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Approximation in Certain Intermediate Spaces

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A theorem of Bojanic gives a precise estimate on the rate of convergence of the Fourier series of a function of boundend variation. While the method of K-functionals is not directly applicable to obtain similar estimates for functions in classes intermediate to $BV[-1, 1]$ and $C[-1, 1]$, we obtain such an estimate in the case of a general class of operators. The result is given in terms of an expression, which for continuous functions, is equivalent to the K-functional. As particular cases, we study the expansions in certain (general) orthogonal polynomials, Lagrange interpolation at the zeros of (general) orthogonal polynomials, and Hermite–Fejér interpolation at the zeros of generalized Jacobi polynomials. When applicable, our result (essentially) includes the previously known results, while many corollaries are new. © 1990 Academic Press, Inc.

1. INTRODUCTION

In recent years, many results have been proved concerning the rate of convergence of various approximation processes for functions of bounded variation [1, 4–12, 16, 17]. A simple, yet typical result, which seems to have motivated most of this research is the following theorem of Bojanic [4].

THEOREM 1.1. *Let f be a 2π-periodic function having bounded variation on $[-\pi, \pi]$ and for integer $n \geq 1$, $s_n(f)$ denote the nth partial sum of the (trigonometric) Fourier series of f. Then, for $x \in [-\pi, \pi]$,*

$$\left| s_n(f, x) - \frac{1}{2}\{f(x+) + f(x-)\} \right| \leq \frac{3}{n} \sum_{k=1}^{n} V\left(g_x, \left[0, \frac{\pi}{k}\right]\right), \qquad (1.1)$$

where

$$g_x(t) := f(x+t) + f(x-t) - f(x+) - f(x-)$$

and $V(g_x, [0, y])$ denotes the total variation of g_x on $[0, y]$, $y \in [0, \pi]$.

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Theorem 1.1 is asymptotically unimprovable. What we find very interesting is also the fact that this theorem provides a link between the two most classical convergence criteria for Fourier series; the Dirichlet–Jordan test and the Dini test. Indeed, if f is a continuously differentiable function, then

$$V(g_x, [0, y]) \leqslant 2y \|f'\|_\infty . \tag{1.2}$$

Thus, for $n \geqslant 3$, (1.1) becomes

$$\|s_n(f) - f\| \leqslant \frac{6\pi}{n} \sum_{k=1}^{n} \frac{1}{k} \|f'\|_\infty \leqslant 12 \frac{\log n}{n} \|f'\|_\infty . \tag{1.3}$$

In view of the equivalence between the modulus of continuity and a K-functional, estimate (1.3) is equivalent to the Dini–Lipschitz test (cf. [13]). This suggests that it would be interesting to examine the rate of convergence of Fourier series for functions in interpolation spaces between the class of continuous functions and the class of functions with bounded variation. Unfortunately, this approach using the K-functional is not applicable for an estimate of the form (1.1). For the classical Fourier series, we used in [18] an explicit evaluation of the relevant K-functional due to Bergh and Peetre [3] and the explicit form of the Dirichlet kernel to obtain an estimate of this nature in terms of the K-functional.

In this paper, we investigate this problem for the aperiodic case, when we do not have an explicit structure afforded by the Dirichlet kernel. We shall consider approximation processes for bounded real valued functions f on $[-1, 1]$ which are of the form

$$U_n(f, x) := \int_{-1}^{1} \kappa_n(x, t) f(t) \, d\mu_n(t), \tag{1.4}$$

where κ_n is a kernel function satisfying certain technical conditions to be described in the next section and μ_n is a positive Borel measure. Typically, μ_n would be either absolutely continuous with respect to the Lebesgue measure or a discrete measure. For a bounded function f and $x \in (-1, 1)$ for which both $f(x+)$ and $f(x-)$ exist, we shall estimate $U_n(f, x) - (f(x+) + f(x-))/2$. For continuous functions f, this estimate will be in terms of the K-functionals between the spaces $C[a, b]$ and $BV[a, b]$ (the class of functions having bounded variation on $[a, b]$) where $[a, b] \subseteq [-1, 1]$. In particular, our estimate will be valid for a large class of orthogonal polynomial expansions, as well as the Lagrange and Hermite–Fejér interpolation processes at certain generalised Jacobi nodes. For functions having bounded variation on $[-1, 1]$, our estimate (essentially) includes the corresponding previously known results in [5, 6]. For functions in intermediate classes such as the Wiener classes V_p or the

Waterman classes $\{n^\alpha\} - BV$, our estimates are new, to the best of our knowledge.

Although we cannot apply the K-functional directly with an estimate of the form (1.1), we will, nevertheless, use similar ideas. Thus, our proof will consist of estimating $U_n(f-g, x)$ and $U_n(g, x)$ for a judiciously chosen function g having bounded variation on $[-1, 1]$. The particular form of the estimates, together with the explicit expression of the proper K-functional given by Bergh and Peetre [3] will then help us to arrive at the final result.

In Section 2, we introduce most of the necessary notations and definitions. The main theorem and the applications are described in Section 3. The main theorem itself is proved in Section 4, while in Section 5, we prove that the various operators belong to the class for which the main theorem is stated. In the process, we shall obtain certain technical estimations and complete the proofs of the remaining theorems.

2. Preliminaries

Throughout the rest of this paper, we adopt the following conventions concerning constants. All constants will be independent of the variables not explicitly listed. The constants denoted by small case letters can have different values at different occurences of the same letter, even within a single formula. The constants denoted by capital letters will retain their values.

The following definition describes the various properties of the approximating operators which we wish to study.

Definition 2.1. Let $n \geqslant 1$ be a positive integer. An operator U_n, acting on bounded, real valued, measurable functions f on $[-1, 1]$ will be said to be of type B if it can be expressed in the form

$$U_n(f, x) = \int_{-1}^{1} \kappa_n(x, t) f(t)\, d\mu_n(t), \tag{2.1}$$

where μ_n is a positive, unit, Borel measure on $[-1, 1]$, κ_n: $[-1, 1] \times [-1, 1] \to \mathbf{R}$ is continuous in x, μ_n-integrable in t and, in addition, satisfies each of the following conditions.

(P1)

$$\int_{-1}^{1} \kappa_n(x, t)\, d\mu_n(t) = 1, \qquad x \in [-1, 1].$$

(P2)

$$|\kappa_n(x,t)| \leqslant \min\left\{ nB_{1,n}(x), \frac{B_{2,n}(x)}{|x-t|}\right\} M_n(t), \qquad x, t \in [-1, 1]. \tag{2.2}$$

(P3) There exists a partition

$$-1 =: y_{n+1,n} < y_{n,n} < \cdots < y_{1,n} < y_{0,n} := 1$$

such that

$$\frac{D_1}{n} \leqslant \int_{y_{j+1,n}}^{y_{j,n}} M_n(t)\, d\mu_n(t) \leqslant \frac{D_2}{n}. \tag{2.3}$$

(P4) We have

$$\int_x^t M_n(u)\, d\mu_n(u) \leqslant C(x)(t-x), \qquad -1 < x < t \leqslant 1 \tag{2.4a}$$

$$\int_t^x M_n(u)\, d\mu_n(u) \leqslant C(x)(x-t), \qquad -1 \leqslant t < x < 1. \tag{2.4b}$$

(P5) Suppose that $x \in (y_{l+1,n}, y_{l,n}] \subseteq [y_{n-2,n}, y_{2,n}]$. Then

$$\left|\int_{-1}^t \kappa_n(x,u)\, d\mu_n(u)\right| \leqslant \frac{A_n(x)}{n\,|x-t|}, \qquad -1 \leqslant t \leqslant y_{l+2,n} \tag{2.5a}$$

$$\left|\int_t^1 \kappa_n(x,u)\, d\mu_n(u)\right| \leqslant \frac{A_n(x)}{n\,|x-t|}, \qquad y_{l-1,n} \leqslant t \leqslant 1. \tag{2.5b}$$

We pause here to describe three examples of operators of type B. We say that $w: [-1, 1] \to [0, \infty)$ is a weight function if

$$\int_{-1}^1 |t|^n\, w(t)\, dt < \infty, \qquad n = 0, 1, \ldots. \tag{2.6}$$

With a weight function w we can construct a unique system of orthogonal polynomials [14]

$$p_n(w, x) =: p_n(x) =: \gamma_n \prod_{k=1}^n (x - x_{kn}), \qquad n = 0, 1, \ldots, \tag{2.7a}$$

where

$$\gamma_n > 0, \; -1 < x_{nn} < \cdots < x_{1n} < 1, \qquad n = 0, 1, \ldots \tag{2.7b}$$

$$\int_{-1}^1 p_n(x)\, p_m(x)\, w(x)\, dx = \delta_{nm}, \qquad n, m = 0, 1, \ldots. \tag{2.7c}$$

If f is a bounded, real valued, measurable function on $[-1, 1]$, then we can define the following three approximating processes for f: the partial sums $s_n(f)$ of the orthogonal expansion of f, the Lagrange interpolation polynomials $L_n(f)$ and the Hermite–Fejér interpolation polynomials $H_n(f)$ at $\{x_{kn}\}$. More explicitly, we set

$$a_k(f) := a_k(w, f) := \int_{-1}^{1} f(t)\, p_k(t)\, w(t)\, dt, \qquad k = 0, 1, \ldots \tag{2.8a}$$

$$s_n(f, x) := s_n(w, f, x) := \sum_{k=0}^{n-1} a_k(f)\, p_k(x), \qquad n = 1, 2, \ldots, \quad x \in \mathbf{R}. \tag{2.8b}$$

The polynomial $L_n(f)$ is the unique polynomial of degree at most $n-1$ such that

$$L_n(f, x_{kn}) = f(x_{kn}), \qquad k = 1, \ldots, n, \quad n = 1, 2, \ldots. \tag{2.9}$$

The polynomial $H_n(f)$ is the unique polynomial of degree at most $2n-1$ such that

$$H_n(f, x_{kn}) = f(x_{kn}),\ H_n'(f, x_{kn}) = 0, \qquad k = 1, \ldots, n, \quad n = 1, 2, \ldots. \tag{2.10}$$

Under suitable conditions on w, which will be described in Section 5, each of the operators s_n, L_n, H_n will be of type B. In particular, these conditions are satisfied when $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta \geq -\frac{1}{2}$ (in the case of H_n, even when $\alpha, \beta > -1$).

We now turn our attention to the description of the K-functional which we will be (indirectly) using. Let $B[a, b]$ denote the class of all bounded, real valued, measurable functions on $[a, b]$. When $f \in B[a, b]$, we write

$$\|f\|_{[a,b]} = \sup\{|f(x)| : x \in [a, b]\} \tag{2.11a}$$

$$V(f, [a, b]) := \sup\left\{\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| : t_0 := a < t_1 < \cdots < t_n := b\right\}. \tag{2.11b}$$

The class $BV[a, b]$ then consists of all $f: [a, b] \to \mathbf{R}$ for which $V(f, [a, b]) < \infty$. The K-functional between $B[a, b]$ and $BV[a, b]$ can then be defined by

$$K(f, \delta, [a, b]) := \inf\{\|f - h\|_{[a,b]} + \delta V(h, [a, b])\}, \qquad \delta > 0,\ f \in B[a, b], \tag{2.12}$$

where the inf is over all $h \in BV[a, b]$.

If I is a subinterval of $[-1, 1]$ and $f: [-1, 1] \to \mathbf{R}$, we set

$$osc(f, I) := \sup\{|f(t) - f(u)| : t, u \in I\}. \tag{2.13}$$

For $[a, b] \subseteq [-1, 1]$ and $\delta > 0$, we put

$$\Omega(f, \delta, [a, b]) := \sup \delta \sum_{I \in \mathscr{I}} osc(f, I), \tag{2.14}$$

where the sup is over all the families $\mathscr{I}$ of subintervals of $[a, b]$ such that

$$\text{card}(\mathscr{I}) := \text{number of intervals in } \mathscr{I} \leqslant \delta^{-1} \tag{2.15a}$$

$$\bigcup_{I \in \mathscr{I}} I = [a, b] \tag{2.15b}$$

$$\text{the members of } \mathscr{I} \text{ are pairwise disjoint.} \tag{2.15c}$$

When $f \in C[-1, 1]$, $[a, b] \subseteq [-1, 1]$, $\Omega(f, \delta, [a, b])$ gives the order of magnitude of the K-functional $K(f, \delta, [a, b])$. More precisely, we have

THEOREM 2.2 [3]. *Let $f \in C[-1, 1]$, $[a, b] \subseteq [-1, 1]$, and $\delta > 0$. Then,*

$$\tfrac{1}{4}\Omega(f, \delta, [a, b]) \leqslant K(f, \delta, [a, b]) \leqslant 4\Omega(f, \delta, [a, b]). \tag{2.16}$$

In [18], we have verified that the constants are, indeed, independent of $[a, b]$.

We will state our results in terms of Ω. In applications, however, we would like an expression which is also increasing. Towards this end, we set

$$\Omega^*(f, \delta, [a, b]) := \sup\{\Omega(f, t, [a, b]), 0 < t \leqslant \delta\}. \tag{2.17}$$

Since the K-functional is increasing, we see from (2.16) that, when $f \in C[-1, 1]$,

$$\tfrac{1}{4}\Omega^*(f, \delta, [a, b]) \leqslant K(f, \delta, [a, b]) \leqslant 4\Omega^*(f, \delta, [a, b]). \tag{2.18}$$

The following proposition summarizes some of the obvious properties of Ω and Ω^*.

PROPOSITION 2.3. *Let $f, g \in B[-1, 1]$, $[a, b] \subseteq [-1, 1]$, $d \in [a, b]$, $0 < \delta_1 < \delta_2$, $\delta > 0$, $\lambda > 0$. Then,*

(a) $\Omega^*(f, \delta, [a, b])$ *is increasing in δ, b and decreasing in a.*

(b) $$\frac{\Omega^*(f, \delta_2, [a, b])}{\delta_2} \leqslant \frac{\Omega^*(f, \delta_1, [a, b])}{\delta_1}$$

(c) $\Omega^*(f, \delta_1 + \delta_2, [a, b]) \leqslant \Omega^*(f, \delta_1, [a, b]) + \Omega^*(f, \delta_2, [a, b])$

(d) $$\Omega^*(f, \lambda\delta, [a, b]) \leqslant (1+\lambda)\,\Omega^*(f, \delta, [a, b])$$

(e) $$\Omega(f+\lambda g, \delta, [a, b]) \leqslant \Omega(f, \delta, [a, b]) + \lambda\Omega(g, \delta, [a, b])$$

(f) $$\Omega(f, \delta, [a, b]) \leqslant \Omega(f, \delta, [a, d]) + \Omega(f, \delta, [d, b]) \leqslant 2\Omega(f, \delta, [a, b]).$$

Next, we describe the order of magnitude of $\Omega^*(f, \delta, [a, b])$ for a few function classes. First, we note that the definition of Ω is directly related to the Chanturiya classes $V(\nu)$ investigated in some detail in connection with the convergence of Fourier series (cf. [1, 2, and the references therein]). Given a nondecreasing, concave sequence ν, the Chanturiya class $V(\nu, [a, b])$ is, in fact, defined to be the class of functions $f \in B[a, b]$ for which

$$\Omega(f, \delta, [a, b]) = O\left(\frac{\nu(n)}{n}\right), \qquad n := \lfloor \delta^{-1} \rfloor. \tag{2.19}$$

Obviously, if $f \in V(\nu, [a, b])$, and $[c, d] \subseteq [a, b]$, then $f \in V(\nu, [c, d])$ also.

DEFINITION 2.4. Let $\Lambda := \{\lambda_k\}$ be an increasing sequence, $1 \leqslant p < \infty$. For f: $[a, b] \to \mathbf{R}$, we set

$$V_\Lambda(f, [a, b]) := \sup\left\{\sum_{k=1}^{n} \frac{osc(f, I_k)}{\lambda_k}\right\} \tag{2.20}$$

$$V_p(f, [a, b]) := \sup\left\{\sum_{k=1}^{n} (osc(f, I_k))^p\right\}^{1/p}, \tag{2.21}$$

where the sup is over all the pairwise disjoint intervals $\{I_k\}_{k=1}^n$ whose union is $[a, b]$. The classes $\Lambda BV[a, b]$ and $V_p([a, b])$ are then defined by

$$\Lambda BV([a, b]) := \{f: [a, b] \to \mathbf{R}: V_\Lambda(f, [a, b]) < \infty\} \tag{2.22}$$

$$V_p([a, b]) := \{f: [a, b] \to \mathbf{R}: V_p(f, [a, b]) < \infty\}. \tag{2.23}$$

We have the following estimates [2].

$$\Omega(f, \delta, [a, b]) \leqslant \left(\sum_{k=1}^{n} 1/\lambda_k\right) V_\Lambda(f, [a, b]), \qquad n := \lfloor \delta^{-1} \rfloor \tag{2.24}$$

$$\Omega(f, \delta, [a, b]) \leqslant \delta^{1/p} V_p(f, [a, b]). \tag{2.25}$$

Finally, we make some observations which will simplify the statement of our main theorem. If $f \in B[-1, 1]$, $x \in (-1, 1)$ and $f(x+)$ and $f(x-)$ exist, then we set

$$g_x(t) := \begin{cases} f(t) - f(x-) & \text{if} \quad -1 \leqslant t < x < 1; \\ 0 & \text{if} \quad t = x; \\ f(t) - f(x+) & \text{if} \quad -1 < x < t \leqslant 1. \end{cases} \tag{2.26}$$

If we then let

$$\psi_x(t) := \begin{cases} -1 & \text{if } -1 \leqslant t < x < 1; \\ 0 & \text{if } t = x; \\ 1 & \text{if } -1 < x < t \leqslant 1, \end{cases} \tag{2.27}$$

then, for $t \neq x$, we have

$$f(t) = \frac{1}{2}(f(x+) + f(x-)) + g_x(t) + \frac{f(x+) - f(x-)}{2}\psi_x(t). \tag{2.28}$$

For convenience, we shall assume that f is regulated at x, i.e., (2.28) holds even for $t = x$. Hence, if U_n is an operator of type B, (2.2) implies that

$$U_n(f, x) - \frac{1}{2}(f(x+) + f(x-))$$
$$= U_n(g_x, x) + \frac{f(x+) - f(x-)}{2} U_n(\psi_x, x). \tag{2.29}$$

The asymptotic behavior of $U_n(\psi_x, x)$ as $n \to \infty$ is perhaps a fairly difficult problem. The solution is known only for a few particular operators (e.g., [5, 17]). Since our objective is to investigate the effect of the smoothness of f (as measured by the quantity Ω defined in (2.14)) on the convergence of $U_n(f, x)$, we concentrate on estimating $U_n(g_x, x)$. Thus, it is enough for us to estimate $U_n(g, x)$ where $g \in B[-1, 1]$, $g(x) = 0$ and x is a point of continuity of g.

3. Main Results

Our main theorem is the following.

THEOREM 3.1. *Let U_n be an operator of type* B, $g \in B[-1, 1]$, $x \in (-1, 1)$ *be a point of continuity of g and $g(x) = 0$. Let l be the integer, $2 \leqslant l \leqslant n-3$ such that $x \in (y_{l+1,n}, y_{l,n}] \subseteq [y_{n-2,n}, y_{2,n}]$. Then,*

$$|U_n(g, x)| \leqslant F_{1,n}(x) \sum_{k=1}^{l-2} \frac{1}{k} \Omega\left(g, \frac{k}{l-1}, [y_{l-1}, y_{l-1-\lfloor (l-1)/k \rfloor}]\right)$$
$$+ F_{2,n}(x)\, \Omega(g, 1, [y_{l+2}, y_{l-1}])$$
$$+ F_{1,n}(x) \sum_{k=l+2}^{n} \frac{1}{k-l-1}$$
$$\times \Omega\left(g, \frac{k-l-1}{n-l}, [y_{\lfloor (n-l)/(k-l-1) \rfloor + l + 1}, y_{l+2}]\right), \tag{3.1}$$

where y_j denotes $y_{j,n}$ and

$$F_{1,n}(x) := \frac{4C(x)}{D_1}(A_n(x) + D_2 B_{2,n}(x)) \tag{3.2a}$$

and

$$F_{2,n}(x) := 3D_2 B_{1,n}(x) + 2A_n(x)\, C(x)/D_1. \tag{3.2b}$$

In applications, when we have a good estimate for the quantities $\{y_{j,n}\}$, we can use Proposition 2.3 to obtain a more elegant estimate. Theorem 3.1 thus prescribes a method to obtain estimates on the rate of convergence of various processes for function classes intermediate to $BV[-1, 1]$ and $B[-1, 1]$; so as to include (directly) both the Dini–Lipschitz type criterion and the Bojanic type estimate. We illustrate this for the operators s_n, L_n, H_n introduced in (2.8), (2.9), (2.10), respectively.

THEOREM 3.2. *Let g, x satisfy the conditions of Theorem* 3.1. *Suppose that*

$$|p_n(t)\, w(t) \sqrt{1-t^2}| \leqslant c, \qquad t \in [-1, 1], \quad n = 0, 1, ... \tag{3.3}$$

Then

$$w(x)\, |s_n(g, x)| \leqslant c_1(x) \sum_{k=1}^{n} \frac{1}{k} \Omega^*\left(g, k/n, \left[x - c\frac{1+x}{k}, x + c\frac{1-x}{k}\right]\right), \tag{3.4}$$

where as a function of x, c_1 is bounded on every closed subinterval of $(-1, 1)$.

Here, and elsewhere in this context, the various constants will depend upon w even though this is not clearly indicated.

For the Lagrange interpolation process, we state our conditions in terms of the Christoffel function

$$\lambda_n(x) := \left\{\sum_{k=0}^{n-1} p_k^2(x)\right\}^{-1} \tag{3.5}$$

and the numbers θ_{kn} defined by $\cos \theta_{kn} := x_{kn}$. We denote the Cotes numbers $\lambda_n(x_{kn})$ by λ_{kn}.

THEOREM 3.3. *Suppose that g, x satisfy the conditions of Theorem* 3.1. *We assume that each of the following conditions holds.*

$$\textit{If } t \in (-1, 1) \textit{ then } n\lambda_n(t) \geqslant W(t), \qquad n = 1, 2, ..., \tag{3.6a}$$

where $W(t)$ denotes a positive "constant" depending on t, but independent of n.

$$n\lambda_{kn} \leqslant c, \quad n\lambda_{kn} p_{n-1}^2(x_{kn}) \leqslant c, \qquad k=1, ..., n, \quad n=1,2,... \tag{3.6b}$$

$$c_1 \leqslant n(\theta_{k+1,n} - \theta_{k,n}) \leqslant c_2, \qquad k=1, ..., n, \quad n=1, 2, ... \tag{3.6c}$$

Then

$$|L_n(g, x)| \leqslant c_1(x) \left\{ |p_n(x)| \sum_{k=1}^{n} \frac{1}{k} \Omega^* \left(g, \frac{k}{n}, \left[x - c\frac{1+x}{k}, x + c\frac{1-x}{k} \right] \right) + \frac{1}{n} \Omega^* \left(g, 1, \left[x - c\frac{1+x}{n}, x + c\frac{1-x}{n} \right] \right) \right\}. \tag{3.7}$$

In particular, the conditions (3.6) are satisfied by all the generalized Jacobi polynomials (to be defined below) when the parameters $\alpha, \beta \geqslant -1/2$.

DEFINITION 3.4. The weight function w is a generalized Jacobi weight if it can be represented as

$$w(x) := \psi(x)(1-x)^\alpha (1+x)^\beta, \qquad \alpha, \beta > -1 \tag{3.8}$$

where $\psi(x) > 0$, $x \in [-1, 1]$, ψ is continuously differentiable on $[-1, 1]$ and ψ' satisfies a Lipschitz condition on $[-1, 1]$:

$$|\psi'(x) - \psi'(t)| = O(|x-t|), \qquad x, t \in [-1, 1].$$

In [19], it has been proved that if w is a generalized Jacobi weight, and $f \in C[-1, 1]$, then the Hermite–Fejér interpolation process $H_n(f)$ at the zeros of $p_n(x)$ (cf. (2.10)) converges uniformly to f on closed subintervals of $(-1, 1)$. Theorem 3.1 applied to H_n in this case gives

THEOREM 3.5. *Let w be a generalized Jacobi weight, g, x as in Theorem 3.1. Then*

$$|H_n(g, x)| \leqslant c_1(x) \left\{ p_n^2(x) \sum_{k=1}^{n} \frac{1}{k} \Omega^* \left(g, \frac{k}{n}, \left[x - c\frac{1+x}{k}, x + c\frac{1-x}{k} \right] \right) + \frac{1}{n} \Omega^* \left(g, 1, \left[x - c\frac{1+x}{n}, x + c\frac{1-x}{n} \right] \right) \right\}. \tag{3.9}$$

We note that g is not necessarily a continuous function on $[-1, 1]$. Our theorem, except for the last term on the right hand side of (3.9), extends a theorem of Bojanic and Cheng [6] which is for the case when w is a

Chebyshev weight (i.e., $\psi \equiv 1$, $\alpha=\beta=-\frac{1}{2}$ in (3.8)). It is proved in [6] that even in this case, $H_n(g, x)$ does not converge to 0 if x is not a point of continuity of g.

4. The Proof of the Main Theorem

A major step in this proof is to obtain an estimate on $U_n(g, x)$ in terms of $\{osc(g, [y_{j+1,n}, y_{j,n}])\}$ where the points $\{y_{j,n}\}$ are defined in the condition (P3) in Definition 2.1. Since n will be fixed throughout the proof, we will omit it as a subscript in this section. For example, $U(g, x)$ means $U_n(g, x)$, y_j means $y_{j,n}$, etc. Suppose that $x \in [y_{l+1}, y_l] \subseteq [y_{n-2}, y_2]$. We set

$$h(t) := \begin{cases} g(y_k) & \text{if} \quad t \in (y_{k+1}, y_k], \quad k \leqslant l-2 \\ 0 & \text{if} \quad t \in [y_{l+2}, y_{l-1}] \\ g(y_{k+1}) & \text{if} \quad t \in [y_{k+1}, y_k), \quad k \geqslant l+2. \end{cases} \tag{4.1}$$

Then h is a function of bounded variation. If we let

$$G(t) := g(t) - h(t), \qquad t \in [-1, 1], \tag{4.2}$$

then, we have, in view of the fact that $g(x)=0$,

$$|G(t)| \leqslant \begin{cases} osc(g, [y_{l+2}, y_{l-1}]), & t \in [y_{l+2}, y_{l-1}] \\ osc(g, [y_{k+1}, y_k]), & t \in [y_{k+1}, y_k], \quad |k-l| \geqslant 2. \end{cases} \tag{4.3}$$

We shall first obtain a preliminary estimate on both $U(G, x)$ and $U(h, x)$.

Lemma 4.1. *We have*

(a)
$$|U(G, x)| \leqslant \frac{D_2}{D_1} B_2(x)\, C(x) \sum_{k=0, |k-l| \geqslant 2}^{n} \frac{osc(g, [y_{k+1}, y_k])}{|k-l-1|} + 3D_2 B_1(x)\, osc(g, [y_{l+2}, y_{l-1}]) \tag{4.4}$$

(b)
$$|U(h, x)| \leqslant \frac{A(x)\, C(x)}{D_1} \left\{ \sum_{k=0, |k-l| \geqslant 2}^{n} \frac{osc(g, [y_{k+1}, y_k])}{|k-l-1|} + 2osc(g, [y_{l+2}, y_{l-1}]) \right\}. \tag{4.5}$$

Proof of Lemma 4.1. (a) We express

$$U(G, x) =: S_1 + S_2 + S_3 \tag{4.6}$$

where,

$$S_1 := \sum_{k=0}^{l-2} \int_{y_{k+1}}^{y_k} \kappa(x, t)\, G(t)\, d\mu(t) \tag{4.7a}$$

$$S_2 := \int_{y_{l+2}}^{y_{l-1}} \kappa(x, t)\, G(t)\, d\mu(t) \tag{4.7b}$$

$$S_3 := \sum_{k=l+2}^{n} \int_{y_{k+1}}^{y_k} \kappa(x, t)\, G(t)\, d\mu(t). \tag{4.7c}$$

In view of (2.2), (2.3), and (4.3),

$$|S_2| \leqslant 3D_2 B_1(x)\, osc(g, [y_{l+2}, y_{l-1}]). \tag{4.8}$$

Next, we estimate S_1. Let $0 \leqslant k \leqslant l-2$. Then using (2.2), (4.3), and (2.3),

$$\begin{aligned} &\left| \int_{y_{k+1}}^{y_k} \kappa(x, t)\, G(t)\, d\mu(t) \right| \\ &\qquad \leqslant B_2(x)\, osc(g, [y_{k+1}, y_k]) \int_{y_{k+1}}^{y_k} \frac{M(t)}{t-x}\, d\mu(t) \\ &\qquad \leqslant \frac{D_2 B_2(x)}{n} \cdot \frac{osc(g, [y_{k+1}, y_k])}{y_{k+1}-x}. \end{aligned} \tag{4.9}$$

In view of (2.4a), (2.3), we get

$$\begin{aligned} y_{k+1} - x &\geqslant \frac{1}{C(x)} \int_x^{y_{k+1}} M(t)\, d\mu(t) \\ &\geqslant \frac{D_1}{nC(x)} \cdot (l-k-1). \end{aligned} \tag{4.10}$$

Substituting from (4.9) and (4.10) into (4.7a), we get

$$|S_1| \leqslant \frac{D_2}{D_1} B_2(x)\, C(x) \sum_{k=0}^{l-2} \frac{osc(g, [y_{k+1}, y_k])}{l-k-1}. \tag{4.11}$$

Similarly,

$$|S_3| \leqslant \frac{D_2}{D_1} B_2(x)\, C(x) \sum_{k=l+2}^{n} \frac{osc(g, [y_{k+1}, y_k])}{k-l-1}. \tag{4.12}$$

In view of (4.8), (4.11), (4.12), the estimate (4.4) is proved.

(b) Using (4.1), we see that

$$U(h, x) = \sum_{k=l+2}^{n} g(y_{k+1}) \int_{y_{k+1}}^{y_k} \kappa(x, t)\, d\mu(t) + \sum_{k=0}^{l-2} g(y_k) \int_{y_{k+1}}^{y_k} \kappa(x, t)\, d\mu(t). \tag{4.13}$$

We estimate the second sum first. In fact, this is the integration by parts argument which is usually used to obtain estimates for functions of bounded variation. Here, it takes the form of a summation by parts. Thus, we set

$$A_k := \sum_{m=0}^{k} \int_{y_{m+1}}^{y_m} \kappa(x, t)\, d\mu(t) = \int_{y_{m+1}}^{1} \kappa(x, t)\, d\mu(t). \tag{4.14}$$

Then, (2.5b) and (4.10) imply that

$$|A_k| \leqslant \frac{A(x)}{n(y_{k+1} - x)} \leqslant \frac{A(x)\, C(x)}{D_1(l-k-1)}; \qquad k = 0, ..., l-2. \tag{4.15}$$

Thus,

$$\begin{aligned}
&\left| \sum_{k=0}^{l-2} g(y_k) \int_{y_{k+1}}^{y_k} \kappa(x, t)\, d\mu(t) \right| \\
&\quad = \left| \sum_{k=0}^{l-2} g(y_k)(A_k - A_{k-1}) \right| \\
&\quad = \left| g(y_{l-2}) A_{l-2} + \sum_{k=0}^{l-3} [g(y_k) - g(y_{k+1})]\, A_k \right| \\
&\quad \leqslant |g(y_{l-1}) A_{l-2}| + \sum_{k=0}^{l-2} |g(y_k) - g(y_{k+1})|\, |A_k|.
\end{aligned} \tag{4.16}$$

Since $g(x) = 0$ and $x \in [y_{l+1}, y_l]$, we see from (4.15) and (4.16) that

$$\begin{aligned}
&\left| \sum_{k=0}^{l-2} g(y_k) \int_{y_{k+1}}^{y_k} \kappa(x, t)\, d\mu(t) \right| \\
&\quad \leqslant \frac{A(x)\, C(x)}{D_1} \left\{ osc(g, [y_{l+2}, y_{l-1}]) + \sum_{k=0}^{l-2} \frac{osc(g, [y_{k+1}, y_k])}{l-1-k} \right\}
\end{aligned} \tag{4.17}$$

We estimate the first term in (4.13) in a similar way to get (4.5).

COROLLARY 4.2. *We have*

$$|U(g,x)| \leqslant |U(G,x)| + |U(h,x)|$$
$$\leqslant \frac{C(x)}{D_1}(A(x)+D_2B_2(x)) \sum_{k=0,\,|k-l|\geqslant 2}^{n} \frac{osc(g,[y_{k+1},y_k])}{|k-l-1|}$$
$$+ [3D_2B_1(x)+2A(x)\,C(x)/D_1]\,osc(g,[y_{l+2},y_{l-1}]). \qquad (4.18)$$

This is the aperiodic version of a corresponding estimate due to Bojanic and Waterman [10] for periodic functions. One may use this to study the convergence of $U(g,x)$ when g is in the class ΛBV introduced by Waterman (cf. [21]). We will, however, proceed in a different direction. The argument in the sequel is similar to the one in [21], but involves more technical details. We write

$$S_1^* := \sum_{k=0}^{l-2} \frac{osc(g,[y_{k+1},y_k])}{l-k-1} \qquad (4.19a)$$

$$S_2^* := \sum_{k=l+2}^{n} \frac{osc(g,[y_{k+1},y_k])}{k-l-1}. \qquad (4.19b)$$

To estimate S_1^*, we introduce

$$F(t) := \sum_{nt\leqslant m\leqslant l-2} osc(g,[y_{k+1},y_k]). \qquad (4.20)$$

Then F is a decreasing function and, in view of (2.14),

$$F(t) \leqslant (l-1-nt)\,\Omega\left(g, \frac{1}{l-1-nt}, [y_{l-1}, y_{[nt]}]\right). \qquad (4.21)$$

A summation by parts yields that

$$S_1^* = \sum_{k=0}^{l-2} \frac{F(k/n)-F((k+1)/n)}{l-1-k}$$
$$= \frac{F(0)}{l-1} + \sum_{k=0}^{l-2} \frac{F(k/n)}{(l-1-k)(l-k)}$$
$$= \frac{F(0)}{l-1} + \sum_{k=1}^{l-1} \frac{F((l-1-k)/n)}{k(k+1)}$$
$$\leqslant \frac{l+1}{l}\frac{F(0)}{l-1} + 2\sum_{k=1}^{l-2} \frac{F((l-1-k)/n)}{(k+1)^2}. \qquad (4.22)$$

Using (4.21), we see that

$$\frac{F(0)}{l-1} \leqslant \Omega\left(g, \frac{1}{l-1}, [y_{l-2}, 1]\right). \tag{4.23}$$

Moreover, since F is decreasing, we get, using (4.21) again,

$$\begin{aligned} S_{1,1}^* &:= \sum_{k=1}^{l-2} \frac{F((l-1-k)/n)}{(k+1)^2} \\ &\leqslant \frac{1}{n} \int_{1/n}^{(l-1)/n} \frac{F((l-1)/n - t)}{t^2}\, dt \\ &\leqslant \frac{1}{l-1} \sum_{k=1}^{l-2} F\left(\frac{l-1}{n}\left(1 - \frac{1}{k}\right)\right) \\ &\leqslant \sum_{k=1}^{l-2} \frac{1}{k}\, \Omega\left(g, \frac{k}{l-1}, [y_{l-1}, y_{\lfloor l-1-(l-1)/k \rfloor}]\right). \end{aligned} \tag{4.24}$$

Substituting from (4.24) and (4.23) into (4.22), we get

$$S_1^* \leqslant 4 \sum_{k=1}^{l-2} \frac{1}{k}\, \Omega\left(g, \frac{k}{l-1}, [y_{l-1}, y_{\lfloor l-1-(l-1)/k \rfloor}]\right). \tag{4.25}$$

We estimate S_2^* in a similar manner to get

$$S_2^* \leqslant 4 \sum_{k=1}^{n-l-1} \frac{1}{k}\, \Omega\left(g, \frac{k}{n-l}, [y_{\lfloor (n-l)/k \rfloor + l+1}, y_{l+2}]\right). \tag{4.26}$$

Since

$$osc(g, [y_{l+2}, y_{l-1}]) \leqslant \Omega(g, 1, [y_{l+2}, y_{l-1}]), \tag{4.27}$$

Theorem 3.1 is proved in view of (4.25), (4.26), (4.27), (4.24), and (4.18).

5. Applications

Let Π_n denote the class of all polynomials of degree at most n, w be a weight function, and $\{p_n\}$, $\{x_{kn}\}$ be as in (2.7).

First, we prove that under the condition (3.3), s_n is of type B. It is well known that [20]

$$s_n(f, x) = \int_{-1}^{1} f(t)\, K_n(x, t)\, w(t)\, dt, \tag{5.1}$$

where

$$K_n(x, t) := \sum_{k=0}^{n-1} p_k(x)\, p_k(t)$$

$$= \frac{\gamma_{n-1}}{\gamma_n} \cdot \frac{p_n(x)\, p_{n-1}(t) - p_n(t)\, p_{n-1}(x)}{x-t}. \tag{5.2}$$

We therefore, show that s_n is of type B with $\kappa_n = K_n$ and $d\mu_n(t) = w(t)\, dt$.

LEMMA 5.1. *Suppose that* (3.3) *holds. Then* s_n *is of type* B. *For* $x \in (-1, 1)$ *and* n *large enough, we have the following estimates.*

$$A_n(x) \leqslant c\{|p_n(x)| + |p_{n-1}(x)|\} \leqslant c\{w(x)\sqrt{1-x^2}\}^{-1} \tag{5.3a}$$

$$B_{1,n}(x) = B_{2,n}(x) = c\{w(x)\sqrt{1-x^2}\}^{-1} \tag{5.3b}$$

$$M_n(t) = \{w(t)\sqrt{1-t^2}\}^{-1} \tag{5.3c}$$

$$y_{j,n} = \cos(j\pi/(n+1)) \tag{5.3d}$$

$$D_1 = \pi/2. \qquad D_2 = \pi \tag{5.3e}$$

$$C(x) \leqslant (1-x^2)^{-1/2}. \tag{5.3f}$$

If $x \in [y_{l+1,n}, y_{l,n}] \subseteq [y_{n-2,n}, y_{2,n}]$, *then*

$$l - 1 \geqslant \frac{n+1}{\pi}\sqrt{(1-x)/2}, \qquad n - l \geqslant \frac{n+1}{\pi}\sqrt{(1+x)/2} \tag{5.3g}$$

$$\begin{aligned} &y_{\lfloor l-1-(l-1)/k \rfloor} \leqslant x + c(1-x)/k, \qquad 1 \leqslant k \leqslant l-2 \\ &y_{\lfloor (n-l)/k \rfloor + l + 1} \geqslant x - c(1+x)/k, \qquad l+2 \leqslant k \leqslant n. \end{aligned} \tag{5.3h}$$

Proof of Lemma 5.1. The estimate (5.3a) is proved in [5] (cf. also [17]). The rest of the estimates involve only elementary computations using (3.3), (5.2), which we omit. ∎

In view of Lemma 5.1, Theorem 3.2 follows as a simple application of Theorem 3.1 and Proposition 2.3.

Next, we show that the Lagrange interpolation operators L_n are of type B.

LEMMA 5.2. *Let* w *satisfy the conditions* (3.6). *Then* L_n *is of type* B *and we have the following estimates.*

$$A_n(x) \leqslant c\, |p_n(x)| \tag{5.4a}$$

$$B_{1,n}(x) \leqslant cW(x)^{-1/2} \tag{5.4b}$$

$$B_{2,n}(x) \leqslant c\,|p_n(x)| \tag{5.4c}$$

$$M_n(t) = 1 \tag{5.4d}$$

$$y_{0,n} = 1, \qquad y_{j,n} = x_{jn}, \qquad j = 1, ..., n, \qquad y_{n+1,n} = -1 \tag{5.4e}$$

$$D_1 = 1, \qquad D_2 = 3 \tag{5.4f}$$

$$C(x) \leqslant c(1-x^2)^{-1/2}. \tag{5.4g}$$

If $x \in [y_{l+1,n}, y_{l,n}] \subseteq [y_{n-2,n}, y_{2,n}]$, *then*

$$l - 1 \geqslant cn\sqrt{1-x}, \qquad n - l \geqslant cn\sqrt{1+x} \tag{5.4h}$$

$$\begin{aligned} y_{\lfloor l-1-(l-1)/k \rfloor} &\leqslant x + c(1-x)/k, \qquad 1 \leqslant k \leqslant l-2 \\ y_{\lfloor (n-l)/k \rfloor + l + 1} &\geqslant x - c(1+x)/k, \qquad l+2 \leqslant k \leqslant n. \end{aligned} \tag{5.4i}$$

Proof of Lemma 5.2. It is well known [20] that

$$L_n(f, x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x) = \int_{-1}^{1} f(t)\, \bar{K}_n(x, t)\, d\mu_n(t), \tag{5.5}$$

where

$$l_{kn}(x) := p_n(x)/\{(x - x_{kn})\, p_n'(x_{kn})\} \tag{5.6}$$

$$\bar{K}_n(x, t) = \begin{cases} n l_{kn}(x) & \text{if} \quad t = x_{kn}, \quad k = 1, ..., n \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

and for Borel subsets B of $[-1, 1]$,

$$\mu_n(B) := \frac{1}{n}\,(\text{number of } x_{kn}\text{'s in } B). \tag{5.8}$$

Property (P1) in Definition 2.1 is immediate. We prove property (P2). An alternative expression for l_{kn} is given by [14]

$$l_{kn} = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)}{x - x_{kn}} \lambda_{kn}\, p_{n-1}(x_{kn}) = \lambda_{kn} K_n(x, x_{kn}). \tag{5.9}$$

We note that, since w is supported on $[-1, 1]$, (cf. [14])

$$\frac{\gamma_{n-1}}{\gamma_n} \leqslant 1. \tag{5.10}$$

Using the Cauchy–Schwartz inequality and (3.6), we see that

$$\begin{aligned} l_{kn}^2(x) &\leqslant \lambda_{kn}^2 K_n(x, x_{kn})^2 \leqslant \lambda_{kn}[\lambda_n(x)]^{-1} \\ &\leqslant cW(x)^{-1}. \end{aligned} \tag{5.11}$$

Thus,

$$|\bar{K}_n(x, t)| \leqslant cnW(x)^{-1} \tag{5.12a}$$

Also, using (5.10) and (3.6b), we see that

$$|\bar{K}_n(x, t)| \leqslant \frac{c}{n\,|x-t|}\,|p_n(x)|. \tag{5.12b}$$

Property (P2) with the estimates (5.4b), (5.4c), and (5.4d) follows from (5.12). The estimates (5.4e) to (5.4i) and the properties (P3) and (P4) now follow by making a few simple calculations.

The verification of property (P5) in Definition 2.1 is perhaps the most difficult. We first estimate

$$\begin{aligned} A_k &:= \sum_{m=k}^{n} \lambda_{mn}\, p_{n-1}(x_{mn}) \\ &= \sum_{m=1}^{n} \lambda_{mn}\, \Gamma_k(x_{mn})\, p_{n-1}(x_{mn}), \end{aligned} \tag{5.13}$$

where

$$\Gamma_k(t) := \begin{cases} 1 & \text{if} \quad x_{kn} \leqslant t \leqslant 1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.14}$$

Freud [15] has shown that there exist polynomials Φ and $\phi \in \Pi_{[n/4]}$ such that

$$\phi(t) \leqslant \Gamma_k(t) \leqslant \Phi(t), \qquad t \in [-1, 1] \tag{5.15a}$$

$$\int_{-1}^{1} (\Phi(t) - \phi(t))(1-t^2)^{-1/2}\, dt \leqslant c/n. \tag{5.15b}$$

Using the quadrature formula [14] and the orthogonality of p_{n-1}, we see that

$$A_k = \sum_{m=1}^{n} \lambda_{mn} (\Gamma_k(x_{mn}) - \phi(x_{mn}))\, p_{n-1}(x_{mn}). \tag{5.16}$$

Using (5.15a), (3.6b), (3.6c), we now get

$$|A_k| \leqslant c \sum_{m=1}^{n} T(\theta_{mn})(\theta_{m+1,n} - \theta_{mn}), \tag{5.17a}$$

where

$$T(\theta) = \Phi(\cos\theta) - \phi(\cos\theta) \tag{5.17b}$$

is a trigonometric polynomial of order not exceeding $n/4$. Now, from the formula

$$\int_a^\theta T(t)\,dt = T(a)(\theta - a) + \int_a^\theta (\theta - t)\,T'(t)\,dt \tag{5.18}$$

we get, using (3.6c) and Bernstein's inequality,

$$\begin{aligned}
\left|\int_0^\pi T(t)\,dt - \sum_{m=1}^{n} T(\theta_{mn})(\theta_{m+1,n} - \theta_{mn})\right|
&\leqslant \sum_{m=1}^{n} \left|\int_{\theta_{mn}}^{\theta_{m+1,n}} (\theta_{m+1,n} - t)\,T'(t)\,dt\right| \\
&\leqslant \frac{1}{n}\int_0^\pi |T'(t)|\,dt \\
&\leqslant c\int_0^\pi |T(t)|\,dt.
\end{aligned} \tag{5.19}$$

We then substitute from (5.19) into (5.17a) and use (5.15) to get

$$|A_k| \leqslant \frac{c}{n}, \qquad k = 1, ..., n. \tag{5.20}$$

Now, let m be any integer such that $x_{mn} < x$. Then,

$$\begin{aligned}
&\left|\sum_{k=m}^{n} \lambda_{kn}\, p_{n-1}(x_{kn})(x - x_{kn})^{-1}\right| \\
&\quad = \left|\sum_{k=m}^{n} (A_k - A_{k+1})(x - x_{kn})^{-1}\right| \\
&\quad = \left|\frac{A_m}{x - x_{mn}} + \sum_{k=m+1}^{n} A_k\left\{\frac{1}{x - x_{kn}} - \frac{1}{x - x_{k-1,n}}\right\}\right|,
\end{aligned} \tag{5.21}$$

where $A_{n+1} := 0$. In view of (5.20), (5.21) implies that

$$\left|\sum_{k=m}^{n} \lambda_{kn}\, p_{n-1}(x_{kn})(x - x_{kn})^{-1}\right| \leqslant \frac{c}{n(x - x_{mn})} \qquad (x_{mn} < x). \tag{5.22}$$

Using the expression (5.9) in (5.7), it is now easy to see that (2.5a) is satisfied with A_n given by (5.4a). The estimate (2.5b) can be verified in exactly the same way.

In order to prove Theorem 3.5, we need to recall certain facts about the Hermite–Fejér interpolation process (cf. [19 and the references therein]).

We write λ'_{kn} for $\lambda'_n(x_{kn})$ where λ_n is defined in (3.5), and w_{kn} for $w(x_{kn})$. Then, it is known that

$$H_n(g, x) = \sum_{k=1}^{n} g(x_{kn})\, v_{kn}(x)\, l_{kn}^2(x), \tag{5.23a}$$

where l_{kn} is given in (5.9) and

$$v_{kn}(x) = 1 - p_n''(x_{kn})[p_n'(x_{kn})]^{-1}(x - x_{kn}) = 1 + \lambda'_{kn}\lambda_{kn}^{-1}(x - x_{kn}). \tag{5.23b}$$

Thus, if we let

$$V_n(x, t) = \begin{cases} n v_{kn}(x)\, l_{kn}^2(x) & \text{if } \; x \in [-1, 1], \quad t = x_{kn} \\ 0 & \text{otherwise} \end{cases} \tag{5.24}$$

and μ_n be the measure defined in (5.8), then

$$H_n(g, x) = \int_{-1}^{1} g(t)\, V_n(x, t)\, d\mu_n(t). \tag{5.25}$$

In order to prove that H_n is of type B, we need the following estimates valid for the generalized Jacobi weights [19]. (Here, and in the sequel, $A \sim B$ means that $c_1 A \leqslant B \leqslant c_2 A$). Let $x_{kn} =: \cos\theta_{kn}$, $\theta_{m+1,n} = \pi$, $\theta_{0n} = 0$. Then,

$$\theta_{k+1,n} - \theta_{k,n} \sim 1/n. \tag{5.26}$$

If l is the index of the zero x_{kn} which is (one of the) closest to x, then

$$|p_n(x)| \sim n\,|x - x_{ln}|\,[w(x)(1-x^2)^{3/2}]^{-1/2} \tag{5.27}$$

$$|p_{n-1}(x_{kn})| \sim w_{kn}^{-1/2}(1 - x_{kn}^2)^{1/4} \tag{5.28}$$

$$\lambda_{kn} \sim n^{-1} w_{kn}(1 - x_{kn}^2)^{1/2} \tag{5.29}$$

$$|\lambda'_{kn}| \leqslant c n^{-1} w_{kn}(1 - x_{kn}^2)^{-1/2} \tag{5.30}$$

$$\begin{aligned} &|v_{kn}(x) - 1 - (1 - x_{kn}^2)^{-1}[\alpha - \beta + (\alpha + \beta + 2)\, x_{kn}](x - x_{kn})| \\ &\quad \leqslant c\,|x - x_{kn}|, \qquad k = 1, ..., n, \quad x \in [-1, 1]. \end{aligned} \tag{5.31}$$

In particular,

$$|v_{kn}(x)| \leqslant c(1 - x_{kn}^2)^{-1}, \qquad k = 1, ..., n. \tag{5.32}$$

We note also that if $t \in (x_{k+1,n}, x_{kn}]$, and $p \in \mathbf{R}$ then

$$w(t)(1 - t^2)^p \sim w_{kn}(1 - x_{kn}^2)^p \sim w_{k+1,n}(1 - x_{k+1,n}^2)^p. \tag{5.33}$$

The following lemma summarizes the estimates needed to prove that H_n is of type B.

LEMMA 5.3. *Let w be a generalized Jacobi weight (cf. Definition 3.4). Then H_n is of type B, and we have the following estimates.*

$$A_n(x) \leqslant c(x)\, p_n^2(x) \tag{5.34a}$$

$$B_{1,n}(x) \leqslant c(1-x^2)^{-1} \tag{5.34b}$$

$$B_{2,n}(x) \leqslant c(1-x^2)^{-1/2}\, p_n^2(x). \tag{5.34c}$$

Moreover, the estimates (5.4d) *to* (5.4i) *are also valid.*

Proof of Lemma 5.3. We choose $\kappa_n = V_n$ where V_n is defined in (5.24) and μ_n to be the measure as in (5.8). Property (P1) in the Definition 2.1 is obvious. If x_{ln} is (one of the) closest zero to x, then, using (in sequence) (5.32), (5.9), (5.10), (5.29), (5.28), (5.27), and (5.33) we see that

$$\begin{aligned} |v_{kn}(x)\, l_{kn}^2(x)| &\leqslant \frac{c}{1-x_{ln}^2}\left[\frac{\gamma_{n-1}}{\gamma_n}\lambda_{kn}\, p_{n-1}(x_{kn})\right]^2 \frac{p_n^2(x)}{(x-x_{ln})^2} \\ &\leqslant c(1-x_{ln}^2)^{-1}\, n^{-2} w_{ln}(1-x_{ln}^2)^{3/2}\, p_n^2(x)(x-x_{ln})^{-2} \\ &\leqslant cn^{-2} w_{ln}(1-x_{ln}^2)^{1/2}\, w^{-1}(x)(1-x^2)^{-3/2}\, n^2 \\ &\leqslant c(1-x^2)^{-1}. \end{aligned} \tag{5.35}$$

If x_{kn} is not the closest zero to x, then an easy computation using (5.26) yields that

$$n\,|x-x_{kn}| \geqslant \sqrt{1-x^2} \tag{5.36a}$$

$$n\sqrt{1-x_{kn}^2} \geqslant c. \tag{5.36b}$$

Also, in view of (5.28), (5.29), and (5.30),

$$\lambda_{kn}\,|p_{n-1}(x_{kn})| \sim n^{-1}\{w_{kn}^{1/2}(1-x_{kn}^2)^{3/4}\} \tag{5.37}$$

$$|\lambda'_{kn}\lambda_{kn}^{-1}| \leqslant c(1-x_{kn}^2)^{-1}. \tag{5.38}$$

Using (5.9), (5.10), (5.37), (5.36a), we get

$$\begin{aligned} l_{kn}^2(x) &\leqslant c\,\frac{p_n^2(x)}{n^2(x-x_{kn})^2}\, w_{kn}(1-x_{kn}^2)^{3/2} \\ &\leqslant c(1-x^2)^{-1/2}\, p_n^2(x)[n\,|x-x_{kn}|]^{-1}. \end{aligned} \tag{5.39}$$

Similarly, using (5.38), (5.36b),

$$\begin{aligned} |\lambda'_{kn}\lambda_{kn}^{-1}(x-x_{kn})\, l_{kn}^2(x)| &\leqslant cp_n^2(x)[n\,|x-x_{kn}|]^{-1}\, n^{-1} w_{kn}(1-x_{kn}^2)^{1/2} \\ &\leqslant cp_n^2(x)[n\,|x-x_{kn}|]^{-1}\, w_{kn}(1-x_{kn}^2) \\ &\leqslant cp_n^2(x)[n\,|x-x_{kn}|]^{-1}. \end{aligned} \tag{5.40}$$

In view of (5.23b), (5.39), and (5.40),

$$|v_{kn}(x)\, l_{kn}^2(x)| \leqslant c(1-x^2)^{-1/2}\, p_n^2(x)[n\,|x-x_{kn}|]^{-1} \tag{5.41}$$

if x_{kn} is not the closest zero to x. Property (P2) in Definition 2.1 and the estimates (5.34b), (5.34c) follow from (5.35), (5.41). Properties (P3) and (P4) are also now evident, as in the proof of Lemma 5.2. To prove the property (P5), let $x \in [x_{l+1,n}, x_{ln}]$ and $-1 \leqslant x_{mn} \leqslant t < x_{m-1,n} \leqslant x_{l+2,n}$. Then, using (5.32), (5.37), (5.29), (5.9), (5.10), we get

$$\begin{aligned}
&\left|\int_{-1}^{t} V_n(x,u)\, d\mu_n(u)\right| \\
&\quad\leqslant \sum_{k=m}^{n} |v_{kn}(x)\, l_{kn}^2(x)| \\
&\quad\leqslant c \sum_{k=m}^{n} (1-x_{kn}^2)^{-1} \frac{p_n^2(x)}{(x-x_{kn})^2}\, n^{-2} w_{kn} (1-x_{kn}^2)^{3/2} \\
&\quad\leqslant c\,\frac{p_n^2(x)}{n} \sum_{k=m}^{n} \frac{\lambda_{kn}}{(x-x_{kn})^2}.
\end{aligned} \tag{5.42}$$

It is elementary to check that if $x_{k+1,n} \leqslant u \leqslant x_{k-1,n}$, then

$$\left|\frac{x-u}{x-x_{kn}}\right| \leqslant c(1-x^2)^{-1/2}. \tag{5.43}$$

Moreover, the Markov–Stieltjes inequalities [14] yield

$$\lambda_{kn} \leqslant \int_{x_{k+1,n}}^{x_{k-1,n}} w(u)\, du. \tag{5.44}$$

In view of (5.43) and (5.44),

$$\begin{aligned}
&\sum_{k=m}^{n} \frac{\lambda_{kn}}{(x-x_{kn})^2} \\
&\quad\leqslant c(1-x^2)^{-1} \int_{-1}^{x_{m-1,n}} \frac{w(u)}{(x-u)^2}\, du \\
&\quad\leqslant c(1-x^2)^{-1} \left[\left\{\int_{-1}^{(x-1)/2} + \int_{(x-1)/2}^{x_{m-1,n}}\right\} \frac{w(u)}{(x-u)^2}\, du\right] \\
&\quad\leqslant c(x) + c_1(x) \int_{x-x_{m-1,n}}^{\infty} v^{-2}\, dv \\
&\quad\leqslant c(x)/(x-x_{m-1,n}) \\
&\quad\leqslant c(x)/(x-x_{m,n}) \\
&\quad\leqslant c(x)/(x-t).
\end{aligned} \tag{5.45}$$

Substituting from (5.45) into (5.42), we get (2.5a) with $A_n(x)$ as in (5.34a). The estimate (2.5b) is proved similarly. ∎

Theorem 3.5 follows from Theorem 3.1 and Lemma 5.3 after a few simple computations involving Proposition 2.3.

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