REMARKS ON THE BLOWUP RATE OF CLASSICAL SOLUTIONS TO QUASILINEAR MULTIDIMENSIONAL HYPERBOLIC SYSTEMS

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ABSTRACT. – We note that for classical solutions of hyperbolic symmetric quasilinear systems, the blowup rate of the gradient at a point \( M = (x, T) \) (\( T \) being the lifespan) cannot be smaller than \( C(M, u(M))(T - t)^{-1} \) (\( C \) is the minimal growth constant). This allows us also to introduce blowup wave fronts \( W F_s(u) \) (in the cotangent to the \((x, t)\) space) and \( W F^*(u) \) (in the tangent to the \( u \) space). Several examples and open questions are discussed.

Keywords: Quasilinear systems, Symmetric hyperbolic systems, Blowup rate, Wave fronts

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RéSUMÉ. – Nous remarquons que, pour des solutions classiques de systèmes symétriques hyperboliques, la vitesse d’explosion du gradient en un point \( M = (x, T) \) (\( T \) étant le temps de vie) ne peut être inférieure à \( C(M, u(M))(T - t)^{-1} \) (\( C \) est la constante d’explosion minimale). Cela nous permet également d’introduire les fronts d’onde à l’explosion \( W F_s(u) \) (dans l’espace cotangent à \((x, t)\)) et \( W F^*(u) \) (dans l’espace tangent à l’espace des \( u \)). Nous discutons divers exemples et questions ouvertes.

Introduction

In this paper, we consider solutions, defined in a neighborhood \( V \) of some point \( M_0 = (x_0, T) \) in \( \{ t \leq T \} \), of hyperbolic symmetric quasilinear first order systems. We assume that the solution \( u \) is continuous in \( V \) and smooth for \( t < T \); this is typically the situation one encounters when studying blowup of classical solutions to such systems (\( T \) being the lifespan in this case). Our first remark is that \( (T - t)^{-1} \) is the minimum rate of blowup for \( \|u'_s(x, t)\|_{L^\infty} \); more precisely, we show that if

\[
\limsup_{(x, t) \to M_0} (T - t)\|u'_s(x, t)\| \]

is strictly smaller than some constant (the “Minimal Growth Constant”) depending only on the system and on \((M_0, u(M_0))\), then \( u \) does not blow up at \( M_0 \) (Theorem 1.1). In Theorems 1.2, 1.3, 1.4, we display lower and upper bounds for this constant in very simple cases.

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To evaluate this constant, it turns out that it is best to measure $u'$ in some appropriate metric on the phase space $\mathbb{R}^N$: we display such a metric in Section 1.2 and consider examples. We also define a "blowup wave front" where $(x, t)$ and $u$ play a symmetric role: the set $WF_u(u)$ is a set of codirections in $R_{x,t}^{n+1}$ along which the solution is "really singular", while $WF^*(u)$ is a set of directions in $R_{u}^{n}$ corresponding to the "really singular" components of $u$.

In part II, we examine the different known examples of solutions which blow up, that is solutions which blow up geometrically with rank one or two, in the sense we have introduced in [1,4]. We compute the wave fronts $WF_u$ and $WF^*$ for these solutions, and discuss the best Minimal Growth Constants. Finally, we propose some open questions which may lead to a better understanding of the phenomenon.

## 1. Minimal growth constants, phase space metrics and blowup wave fronts

In this paper, we consider for simplicity symmetric hyperbolic systems in $\mathbb{R}^{n+1}$:

$$L(u) = A_0(x, t, u)\partial_t u + \sum_{1 \leq i \leq n} A_i(x, t, u)\partial_i u + B(x, t, u) = 0,$$

where $u \in \mathbb{R}^N$, the matrices $A_j$ and $B$ are real and smooth, and the matrices $A_j$ are symmetric, with $A_0$ positive definite.

We will denote generically by $V$ a neighborhood of the point $M_0 = (x_0, T)$ in $\{t \leq T\}$, and will only consider solutions $u$ of $L(u) = 0$ which are continuous in $V$ and $C^\infty$ (in $V$) for $t < T$. The value $u(x_0, T)$ will be denoted by $u_0$. This choice seems relevant at least for the study of blowup of classical solutions. Some examples of unbounded solutions $u$ are given in [5,6], but we believe that the phenomenon involved is rather artificial and will not consider it here.

In the sequence, $u'_x(x, t)$ will be considered as a map from $\mathbb{R}^n$ to $\mathbb{R}^N$, and $\|u'_x(x, t)\|$ will denote its norm as a linear map.

### 1.1.

Our first theorems display the existence of a minimal growth constant and give examples.

**Theorem 1.1.** Let $L$ be a hyperbolic symmetric system (1.1). There exists a function $C(x, t, u) > 0$ such that, if $u \in C^0(V)$ is a solution of $L(u) = 0$ which is $C^\infty$ for $t < T$ and

$$\limsup_{(x, t) \to M_0} (T - t)\|u'(x, t)\| < C(x_0, T, u(M_0)),$$

then $u \in C^\infty$ in some neighborhood of $M_0$ in $\{t \leq T\}$.

**Proof.** (a) It is a simple application of the energy method. We assume $x_0 = 0$, $T = 0$ and $A_0(0, 0, u_0) \geq \gamma > 0$. Let us first define a family of domains:

$$V_{\varepsilon} = \{(x, t), -\varepsilon \leq t \leq T \leq 0, |x| \leq \varepsilon - \mu t \},$$

where $\mu > 0$ is chosen in such a way that

$$\mu \gamma + \sum A_i(0, 0, u_0)\omega_i \geq 0, \quad |\omega| = 1.$$
For the linearized equation
\[ \mathcal{L} = A_0(x,t,u)\partial_t + \sum A_i(x,t,u)\partial_i, \]
the classical energy inequality goes as follows:
\[ \partial_t (\nu A_0 v) + \sum \partial_i (\nu A_i v) = \nu D v + \nu^2 v L v, \]
with \( D = \partial_t (A_0(x,t,u)) + \sum \partial_i (A_i(x,t,u)). \) For all \( \eta > 1, \) provided \( \varepsilon \leq \varepsilon_0, \) we get the energy inequality in \( V^2_\varepsilon (\varepsilon > 0, T \leq 0): \)
\[
\frac{\gamma}{\eta} \| v(\cdot, T) \|_{L^2}^2 \leq C \| v(\cdot, -\varepsilon) \|_{L^2}^2 + 2 \int_{-\varepsilon}^T \| L v(\cdot, s) \|_{L^2} \| v(\cdot, s) \|_{L^2} \, ds 
+ \int_{-\varepsilon}^T \left( C + C \| u'_s (\cdot, t) \|_{L^\infty} \right) \| v(\cdot, t) \|_{L^2}^2 \, dt,
\]
where \( C \) (or \( C_j \)) denotes generally a constant which depends only on \( u_0. \) We emphasize the fact that here \( \varepsilon_0 \) depends on \( \eta \) and on the function \( u \) itself (not just on \( u_0). \)

(b) For a derivation \( \partial_x^\alpha \), \( |\alpha| = s, \) we write as usual
\[ A_0 \partial_x^\alpha (A_0^{-1} L(u)) = \mathcal{L}(\partial_x^\alpha u) + F_\alpha, \]
with
\[ F_\alpha = A_0 \sum_{|\alpha - \beta| \geq 1} C_\beta \partial_x^{\alpha - \beta} (A_0^{-1} A_1) \partial_x^\beta \partial_t u + A_0 \partial_x^\alpha (A_0^{-1} B). \]

To estimate \( F_\alpha \) in \( L^2 \)-norm for fixed \( t, \) we proceed as in [7], using the Gagliardo–Nirenberg inequality. The problem is that this \( L^2 \)-norm is taken on the ball \( B(0, \varepsilon + \mu(-t)) \), hence the constants in the Gagliardo–Nirenberg inequality depend on \( \varepsilon \). More precisely, we have:
\[ \| \partial_k^\alpha w \|_{L^2} \leq C \| w \|_{L^\infty} |k| \| w \|_{L^2}^{1/|k|} + \| \partial_k \alpha \|_{L^2}^{1/|k|}, \quad 0 \leq k \leq s, \]
where \( C \) is a constant independent of \( \varepsilon. \) The essential point is that the constant in front of the higher derivative terms does not depend on \( \varepsilon. \) Changing the variable \( x \) to work on a fixed ball, estimating \( F_\alpha \) as usual there and changing back, we obtain that for \( \varepsilon \leq \varepsilon_0, \)
\[ \| F_\alpha \|_{L^2} \leq C \| u \|_{L^\infty} \| u \|_{L^2} + C \| u_s \|_{L^\infty} \| u \|_{L^2} \]
for some big constant \( C \) and again, \( \varepsilon_0 \) depends on \( u \) itself and not just on \( u_0. \)

We apply now the energy inequality of (a) first with \( \alpha = 0, \) then with all \( \alpha, |\alpha| = s; \) multiplying the first inequality by \( C_\varepsilon^2 \) and summing, we have finally:
\[ \| u(\cdot, T) \|_{L^2}^2 \leq C \| u(\cdot, -\varepsilon) \|_{L^2}^2 + C_2 \int_{-\varepsilon}^T \left( C_\varepsilon + \| u'_s (\cdot, t) \|_{L^\infty} \right) \| u(\cdot, t) \|_{L^2}^2 \, dt, \]
with $C, C_j$ depending only on $u_0$.

(c) We fix $s > n/2 + 1$. For the given function $u$, we already have the constraint $\varepsilon \leq \varepsilon_0$ ensuring the above inequality. By assumption, for some $\eta > 1$, we can choose $\varepsilon \leq \varepsilon_1$ so that, in $V^0_t$,

$$\left\| u'(\cdot, t) \right\|_{L^\infty} \leq C(0, 0, u_0)/\eta(-t).$$

We take now $C(u_0) = 2(C_2)^{-1}$. Gronwall lemma applied to the final inequality of (b) yields

$$\left\| u(\cdot, T) \right\|_{L^\infty} \leq C\left\| u(\cdot, -\varepsilon) \right\|_{L^\infty} C(-T)^{-1/\eta}.$$

Since $(-T)^{-1/\eta}$ is integrable near $T = 0$, the function $u$ is in fact smooth in $V^0_t$. □

**Corollary 1.1.** – In the case $n = 1$, consider instead of (1.1) a system:

$$\partial_t u + A(x, t, u)\partial_x u + B(x, t, u) = 0,$$

where $A$ is not necessarily symmetric and assume that the eigenvalues of $A(x_0, T, u_0)$ are real and distinct. Then the conclusion of Theorem 1.1 holds.

**Example.** – The compressible Euler system. This is the system (with $D_t = \partial_t + u \nabla$)

$$D_t \rho + \rho \text{ div } u = 0,$$

$$D_t u + \frac{\nabla p}{\rho} = 0.$$

Here we assume for simplicity that the entropy is constant and

$$p(\rho) = A\rho^\gamma, \quad A > 0, \quad \gamma > 1.$$

Consider as in Theorem 1.1 a continuous solution $(u, \rho)$ (smooth for $t < T$) with $\rho(M_0) = \rho_0 > 0$. Setting as usual

$$q = 2/(\gamma - 1)(\gamma A)^{1/2}\rho^{(\gamma - 1)/2},$$

we obtain the following symmetric system with affine coefficients:

$$D_t q + [(\gamma - 1)/2] q \text{ div } u = 0,$$

$$D_t u + [(\gamma - 1)/2] q \nabla q = 0.$$

Using the proof of Theorem 1.1 for this system yields a constant $C(n)$ such that

$$\lim \sup (T - t) \left( \left\| u'_x(x, t) \right\| + \left\| q'_x(x, t) \right\| \right) \leq C(n)$$

implies that we are dealing with a smooth solution near $M_0$. Translating this back to the original unknowns, we obtain two constants $C_1(n) > 0$ and $C_2(n, \rho_0) > 0$ such that if

$$\lim \sup_{(x, t) \to M_0} (T - t) \left\| u'_x(x, t) \right\| < C_1(n), \quad \lim \sup_{(x, t) \to M_0} (T - t) \left\| \nabla \rho(x, t) \right\| < C_2(n, \rho_0),$$

the considered solution is smooth near $M_0$. □
The proof of Theorem 1.1 yields in principle a strictly positive value of $C(x_0, T, u_0)$, which is certainly not the best. This is why we introduce the following definition.

**Definition 1.1 (Minimal Growth Constant).** For any fixed $(M_0, u_0)$ and any continuous but non smooth solution $u$ in some neighborhood $V$ of $(x_0, T)$, with $u(x_0, T) = u_0$ (if any), one can compute

$$\limsup_{(x,t) \to M_0} (T-t)\|u'_x(x,t)\|,$$

which, by Theorem 1.1, is certainly not the best. This is why we introduce the following definition.

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$$\limsup_{(x,t) \to M_0} (T-t)\|u'_x(x,t)\|,$$

which is, by Theorem 1.1, $+\infty$ or a number greater than or equal to $C(x_0, T, u_0)$. The infimum of all such numbers corresponding to all possible $u$ is the best Minimal Growth (MG) Constant $\overline{C}(x_0, T, u_0)$ (possibly $C_1$). Note that $C > 0$.

For nonlinear systems, it is easy to obtain upper bounds for the MG constants by constructing special solutions to the given system.

**Theorem 1.2.** – Assume that system (1.1) is homogeneous ($B \equiv 0$) with coefficients independent of $(x,t)$. Let $\xi \in \mathbb{R}^n$ $(|\xi| = 1)$, and denote by $\lambda_j(v, \xi)$, $r_j(v, \xi)$ the eigenvalues and corresponding (right) eigenvectors of the matrix $A_0^{-1}(v) \sum A_i(v)\xi_i$. For a fixed $u_0$, let $E$ be the set of those $(j, \xi)$ for which

$$(r_j\nabla \lambda_j)(u_0, \xi) \neq 0.$$  

For $(j, \xi) \in E$ and $v$ close to $u_0$, we assume that $r_j$ has been normalized to have

$$r_j(v, \xi)\nabla \lambda_j(v, \xi) \equiv 1.$$  

Then

$$\overline{C}(x_0, T, u_0) \equiv \overline{C}(u_0) \leq \inf_{(j,\xi) \in E} \|r_j(u_0, \xi)\|.$$  

If $E$ is empty, the right-hand side is $+\infty$ and the theorem says nothing.

**Proof.** – The proof uses the classical simple wave construction (see, for instance, [7]). Let $\xi$, $|\xi| = 1$, and consider a solution $u(x, t) = v(x, \xi, t)$. Then $v$ has to satisfy the 1D system:

$$A_0(v)\partial_s v + \tilde{A}(v)\partial_x v = 0, \quad \tilde{A}(v) = \sum A_i(v)\xi_i.$$  

Trying a simple wave

$$v(s, t) = V(\phi(s, t)), \quad s = x, \xi, \phi \in \mathbb{R},$$  

we take for some $j$ such that $(j, \xi) \in E$,

$$V'(\sigma) = r_j(V(\sigma), \xi), \quad V(0) = u_0, \quad \partial_t \phi + \lambda_j(V(\phi), \xi)\partial_x \phi = 0.$$  

Since $\lambda_j(V(\sigma), \xi) = \sigma + \lambda_j(u_0, \xi)$, the function $\psi = \phi + \lambda_j(u_0, \xi)$ will be a solution of Burgers’ equation. It is easy to construct $\psi$ in a neighborhood of $(s_0 = x_0, T)$ with the value $\psi(s_0, T) = \lambda_j(u_0, \xi)$ and $\psi$ blowing up exactly at $(s_0, T)$. We have then:

$$u'_x(x, t)\eta = v'_x(s, t)\xi, \quad v'_x(s, t) = r_j(V(\phi(s, t)), \xi)\phi'_s,$$

hence

$$\|u'_x\| = \|v'_x\| = \|r_j(V(\phi), \xi)\| \cdot |\psi'_s|.$$
For the solution \( \psi \) of Burgers’ equation just constructed, it is easy to see that

\[
\limsup_{(s,t) \to (s_0,T)} |\psi'(s,t)| = 1,
\]

so finally

\[
\limsup_{(x,t) \to M_0} |u'_x(x,t)| = \left\| r_j(u_0, \xi) \right\|,
\]

what was to be proved. \( \square \)

Despite the fact that the proof uses simple waves, we believe that the theorem is also true for variable coefficients systems: one has to use rank one geometric blowup solutions instead of simple waves, but this requires some more work (see [3] for instance).

To obtain relevant lower bounds for the MG constants seems more difficult; it is possible though in some very simple cases.

**Theorem 1.3.** – Let the system be a scalar homogeneous equation:

\[
\partial_t u + \sum A_i(u) \partial_i u = 0.
\]

Then

\[
\overline{C}(u_0) = \left\| A'(u_0) \right\|^{-1}, \quad A(u) = (A_1(u), \ldots, A_n(u)).
\]

**Proof.** – The assumption that \( u \) is continuous in a neighborhood \( V \) of \((x_0, T)\) allows us to capture all characteristics closely and use the method of characteristics to represent the solution. Let \( \Delta \) be the characteristic reaching \((x_0, T)\) and \( x_1 \) its intersection with the plane \( t = T_1 \) for some \( T_1 < T \) close enough to \( T \). We set

\[
\tilde{u}(x) = u(x, T_1), \quad \tilde{u}(x_1) = u_0.
\]

for \( x \) close to \( x_1 \). Since \( u \) actually blows up at \((x_0, T)\), we have, considering as usual the divergence of the field \( A(u(\cdot, t)) \),

\[
A'(u_0) \cdot \tilde{u}'(x_1) = -(T - T_1)^{-1}.
\]

By the method of characteristics, we have

\[
u(\Phi(x, t), t) = \tilde{u}(x), \quad \Phi(x, t) = x + (t - T_1) A(\tilde{u}(x)).
\]

Differentiating this, we obtain

\[
u'_x(\Phi, t) \Phi'_x = \tilde{u}'_x, \quad \Phi'_x = \text{id} + (t - T_1) A'(\tilde{u}(x)) \tilde{u}'(x).
\]

We fix now \( x = x_1 \). In order to evaluate \( \|u'_x\| \) along \( \Delta \), we solve for a fixed \( \xi \) the equation:

\[
\xi = \Phi'_x(x_1, t) \eta.
\]

Decomposing \( \mathbb{R}^n \) in the direct sum of \( \mathbb{R} A'(u_0) \) and \( \Pi \) (the orthogonal plane to \( \tilde{u}'(x_1) \)), we set:

\[
\xi = \beta A' + \xi', \quad \eta = \alpha A' + \eta'.
\]
We thus obtain
\[ \xi' = \eta', \quad \alpha = \beta(T - T_1)/(T - t), \]
and
\[ |u' \xi| = |\beta|/(T - t). \]
For fixed $\beta$, the minimum of $\|\xi\|$ is reached when $\xi$ is colinear to $\tilde{u}'(x_1)$, hence on $\Delta$
\[
\|u' \|/(T - t) = \|\tilde{u}'(A' \cdot \tilde{u}')/\|A'(u_0)\|.
\]
and
\[
\limsup\|u' \|/(T - t) \geq \|\tilde{u}'(x_1)\|/\|A'(u_0) \cdot \tilde{u}'(x_1)\| \geq 1/\|A'(u_0)\|.
\]
This gives a lower bound for $C$ which turns out to be also the upper bound found in Theorem 1.2.
The proof is complete. \qed

In the case of diagonal $2 \times 2$-systems, we cannot prove that Theorem 1.2 gives actually the good constant, but only a slightly weaker result.

**Theorem 1.4.** – Consider a $2 \times 2$ system in diagonal form:
\[
\partial_t u + \Lambda(u) \partial_x u = 0,
\]
where $\Lambda$ is diagonal with elements $\lambda_1(u)$, $\lambda_2(u)$ and
\[
\lambda_1(u_0) < \lambda_2(u_0), \quad \partial_1 \lambda_1(u_0) \neq 0, \quad \partial_2 \lambda_2(u_0) \neq 0.
\]
Let $u$ be a solution continuous in a neighborhood $V$ of $M_0 = (x_0, T)$ ($u(M_0) = u_0$), smooth for $t < T$, but not smooth in any neighborhood of $M_0$. Let $\gamma_1$ and $\gamma_2$ be characteristic curves respectively for $\lambda_1$ and $\lambda_2$, reaching $M_0$. Choose $T_1$ close to $T$ such that $\gamma_j$ cuts $t = T_1$ at a point $M_j$. Assume moreover that the $j$-characteristic curves starting from points close to $M_j$ on $t = T_1$ reach all points of a neighborhood $V'$ of $M_0$ ($j = 1, 2$). Then
\[
\limsup_{(x, t) \to M_0} (T - t) \|u'_x(x, t)\| \geq \inf(\|\partial_1 \lambda_1(u_0)\|^{-1}, \|\partial_2 \lambda_2(u_0)\|^{-1}).
\]

**Proof.** – Setting
\[
D_j = \partial_t + \lambda_j(u) \partial_x, \quad j = 1, 2,
\]
we have as usual (see, for instance, [7])
\[
D_1 \tilde{q}_1 + (\partial_1 \lambda_1)(\exp -H_1)\tilde{q}_1^2 = 0,
D_2 \tilde{q}_2 + (\partial_2 \lambda_2)(\exp -H_2)\tilde{q}_2^2 = 0.
\]
Here, the functions $H_j$ have been chosen so that
\[
\partial_2 H_1 = (\partial_2 \lambda_1)/(\lambda_1 - \lambda_2), \quad H_1(u_0) = 0,
\partial_1 H_2 = -(\partial_1 \lambda_2)/(\lambda_1 - \lambda_2), \quad H_2(u_0) = 0,
\]
and we have set
\[
\tilde{q}_j = (\partial_x u_j) \exp H_j, \quad j = 1, 2.
\]
Considering now \( q_j \) on \( \gamma_j \) as a function of \( t \), it satisfies the differential equation:

\[
q_j' + a_j(t)q_j^2 = 0, \quad j = 1, 2.
\]

This equation can be solved explicitly, yielding

\[
-(q_j(t))^{-1} + (q_j(T_1))^{-1} = \int_{T_1}^t a_j(s) \, ds.
\]

If, for \( j = 1 \) and \( j = 2 \),

\[
(q_j(T_1))^{-1} \neq \int_{T_1}^t a_j(s) \, ds, \quad T_1 \leq t \leq T,
\]

this is also true for the neighboring curves, hence, by the geometric assumption on \( \gamma_1 \) and \( \gamma_2 \), \( |\partial_t u| \) remains bounded in a neighborhood of \( M_0 \), which implies that no blowup occurs, in contradiction with our assumption. It follows that for some \( j \), say \( j = 1 \),

\[
(q_1(T_1))^{-1} = \int_{T_1}^T a_1(s) \, ds, \quad (q_j(t))^{-1} = \int_{t}^T a_1(s) \, ds.
\]

This shows

\[
\lim (T - t)q_1(t) = \lim (T - t)q_1(t) = (a_1(T))^{-1} = (\partial_1\lambda_1(u_0))^{-1}.
\]

Finally, in this case,

\[
\limsup \|u'_t\| (T - t) \geq |\partial_1\lambda_1(u_0)|^{-1}
\]

which completes the proof. \( \square \)

It seems that the problem of finding the best constant \( C \) for systems is not really relevant. In fact, \( \partial_t u(x, t) \) lies in the tangent space to \( \mathbb{R}^N \) at \( u(x, t) \): it should be measured in an appropriate metric on \( \mathbb{R}^N \), depending on the system. An attempt of doing so is offered in the next section.

1.2. A metric approach to measure growth constants

For simplicity, consider now normalized systems of type (1.1):

\[
(1.2) \quad \partial_t u + \sum A_i(x, t, u)\partial_i u + B(x, t, u) = 0,
\]

where the matrices \( A_i \) are no longer assumed to be symmetric. Fix a point \( M_0 = (x_0, T) \) and a state \( u_0 \), and assume that for \( (x, t, u) \) close to \( (M_0, u_0) \) and all \( \xi \), the matrix

\[
A(x, t, u, \xi) = \sum A_i(x, t, u)\xi_i
\]
has real distinct eigenvalues $\lambda_j(x, t, u, \xi)$. We assume now that all these eigenvalues are genuinely nonlinear, and choose the corresponding eigenvectors $r_j(x, t, u, \xi)$ to be normalized, that is

$$(r_j \nabla_u \lambda_j)(x, t, u, \xi) = 1.$$ 

For a given function $u$ with values close to $u_0$, we set:

$$u'_j(x, t)\xi = \sum w_j r_j(x, t, u(x, t), \xi),$$
$$N(u'_j)(x, t) = \sup_{|\xi|=1} \|u\|.$$ 

**THEOREM 1.5.** – If $u$ is a solution of (1.2), $N(u'_j)$ is invariant under changes of unknown functions and conformal changes of coordinates. More precisely:

(i) If $u = \Phi(v)$ is a local $C^1$ diffeomorphism, then

$$N(u'_j)(x, t) = N(v'_j)(x, t).$$

(ii) If

$$(x, t) = (\psi(y, t), t), \quad \psi(y_0, T) = x_0$$

is a local $C^1$ diffeomorphism satisfying, for some $\delta = \delta(y, t)$,

$$'\left(\psi'_j\right)(\psi'_j) = \delta^2 \text{Id},$$

we have

$$N(u'_j)(x, t) = N(v'_j)(y, t).$$

**Proof.** – If $u = \Phi(v)$, $v$ satisfies the new normalized system:

$$\partial_t v + \sum A_i(x, t, v) \partial_i v + B(x, t, v) = 0,$$

where

$$A(x, t, v) = (\Phi'(v))^{-1} A_i(x, t, \Phi(v)) \Phi'(v), \quad B(x, t, v) = (\Phi'(v))^{-1} B(x, t, \Phi(v)).$$

The new eigenvalues and eigenvectors are

$$\tilde{r}_j(x, t, v, \xi) = (\Phi'(v))^{-1} r_j(x, t, \Phi(v), \xi), \quad \tilde{\lambda}_j(x, t, v, \xi) = \lambda_j(x, t, \Phi(v), \xi),$$

since $\tilde{r}_j$ is an eigenvector and

$$\tilde{r}_j \nabla_v \tilde{\lambda}_j = \tilde{\lambda}_j \tilde{r}_j = \lambda'_j \Phi' \tilde{r}_j = r_j \nabla_u \lambda_j = 1.$$ 

Thus, for any $\xi$,

$$v'_j(x, t)\xi = \sum w_{j'} \tilde{r}_j(x, t, v, \xi)$$

implies

$$u'_j(x, t)\xi = \Phi'(v) v'_j\xi = \sum w_{j'} \tilde{r}_j(x, t, u, \xi) = \sum w_j r_j(x, t, u, \xi).$$
Hence the coordinates $w$ are the same, which implies (i).

If we change the variables as in (ii), we obtain the new system satisfied by $v(y, t) = u(\psi(y, t), t)$

$$
\partial_t v + \sum \tilde{A}_i(y, t, v) \partial_i v + \tilde{B}(y, t, v) = 0.
$$

Setting

$$
\left(\psi'(y, t)\right)^{-1} = C(y, t), \ C(y, t) \partial \psi(y, t) = d(y, t),
$$

we have explicitly

$$
\tilde{A}_i(y, t, v) = \sum C_{ij}(y, t) \tilde{A}_j(\psi(y, t), t, v) - d_i(y, t), \ \tilde{B}(y, t, v) = B(\psi, t, v).
$$

At corresponding points $(x, t)$ and $(y, t)$, we have, with $\eta = \delta C \xi$, $\nu' \eta = \delta u' \xi$. Moreover, since $C^{-1} = \delta^2(C)$,

$$
\tilde{\lambda}_j(y, t, v, \eta) = \delta^{-1} \lambda_j(x, t, v, \xi) = \sum d_i \eta_i.
$$

This shows that the corresponding eigenvectors are colinear, and

$$
\tilde{\lambda}_j(y, t, v, \eta) = \delta^{-1} \lambda_j(x, t, v, \xi) = \sum d_i \eta_i.
$$

Since $d$ is independent of $v$,

$$
r_j(x, t, u, \xi) \nabla_{\xi} \tilde{\lambda}_j = \delta^{-1} r_j \nabla_{\xi} \lambda_j,
$$

which shows that in fact

$$
\tilde{r}_j(y, t, v, \eta) = \delta r_j(x, t, u, \xi).
$$

It follows that again the $w$ coordinates are the same, which proves (ii). \[]

Remark that for $n = 1$, all changes are conformal, so Theorem 1.5 yields a true invariance in this case. The effect of measuring $u_\tau'$ in this way is also to normalize the MG constants: for instance, a simple wave solution (see proof of Theorem 1.2) will correspond to a MG constant greater than or equal to one.

1.3. Blowup wave fronts

It is easily seen on examples that not all space directions or all components of $u$ play the same role. In the framework and with the notations of Theorem 1.1, this leads to the following definitions:

**Definition 1.2 (“good directions”).** – A direction $(\eta, \tau) \in \mathbb{R}^{n+1}$ is a “good direction” for the solution $u$ (above $M_0$) if

$$
\lim_{(x, t) \to M_0} \left\| u_\tau'(x, t) \eta + \tau u_\tau'(x, t) \right\| (T - t) = 0.
$$

**Definition 1.3 (“good components”).** – A component $\xi \in \mathbb{R}^N$ is a “good component” for the solution $u$ if

$$
\lim_{(x, t) \to M_0} \left\| \xi u_\tau'(x, t) \right\| (T - t) = 0.
The dual definitions are the following:

**Definition 1.4** ("wave front \(WF_\nu\)). — The (blowup) wave front \(WF_\nu(u)\) (above \(M_0\)) is the orthogonal in \(\mathbb{R}^{n+1}\) of the space of good directions.

**Definition 1.5** ("wave front \(WF_\nu\)). — The (blowup) wave front \(WF_\nu(u)\) (above \(u_0\)) is the orthogonal in \(\mathbb{R}^N\) of the space of good components.

It is important to remark that \(WF_\nu\) lives in the cotangent space of \(\mathbb{R}^n_{\nu,\lambda_0}\) (above \(M_0\)), while \(WF_\nu\) lives in the tangent space to \(\mathbb{R}^N\) (above \(u_0\)). A corollary to Theorem 1.1 is the following:

**Theorem 1.6.** — Let \(u\) be a solution of (1.1), continuous in \(V\) and smooth for \(t<T\). If \(WF_\nu(u)\) or \(WF_\nu(u)\) is reduced to zero, then \(u\) is \(C^\infty\) in a neighborhood of \(M_0\).

In the next section, we will compute these wave fronts for some simple examples.

### 2. Examples and open questions

#### 2.1. Rank one geometric blowup

We refer the reader to [1] or [4] for a detailed analysis of this case. Let us recall here some simple facts. We say that \(u\) blows up geometrically if there is a smooth map 

\[
\Phi(X, T) = (\Phi_1(X, T), T), \quad \Phi(0, 0) = M_0 = (0, 0),
\]

such that \(u(\Phi) = v\) is smooth. If \(\Phi'(0, 0)\) has a one-dimensional kernel (this is what we call "rank one"), we can assume for a scalar function \(\phi\)

\[
X = (X_1, X'), \quad \Phi(X, T) = (\phi(X, T), X', T), \quad \partial_1 \phi(0) = 0.
\]

We have then

\[
u'(\Phi) = (\partial_1 \phi)^{-1}(\partial_1 v)(1, -\partial_X \phi, -\partial_T \phi) + R,
\]

with \(R\) smooth. In [1,3], we explain how such solutions can be constructed for systems (1.1). We require that for some characteristic point

\[
(\xi^0, \tau^0), \quad \tau^0 = -\lambda_j(0, u_0, \xi^0).
\]

the function \(\phi\) should satisfy an eikonal equation

\[
\phi_T = \lambda_j(\Phi, v, 1, -\partial_X \phi).
\]

Moreover, at the origin, the vector \(\partial_1 v(0)\) should be colinear to the corresponding eigenvector \(r_j(0, u_0, \xi^0)\):

\[
\partial_1 v(0) = \alpha r_j, \quad \alpha \neq 0,
\]

and \((1, -\partial_X \phi, -\partial_T \phi)\) should be colinear to \((\xi^0, \tau^0)\). If we assume that \(\lambda_j\) is genuinely nonlinear, we deduce from the eikonal equation and the requirements that \(\partial_1 \phi > 0\) for \(T < 0\)

\[
\partial_T (\partial_1 \phi)(0) = \alpha r_j \nabla u \lambda_j.
\]
Hence
\[
\limsup (-T)(\partial_t \phi)^{-1} = (ar_j \nabla_a \lambda_j)^{-1},
\]
and the solution \( u \) blows up with the minimal rate \((-t)^{-1}\). It follows that
\[
WF_*(u) = R(\xi^0, \tau^0),
\]
\[
WF^*(u) = Rr_j(0, u_0, \xi^0).
\]
We also have
\[
\limsup (T-t) \| u'_t \eta \| = \| r_j / (r_j \nabla_a \lambda_j)(0, u_0, \xi^0) \| \cdot |\xi^0 \cdot \eta|.
\]
This implies
\[
\limsup N(u'_t)(T-t) \geq 1.
\]
One special case is the scalar equation:
\[
\partial_t u + \sum A_i(u) \partial_i u = 0, \quad u(x, 0) = \bar{u}(x).
\]
The blowup of any \( u \) is then a geometric blowup of rank one, since, by the method of characteristics (see for instance the proof of Theorem 1.3)
\[
u(\Psi(x, t), t) = \bar{u}(x), \quad \Psi(x, t) = x + tA(\bar{u}(x)),
\]
and \( \Psi' \) has at most a one dimensional kernel. In this case, \( WF_u \) is just the characteristic direction lying above \( d\bar{u} \).

Another special case of interest is a one-dimensional \( 2 \times 2 \)-system in diagonal form: geometric blowup of rank one is not then automatic, it means that only \( \partial_x u_j \) blows up, the gradient of the other component staying bounded. In this case of course
\[
WF^*(u) = R e_j.
\]

2.2. Rank two geometric blowup

(a) Let us first consider the extremely simple case of two uncoupled Burger’s equations in the plane, the solutions \( u_1 \) and \( u_2 \) of which we have arranged to blowup at the same point. Thus the wave \( u = (u_1, u_2) \) is truly a superposition of the two simple waves \( (u_1, 0) \) and \((0, u_2)\). If the propagation speeds of \( u_1 \) and \( u_2 \) are different, we have:
\[
WF_u(u) = R^2, \quad WF^*(u) = R^2,
\]
but
\[
\limsup (T-t) N(u'_t) = 1.
\]
This is due to the fact that, for each time \( t \), \( \partial_t u_1 \) and \( \partial_t u_2 \) are big at different points: the two waves can be considered as “uncoupled”.

If the special case of two waves with the same speed \( \lambda \), we have in general:
\[
WF_u(u) = R(-1, \lambda), \quad WF^*(u) = R^2,
\]
the last equality being false in some exceptional cases (for instance, $u_1 \equiv u_2$). Also in this case,
\[
\limsup (T-t) N(u'_i) \geq \sqrt{2}
\]
and this reflects the coupling of the two waves.

(b) Let us turn now to rank two singularities such as those constructed in [4] for quasilinear wave equations in $\mathbb{R}^{n+1}$:
\[
L(u) = \sum \ell_{ij}(\partial u) \partial_{ij}^2 u + c(\partial u) = 0, \quad \partial u = (\partial_1 u, \ldots, \partial_n u, \partial_{n+1} u = \partial_t u).
\]
We denote by $\tau_j(\partial u, \xi), \ j = 1, 2$, the two values which make the vectors
\[
(\xi, \tau_1), (\xi, \tau_2)
\]
characteristic for the symbol
\[
\ell(\partial u, \xi, \tau = \xi_{n+1}) = \sum \ell_{ij}(\partial u) \xi_i \xi_j
\]
of $L$. If we set
\[
U = (\partial_1 u, \ldots, \partial_n u, \partial_{n+1} u = \partial_t u),
\]
we can think of $L(u)$ as a first order nonlinear $(n+1) \times (n+1)$-system, for which the eigenvalues $\lambda_j(U, \xi)$ are 0 (with multiplicity $n - 1$) and $-\tau_1, -\tau_2$. The eigenvector corresponding to $\lambda_j$ is
\[
r_j(U, \xi) = (\xi, -\lambda_j(U, \xi)) = (\xi, \tau_j(U, \xi)).
\]
We have then
\[
(r_j \nabla \lambda_j)(U, \xi) = [(\partial_\tau \ell)^{-1} (r_j \nabla \partial_\tau \ell)](\partial u, \xi, \tau_j).
\]
We summarize now roughly what has been done in [4]: we assume that we are given two numbers $\alpha_1, \alpha_2$ such that the plane
\[
\Pi = \{(\xi, \tau), \ \tau + \alpha_1 \xi_1 + \alpha_2 \xi_2 = 0, \ \xi_3 = \cdots = \xi_n = 0\}
\]
intersects the characteristic cone at $(\partial u(0) = U_0)$ along two independent vectors
\[
\bar{e}^1 = (-\mu_1, -\mu_2, 0, \alpha_1 \mu_1 + \alpha_2 \mu_2), \quad \bar{e}^2 = (-v_1, -v_2, 0, \alpha_1 v_1 + \alpha_2 v_2).
\]
From what has been said above, we can assume
\[
\bar{e}^1 = r_1(U_0, -\mu_1, -\mu_2, 0), \quad \bar{e}^2 = r_2(U_0, -v_1, -v_2, 0).
\]
For a scalar function
\[
h(y, t), \quad y = (y_1, \ldots, y_n), \ \partial_1 h(0) = \partial_2 h(0) = 0,
\]
we define functions $\phi_j$ by
\[
\partial_1 h = \mu_1 \phi_1 + \mu_2 \phi_2, \quad \partial_2 h = v_1 \phi_1 + v_2 \phi_2.
\]
and a map $\Phi$

$$\Phi(y, t) = (\phi_1(y, t), \phi_2(y, t), y_3, \ldots, y_n, t), \quad \Phi(0) = 0.$$ 

It is possible to construct $\theta$ such that:

(i) $\Phi$ is a local homeomorphism.

(ii) For some smooth $w$, the function $u$ defined by

$$u(\Phi) = w$$

is a solution of $L(u) = 0$.

(iii) The function $u$ and its gradient are continuous, while

$$j(u'')(\Phi) = q/j e^1(\xi^1) - m/j (e^1(\xi^2) + e^2(\xi^1)) + p/j e^2(\xi^3) + R,$$

where

$$p = \partial_1^2 h, \quad m = \partial_2^2 h, \quad q = \partial_2^2 h, \quad j = pq - m^2, \quad j(0) = 0,$$

$$e^1(0) = \bar{e}^1, \quad e^2(0) = \bar{e}^2,$$

and $R$ is a smooth matrix.

Only the structure of the derivatives which blow up is important here. According to (iii), we have

$$W F_*(U) = R\bar{e}^1 + R\bar{e}^2.$$ 

Similarly,

$$(\xi, \tau)U'_{x,t} = [q/j (e^1(\xi, \tau)) - m/j (e^2(\xi, \tau))]'(\xi^1)$$

$$+ [-(m/j (e^1(\xi, \tau)) + p/j (e^2(\xi, \tau)))]'(\xi^2) + \text{bounded terms},$$

hence

$$W F^*(U) = Rr_1 + Rr_2.$$ 

From the above formula, we also see that at the points $\Phi(0, t)$, we have simply

$$U'_{x,t} = (-t)^{-1} \left[ \frac{r_1}{r_1 \nabla \lambda_1} + r_2 \frac{r_2}{r_2 \nabla \lambda_2} \right] + o((-t)^{-1}).$$
By comparing with the structure of \( u^0 \) in (a), we can interpret this formula by thinking of \( u \) as some sort of “superposition” of the two simple waves with \( WF^* \) respectively the vectors \( r_1 \) and \( r_2 \). These would be, just as in the second case of (a), coupled waves.

2.3. Open questions

We comment here on the results of Section 1 and ask questions we were not able to clarify.

(a) In all constructions of singular solutions we know (rank one and rank two geometric blowup), \( WF_u \) is spanned by characteristic (co-)directions, while \( WF^* \) is spanned by the corresponding eigenvectors. Are there examples of solutions which blow up and for which this is not the case?

(b) The rough definition of \( WF_u \) we have adopted correspond to the fact that pseudodifferential operators do not operate on \( L^1 \). We don’t know if it is possible to discuss minimal growth solutions by considering the growth of

\[
\|u^0(x,t)\|_X
\]

for some appropriate space \( X \) (BMO, \( C^0 \) or other substitutes, see [9]).

(c) Considering that \( u \) behaves essentially as if it were depending only on \( \dim WF_u \) variables, one could be tempted to think that

\[
\dim WF^*(u) \leq \dim WF_u(u).
\]

We have seen in 1.1(a) that this is not true. Is there in general some relation between \( WF_u(u) \) and \( WF^*(u) \)?

(d) For each \( \eta \), there is a set of good directions for the component \( \eta \cdot u \), and for each \( \xi \), there is a set of good components for \( u^\xi \). What geometric objects would describe some \( WF(u) \) in this more precise analysis?

(e) In 1.2, we gave a metric on \( \mathbb{R}^N \) to evaluate \( u^\xi(x,t) \) when \( u \) is a solution of a first order system with genuinely nonlinear eigenvalues. In the multidimensional case, the definition of this metric involves some identification between vectors and forms, since \( u^\xi \) (\( \xi \) being a vector) is measured using the eigenvectors \( r_j(\xi) \) (\( \xi \) being a form then): this is the reason why it is only conformally invariant. Is there a better definition which would be invariant for general changes of variables? A better formulation would be at least to measure \( u^\xi \) only for \( \xi \in WF(u) \), since only these directions contribute to

\[
\lim \sup (T-t)N(u^\xi).
\]

(f) For all the examples of blowup solutions with minimal growth we have considered, we found that

\[
\lim \sup (T-t)N(u^\xi)(x,t) \geq 1.
\]

Are there blowup solutions for which

\[
\lim \sup (T-t)N(u^\xi)(x,t) < 1?
\]

If not, what is the structure of the solutions for which

\[
\lim \sup (T-t)N(u^\xi)(x,t) = 1?
\]
This question is similar to the question about minimal energy blowup solutions of the conformal Schrödinger equation, solved in [8].

REFERENCES