1 - t)}]y$. By the convexity of $f$ on $I \cap (-\infty, b]$ and $I \cap [a, \infty)$, respectively, we have

$$f(z) \leq tf(b) + (1 - r)f(y), \quad f(b) \leq sf(a) + (1 - s)f(y), \quad f(a) \leq tf(x) + (1 - t)f(b).$$

After computations, we have 

$$[1 - s(1 - t)]f(b) \leq stf(x) + (1 - s)f(y),$$

and so

$$f(z) \leq tf(b) + (1 - r)f(y) \leq \frac{rst}{1 - s(1 - t)}f(x) + \frac{(1 - r) + \frac{r(1 - s)}{1 - s(1 - t)}}{f(y)} = tf(x) + (1 - t)f(y).$$

Case 3. If $x < z < a$, then the proof of this case is similar to that of Case 2, so we omit the detail.

Thus, our proof is complete. $\square$

In Lemma 2.2, if $f$ is convex on $I \cap (a, \infty)$ and $I \cap (-\infty, b)$, respectively, then there are $a < a' < b' < b$ such that $f$ is convex on $I \cap (a', \infty)$ and $I \cap (-\infty, b')$, respectively. Therefore $f$ is still convex on $I$.

Theorem 2.3. If $f$ is locally convex on $I$, where $I$ is an interval of $\mathbb{R}$, then $f$ is convex on $I$.

Proof. Since every convex function on an open interval of $\mathbb{R}$ is continuous, a locally convex function on such an open interval is continuous. Fix any point $z \in I$. We assume that $z$ is not the right endpoint of $I$. Let

$$d := \sup \left\{ y \in [z, b] \mid f \text{ is convex on } I \cap [z, y] \right\}.$$
Similarly, we can prove that \( f \) is convex on \( I \cap (-\infty, z] \). Since \( f \) is also convex on a neighborhood of \( z \), that is, \( f \) is convex on \( I \cap (z - s, z + s) \) for some \( s > 0 \), \( f \) is convex on \( I \cap (-\infty, z], I \cap (z - s, z + s), I \cap (-\infty, z), I \cap [z, \infty) \), respectively. By Lemma 2.2 again, we must have that \( f \) is convex on \( I \). This completes the proof. \( \square \)

The following result follows directly from Theorem 2.3.

**Corollary 2.4.** Every locally convex function on a convex subset of a normal linear space \( X \) is convex.

**Lemma 2.5.** Every connected and locally convex set \( \Omega \) in a normal linear space is polygon connected; that is, any two points in \( \Omega \) can be joined by finite line segments in \( \Omega \).

**Proof.** Let \( \Omega \) be a connected (see [2, p.143]) and locally convex set in a normed linear space. Fix a point \( A \in \Omega \). Let

\[
I' \equiv \{ B \in \Omega \mid B \text{ can be joined with } A \text{ by finite line segments} \}.
\]

Then \( A \in \Omega \) by local convexity. Let \( P \) be a boundary point of \( I' \) in \( \Omega \). Since \( \Omega \) is locally convex, there is some \( r > 0 \) such that \( B(P; r) \cap \Omega \) is convex. Since \( P \) is a boundary point of \( I' \), there is some \( Q \in B(P; r) \cap \Omega \) but \( Q \not\in I' \). By the convexity of \( B(P; r) \cap \Omega \) and the definition of \( I' \), we have \( Q \in B(P; r) \cap \Omega \subset I' \). Therefore \( P \) is an interior point of \( I' \) relative to \( \Omega \). This proves that \( I' \) is relative both open and closed to \( \Omega \). Since \( \Omega \) is connected and \( I' \neq \emptyset \), \( I' = \Omega \). The proof is complete. \( \square \)

For three points \( A, B, C \in \mathbb{R}^2 \), we shall denote the line segment joined the points \( A \) and \( B \) by \( AB \), the triangle with three vertices \( A, B \) and \( C \) by \( \triangle ABC \), and the length of \( AB \) by \( \ell(AB) \).

**Lemma 2.6.** Let \( \Omega \) be a nonempty closed and locally convex subset of \( \mathbb{R}^2 \) and let \( A, B, C \) be three points in \( \Omega \) such that \( \Omega \) contains the two line segments \( AB \) and \( AC \). If \( \Omega \) is locally convex, then \( \Omega \) contains the line segment \( BC \).

**Proof.** Since \( \Omega \cap \triangle ABC \) is still locally convex, we may replace \( \Omega \) by \( \Omega \cap \triangle ABC \) so that \( \Omega \) is compact. It follows from the local convexity of \( \Omega \) that, for any point \( P \in \Omega \), there is an \( r_P > 0 \) such that \( \Omega \cap B(P; r_P) \) is convex. Since \( \{B(P; r_P)\}_{P \in \Omega} \) is an open covering of \( \Omega \) and \( \Omega \) is compact, there are finite points \( A_1, A_2, \ldots, A_n \in \Omega \) such that \( B(A_i; r_{A_i}), i = 1, 2, \ldots, n \) form a cover of \( \Omega \). Therefore there is a Lebesgue number \( r > 0 \) (see [2, Theorem 24.C]) such that, for any \( P \in \Omega \), \( B(P; r) \subset B(A_i; r_{A_i}) \) for some \( 1 \leq i \leq n \). That is, \( \Omega \cap B(P; r) \) is convex for any point \( P \in \Omega \). Define

\[
d \equiv \inf(\ell(OR) + \ell(PQ) + \ell(RQ) \mid O, P, Q \in \Omega \text{ such that } OR, PQ \subset \Omega \text{ but } RQ \not\subset \Omega \}.
\]

If we can show \( d = 0 \), this will deduce a contradiction to \( r > 0 \).

**Claim.** \( d = 0 \).

Suppose \( d > 0 \). Then there are three points \( O, R, Q \in \Omega \) such that the length of the three edges of the triangle \( \triangle ORQ \) is less than \( \frac{2}{3}d \) and \( \overline{OR}, \overline{RQ} \subset \Omega \) but \( \overline{OQ} \not\subset \Omega \). Let

\[
k = \inf(\ell(\overline{OB}) + \ell(\overline{BQ}) + \ell(\overline{OQ}) \mid B \in \Delta OAQ \text{ such that } \overline{OB}, \overline{BQ} \subset \Omega \}.
\]

Since \( \Omega \cap \triangle ORQ \) is compact, there is a \( P \in \Delta ORQ \) such that \( \overline{OP}, \overline{PQ} \subset \Omega \) and

\[
k = \ell(\overline{OP}) + \ell(\overline{PQ}) + \ell(\overline{QP}).
\]

Clearly, \( k < \frac{4}{3}d \). Without loss of generality, we may assume that \( \ell(\overline{OP}) \leq \ell(\overline{PQ}) \). Then we have

\[
4\ell(\overline{PQ}) \geq 2\ell(\overline{OP}) + \ell(\overline{PQ}) + \ell(\overline{QP}) \\
\geq 2\ell(\overline{OP}) + \ell(\overline{OQ}) \\
\geq \ell(\overline{OP}) + \ell(\overline{PQ}) + \ell(\overline{QP}) \equiv k.
\]

Let \( M \) and \( N \) be the mid-points of the line segments \( \overline{PQ} \) and \( \overline{OP} \), respectively. By the choice of \( P \), we have \( \overline{OM} \not\subset \Omega \); otherwise, the length of the three edges of the triangle \( \triangle OMQ \) < \( k \), which contradicts the definition of \( k \). Therefore \( \overline{OM} \subset \Omega \). If \( \overline{NM} \not\subset \Omega \), then

\[
\ell(\overline{NP}) + \ell(\overline{PM}) + \ell(\overline{MN}) = \frac{1}{2}(\ell(\overline{OP}) + \ell(\overline{PQ}) + \ell(\overline{QP})) < \frac{2}{3}d < d.
\]

This is a contradiction. So, we assume that \( \overline{NM} \subset \Omega \). Then

\[
\ell(\overline{ON}) + \ell(\overline{MN}) + \ell(\overline{OM}) < \ell(\overline{ON}) + \ell(\overline{MN}) + (\ell(\overline{ON}) + \ell(\overline{MN})) = \ell(\overline{OP}) + \ell(\overline{OQ})
\]

\[
= k - \ell(\overline{PQ}) \leq \frac{3}{4}k < d.
\]

This contradicts the definition of \( d \). Therefore \( d = 0 \) and the proof is complete. \( \square \)
Theorem 2.7. Every closed, connected, and locally convex set in a normed linear space is convex.

Proof. Let $\Omega$ be a closed, connected, and locally convex set in a normed linear space $X$. We show that $\Omega$ is convex. Let $A, B \in \Omega$ be arbitrary. By Lemma 2.5, $\Omega$ is polygon connected. Therefore, there are finite points $A_1, A_2, \ldots, A_n \in \Omega$ such that

$$\overline{A_1A_2}, \ldots, \overline{A_{n-1}A_n}, \overline{A_nB} \subset \Omega.$$ 

Since $\Omega \cap \Delta A_1A_2$ is also locally convex and it can be considered as a subset of $\mathbb{R}^2$, it follows from Lemma 2.6 that $\overline{A_1A_2} \subseteq \Omega$. Applying Lemma 2.6 $n$ times, we get $\overline{AB} \subseteq \Omega$. This proves that $\Omega$ is convex. □

As we mentioned above, every convex function or convex set is locally convex; we can characterize a locally convex function and locally convex set in the following:

Theorem 2.8. (a) Let $f : A \to \mathbb{R}$ be a function, where $A$ is a convex subset of a normed linear space $X$. Then $f$ is locally convex on $A$ if and only if $f$ is convex on $A$.

(b) Let $A$ be a closed and connected subset of a normed linear space $X$. Then $A$ is locally convex in $X$ if and only if $A$ is convex.

3. Characterizations of a convex function on an open interval

In this section, we shall study some characterizations of a convex function on an open interval.

Lemma 3.1. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. If, for any $x$, $y$ with $a < x < y < b$, there is a $t \in (0, 1)$ such that $f((tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$, then $f$ is convex on $(a, b)$.

Proof. For $a < x < y < b$, let

$$A_{x,y} := \{ t \in [0, 1] | f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \}.$$ 

By the assumption, $A_{x,y}$ is not empty. Since $f$ is continuous on $(a, b)$, $A_{x,y}$ is closed in $[0, 1]$. Suppose $A_{x,y}$ is a proper subset of $[0, 1]$, then there is an open interval $(u, v) \subset (a, b)$ such that $u, v \in A_{x,y}$. We set

$$\alpha := ux + (1 - u)y \quad \text{and} \quad \beta := vx + (1 - v)y.$$ 

(2)

It follows from the assumption that there is an $s \in (0, 1)$ such that

$$f(su + (1 - s)v) \leq sf(\alpha) + (1 - s)f(\beta).$$ 

(3)

Clearly, $su + (1 - s)v \in (u, v)$. It follows from (2) and $u, v \in A_{x,y}$ that

$$f(\alpha) \leq uf(x) + (1 - u)f(y) \quad \text{and} \quad f(\beta) \leq vf(x) + (1 - v)f(y).$$ 

(4)

By choosing $u$ and $v, r := su + (1 - s)v \not\in A_{x,y}$. Since

$$su + (1 - s)v = (s(ux + (1 - u)y) + (1 - s)(ux + (1 - u)y)) = [su + (1 - s)v]x + [s(1 - u) + (1 - s)(1 - v)]y = rx + (1 - r)y,$$

it follows from $r \not\in A_{x,y}$, (3) and (4), that

$$f((rx + (1 - r)y) > tf(x) + (1 - r)f(y) = [su + (1 - s)v]f(x) + [s(1 - u) + (1 - s)(1 - v)]f(y) = s[uf(x) + (1 - u)f(y)] + (1 - s)[vf(x) + (1 - v)f(y)] \geq sf(\alpha) + (1 - s)f(\beta) \geq f(su + (1 - s)v) \geq f(su + (1 - s)v).$$

This is a contradiction, so $A_{x,y} = [0, 1]$. Since $x, y \in (a, b)$ with $x < y$ being arbitrary, this proves that $f$ is convex on $(a, b)$. Thus, our proof is complete. □

The following lemma can be found in [3, p. 15] or [4, p. 141].

Lemma 3.2. Let $f : (a, b) \to \mathbb{R}$ be a continuous function such that

$$\frac{1}{y - x} \int_x^y f(t)dt \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) \quad \text{for all} \quad x, y \in (a, b) \text{ with } x \neq y.$$ 

Then $f$ is convex on $(a, b)$.

Proof. Suppose $f$ is not convex on $(a, b)$. Then, by Lemma 3.1, there are $x, y \in (a, b)$ with $x < y$ such that

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y) \quad \text{for all} \quad 0 < t < 1.$$
For such \( x \) and \( y \),
\[
\frac{1}{y-x} \int_x^y f(t) \, dt = \int_0^1 [f(tx + (1-t)y)] \, dt > \int_0^1 [tf(x) + (1-t)f(y)] \, dt = \frac{1}{2}f(x) + \frac{1}{2}f(y).
\]

This is a contradiction. Hence \( f \) is convex on \((a, b)\). \( \square \)

**Lemma 3.3.** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function such that \( f(a) = f(b) = 0 \) and \( f(u) > 0 \) for some \( u \in (a, b) \). Then, there is \( a < z \in (a, b) \) such that
\[
f(z) = \max_{a < x < b} f(x) \tag{6}
\]
and
\[
f(z) > f(x) \quad \text{for all } a < x < z. \tag{7}
\]

**Proof.** Since \( f \) is continuous on \([a, b]\) and \([a, b]\) is compact, \( f \) attains its maximum \( m \) in \([a, b]\). By the assumption, \( m \geq f(u) > 0 \). Put \( A := \{x \in [a, b] \mid f(x) = m\} \). Since \( f \) is continuous, \( A \) is a nonempty compact subset of \([a, b]\) and \( a, b \not\in A \). This proves \( (6) \). Let \( z := \min \, A \). Then \( a < z < b \), so \( (7) \) holds. This completes the proof. \( \square \)

**Lemma 3.4.** Let \( f : (a, b) \to \mathbb{R} \) be a continuous function. Suppose that, for any \( z \in (a, b) \) and \( \delta > 0 \), there are \( x, y \in (z-\delta, z+\delta) \cap (a, b) \) with \( x < z < y \) and \( z = \lambda x + (1-\lambda)y \) for some \( 0 < \lambda < 1 \) such that
\[
f(z) \leq \lambda f(x) + (1-\lambda)f(y).
\]
Then \( f \) is convex on \((a, b)\).

**Proof.** Suppose \( f \) is not convex. Then, by **Lemma 3.1**, there are \( x_0, y_0 \in (a, b) \) with \( x_0 < y_0 \) such that
\[
f(tx_0 + (1- t)y_0) > tf(x_0) + (1- t)f(y_0) \quad \text{for all } 0 < t < 1. \tag{8}
\]
Define \( g : (a, b) \to \mathbb{R} \) by
\[
g(x) := f(x) - f(x_0) - \frac{f(y_0) - f(x_0)}{y_0 - x_0} (x - x_0).
\]
Then \( g \) is continuous on \((a, b)\), \( g(x_0) = g(y_0) = 0 \) and, by \( (8) \),
\[
g(tx_0 + (1- t)y_0) > tg(x_0) + (1- t)g(y_0) = 0 \quad \text{for all } 0 < t < 1.
\]
This and **Lemma 3.3** imply that there is a \( z \in (x_0, y_0) \) such that
\[
g(z) = \max_{a \leq x \leq b} g(x)
\]
and
\[
g(z) > g(x) \quad \text{for all } x_0 \leq x < z.
\]
Then \( z = s x_0 + (1-s)y_0 \) for some \( 0 < s < 1 \). Let \( x_1, y_1 \in [x_0, y_0] \) be arbitrary such that \( x_0 \leq x_1 < z < y_1 \leq y_0 \). Then, for every \( t \in (0, 1) \),
\[
g(z) = tg(z) + (1-t)g(z) > tg(x_1) + (1-t)g(y_1).
\]
Therefore,
\[
f(z) = tf(z) + (1-t)f(z) > tf(x_1) + (1-t)f(y_1) \quad \text{for all } 0 < t < 1.
\]
Since the choices of \( x_1, y_1 \) and \( t \) are arbitrary, this contradicts the assumption, and hence \( f \) is convex on \((a, b)\). Thus, the proof is complete. \( \square \)

**Lemma 3.5.** Let \( f : (a, b) \to \mathbb{R} \) be a continuous function. Suppose that, for any \( z \in (a, b) \) and for any \( \delta > 0 \), there is an \( r \in (0, \delta) \) such that
\[
f(z) \leq \frac{1}{2}f(z-r) + \frac{1}{2}f(z+r).
\]
Then \( f \) is convex on \((a, b)\).
Proof. The proof follows from Lemma 3.4 by taking $\lambda := \frac{1}{2}, x := z - r$, and $y := z + r$. □

Lemma 3.6. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. If, for any $z \in (a, b)$ and $\delta > 0$, there is an $r \in (0, \delta)$ such that
\[
f(z) \leq \frac{1}{2r} \int_{z-r}^{z+r} f(t) \, dt,
\]then $f$ is convex on $(a, b)$.

Proof. Suppose $f$ is not convex on $(a, b)$. Then, it follows from Lemma 3.5 that there are $a \in (a, b)$ and a $\delta > 0$ such that $a < z - \delta < z + \delta < b$ and
\[
f(z) > \frac{1}{2} f(z - r) + \frac{1}{2} f(z + t)
\]for all $0 < r < \delta$.

Integrating both sides of the last inequality with respect to $r$, we have, for every $r \in (0, \delta)$,
\[
2rf(z) = 2 \int_0^r f(z) \, dr
\]
\[
> \int_0^r f(z - r) \, dr + \int_0^r f(z + r) \, dr
\]
\[
= \int_{z-r}^{z+r} f(u) \, du.
\]
This contradicts assumption (9), and hence $f$ is convex on $(a, b)$. □

For any $x, y \in (a, b)$ with $x \neq y$, we shall denote $\frac{f(y) - f(x)}{y - x}$ by $f(x, y)$ (see [5, p. 84]) in the following Lemmas 3.7–3.9, Theorems 3.10 and 3.12. Clearly, $f(x, y) = f(y, x)$.

Lemma 3.7. Let $f : (a, b) \to \mathbb{R}$ be a continuous function on $(a, b)$. Then the following are equivalent:
(a) $f$ is a convex function;
(b) for any $a < x < y < z < b$,
\[f(x, y) \leq f(x, z) \leq f(y, z);
\]
(c) for any $a < c < b$ and $t \in (0, 1)$,
\[tf(x) + (1 - t)f(c) - f(tx + (1 - t)c)
\]
is nondecreasing on $[c, b]$.

Proof. (a) $\Rightarrow$ (b): (See [5, Theorem 6.2.2]) Let $a < x < y < z < b$ be arbitrary. Since $f$ is convex,
\[f(y) \leq \frac{z - y}{z - x} f(x) + \frac{y - x}{z - x} f(z).
\]
So
\[(z - x)(f(y) - f(x)) \leq (y - x)(f(z) - f(x)),
\]
or
\[f(x, y) \leq f(x, z).
\]
Similarly,
\[
f(x, z) = \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y} = f(y, z).
\]

This proves (b).

(b) $\Rightarrow$ (c). Let $a < c < x < y < b$ and $t \in (0, 1)$ be arbitrary. It follows from $y > ty + (1 - t)c > c$ and $x > tx + (1 - t)c > c$ that
\[
[tf(y) + (1 - t)f(c) - f(tx + (1 - t)c)] - [tf(x) + (1 - t)f(c) - f(tx + (1 - t)c)]
\]
\[
= t[y - f(x)] - [f(tx + (1 - t)c) - f(tx + (1 - t)c)]
\]
\[
= t(y - x)[f(x, y) - f(tx + (1 - t)c, y)] + t(y - x)[f(tx + (1 - t)c, y) - f(tx + (1 - t)c, ty + (1 - t)c)]
\]
\[\geq 0.
\]

This proves (c).

(c) $\Rightarrow$ (a) is clear.

Thus, our proof is complete. □
Lemma 3.8. Let \( f: (a, b) \to \mathbb{R} \) be a convex function on \((a, b)\). Then

(a) \( f \) is Lipschitz continuous on any compact subinterval of \((a, b)\).
(b) \( f' \) exists almost everywhere, and if both \( f'(x) \) and \( f'(y) \) exist for some \( a < x < y < b \), then \( f'(x) \leq f'(y) \).
(c) If \( a < c < x < b \) and \( f'(x) \) exists, then \( f'(x) \geq f(x, c) \).

Proof. (see [4, Theorem 1.4.4] for the proofs of (a) and (b)). (a) Let \( a < c < d < b \) be arbitrary. For any fixed \( u \in (a, c) \) and \( v \in (d, b) \), if \( c < x < y < d \), then it follows from (b) of Lemma 3.7 that

\[
\int_{x}^{y} f'(t)dt = f(y) - f(x) \leq f(d) - f(u) = \int_{u}^{d} f'(t)dt.
\]

This proves (a).

(b) It follows from (a) that \( f \) is absolutely continuous on compact subintervals of \((a, b)\). Therefore, \( f'(x) \) exists almost everywhere on \((a, b)\) by Corollaries 2 and 3 of [5, p. 162]. Suppose \( a < x < y < b \) are such that both \( f'(x) \) and \( f'(y) \) exist. It follows from the proof of part (a) that, for every \( a < u < x < y < v < d \),

\[
f(u, x) \leq f(y, v).
\]

Therefore,

\[
f'(x) = \lim_{u \to x} f(u, x) \leq \lim_{v \downarrow y} f(y, v) = f'(y).
\]

This proves (b).

(c) Let \( a < c < x < b \) be such that \( f'(x) \) exists. If \( c < y < x \), then, by (b) of Lemma 3.7,

\[
f(c, x) \leq f(y, x) \to f'(x) \quad \text{as } y \uparrow x.
\]

This proves (c). Thus the proof is complete. \( \square \)

By Corollary 3 of [5, Theorem 9.3.8, p. 162], if \( f: (a, b) \to \mathbb{R} \) is absolutely continuous on compact subsets of \((a, b)\), then

\[
f(x) - f(c) = \int_{c}^{x} f'(t)dt \quad \text{for } c, x \in (a, b).
\]

By taking this together with Lemma 3.8, we have the following lemma.

Lemma 3.9. Let \( f: (a, b) \to \mathbb{R} \) be a continuous function on \((a, b)\). Then the following are equivalent:

(a) \( f \) is convex on \((a, b)\);
(b) \( f \) is absolutely continuous on \((a, b)\), and if \( a < c < x < b \) are such that \( f'(x) \) exists, then \( f'(x) \geq f(x, c) \);
(c) \( \int_{c}^{x} f'(t)dt \leq (x-c)^{f(x)+f(c)} \) for all \( a < c < x < b \).

Proof. (a) \( \Rightarrow \) (b) follows from Lemma 3.8.

(b) \( \Rightarrow \) (c). Let \( c \in (a, b) \) be arbitrary. Define \( F(x) = (x-c)f(x) - \int_{c}^{x} f(t)dt, x \in (c, b). \) Since \( f \) is absolutely continuous on compact subsets of \((a, b)\), \( f'(x) \) exists almost everywhere on \((a, b)\) and \( F \) is also absolutely continuous on compact subsets of \((a, b)\). Therefore we have

\[
F'(x) = (x-c)f'(x) \geq (x-c)f(x, c) = f(x) - f(c) \quad \text{almost everywhere on } (a, b)
\]

and

\[
F(x) = (x-c)f(x) - \int_{c}^{x} f(t)dt \geq \int_{c}^{x} f(t)dt - f(c)dt \quad \text{for } x \in (c, b).
\]

This proves \( \int_{c}^{x} f(t)dt \leq (x-c)^{f(x)+f(c)} \) for \( x \) in \((c, b)\). Since \( c \in (a, b) \) is arbitrary, this proves (c).

(c) \( \Rightarrow \) (a) follows from Lemma 3.2.

Thus, the proof is complete. \( \square \)

We now list some characterizations of the convex function in the following:

Theorem 3.10. Let \( f: (a, b) \to \mathbb{R} \) be a continuous function. Then the following statements are equivalent:

(R1) \( f \) is convex on \((a, b)\);
(R_2) for any \(a < c < b\) and \(t \in (0, 1)\)
\[
 tf(x) + (1 - t)f(c) - f(tx + (1 - t)c) \text{ is nondecreasing on } [c, b);
\]

(R_3) for any \(a < x < y < z < b\),
\[
 f(x, y) \leq f(x, z) \leq f(y, z);
\]

(R_4) \(f\) is locally convex on \((a, b)\);

(R_5) for any \(x, y \in (a, b)\) with \(x < y\), there is a \(t \in (0, 1)\) such that \(f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)\);

(R_6) \(\frac{1}{2} \int_x^y f'(t) \, dt \leq \frac{1}{2} f(x) + \frac{1}{2} f(y)\) for all \(x, y \in (a, b)\) with \(x \neq y\);

(R_7) \(g(y) = (f(x) + f(y))\, (y - x) - 2 \int_x^y f'(t) \, dt\) is a nondecreasing function on \((a, b)\) for any fixed \(x \in (a, b)\);

(R_8) \(f'(x)\) exists almost everywhere on \((a, b)\), \(f'\) is locally integrable on \((a, b)\), and if \(a < c < x < b\) are such that \(f'(x)\) exists, then
\[
 f'(x) \geq f(x, c);
\]

(R_9) for any \(z \in (a, b)\) and \(\delta > 0\), there are \(x, y \in (z - \delta, z + \delta) \cap (a, b)\) with \(x < z < y\) and \(z = \lambda x + (1 - \lambda)y\) for some \(0 < \lambda < 1\) such that
\[
 f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);
\]

(R_10) for any \(z \in (a, b)\) and \(\delta > 0\), there is an \(r \in (0, \delta)\) with \(a < z - r < z + r < b\) such that
\[
 f(z) \leq \frac{1}{2} f(z - r) + \frac{1}{2} f(z + r);
\]

(R_11) for any \(z \in (a, b)\) and \(\delta > 0\), there is an \(r \in (0, \delta)\) such that
\[
 f(z) \leq \frac{1}{2r} \int_{z-r}^{z+r} f(t) \, dt;
\]

(R_{12}) \(f(\frac{1}{4}x + \frac{1}{2}y) \leq \frac{1}{3} \int_{\frac{1}{4}x}^{\frac{1}{2}y} f(t) \, dt\) for all \(x, y \in (a, b)\) with \(x \neq y\);

(R_{13}) \(F(y) := \int_x^y f(t) \, dt - (y - x)f(\frac{1}{2}x + \frac{1}{2}y)\) is a nondecreasing function on \((a, b)\) for any fixed \(x \in (a, b)\), see Hu [6].

(R_{14}) Jensen–Steffensen’s inequality [4]: \(f\left(\frac{1}{n} \sum_{i=1}^n t_i x_i\right) \leq \frac{1}{n} \sum_{i=1}^n t_i f(x_i)\) if \((x_1, x_2, \ldots, x_n)\) is a real monotonic \(n\)-tuple such that \(x_i \in (a, b), i = 1, 2, \ldots, n\), and \(t_i \in \mathbb{R}\) with \(0 \leq T_i = \frac{1}{n} \sum_{j=1}^n t_j \leq T_n\) for \(i = 1, 2, \ldots, n\) such that \(T_n > 0\);

(R_{15}) \(f\left(\frac{1}{n} \sum_{i=1}^n t_i x_i\right) \leq \frac{1}{T_n} \sum_{i=1}^n t_i f(x_i)\) if \(t_i \in (0, \infty), x_i \in (a, b)\) for \(i = 1, 2, \ldots, n\) and \(T_n = \sum_{i=1}^n t_i\);

(R_{16}) \(g(n) = \sum_{i=1}^n t_i f(x_i) - T_n f\left(\frac{1}{n} \sum_{i=1}^n t_i x_i\right)\) is an increasing function of \(n\) if \(t_i \in (0, \infty), x_i \in (a, b)\) for \(i = 1, 2, \ldots, n\) and \(T_n = \sum_{i=1}^n t_i\);

(R_{17}) Hermite–Fejér’s inequality [7] (1):
\[
 f(pq + qd) \leq \frac{p f^k_x f(x) g(x) \, dx}{\int_c^d g(x) \, dx} + \frac{q f^d_x f(x) g(x) \, dx}{\int_c^d g(x) \, dx} \leq pf(c) + qf(d),
\]
where \(p + q = 1\) with \(p, q \in (0, 1), g: (a, b) \to [0, \infty)\) is integrable and symmetric to \(x = pa + qb = pc + qd := k\) with \(a < c < d < b\); that is, for each \(t \in [0, d - c], g(k + pt) = g(k - qt);

(R_{18}) Hermite–Fejér’s inequality [8] (II):
\[
 f\left(\frac{c + d}{2}\right) \int_c^d g(t) \, dt \leq \int_c^d f(t) g(t) \, dt \leq \int_c^d g(t) \, dt \leq f(c) + f(d)\left(\frac{d - c}{2}\right)
\]
for \(a < c < d < b\) with \(a + b = c + d\), where \(g: (a, b) \to [0, \infty)\) is integrable and symmetric to \(x = \frac{a + b}{2}\);

(R_{19}) Hermite–Hadamard’s inequality [9–11]:
\[
 f\left(\frac{c + d}{2}\right) \leq \frac{1}{d - c} \int_c^d f(t) \, dt \leq \frac{f(c) + f(d)}{2}
\]
for all \(c, d \in (a, b)\) with \(c \neq d\).

**Proof.** Clearly, (R_1) implies (R_4)–(R_6).

(R_1) \(\iff\) (R_2) \(\iff\) (R_3) follows from Lemma 3.7.

(R_4) \(\implies\) (R_1) follows from Theorem 2.3.

(R_5) \(\implies\) (R_1) follows from Lemma 3.1.

(R_6) \(\implies\) (R_1) follows from Lemma 3.2.
Lemma 3.7 follows from Lemma 3.9.

(R_6) \iff (R_8): Since (R_6) \imp (R_1), we see that \( f \) is convex on \((a, b)\). Define, for every \(x, y \in (a, b)\),
\[
L_1(x, y) := (y - x)(f(y) + f(x)) - 2 \int_x^y f(t) \, dt.
\]

Clearly, \( L_1 \) satisfies that \( L_1(x, y) = -L_1(y, x) \) for \( x, y \in (a, b) \) and \( L_1(x, y) \geq 0 \) for \( a < x < y < b \). By (b) of Lemma 3.7, for every \( a < x < y < z < b \),
\[
L_1(x, z) - L_1(x, y) = \left[ (z - x)(f(z) + f(x)) - 2 \int_x^z f(t) \, dt \right] - \left[ (y - x)(f(y) + f(x)) - 2 \int_x^y f(t) \, dt \right]
= (z - y)(f(z) + f(y)) - 2 \int_x^y f(t) \, dt \\
+ (x - z)(f(x) + f(z)) - (y - x)(f(y) + f(x)) - (z - y)(f(z) + f(y))
= L_1(y, z) + (z - y)\left[ f(x) - f(y) \right] + (y - x)\left[ f(z) - f(y) \right]
= L_1(y, z) - (z - y)(y - x)f(x, y) + (y - x)(z - y)f(y, z)
= L_1(y, z) + (z - y)(y - x)\left[ f(y, z) - f(x, y) \right]
\geq L_1(y, z) \geq 0.
\]

Thus (R_7) holds.

(R_7) \Rightarrow (R_6) is clear.

(R_1) \Rightarrow (R_6), (R_10) are clear.

(R_6) \Rightarrow (R_1) follows from Lemma 3.4.

(R_10) \Rightarrow (R_1) follows from Lemma 3.5.

(R_11) \Rightarrow (R_1) follows from Lemma 3.6.

(R_12) \Rightarrow (R_11) is obvious.

(R_13) \Rightarrow (R_12) is obvious.

(R_12) \Rightarrow (R_13): We have shown that (R_12) implies that \( f \) is convex on \((a, b)\) (see Lemma 3.6). Define, for every \(x, y \in (a, b)\),
\[
L_2(x, y) := \int_x^y f(t) \, dt - (y - x)f \left( \frac{y + x}{2} \right).
\]

Clearly, \( L_2 \) satisfies that \( L_2(x, y) = -L_2(y, x) \) for \( x, y \in (a, b) \) and \( L_2(x, y) \geq 0 \) for \( a < x < y < b \). By (b) of Lemma 3.7, for every \( a < x < y < z < b \),
\[
L_2(x, z) - L_2(x, y) = \left[ \int_x^z f(t) \, dt - (z - x)f \left( \frac{z + x}{2} \right) \right] - \left[ \int_x^y f(t) \, dt - (y - x)f \left( \frac{y + x}{2} \right) \right]
= \int_x^y f(t) \, dt - (z - y)f \left( \frac{z + y}{2} \right) + (z - y)f \left( \frac{z + x}{2} \right) - (z - x)f \left( \frac{z + x}{2} \right)
= L_2(y, z) + (z - y)\left[ f \left( \frac{z + y}{2} \right) - f \left( \frac{z + x}{2} \right) \right] + (y - x)\left[ f \left( \frac{y + x}{2} \right) - f \left( \frac{z + x}{2} \right) \right]
= L_2(y, z) + (z - y)\frac{y - x}{2}f \left( \frac{z + y}{2}, \frac{z + x}{2} \right) + (y - x)\frac{y - z}{2}f \left( \frac{y + x}{2}, \frac{z + x}{2} \right)
\geq L_2(y, z) \geq 0.
\]

Thus (R_13) holds.

(R_13) \iff (R_1) follows from Lemma 3.6.

(R_1) \Rightarrow (R_{14}); see [4, Theorem 2.19].

(R_{14}) \Rightarrow (R_{15}) is clear.

(R_{15}) \Rightarrow (R_1) is by the definition of the convexity.

(R_1) \Rightarrow (R_{16}): The proof of this implication in [6, p. 127] is not easily available, so will be given here for completeness. It follows from (R_1) that
\[
f \left( \frac{1}{T_n} \sum_{i=1}^{n} t_i \right) \leq \frac{T_{n-1}}{T_n} f \left( \frac{1}{T_{n-1}} \sum_{i=1}^{n-1} t_i \right) + \frac{t_n}{T_n} f(x_{n}).
\]
Adding \( \sum_{i=1}^{n-1} t_i f(x_i) \) in both sides of the above inequality, we prove that (R_{16}) holds.
(R_{16}) \Rightarrow (R_{15}) because g(n) \geq g(n - 1) \geq \cdots \geq g(1) = 0 if n \geq 2; see Hu [6].

(R_1) \Rightarrow (R_{17}); see Yu and Liu [7]. The proof of this implication is too long, so we do not include its proof here.

(R_{17}) \Rightarrow (R_{18}) follows by taking p = q in (R_{17}) and noting \( \int_b^a g(x)dx = \int_b^a g(x)dx \) for \( p = q \).

(R_{18}) \Rightarrow (R_{19}) follows by taking \( g(t) \equiv 1 \) on \( (a, b) \) in (R_{18}).

(R_{19}) \Rightarrow (R_6) and (R_{12}) is obvious.

Thus, our proof is complete. \( \Box \)

The following example explains that the representations of (R_6) and (R_{12}) (that is, Hermite–Hadamard’s inequality) are best.

**Example.** Let \( f(x) := x \) for \( x \in \mathbb{R} \). Then, for every \( x, y \in \mathbb{R} \) with \( x < y \) and \( 0 < \lambda < \frac{1}{2} \),

\[
\begin{align*}
  f(\lambda x + (1 - \lambda)y) - \frac{1}{y - x} \int_x^y f(t)dt &= \lambda x + (1 - \lambda)y - \frac{1}{y - x} \int_x^y (\lambda - \frac{1}{2})x + \left(\frac{1}{2} - \lambda\right)y\ dt \\
  &= \left(\frac{1}{2} - \lambda\right)(y - x) \begin{cases} 
  > 0 & \text{for } 0 < \lambda < \frac{1}{2} \\
  = 0 & \text{for } \lambda = \frac{1}{2} \\
  < 0 & \text{for } \frac{1}{2} < \lambda < 1.
\end{cases}
\end{align*}
\]

Therefore \( \lambda = \frac{1}{2} \) in (R_6) and (R_{12}) is best.

**Theorem 3.11.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be measurable and let \( h : [c, d] \rightarrow [0, \infty) \) be integrable such that \( \int_c^d h(x)dx = 1 \) and \( hf \) is integrable on \( [a, b] \). Then \( f \) is convex if and only if

\[
f\left(\int_c^d h(x)g(x)dx\right) \leq \int_c^d h(x)f(g(x))dx
\]

for any measurable function \( g : [c, d] \rightarrow [a, b] \).

**Proof.** "If part". Let \( \alpha, \beta \in [a, b] \) be arbitrary. We define \( g_\alpha(t) := \alpha l_{[c, a]}(t) + \beta l_{(s, d)}(t) \) for \( t \in [c, d] \), where \( l_{A} \) denotes the indicator function on \( A \). Then,

\[
\int_c^d h(x)g_\alpha(x)dx = \alpha \int_c^s h(x)dx + \beta \int_s^d h(x)dx
\]

and

\[
\int_c^d h(x)f(g)dx = \int_c^s h(x)f(\alpha)dx + \int_s^d h(x)f(\beta)dx
\]

\[
= \int_c^s h(x)dxf(\alpha) + \int_s^d h(x)dxf(\beta).
\]

These and (10) show that

\[
f\left(\alpha \int_c^s h(x)dx + \beta \int_s^d h(x)dx\right) \leq \int_c^s h(x)dxf(\alpha) + \int_s^d h(x)dxf(\beta).
\]

Since \( \int_c^s h(x)dx \) runs over \( [0, 1] \) and \( \alpha, \beta \in [a, b] \) are arbitrary, \( f \) is convex on \( [a, b] \).

"Only if part". Suppose \( f \) is convex. Define \( \mu(A) := \int_A h(x)dx \) for all Borel subset \( A \) of \( [c, d] \). Then \( \mu \) is a probability measure on \( [c, d] \). It follows from Jensen’s Inequality (see [5, p. 111]) that, for any measurable function \( g : [c, d] \rightarrow [a, b] \),

\[
f\left(\int_c^d g\,d\mu\right) \leq \int_c^d f(g)\,d\mu.
\]

Therefore

\[
f\left(\int_c^d h(x)g(x)dx\right) = f\left(\int_c^d g\,d\mu\right) \leq \int_c^d f(g)\,d\mu = \int_c^d h(x)f(g(x))dx.
\]

This proves (10), and the proof is complete. \( \Box \)
Theorem 3.12. Let $f : I \to \mathbb{R}$ be a continuous function, where $I$ is an interval of $\mathbb{R}$. Then the following are equivalent:

(a) $f$ is convex;
(b) for every $c \in I$, the function $F_c(x) := f(x) + f(2c - x)$ is nondecreasing on $x \in I \cap [c, \infty)$ with $2c - x \in I$;
(c) $f(x) + f(y) \leq f(x + y - a) + f(a)$
   for all $x, y \in I$ with $x + y - a \in I$ and $(x - a)(y - a) \geq 0$;
(d) $f(x) + f(y) \geq f(x + y - a) + f(a)$
   for all $x, y, x + y - a \in I$ with $(x - a)(y - a) \leq 0$;
(e) $\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i - (n - 1)a) + (n - 1)f(a)$
   for $x_i, a \in I$ with $x_i \geq a$ or $x_i \leq a$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} x_i - (n - 1)a \in I$, where $n \geq 2$ is an integer.

Proof. (a) $\Rightarrow$ (b): Let $c \leq x < y$ be with $2c - y \in I$. Then $x + y > 2c$ and it follows from (b) of Lemma 3.7 that

$$F_c(x) - F_c(y) = \left[f(y) + f(2c - y)\right] - \left[f(x) + f(2c - x)\right]$$

$$= f(y - f(2c - y)) - f(x) + f(2c - x)$$

$$= (x + y - 2c)f(y, 2c - x) - (x + y - 2c)f(x, 2c - y)$$

$$= (x + y - 2c)f(y, 2c - x) - f(2c - x) - f(x, 2c - y)$$

$$\geq (x + y - 2c)[f(x, 2c - x) - f(x, 2c - y)] \geq 0.$$

This proves (b).

(b) $\Rightarrow$ (c): Let $a, x, y \in I$ with $x + y - a \in I$ and $(x - a)(y - a) \geq 0$. Let $c := \frac{x + y}{2}$. Then $c \in I, x - c = c - y$, and

$$0 \leq (x - a)(y - a) = [(x - c) - (a - c)][(y - c) - (a - c)] = (a - c)^2 - (x - c)^2.$$

Since $F$ is symmetric to $x = c$, we may assume that $c \leq x < a$. By part (b), we have

$$f(x) + f(y) = F_c(x) \leq F_c(a) = f(a) + f(x + y - a).$$

This proves (c).

(c) $\Rightarrow$ (d): Let $a, x, y \in I$ with $x + y - a \in I$ and $(x - a)(y - a) \leq 0$. Let $b = x + y - a$. Then $0 \leq (a - x)(y - a) = (a - x)(b - x)$.

Replacing $a, x, y$ by $a, b, c$ in (c), respectively, we have

$$f(a) + f(x + y - a) = f(a) + f(b) \leq f(a + b - x) + f(x) = f(y) + f(x).$$

This proves (d).

(d) $\Rightarrow$ (a): Let $x, y \in I$ be arbitrary. Take $a := \frac{x + y}{2}$. Then $x + y - a = a \in I$ and $(x - a)(y - a) = \frac{(x - y)^2}{2} \leq 0$. By part (d), we have

$$f(x) + f(y) \geq f(a) + f(x + y - a) = 2f\left(\frac{x + y}{2}\right).$$

Therefore, $f$ is mid-point convex on $I$. Since $f$ is continuous on $I$, it must be convex (see also $R_3$ of Theorem 3.10). This proves that (a) holds.

(c) $\Rightarrow$ (e) by induction.

(e) $\Rightarrow$ (c) is clear.

Thus, our proof is complete. \qed

Remarks. (a) Theorem 3.12 improves some results in Shieh and Wen [12], in which $f$ is required to be twice differentiable, but in our Theorem 3.12, $f$ is required only to be continuous in $I$. Moreover, if $f$ is not continuous, then (b) $\Rightarrow$ (a) may fail.

To see this, we can choose a Hamel basis $\{\alpha_j\}_{j=1}^{\infty}$ for $\mathbb{R}$ over $\mathbb{Q}$ (the set of all rational numbers) and $\alpha_j = 1$ for some $j \in I$. If $f$ is a function mapping from the $\alpha_j$ to $\mathbb{Q}$, then $f$ uniquely determines a linear functional. If we take $f(1) = 1$, then $f$ must be discontinuous and mid-point convex; thus $f$ is not convex. But $f$ satisfies (b).

(b) Taking $a = 0$ in (e) of Theorem 3.12, we obtain Petrović’s inequality; see [13, p. 22].

References