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Existence of Multiple Solutions for Second-Order Discrete Boundary Value Problems

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Abstract—We give conditions on f involving pairs of discrete lower and discrete upper solutions which lead to the existence of at least three solutions of the discrete two-point boundary value problem $y_{k+1} - 2y_k + y_{k-1} + f(k, y_k, v_k) = 0$, for $k = 1, \dots, n-1$, $y_0 = 0 = y_n$, where f is continuous and $v_k = y_k - y_{k-1}$, for $k = 1, \dots, n$. In the special case $f(k, t, p) = f(t) \geq 0$, we give growth conditions on f and apply our general result to show the existence of three positive solutions. We give an example showing this latter result is sharp. Our results extend those of Avery and Peterson and are in the spirit of our results for the continuous analogue.

Keywords—Brouwer degree, Discrete two-point boundary value problems, Discrete lower solutions, Discrete upper solutions.

1. INTRODUCTION

In this paper, we consider two-point boundary value problems for second-order difference equations of the form

$$\Delta^2 y_{k+1} + f(k, y_k, v_k) = 0, \qquad \text{for } k = 1, \dots, n-1, \tag{1.1}$$

$$y_0 = 0 = y_n, \tag{1.2}$$

where $f : \{1, \dots, n-1\} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, $\Delta^2 y_{k+1} = y_{k+1} - 2y_k + y_{k-1}$, for $k = 1, \dots, n-1$, and $v_k = y_k - y_{k-1}$, for $k = 1, \dots, n$.

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We also consider the special case of (1.1)

$$\Delta^2 y_{k+1} + f(y_k) = 0, \qquad \text{for } k = 1, \dots, n-1, \tag{1.3}$$

where $f \geq 0$.

We give sufficient conditions on f for these boundary value problems to have three solutions; by a solution y of (1.1), we mean a vector $y = (y_0, \dots, y_n)$ satisfying (1.1) for $k = 1, \dots, n-1$.

Leggett and Williams [1] developed a fixed-point theorem using the fixed-point index in ordered Banach spaces. They applied their fixed-point theorem to prove existence of three positive solutions for Hammerstein integral equations of the form $y = \int_\Omega G(x,s) f(s, y(s))\, ds$, $\Omega \subset \mathbb{R}^n$, by making use of suitable inequalities they imposed on the kernel G and on f. Green's functions for differential operators closely related to our problem satisfy these inequalities. Avery [2] used the Leggett and Williams approach to study problem (1.3),(1.2). Sun and Sun [3] gave an extension of the Leggett-Williams multiple fixed-point theorem on ordered Banach spaces. They also used the fixed-point index in ordered Banach spaces but gave no applications to differential equations.

Motivated by the papers of Leggett and Williams and of Sun and Sun, Anderson [4] applied the integral equation approach to a third-order problem $-x'''(t) + f(x(t)) = 0$, $x(0) = x'(0) = x''(1) = 0$. Again the Green's function satisfies inequalities similar to those in [1]. In [5], we used this approach to study the n^{th}-order equation $y^{(n)} + f(y) = 0$, together with the boundary conditions $y^{(i)}(0) = 0 = y(1)$, for $i = 0, \dots, n-2$, and also with the boundary conditions $y^{(i)}(0) = 0 = y^{(n-2)}(1)$, for $i = 0, \dots, n-2$.

In the current work, we show there are three solutions if there exist two discrete lower solutions α_1 and α_2 and two discrete upper solutions β_1 and β_2 for problem (1.1),(1.2) satisfying $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$. In the special case $f(k,t,p) = f(t) \geq 0$, we give growth conditions on f which guarantee the existence of three positive solutions. We give an example showing this latter result is sharp.

We follow the approach we adopted in [6] for the continuous analogue of problem (1.1),(1.2) and of problem (1.3),(1.2). There we modified f for y outside of $[\alpha_1, \beta_2]$ and formulated the modified continuous analogues as integral equations. We used Schauder degree on a suitable open set in function space to show there are three solutions. Here we proceed similarly modifying f for t outside of $[\alpha_1, \beta_2]$ and formulate the modified problem (1.1),(1.2) as a summation equation. We use Brouwer degree on suitable open set in $\mathbb{R}^{n+1}$ to show there are three solutions. In the special case of problem (1.3),(1.2), we construct discrete lower and discrete upper solutions and apply our general result to show there are three positive solutions. A novel feature of our work is that we do not require that $\beta_1 \leq \alpha_2$ on $\{0, \dots, n\}$. Further, we use Brouwer degree theory rather than the Leggett-Williams or the Sun-Sun fixed-point theorems and allow the right-hand side to depend on k and v_k. Moreover, it would have been possible to give a proof modelled on that in [7, Example 2.4.2] which uses [7, Corollary 2.4.2] and is based on monotone mappings in ordered Banach spaces. Using this approach, we could not allow f to depend on v_k, although we could allow it to depend on k. On the other hand, the monotone mappings approach has the advantage of providing a convergent sequence of approximate solutions which provide reasonable accuracy.

Our results extend those of Avery and Peterson [8] who studied problem (1.3),(1.2) using the Sun-Sun fixed-point theorem. For more information on multiple solutions of problem (1.1),(1.2), its continuous analogue, and related results, see [2,7,8] and the references therein. For more information on difference equations, see the books by Agarwal [9], Elaydi [10], and Kelly and Peterson [11], and the references therein.

2. BACKGROUND NOTATION AND DEFINITIONS

In order to state our results, we need some notation.

We denote the closure of a set T by $\bar{T}$ and its boundary by ∂T. As usual, $C^m(A; B)$ denotes the space of m times continuously differentiable functions from A to B endowed with the maximum norm. In the case of continuous functions, we abbreviate this to $C(A; B)$. In the case $B = \mathbb{R}$, we omit the B. Let $\mathcal{N}$ be the nonnegative integers. If $n \in \mathcal{N}$ and $I \subseteq \mathbb{R}$ is an interval, then by $n \in I$ we mean $n \in I \cap \mathcal{N}$. We let $\mathcal{N}_{i,j} = \{k \in \mathcal{N} : i \leq k \leq j\}$ and $\mathcal{N}_n = \mathcal{N}_{0,n}$. For any vector $s = (s_0, \dots, s_n) \in \mathbb{R}^{n+1}$, we set $\overline{s} = \max\{s_k : k \in \mathcal{N}_n\}$, $\underline{s} = \min\{s_k : k \in \mathcal{N}_n\}$, $\Delta s_k = s_k - s_{k-1}$, for $k \in \mathcal{N}_{1,n}$, and $\Delta^2 s_{k+1} = \Delta(\Delta s_{k+1}) = s_{k+1} - 2s_k + s_{k-1}$, for $k \in \mathcal{N}_{1,n-1}$. We write $s \leq z$ if $s_k \leq z_k$, for $k \in \mathcal{N}_n$, where $z = (z_0, \dots, z_n) \in \mathbb{R}^{n+1}$. If $c \in \mathbb{R}$ is a constant, then we identify c and $(c_0, \dots, c_n)$, where $c_k = c$, for all $k \in \mathcal{N}_n$, and the meaning is clear from the context. For $y \in \mathbb{R}^{n+1}$ we define the maximum norm, $\|y\|$, by $\|y\| = \max\{|y_k| : y = (y_0, \dots, y_n),\ k \in \mathcal{N}_n\}$. Let $\mathbb{R}_0^{n+1} = \{y \in \mathbb{R}^{n+1} : y = (0, y_1, \dots, y_{n-1}, 0)\}$. If A is a bounded open subset of $\mathbb{R}^{n+1}$, $p \in \mathbb{R}^{n+1}$, $F \in C(A; \mathbb{R}^{n+1})$, and $p \notin F(\partial A)$, we denote the Brouwer degree of F on A at p by $d(F, A, p)$.

It is common in the proofs of existence of solutions of two-point boundary value problems for (1.1) to modify f. We will do this making use of the following functions (see [6]).

If $c \leq d$ are given, let $\pi : \mathbb{R} \to [c, d]$ be the retraction given by

$$\pi(y, c, d) = \max\{\min\{d, y\}, c\}. \tag{2.1}$$

For each $\epsilon > 0$, let $K \in C(\mathbb{R})$ satisfy

(i) $tK(t, \epsilon) < 0$, for all $t \neq 0$,
(ii) $K(0, \epsilon) = 0$, and
(iii) $|K(t, \epsilon)| \leq \epsilon$ for all t.

If $c \leq d$ and $\epsilon > 0$ are given, let $T \in C(\mathbb{R})$ be given by

$$T(y, c, d, \epsilon) = K(y - \pi(y, c, d), \epsilon). \tag{2.2}$$

We will need the Greens function for the problem

$$\Delta^2 y_{k+1} + g_k = 0, \qquad \text{for } k \in \mathcal{N}_{1,n-1}, \tag{2.3}$$

$$y_0 = 0, \qquad y_n = 0, \tag{2.4}$$

where $g = (g_1, \dots, g_{n-1}) \in \mathbb{R}^{n-1}$ and the solution $y \in \mathbb{R}^{n+1}$. The Greens function, $\mathcal{Q} : \mathcal{N}_n \times \mathcal{N}_{1,n-1} \to \mathbb{R}$, is given by

$$\mathcal{Q}(k, i) = \begin{cases} (n-k)\dfrac{i}{n}, & \text{for all } 0 \leq i \leq k \leq n, \\ (n-i)\dfrac{k}{n}, & \text{for all } 0 \leq k \leq i \leq n. \end{cases}$$

Thus, we define $\mathcal{C} : \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$ by

$$\mathcal{C}(\phi)_k = \sum_{i=1}^{n-1} \mathcal{Q}(k, i)\phi_i, \qquad \text{for } k \in \mathcal{N}_n, \tag{2.5}$$

for all $\phi \in \mathbb{R}^{n-1}$. Thus, y is a solution of problem (2.3),(2.4) if and only if

$$y - \mathcal{C}(g) = 0. \tag{2.6}$$

Moreover, $\mathcal{C}$ is continuous.

DEFINITION 2.1. *We call α a strict discrete lower solution for (1.1) if there is $\gamma > 0$ such that*

$$\Delta^2 \alpha_{k+1} + f(k, \alpha_k, u) \geq \gamma, \tag{2.7}$$

for all $k \in \mathcal{N}_{1,n-1}$ and $u \leq u_k$, where $u_k = \Delta\alpha_k$ for all $k \in \mathcal{N}_{1,n}$.

Similarly, we call β a strict discrete upper solution for (1.1) if we replace (2.7) by

$$\Delta^2 \beta_{k+1} + f(k, \beta_k, w) \leq -\gamma, \tag{2.8}$$

for all $k \in \mathcal{N}_{1,n-1}$ and $w \leq w_k$, where $w_k = \Delta\beta_k$, for all $k \in \mathcal{N}_{1,n}$.

We say α (β) is a strict discrete lower (a strict discrete upper) solution for (1.1),(1.2) if in addition $\alpha_0 \le 0$ and $\alpha_n \le 0$ ($\beta_0 \ge 0$ and $\beta_n \ge 0$). If $\gamma = 0$, we omit the word "strict".

If there exist a discrete lower solution, α, and discrete upper solution, β, for (1.1),(1.2) satisfying $\alpha \le \beta$, then we define $\overline{\beta}$ and $\underline{\alpha}$ by

$$\overline{\beta} = \max\{\beta_k : k \in \mathcal{N}_n\}, \qquad \text{and}$$
$$\underline{\alpha} = \min\{\alpha_k : k \in \mathcal{N}_n\},$$

respectively.

We need the discrete maximum principle and the remark following it; we include them for clarity.

THEOREM 2.2. *Let $f : \mathcal{N}_{1,n-1} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy*

(i) $f(k, \alpha_k, u) < f(k, t, u)$, *for all* $t < \alpha_k$, $u \le u_k$,
(ii) $f(k, \beta_k, u) > f(k, t, u)$, *for all* $t > \beta_k$, $u \ge w_k$,

where α is a strict discrete lower solution and β is a strict discrete upper solution for problem (1.1),(1.2) satisfying $\alpha \le \beta$, $u_k = \Delta\alpha_k$, and $w_k = \Delta\beta_k$, for $k \in \mathcal{N}_{1,n-1}$. If y is a solution of problem (1.1),(1.2), then $\alpha \le y \le \beta$ on $\mathcal{N}_n$.

PROOF. Suppose that y is a solution of (1.1),(1.2). We show that $\alpha \le y \le \beta$. Suppose, for example, that $y_j < \alpha_j$ for some $j \in \mathcal{N}_{1,n-1}$. From the boundary conditions, we may assume that $\alpha - y$ attains its positive maximum at $k \in \mathcal{N}_{1,n-1}$. Thus, $u_k = \Delta\alpha_k \ge \Delta y_k = v_k$, and $u_{k+1} = \Delta\alpha_{k+1} \le \Delta y_{k+1} = v_{k+1}$, so that $u_k - v_k \ge 0$, $u_{k+1} - v_{k+1} \le 0$, so that $\Delta^2 y_{k+1} = y_{k+1} - 2y_k + y_{k-1} \ge \alpha_{k+1} - 2\alpha_k + \alpha_{k-1} = \Delta^2\alpha_{k+1}$.

Since α is a strict discrete lower solution for (1.1), $y_k < \alpha_k$, and $u_k \ge v_k$, it follows that

$$\begin{aligned}
\Delta^2 y_{k+1} = y_{k+1} - 2y_k + y_{k-1} &= -f(k, y_k, v_k) && (2.9)\\
&< -f(k, \alpha_k, v_k) && (2.10)\\
&\le \alpha_{k+1} - 2\alpha_k + \alpha_{k-1} = \Delta^2\alpha_{k+1}, && (2.11)
\end{aligned}$$

a contradiction. Thus, $\alpha \le y$. Similarly, $y \le \beta$, and the result follows.

By an almost identical proof, we have the following remark.

REMARK 2.3. Let $f : \mathcal{N}_{1,n-1} \times \mathbb{R}^2 \to \mathbb{R}$, and let y be a solution of problem (1.1),(1.2). If α is a strict discrete lower solution for problem (1.1),(1.2) with $\alpha \le y$ on $\mathcal{N}_n$, then $\alpha < y$ on $\mathcal{N}_{1,n-1}$. Similarly, if β is a strict upper solution of problem (1.1),(1.2) with $y \le \beta$ on $\mathcal{N}_n$, then $y < \beta$ on $\mathcal{N}_{1,n-1}$.

3. EXISTENCE OF SOLUTIONS

THEOREM 3.1. *Let $f : \mathcal{N}_{1,n-1} \times \mathbb{R}^2 \to \mathbb{R}$ and assume that there exist two strict discrete lower solutions α_1 and α_2 and two strict discrete upper solutions β_1 and β_2 for problem (1.1),(1.2) satisfying*

(i) $\alpha_1 \le \alpha_2 \le \beta_2$,
(ii) $\alpha_1 \le \beta_1 \le \beta_2$, *and*
(iii) $\alpha_2 \not\le \beta_1$.

Then problem (1.1),(1.2) has at least three solutions x, y, and z satisfying $\alpha_1 \le x \le \beta_1$, $\alpha_2 \le y \le \beta_2$, and $z \not\le \beta_1$ and $z \not\ge \alpha_2$.

PROOF. We modify f for y not between α_1 and β_2 to obtain a second difference equation and reformulate the new problem as a summation equation. We show that solutions of the modified problem lie in the region where f is unmodified and hence are solutions of our problem. We use Brouwer degree theory to prove existence of three solutions for the modified problem and compute the required Brouwer degrees using a homotopy and further modifications.

Let $L = \overline{\beta}_2 - \underline{\alpha}_1 + 3$ so that $L > \max\{|\Delta\alpha_{i,k}|, |\Delta\beta_{i,k}| : k \in \mathcal{N}_{1,n},\ i = 1,2\}$. Let

$$g(k, y_k, p) = f(k, \pi(y_k, \alpha_{1,k}, \beta_{2,k}), \pi(p, -L, L)) + T(y, \alpha_{1,k}, \beta_{2,k}, 1), \tag{3.1}$$

where π and T are given by (2.1) and (2.2), respectively. Thus, $k : \mathcal{N}_{1,n-1} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$g(k,t,p) > f(k, \alpha_{1,k}, \pi(p,-L,L)), \qquad \text{for } t < \alpha_{1,k}, \quad p \in \mathbb{R}, \tag{3.2}$$

$$g(k,t,p) < f(k, \beta_{2,k}, \pi(p,-L,L)), \qquad \text{for } t > \beta_{2,k}, \quad p \in \mathbb{R}, \quad \text{and} \tag{3.3}$$

$$|g(k,t,p)| \le M, \qquad \text{for } (k,t,p) \in \mathcal{N}_{1,n-1} \times \mathbb{R}^2, \tag{3.4}$$

and some constant M. Moreover, we may choose M so that $|\alpha_1|, |\beta_2| < M$ on $\mathcal{N}_n$.

Consider

$$y_{k+1} - 2y_k + y_{k-1} + g(k, y_k, v_k) = 0, \qquad \text{for all } k \in \mathcal{N}_{1,n-1}, \tag{3.5}$$

together with (1.2). It suffices to show that problem (3.5),(1.2) has three solutions x, y, and z satisfying $\alpha_1 \le x \le \beta_1$, $\alpha_2 \le y \le \beta_2$, and $z \not\le \beta_1$, $z \not\ge \alpha_2$ and $\alpha_1 \le z \le \beta_2$ and $|\Delta x_k|$, $|\Delta y_k|$, $|\Delta z_k| \le L$ for all $k \in \mathcal{N}_{1,n}$, since f and g coincide in this region.

Suppose that y is a solution of (3.5),(1.2). We show that y is a solution of (1.1). It suffices to show that $\alpha_1 \le y \le \beta_2$ and that $|\Delta y_k| < L$, for all $k = 1, \dots, n$.

From (3.2) and (3.3), it follows that g satisfies the assumptions of Theorem 2.2 with $\alpha_1 = \alpha$ and $\beta_2 = \beta$. It follows that $\alpha_1 \le y \le \beta_2$. Thus, $|\Delta y_k| = |y_k - y_{k-1}| \le \overline{\beta}_2 - \underline{\alpha}_1 < L$ for all $k \in \mathcal{N}_{1,n}$, so that y is the required solution. Similarly, x and z satisfy $\alpha_1 \le x$, $z \le \beta_2$ on $\mathcal{N}_n$.

Let $\mathbf{\Omega} = \{y \in \mathbb{R}_0^{n+1} : \|y\| < n^2M + L\}$ and define $\mathcal{K} : \mathbb{R}^{n+1} \to \mathbb{R}^{n-1}$ at $k \in \mathcal{N}_{1,n-1}$ by

$$\mathcal{K}(\phi)_k = g(k, \phi_k, \Delta\phi_k).$$

Thus, $y \in \mathbb{R}^{n+1}$ is a solution of (3.5) and (1.2) iff $(I - \mathcal{CK})(y) = 0$. Moreover, it is easy to see that $\mathcal{CK}(\bar{\mathbf{\Omega}}) \subset \mathbf{\Omega}$. Thus, $d(I - \mathcal{CK}, \mathbf{\Omega}, 0) = 1$.

Let $\mathbf{\Omega}_{\alpha_2} = \{y \in \mathbf{\Omega} : y > \alpha_2 \text{ on } \mathcal{N}_{1,n-1}\}$ and $\mathbf{\Omega}^{\beta_1} = \{y \in \mathbf{\Omega} : y < \beta_1 \text{ on } \mathcal{N}_{1,n-1}\}$. Since $\alpha_2 \not\le \beta_1$, $\alpha_2 > -M$, and $\beta_1 < M$, it follows that $\mathbf{\Omega}^{\beta_1} \ne \emptyset \ne \mathbf{\Omega}_{\alpha_2}$, $\overline{\mathbf{\Omega}}^{\beta_1} \cap \overline{\mathbf{\Omega}}_{\alpha_2} = \emptyset$, and $\mathbf{\Omega} \setminus \overline{\{\mathbf{\Omega}_{\alpha_2} \cup \mathbf{\Omega}^{\beta_1}\}} \ne \emptyset$.

By Remark 2.3, there are no solutions $y \in \partial\mathbf{\Omega}_{\alpha_2} \cup \partial\mathbf{\Omega}^{\beta_1}$. Thus,

$$d(I-\mathcal{CK}, \mathbf{\Omega}, 0) = d\left(I - \mathcal{CK}, \mathbf{\Omega} \setminus \overline{\{\mathbf{\Omega}_{\alpha_2} \cup \mathbf{\Omega}^{\beta_1}\}}, 0\right) + d\left(I - \mathcal{CK}, \mathbf{\Omega}^{\beta_1}, 0\right) + d\left(I - \mathcal{CK}, \mathbf{\Omega}_{\alpha_2}, 0\right). \tag{3.6}$$

We show that $d(I-\mathcal{CK}, \mathbf{\Omega}^{\beta_1}, 0) = d(I-\mathcal{CK}, \mathbf{\Omega}_{\alpha_2}, 0) = 1$. Then $d(I-\mathcal{CK}, \mathbf{\Omega} \setminus \overline{\{\mathbf{\Omega}_{\alpha_2} \cup \mathbf{\Omega}^{\beta_1}\}}, 0) = -1$, and there are solutions in $\mathbf{\Omega} \setminus \overline{\{\mathbf{\Omega}_{\alpha_2} \cup \mathbf{\Omega}^{\beta_1}\}}$, $\mathbf{\Omega}^{\beta_1}$, and $\mathbf{\Omega}_{\alpha_2}$, as required.

We show that $d(I - \mathcal{CK}, \mathbf{\Omega}_{\alpha_2}, 0) = 1$. The proof that $d(I - \mathcal{CK}, \mathbf{\Omega}^{\beta_1}, 0) = 1$ is similar and hence omitted. We define $I - \mathcal{CL}$, the extension to $\bar{\mathbf{\Omega}}$ of the restriction of $I - \mathcal{CK}$ to $\bar{\mathbf{\Omega}}_{\alpha_2}$, as follows.

Let

$$l(k,t,p) = f(k, \pi(t, \alpha_{2,k}, \beta_{2,k}), \pi(p, -L, L)) + T(t, \alpha_{2,k}, \beta_{2,k}, 1), \tag{3.7}$$

for all $(k,t,p) \in \mathcal{N}_{1,n-1} \times \mathbb{R}^2$, where π and T are given by (4) and (5), respectively. Thus, $l : \mathcal{N}_{1,n-1} \times \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$l(k,t,p) > f(k, \alpha_{2,k}, \pi(p,-L,L)), \qquad \text{for } t < \alpha_{2,k}, \quad p \in \mathbb{R}, \tag{3.8}$$

$$l(k,t,p) < f(k, \beta_{2,k}, \pi(p,-L,L)), \qquad \text{for } t > \beta_{2,k}, \quad p \in \mathbb{R}, \quad \text{and} \tag{3.9}$$

$$|l(k,t,p)| \le M, \qquad \text{for } (k,t,p) \in \mathcal{N}_{1,n-1} \times \mathbb{R}^2, \tag{3.10}$$

where M is given above.

Define $\mathcal{L} : \mathbb{R}^{n+1} \to \mathbb{R}^{n-1}$ at $k \in \mathcal{N}_{1,n-1}$ by

$$\mathcal{L}(\phi)_k = l(k, \phi_k, \Delta\phi_k).$$

Thus, $y \in \mathbb{R}^{n+1}$ is a solution of $(I - \mathcal{CL})(y) = 0$ iff y is a solution of

$$y_{k+1} - 2y_k + y_{k-1} + l(k, y_k, v_k) = 0, \qquad \text{for all } k \in \mathcal{N}_{1,n-1}, \tag{3.11}$$

together with (1.2). Arguing as before, y is a solution of (3.11),(1.2) iff $y \in \mathbf{\Omega}_{\alpha_2}$. Thus, $d(I - \mathcal{CL}, \mathbf{\Omega} \setminus \overline{\mathbf{\Omega}_{\alpha_2}}, 0) = 0$. Moreover, it is easy to see that $\mathcal{CL}(\bar{\mathbf{\Omega}}) \subset \mathbf{\Omega}$. Thus, $d(I - \mathcal{CL}, \mathbf{\Omega}, 0) = 1$. Thus,

$$\begin{aligned} d(I - \mathcal{CK}, \mathbf{\Omega}_{\alpha_2}, 0) &= d(I - \mathcal{CL}, \mathbf{\Omega}_{\alpha_2}, 0) \\ &= d\left(I - \mathcal{CL}, \mathbf{\Omega} \setminus \overline{\mathbf{\Omega}}_{\alpha_2}, 0\right) + d(I - \mathcal{CL}, \mathbf{\Omega}_{\alpha_2}, 0) \\ &= d(I - \mathcal{CL}, \mathbf{\Omega}, 0) = 1. \end{aligned}$$

Thus, there are three solutions, as required.

As an application of Theorem 3.1, we have the following generalisation of Avery and Peterson [8].

THEOREM 3.2. *Assume there exist real numbers a, b, c, natural numbers e, n, a nonnegative, continuous function f, and let α_2 be given by*

$$\alpha_{2,k} = \begin{cases} \dfrac{kb}{e}, & k \in \mathcal{N}_{0,e}, \\ \gamma_k, & k \in \mathcal{N}_{e,n-e}, \\ \dfrac{b(n-k)}{e}, & k \in \mathcal{N}_{n-e,n}, \end{cases} \tag{3.12}$$

where $\gamma_k = b(-e^2 + e + nk - k^2)/(e(n - 2e + 1))$, for all $k \in \mathcal{N}_n$. Assume that

(i) $0 < a < b < c$, $0 < e < n/2$;
(ii) $f(y) < 8an^{-2}$ *for all* $y \in [4a(n-1)n^{-2}, a]$, *if n is even, and*
(ii)′ $f(y) < 8a(n^2-1)^{-1}$ *for all* $y \in [4a(n+1)^{-1}, a]$, *if n is odd;*
(iii) $f(y) \geq 2b/(e(n - 2e + 1))$, $y \in [b, b(n^2 - 4e^2 + 4e)/(4e(n - 2e + 1))]$, *if n is even, and*
(iii)′ $f(y) \geq 2b/(e(n - 2e + 1))$, $y \in [b, b(n^2 - 4e^2 + 4e - 1)/(4e(n - 2e + 1))]$, *if n is odd;*
(iv) $f(y) \leq 8cn^{-2}$ *for all* $y \in [4c(n-1)n^{-2}, c]$, *if n is even, and*
(iv)′ $f(y) \leq 8c(n^2-1)^{-1}$ *for all* $y \in [4c(n+1)^{-1}, c]$, *if n is odd;*
(v) α_2 *is not a solution of (1.2),(1.3).*

Then problem (1.3),(1.2) has at least three solutions y_1, y_2, and y_3 satisfying $\|y_1\| < a$, $\alpha_2 \leq y_2$, and $\|y_3\| > a$ and $y_3 \not\geq \alpha_2$.

PROOF. Let $\alpha_1 \equiv 0$ and α_2 be as given above. For even n, let $\beta_{1,k} = 4ak(n-k)n^{-2}$, and $\beta_{2,k} = 4ck(n-k)n^{-2}$, for $k \in \mathcal{N}_n$. For odd n, let $\beta_{1,k} = 4ak(n-k)(n^2-1)^{-1}$, and $\beta_{2,k} = 4ck(n-k)(n^2-1)^{-1}$, for $k \in \mathcal{N}_n$.

Let $u_{i,k} = \Delta\alpha_{i,k}$ and $w_{i,k} = \Delta\beta_{i,k}$, for $i = 1, 2$ and $k \in \mathcal{N}_{1,n}$. It is easy to check that $0 \leq \beta_{1,k} \leq a$ and $0 \leq \beta_{2,k} \leq c$, for $k \in \mathcal{N}_{1,n}$. Moreover, $\Delta^2\beta_{1,k+1} = -8an^{-2}$, and $\Delta^2\beta_{2,k+1} = -8cn^{-2}$, for n even and $k \in \mathcal{N}_{1,n-1}$, while $\Delta^2\beta_{1,k+1} = -8a(n^2-1)^{-1}$, and $\Delta^2\beta_{2,k+1} = -8c(n^2-1)^{-1}$, for n odd and $k \in \mathcal{N}_{1,n-1}$.

It follows that β_1 is a strict discrete upper solution and β_2 is a discrete upper solution for problem (1.3),(1.2) such that $\beta_1 < \beta_2$ on $\mathcal{N}_{1,n-1}$. Now $\gamma_k = kb/e$ for $k \in \{e-1, e\}$ and $\gamma_k = b(n-k)/e$ for $k \in \{n-e, n-e+1\}$ so that α_2 satisfies $\Delta^2\alpha_{2,k+1} = 0 \geq -f(\alpha_{2,k})$, on $\mathcal{N}_{1,e-1} \cup \mathcal{N}_{n-e+1,n-1}$, and $\Delta^2\alpha_{2,k+1} = -2b/(e(n-2e+1)) \geq -f(\alpha_{2,k})$ on $\mathcal{N}_{e,n-e}$ so that α_2 is a discrete lower solution for problem (1.3),(1.2).

We show that $\alpha_{2,n/2} > \beta_{1,n/2}$ if n is even and that $\alpha_{2,(n-1)/2} > \beta_{1,(n-1)/2}$ if n is odd. Since $1 \le e \le (n-1)/2$, it follows that $(n-2e)^2 - 1 \ge 0$, and thus $1 \le (n^2-4e^2+4e-1)/(4e(n-2e+1))$. If n is even, then $\alpha_{2,n/2} \ge b(n^2-4e^2+4e-1)/(4e(n-2e+1)) \ge b > a = \beta_{1,n/2}$. If n is odd, then $\alpha_{2,(n-1)/2} \ge b(n^2-4e^2+4e-1)/(4e(n-2e+1)) \ge b > a = \beta_{1,(n-1)/2}$.

Next we show that $\alpha_2 \le \beta_2$. First we show that $\gamma_k \le \beta_{2,k}$ for $k \in \mathcal{N}_{e-1,n-e+1}$, when $1 \le e \le (n-1)/2$. Now $e - e^2 \le 0$ and $n - 2e + 1 \ge 1$, so that

$$\gamma_k \le b\frac{k(n-k)}{e(n-2e+1)}, \qquad \text{for } k \in \mathcal{N}_{e-1,n-e+1}. \tag{3.13}$$

Moreover,

$$0 < \frac{n-1}{e(n-2e+1)}, \qquad \text{for } e \in \mathcal{N}_{1,(n-1)/2}. \tag{3.14}$$

Assume first that n is even. By (3.13) and the definition of β_2 for n even, it suffices to show that

$$\frac{b}{e(n-2e+1)} \le \frac{4c}{n^2}. \tag{3.15}$$

If $b \ge 4c(n-1)/n^2$, then

$$\frac{2b}{e(n-2e+1)} \le f(b) \le \frac{4c}{n^2},$$

by (ii) and (iv), and (3.15) follows. If $b < 4c(n-1)/n^2$, then

$$\frac{b}{e(n-2e+1)} \le \frac{4c(n-1)}{n^2e(n-2e+1)} \le \frac{4c}{n^2},$$

by (3.14), and the result follows for n even.

Assume that n is odd. By (3.13) and the definition of β_2 for n odd, it suffices to show that

$$\frac{b}{e(n-2e+1)} \le \frac{4c}{n^2-1}. \tag{3.16}$$

If $b \ge 4c/(n+1)$, then

$$\frac{2b}{e(n-2e+1)} \le f(b) \le \frac{4c}{(n^2-1)},$$

by (ii)$'$ and (v)$'$, and (3.16) follows. If $b < 4c/(n+1)$, then

$$\frac{b}{e(n-2e+1)} \le \frac{4c}{(n+1)e(n-2e+1)} \le \frac{4c}{n^2-1}\frac{(n-1)}{e(n-2e+1)} \le \frac{4c}{(n^2-1)},$$

by (3.14), and the result follows for n odd. It follows that $\alpha_{2,k} = \gamma_k \le \beta_{2,k}$ on $\mathcal{N}_{e-1,n-e+1}$.

Next we show that $\alpha_{2,k} = \gamma_k \le \beta_{2,k}$ on $\mathcal{N}_{1,e}$ and on $\mathcal{N}_{n-e,n}$. Since α_2 and β_2 are symmetric, it suffices to consider $k \in \mathcal{N}_{0,e}$. Now $\alpha_{2,0} = 0 = \beta_{2,0}$, $\alpha_{2,e} = b = \gamma_e \le \beta_{2,e}$, and

$$\Delta^2(\beta_{2,k+1} - \alpha_{2,k+1}) \le 0, \qquad \text{for } k \in \mathcal{N}_{1,e-1},$$

so $\beta_{2,k} - \alpha_{2,k} \ge 0$, for $k \in \mathcal{N}_{0,e}$, by the discrete maximum principle. Thus, $\alpha_2 \le \beta_2$ on $\mathcal{N}_n$.

We show that there is no solution y of problem (1.3),(1.2) with $y \ge \alpha_2$ on $\mathcal{N}_n$, and $y_k = \alpha_{2,k}$, for some $k \in \mathcal{N}_{1,n-1}$. Assume there is such a solution y. Thus,

$$\Delta^2 y_{k+1} = -f(y_k) = -f(\alpha_{2,k}) \le \Delta^2\alpha_{2,k+1},$$

and since $y_k = \alpha_{2,k}$ and $y \ge \alpha_2$, it follows that $y_{k-1} = \alpha_{2,k-1}$ and $y_{k+1} = \alpha_{2,k+1}$. Iterating this argument, it follows that $y = \alpha_2$, contradicting Assumption (v). Thus, $y_k \ne \alpha_{2,k}$ for any $k \in \mathcal{N}_{1,n-1}$, as required.

Thus, the conditions of Theorem 3.1 are satisfied and there are three solutions of problem (1.3),(1.2), as required.

REMARK 3.3. Now $\Delta\beta_{1,1} = 4a(n-1)/n^2$, for n even, and $\Delta\beta_{1,1} = 4a/(n+1)$, for n odd, and $\Delta\alpha_{2,1} = b/e$. If $b/e < 4a(n-1)/n^2$, and n is even, or if $b/e < 4a/(n+1)$, and n is odd, then $\Delta\alpha_{2,1} < \Delta\beta_{1,1}$ so that $\alpha_2 \not\geq \beta_1$ on $\mathcal{N}_n$, even though $\max\{\alpha_{2,k} : k \in \mathcal{N}_n\} > \max\{\beta_{1,k} : k \in \mathcal{N}_n\}$. If $b/e \geq 4a(n-1)/n^2$, and n is even, or if $b/e \geq 4a/(n+1)$, and n is odd, then $\Delta\alpha_{2,1} \geq \Delta\beta_{1,1}$, and we can show that $\alpha_2 \geq \beta_1$ on $\mathcal{N}_n$.

Thus, assuming that f is Lipschitz and independent of Δy_k, our existence result follows from [7, Corollary 2.4.2] by an argument similar to that in [7, Example 2.4.2], if $b/e \geq 4a(n-1)/n^2$, and n is even, or if $b/e \geq 4a/(n+1)$, and n is odd. It does not appear to follow by this argument if $b/e < 4a(n-1)/n^2$, and n is even, or if $b/e < 4a/(n+1)$, and n is odd.

REMARK 3.4. The conditions of the preceding theorem are sharp, as can be seen from the following example.

EXAMPLE 3.5. We consider the case n and e are positive integers satisfying $e = n/4$.

Let $\eta > 0$ be given and ϵ satisfy $0 < \epsilon < \min\{\eta/8, 4/[n(n+4)]\}$. Let $a = 1$, $b = 1+\epsilon$, and $c \geq 2n(1+\epsilon)/(n+2)$. Let α_2 be given in Theorem 3.2, $\gamma = 2b/[e(n-2e+1)] = 16(1+\epsilon)/[n(n+2)]$, and $\tau = \max\{\alpha_{2,k} : 0 \leq k \leq n\} = (1+\epsilon)(3n+4)/(2n+4)$. Thus, the conditions of Theorem 3.2 hold everywhere except on $[\tau - \eta/2, \tau]$, which is an interval of length less than η, where $\eta > 0$ was arbitrary.

Let

$$f(y) = \begin{cases} 0, & \text{for all } y \in (-\infty, 1], \\ \dfrac{\gamma(y-1)}{\epsilon}, & \text{for all } y \in [1, 1+\epsilon], \\ \gamma, & \text{for all } y \in \left[1+\epsilon, \tau - \dfrac{\eta}{2}\right], \\ \dfrac{\gamma(4\tau - \eta - 4y)}{\eta}, & \text{for all } y \in \left[\tau - \dfrac{\eta}{2}, \tau - \dfrac{\eta}{4}\right], \\ 0, & \text{for all } y \in \left[\tau - \dfrac{\eta}{4}, \infty\right). \end{cases} \tag{3.17}$$

Thus, $f \geq 0$ is Lipschitz continuous. Moreover, $f(y) \geq \gamma$, for $b \leq y \leq \tau - \eta/2$, $f(y) = 0 < 8an^{-2}$, for $0 \leq y \leq a$, and $f(y) \leq \gamma \leq 8cn^{-2}$, for $0 \leq y \leq c$. Thus, all the conditions of Theorem 3.2 are satisfied with $e = n/4$ except condition (iii), which fails on a subinterval of $(\tau - \eta/2, \tau - \eta)$, where $\eta > 0$ may be chosen as small as we please.

We show that $y \equiv 0$ is the only solution of problem (1.3),(1.2).

Clearly, $y \equiv 0$ is the only solution of problem (1.3),(1.2) with $\|y\| \leq 1$. Assume that $y \not\equiv 0$ is a second solution with $\Delta y_1 = l$. By the discrete maximum principle $y_k \geq 0$ for all k, since $f \geq 0$, so $y_s = \max\{y_j : 0 \leq j \leq n\} > 0$. First, we show that $1 < y_s \leq \tau - \eta/4$. If $y_s \leq 1$, then $0 \leq y_k \leq 1$, for all $k = 0, \ldots, n$, so $\Delta^2 y_{k+1} = 0$, for all $k = 1, \ldots, n-1$, so $y_k = lk$, for all $k = 0, \ldots, n$, a contradiction.

If $y_s > \tau - \eta/4$, then $\Delta y_s \geq 0$ and $\Delta^2 y_{k+1} = 0$, while $y_k \geq \tau - \eta/4$. It follows that $y_k = y_s + \Delta y_s(k-s)$, for $k \geq s$, a contradiction.

Thus, $1 < y_s \leq \tau - \eta/4$, and we may choose $m > 0$ such that $y_k < 1$ for $k \leq m-1$ and $y_m \geq 1$. It follows as above that

$$\begin{aligned} \Delta y_k &= l, & &\text{for } 1 \leq k \leq m, \\ y_k &= lk, & &\text{for } 0 \leq k \leq m. \end{aligned}$$

Moreover,

$$\Delta^2 y_{k+1} = -f(y_k) \geq -\gamma, \tag{3.18}$$

so that

$$\Delta y_k \geq l - (k-m)\gamma, \qquad \text{and} \tag{3.19}$$

$$y_k \geq lk - (k-m)(k-m+1)\frac{\gamma}{2}, \qquad \text{for all } m \leq k \leq n. \tag{3.20}$$

Let w satisfy

$$\begin{aligned} \Delta^2 w_{k+1} &= 0, &&\text{for } 1 \leq k \leq \frac{n}{4} - 1, \\ \Delta^2 w_{k+1} &= -\gamma, &&\text{for } \frac{n}{4} \leq k, \\ w_0 &= 0, &&\Delta w_1 = \frac{4}{n} \end{aligned}$$

Thus,

$$\begin{aligned} w_k &= \frac{4k}{n}, &&\text{for } 0 \leq k \leq \frac{n}{4}, \\ \Delta w_k &= \frac{4}{n} - \left(k - \frac{n}{4}\right)\gamma, &&\text{for } k \geq \frac{n}{4}, \qquad \text{and} \\ w_k &= \frac{4k}{n} - \left(k - \frac{n}{4}\right)\frac{(k+1-n/4)}{2}, &&\text{for } k \geq \frac{n}{4}. \end{aligned}$$

Therefore,

$$\begin{aligned} w_{n/2} &= \left\{(2+\epsilon) - n(n+4)\frac{\gamma}{32}\right\} - 2\epsilon \\ &= \tau - 2\epsilon > \tau - \frac{\eta}{4}, \end{aligned}$$

as $\epsilon < \eta/8$. Moreover,

$$\begin{aligned} \Delta w_{n/2} &= \left\{\frac{4(1+\epsilon)}{n} - \frac{n\gamma}{4}\right\} - \frac{4\epsilon}{n} \\ &= \frac{\gamma}{2} - \frac{4\epsilon}{n} \end{aligned}$$

We consider the cases $l \geq n/4$ and $l < 4/n$ separately.

Assume $l \geq n/4$ so that $m \leq n/4$. We show that this leads to the contradiction $y_{m+n/4} > \tau - \eta/4$. Since $l(m+n/4) \geq 2$,

$$\begin{aligned} y_{m+n/4} &\geq l\left(m + \frac{n}{4}\right) - \frac{n(n/4+1)\gamma}{8} \\ &\geq w_{n/2} > \tau - \frac{\eta}{4}, \end{aligned}$$

a contradiction. Thus, $2/n \leq l < 4/n$, so that $4/(n+4t) \leq l < 4/(n+(t-1)4)$, for t satisfying $1 \leq t \leq n/4$ and, in particular, $l(t-1+n/4) < 1 \leq l(t+n/4)$. If $y_k > \tau - \eta/4$, for some $k \leq n$ we are through, so assume that $y_k \leq \tau - n/4$ for all k with $0 \leq k \leq n$. We show that $\Delta y_{n/2} > \gamma/2$. Now $\Delta y_{n/2} = (\Delta y_{n/2} - \Delta w_{n/2}) + \Delta w_{n/2}$, so that it suffices to show that $\Delta y_{n/2} - \Delta w_{n/2} - 4\epsilon/n > 0$.

Now

$$\Delta y_{n/2} - \Delta w_{n/2} - \frac{4\epsilon}{n} \geq \left\{l - \left[\frac{n}{2} - \left(t + \frac{n}{4}\right)\right]\gamma\right\} - \left\{\frac{4}{n} - \left(\frac{n}{2} - \frac{n}{4}\right)\gamma\right\} - \frac{4\epsilon}{n}$$

$$\text{by (3.19) and (3.21) as } m = t + \frac{n}{4},$$

$$\begin{aligned}
&\geq \frac{4}{(n+4t)} - \frac{4}{n} = t\gamma - \frac{4\epsilon}{n}, \qquad \text{as } l \geq \frac{4}{(n+4t)}, \\
&\geq \frac{32t(2t-1) - 4\epsilon n(n+4t)}{n(n+4t)(n+2)} \\
&\geq \frac{32 - 8\epsilon n^2}{n(n+4t)(n+2)}, \qquad \text{since } 4t \leq n \quad \text{and} \quad t \geq 1, \\
&> 0, \qquad \text{as } \epsilon < \frac{4}{n^2}.
\end{aligned}$$

Thus, $\Delta y_{n/2} > \gamma/2$, for $2/n \leq l < 4/n$. If y_k is a solution with $y_k \leq \tau - n/4$ for all k with $0 \leq k \leq n$, and $v_k = y_{n-k}$, for all k with $0 \leq k \leq n$, then v is a solution of problem (1.2),(1.3) satisfying $v_0 = y_n = 0$, $v_k \leq \tau - \eta/4$, and

$$\begin{aligned}
\Delta v_{n/2} &= -\Delta y_{n/2+1} \\
&= -\Delta y_{n/2} + f(y_{n/2}) \\
&< -\frac{\gamma}{2} + \gamma = \frac{\gamma}{2},
\end{aligned}$$

contradicting the previous argument. Since there are no solutions with $y_s \geq \tau - \eta/4$, it follows that there are no solutions $y \not\equiv 0$, as required.

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