Exponential stability of energy solutions to stochastic partial differential equations with variable delays and jumps

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\textbf{Abstract}


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\textbf{1. Introduction}

In recent years, existence, uniqueness, stability, invariant measures, and other quantitative and qualitative properties of solutions to stochastic partial differential equations in a separable Hilbert space have been studied by many authors (e.g. [1–4,7–14] and the references therein). Specially, many scholars pay much attention to stochastic partial differential equations with delays. By introducing approximate system, several criteria for the asymptotic exponential stability of a class of Hilbert space-valued, non-autonomous stochastic evolution equations with variable delays were presented [7]. Caraballo [1], Govindan [4] gave sufficient conditions for the exponential stability in $p$-th mean of mild solutions to stochastic partial differential equations with variable delays by using properties of the stochastic convolution and the comparison principle, respectively. Taniguchi [13], Duan [14] considered by the energy equality the exponential stability of strong solutions to non-autonomous stochastic partial differential equations with finite memory.

In contrast, there has not been very much study of stochastic partial differential equations driven by jump process. However, it begun to gain attention recently. Röckner and Zhang [12] showed by successive approximations the existence, uniqueness and large deviation principle of stochastic evolution equations of jumps. In Dong [2], the global existence and uniqueness of the strong, weak and mild solutions to one-dimensional Burgers equation in $[0, 1]$, with a random and perturbation of the body forces in the form of Poisson and Brownian motion, were studied. By introducing approximate system, Luo and Liu [9] established the existence-and-uniqueness theory of mild solutions to stochastic partial functional differential equations with Markovian switching and Poisson jumps, meanwhile, the exponential stability of mild solutions was investigated by the Razumikhin–Lyapunov type function methods.

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As it is well known, in the case without delay, Lyapunov’s method is in general available to obtain sufficient conditions for the stability of solutions. However, in the case of differential equations with retarded arguments, even with constant delays, Lyapunov’s method is not suitable as Krasovkii [5] pointed out for the ordinary differential equations, and Kushner [6] (among others) also did for stochastic differential equations, since the history of the process must be taken into account.

In this work, we consider by estimating the coefficients functions in the stochastic energy equality the exponential stability and almost sure exponential stability of energy solutions to stochastic partial differential equations with variable delays and jumps. To the best of our knowledge to date, so far little is known concerned with this case, and the aim of this work is to close the gap.

2. Preliminary

First of all, we introduce the framework in which our analysis is going to be carried out. Let $V$, $H$, $K$ be separable Hilbert spaces such that

$$V \leftrightarrow H \equiv H^* \leftrightarrow V^*,$$

where $V^*$ is the dual of $V$ and the injections are continuous and dense. We denote by $\| \cdot \|$, $\| \cdot \|_H$ and $\| \cdot \|_\omega$ the norms in $V$, $H$ and $V^*$, respectively, by $\langle \cdot, \cdot \rangle$ the duality product between $V^*$, $V$ and by $\langle \cdot, \cdot \rangle_H$ the scalar product in $H$. Furthermore, assume that for $\lambda_1 > 0$

$$\lambda_1 \| u \|^2_H \leq \| u \|^2.$$

(2.1)

Let $\tau > 0$ and $D := D([-\tau, 0]; H)$ denote the family of all right-continuous functions with left-hand limits $\psi$ from $[-\tau, 0]$ to $H$. The space $D([-\tau, 0]; H)$ is assumed to be equipped with the norm $\| \psi \|_D = \sup_{-\tau \leq \theta \leq 0} \| \psi(\theta) \|_H$. $D^0_{\tau, 0}(H)$ denotes the family of all almost surely bounded, $\mathcal{F}_\tau$-measurable, $D([-\tau, 0]; H)$-valued random variables. Given $\tau > 0$ and $T > 0$, we denote by $I^2(\tau, T; V)$ the space of all $V$-valued processes $(x(t))_{t\in[-\tau, T]}$ (we will write $x(t)$ for short) which is $\mathcal{F}_t$ measurable and satisfies $\int_0^T \mathbb{E}\|x(t)\|^2 dt < +\infty$.

Let $W(t)$ be a Wiener process defined on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets) and taking values in the separable Hilbert space $K$, with incremental covariance operator $Q$, i.e.,

$$E[W(t), x]_K = (t \wedge s)(Q x, y)_K, \quad \forall x, y \in K,$$

where $Q$ is a positive, self-adjoint, trace class operator on $K$. In particular, we call such $(W(t), t \geq 0)$ a $K$-valued $Q$-Wiener process with respect to $[\mathcal{F}_t]_{t \geq 0}$. According to D. Prato [10], $W(t)$ is defined by

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad t \geq 0,$$

where $\beta_n(t) (n = 1, 2, 3, \ldots)$ is a sequence of real-valued standard Brownian motions mutually independent on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $(\lambda_n, n \in N)$ are the eigenvalues of $Q$ and $(e_n, n \in N)$ are the corresponding eigenvectors. That is,

$$Q e_n = \lambda_n e_n, \quad n = 1, 2, 3, \ldots.$$

For an operator $G \in \mathcal{L}(K, H)$, the space of all bounded linear operators from $K$ into $H$, we denote by $\| G \|_{\mathcal{L}(K,H)}^2$ its Hilbert–Schmidt norm, i.e.

$$\| G \|_{\mathcal{L}(K,H)}^2 = \text{tr}(G Q G^*).$$

Let $p = (p(t)), t \in D_p$ be a stationary $\mathcal{F}_t$-Poisson point process with characteristic measure $\lambda$. Denote by $N(dt, du)$ the Poisson counting measure associated with $p$, i.e., $N(dt, Z) = \sum_{s \in D_p, s \leq t} l_2(p(s))$ with measurable set $Z \in \mathcal{B}(K - \{0\})$ which denotes the Borel $\sigma$-field of $K - \{0\}$. Let $\tilde{N}(dt, du) := N(dt, du) - dt \lambda(du)$ be the compensated Poisson measure which is independent of $W(t)$.

In this work, we are concerned with stochastic partial functional differential equation with jumps in the following form:

$$dX(t) = \left[ A(t, X(t)) dt + F(t, X_t) \right] dt + G(t, X_t) dW(t) + \int_{X} J(t, X_t, u) \tilde{N}(dt, du), \quad t \geq 0,$$

(2.2)

with the initial datum $X(\theta) = \xi(\theta) \in L^2(\Omega, D([-\tau, 0]; H)), \theta \in [-\tau, 0]$, where $A : [0, \infty) \times V \to V^*$, $F : [-\tau, \infty) \times D \to V^*$, $G : [-\tau, \infty) \times D \to \mathcal{L}(K, H)$ and $J : [-\tau, \infty) \times D \times X \to H$.

Now, we introduce the definition of energy solution to (2.2) following from Taniguchi [13].
Definition 2.1. An $\mathcal{F}_t$-adapted stochastic process $X(t)$ on the probability space $(\Omega, \mathcal{F}, P)$ is said to be the energy solution to (2.2) if the following conditions are satisfied:

(I) $X(t) \in L^2(-\tau, T; V) \cap L^2(\Omega; D(-\tau, T; H))$, $T > 0$;
(II) the following equation holds in $V^*$ almost surely, for $t \in [0, T]$,
\[
X(t) = X(0) + \int_0^t [A(s, X(s)) + F(s, X_s)] ds + \int_0^t \int J(s, X_s, u) \tilde{\mathcal{N}}(ds, du), \quad t \geq 0,
\]
\[
X(t) = \xi(t), \quad -\tau \leq t \leq 0;
\]
(III) the following stochastic energy equality holds: for $t \in [0, T]$
\[
\|X(t)\|_H^2 = \|X(0)\|_H^2 + 2 \int_0^t \langle X(s), A(s, X(s)) + F(s, X_s) \rangle ds + \int_0^t \|G(s, X_s)\|_Q^2 ds
\]
\[
+ 2 \int_0^t \langle X(s), G(s, X_s) dW(s) \rangle + \int_0^t \sum_{\xi(\omega, s) \in D} \|J(s, X_s, u)\|_{H, \Lambda}^2 d\lambda(du)
\]
\[
+ \int_0^t \int \|J(s, X_s, u)\|_H^2 + 2\|X_{s-}\|_H \|J(s, X_s, u)\|_H \tilde{\mathcal{N}}(ds, du).
\]

Furthermore, for the existence and uniqueness of energy solution to (2.2), we impose the following assumptions:

(H1) (Monotonicity and coercivity) There is a pair of constants $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that for a.e. $t \in (0, T)$, $\forall u, v \in V$
\[
-2 \langle A(t, u) - A(t, v), u - v \rangle + \lambda \|u - v\|_H^2 \geq \alpha \|u - v\|^2.
\]

(H2) (Measurability) $\forall v \in V$, the map $t \in (0, T) \rightarrow A(t, v) \in V^*$ is measurable.

(H3) (Hemicontinuity) The map
\[
\theta \in \mathbb{R} \rightarrow \langle A(t, u + \theta v), w \rangle \in \mathbb{R}
\]
is continuous $\forall u, v, w \in V$ and a.e. $t \in (0, T)$.

(H4) (Boundedness) There exists $c > 0$ such that $\|A(t, v)\|_e \leq c \|v\|, \forall v \in V$ and a.e. $t \in (0, T)$.

(H5) (Lipschitz condition and linear growth condition) There exists a pair of positive constants $c_1, c_2$ satisfying that for $\xi, \eta, \zeta \in D$
\[
\|F(t, \xi) - F(T, \eta)\|_e + \|G(t, \xi) - G(t, \eta)\|_Q^2 + \int_X \|J(t, \xi, u) - J(t, \eta, u)\|_{H, \Lambda}^2 \leq c_1 \|\xi - \eta\|_D^2
\]

and
\[
\|F(t, \zeta)\|_e^2 + \|G(t, \zeta)\|_Q^2 + \int_X \|J(t, \zeta, u)\|_{H, \Lambda}^2 \leq c_2 (1 + \|\zeta\|_D^2).
\]

Theorem 2.1. Under the assumptions (H1)-(H5), (2.2) admits a unique energy solution $X(t)$, $t \geq 0$. Moreover, it holds that
\[
\frac{d}{dt} \mathbb{E}\|X(t)\|_H^2 = 2\mathbb{E}\langle X(t), A(t, X(t)) + F(t, X_t) \rangle + \mathbb{E}\|G(t, X_t)\|_Q^2 + \mathbb{E}\int_X \|J(t, X_t, u)\|_{H, \Lambda}^2 dt.
\]

Proof. Following the proof of [12, Theorem 3.2], we can complete it. □

For convenience, we need further to introduce two concepts following from Taniguchi [13].

Definition 2.2. If an energy solution $X(t)$ to (2.2) satisfies the following inequality:
\[
\mathbb{E}\|X(t)\|_H^2 \leq Be^{-\alpha t}, \quad t \geq 0,
\]
where \( \eta > 0 \) and \( B = B(X(0)) > 0 \), then it is said that the energy solution \( X(t) \) to (2.2) converges to the zero exponentially in mean square as \( t \to \infty \). Furthermore, if any energy solution converges to zero exponentially in mean square as \( t \to \infty \) and the zero is the solution to (2.2), then it is said that the zero solution is exponentially stable in mean square.

**Definition 2.3.** If there exist positive constants \( B = \delta(\epsilon) > 0, \eta > 0 \) and a subset \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 0 \), and for each \( \omega \in \Omega - \Omega_0 \), there exists a positive random number \( T(\omega) \) such that

\[
\|X(t)\|_H^2 \leq Be^{-\eta t}, \quad t \geq T(\omega),
\]

then it is said that the energy solution \( X(t) \) to (2.2) converges to the zero exponentially almost surely as \( t \to \infty \). Furthermore, if any energy solution converges to zero exponentially almost surely as \( t \to \infty \) and the zero is the solution to (2.2), then it is said that the zero solution is exponentially stable almost surely.

3. **Exponential stability in mean square**

In this section, we start to consider the exponential stability in mean square of energy solution to (2.2). For \( \tau > 0 \) let \( \rho(t), r(t), \delta(t) : [0, \infty) \to [0, \tau] \) be continuous functions. Assume that \( A : [0, \infty) \times V \to V^* \), \( A(t, 0) = 0 \) and \( f : [0, \infty) \to V^* \). Let \( h : [0, \infty) \times H \to H, G : [0, \infty) \times H \to L(K, H) \) and \( k : [0, \infty) \times H \times X \to H \) satisfy the assumption (H5).

We investigate the following stochastic partial differential equation with variable delays and jumps

\[
dX(t) = \left[ A(t, X(t)) + f(t) \right] dt + h(t, X(t - \rho(t))) dt + g(t, X(t - r(t))) dW(t)
\]

with the initial condition \( X(\theta) = \xi(\theta) \in L^2(\Omega, D([-\tau, 0]; H)), \theta \in [-\tau, 0] \), where \( f \in L^2([0, \infty), V^*) \). Set \( F_1(t, \phi) = h(t, \phi(-\rho(t))), G(t, \phi) = g(t, \phi(r(t))) \) and \( J(t, \phi, u) = k(t, \phi(-\delta(t)), u) \) for any \( \phi \in D \). Then (3.1) can be regarded as a stochastic partial functional differential equation (2.2) with \( F(t, \phi) = F_1(t, \phi) + f(t) \).

**Theorem 3.1.** Assume that conditions (H2)-(H4) hold. Moreover, (3.1) satisfies the following conditions:

(a) there are constant \( \delta_1 > 0 \) and continuous, integrable function \( \alpha_1(t) > 0 \) such that, for a.e. \( t \in (0, T) \) and \( \forall u, v \in V \),

\[
-2[A(t, u) - A(t, v), u - v] + \alpha_1(t)\|u - v\|_H^2 \geq \delta_1\|u - v\|^2;
\]

(b) there exist integrable functions \( \alpha_2, \beta_2 : [0, \infty) \to R^+ \) such that, for certain constant \( \delta_2 \geq 0 \) and \( u \in H \),

\[
\|h(t, u)\|_H^2 \leq (\delta_2 + \alpha_2(t))\|u\|_H^2 + \beta_2(t);
\]

(c) there exist integrable functions \( \alpha_3, \beta_3 : [0, \infty) \to R^+ \) such that, for certain constant \( \delta_3 \geq 0 \) and \( u \in H \),

\[
\|g(t, u)\|_{L_0}^2 \leq (\delta_3 + \alpha_3(t))\|u\|_H^2 + \beta_3(t);
\]

(d) there exist integrable functions \( \alpha_4, \beta_4 : [0, \infty) \to R^+ \) such that, for certain constant \( \delta_4 \geq 0 \) and \( u \in H \),

\[
\int_X \|k(t, u, x)\|_H^2 \lambda(dx) \leq (\delta_4 + \alpha_4(t))\|u\|_H^2 + \beta_4(t);
\]

(e) there exists \( \sigma_1 > 0 \) such that

\[
\int_0^\infty e^{\sigma_1 t}\|f(t)\|_u^2 dt < \infty, \quad \int_0^\infty e^{\sigma_1 t}\beta_i(t) dt < \infty, \quad i = 2, 3, 4;
\]

(f) for given \( \lambda_1 \) defined by (2.1),

\[
\delta_1 \lambda_1 > 2\sqrt{\delta_2 + \delta_3 + \delta_4}.
\]

Then for any energy solution \( X(t) \) to (3.1), there exist \( \sigma \in (0, \sigma_1) \) and \( M \geq 1 \) such that

\[
\mathbb{E}\|X(t)\|_H^2 \leq Me^{-\sigma t}, \quad t \geq 0.
\]

In other words, the energy solution \( X(t) \) to (3.1) is exponentially stable in mean square.
Proof. From (3.7), there exists a $\gamma_1 \in (0, \delta_1)$ such that
\[
(\delta_1 - \gamma_1)\lambda_1 > 2\sqrt{\delta_2} + \delta_3 + \delta_4.
\]
Then by elemental inequality $2ab \leq \lambda a^2 + \frac{b^2}{\lambda}$ for any $a, b, \lambda > 0$ we can take $\gamma_2 > 0$ obeying
\[
(\delta_1 - \gamma_1)\lambda_1 > \gamma_2 + \frac{\delta_2}{\gamma_2} + \delta_3 + \delta_4 > 2\sqrt{\delta_2} + \delta_3 + \delta_4.
\]
Furthermore, there exists a $\sigma \in (0, \sigma_1)$ sufficiently small such that
\[
(\delta_1 - \gamma_1)\lambda_1 > \sigma + \gamma_2 + e^{\sigma\tau} \frac{\delta_2}{\gamma_2} + e^{\sigma\tau} \delta_3 + e^{\sigma\tau} \delta_4.
\]
(3.9)
Now, in view of (3.2) and $f \in L^2([0, \infty), V^*)$, we derive that, for any $v \in V$ and $t \geq 0$,
\[
2\langle v, A(t, v) + f(t) \rangle = 2\langle v, A(t, v) \rangle + 2\langle v, f(t) \rangle
\leq \alpha_1(t)\|v\|_{H^1}^2 - (\delta_1 - \gamma_1)\|v\|_H^2 + \gamma_1^{-1}\|f(t)\|_V^2
\leq \left[\alpha_1(t) - (\delta_1 - \gamma_1)\lambda_1\right]\|v\|_H^2 + \gamma_1^{-1}\|f(t)\|_V^2.
\]
Setting $(\delta_1 - \gamma_1)\lambda_1 = a$ and $\beta_1(t) = \gamma_1^{-1}\|f(t)\|_V$, then it follows that
\[
2\langle v, A(t, v) + f(t) \rangle \leq \left[\alpha_1(t) - a\right]\|v\|_{H^1}^2 + \beta_1(t).
\]
(3.10)
For convenience, denote
\[
\theta(t) = \alpha_1(t) + e^{\sigma\tau} \frac{\alpha_2(t)}{\gamma_2} + e^{\sigma\tau} \alpha_3(t) + e^{\sigma\tau} \alpha_4(t), \quad \beta(t) = \beta_1(t) + \frac{\beta_2(t)}{\gamma_2} + \beta_3(t) + \beta_4(t).
\]
Since $\alpha_i(t), i = 1, 2, 3, 4$, is integrable, together with (3.6) and $f \in L^2([0, \infty), V^*)$, this yields that
\[
R_1 = \int_0^\infty \theta(s)\,ds < \infty, \quad R_2 = \int_0^\infty \beta(s)\,ds \leq R_3 = \int_0^\infty e^{\rho s} \beta(s)\,ds < \infty.
\]
(3.11)
Let
\[
K(t) = \begin{cases} E\|X(t)\|_H^2 e^{\sigma\tau} \exp(-\int_0^t [\theta(s) + e^{\sigma s} \beta(s)]\,ds), & \text{if } t \geq 0, \\ E\|X(t)\|_H^2 e^{\sigma\tau}, & \text{if } -\tau \leq t \leq 0. \end{cases}
\]
Clearly, $K(t)$ is continuous on $[-\tau, \infty)$ and
\[
\frac{dK(t)}{dt} = e^{\sigma\tau} \exp\left(-\int_0^t \left[\theta(s) + e^{\sigma s} \beta(s)\right]\,ds\right)\left\{\sigma E\|X(t)\|_H^2 - [\theta(t) + e^{\sigma t} \beta(t)]E\|X(t)\|_H^2 \\
+ 2E\|X(t), A(t, X(t)) + f(t)\|_V^2 + 2E\|X(t), h(t, X(t - \rho(t)))\|_V^2 \\
+ E\|g(t, X(t - r(t)))\|_2^2 + E\int_X \|k(t, X(t - \delta(t)), u)\|_H^2 \lambda(\,du)\right\}. \]
(3.12)
By (3.3)-(3.5) and (3.10), it immediately follows that for $t \geq 0$
\[
\frac{dK(t)}{dt} \leq e^{\sigma\tau} \exp\left(-\int_0^t \left[\theta(s) + e^{\sigma s} \beta(s)\right]\,ds\right)\left\{\sigma E\|X(t)\|_H^2 - [\theta(t) + e^{\sigma t} \beta(t)]E\|X(t)\|_H^2 \\
+ \left[\alpha_1(t) - a\right]E\|X(t)\|_H^2 + \gamma_2 E\|X(t)\|_H^2 \beta_1(t) \\
+ E\|g(t, X(t - r(t)))\|_2^2 + E\int_X \|k(t, X(t - \delta(t)), u)\|_H^2 \lambda(\,du)\right\}
\leq \left[\sigma - \theta(t) + \alpha_1(t) - a + \gamma_2\right]K(t) + e^{\sigma\tau} \beta(t) - e^{\sigma\tau} \beta(t)K(t)
\]
\[
\leq \left[\sigma - \theta(t) + \alpha_1(t) - a + \gamma_2\right]K(t) + e^{\sigma\tau} \beta(t) - e^{\sigma\tau} \beta(t)K(t)
\]
which contradicts with (3.16). That is, the desired assertion (3.14) must hold.

\begin{equation}
\begin{split}
+ e^{\sigma t} \exp \left( - \int_0^t \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \frac{\delta_2 + \alpha_2(t)}{\gamma_2} E \| X(t - \rho(t)) \|_H^2 \\
+ e^{\sigma t} \exp \left( - \int_0^t \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_3 + \alpha_3(t) \right) E \| X(t - r(t)) \|_H^2 \\
+ e^{\sigma t} \exp \left( - \int_0^t \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_4 + \alpha_4(t) \right) E \| X(t - \delta(t)) \|_H^2.
\end{split}
\end{equation}

(3.13)

In what follows, we justify that for any $t \geq 0$

\begin{equation}
K(t) \leq \bar{M} := 1 + \sup_{t \in [-\tau, 0]} E \| X(t) \|_H^2.
\end{equation}

(3.14)

We show (3.14) by contradiction. If (3.14) is not true, we must have some $t_1 > 0$ and $\epsilon > 0$ such that

\begin{equation}
K(t) < \bar{M}, \quad 0 \leq t < t_1, \quad K(t_1) = \bar{M}, \quad K(t) > \bar{M}, \quad t_1 \leq t \leq t_1 + \epsilon.
\end{equation}

(3.15)

This, in addition to (3.12), immediately implies that

\begin{equation}
\frac{dK(t_1)}{dt} \geq 0.
\end{equation}

(3.16)

Substituting (3.15) into (3.13) and combining $M \geq 1$ yield

\begin{equation}
\begin{split}
\frac{dK(t_1)}{dt} &\leq \left[ \sigma - \theta(t_1) + \alpha_1(t_1) - a + \gamma_2 \right] K(t_1) \\
+ &e^{\sigma t_1} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \frac{\delta_2 + \alpha_2(t_1)}{\gamma_2} E \| X(t_1 - \rho(t_1)) \|_H^2 \\
+ &e^{\sigma t_1} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_3 + \alpha_3(t_1) \right) E \| X(t_1 - r(t_1)) \|_H^2 \\
+ &e^{\sigma t_1} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_4 + \alpha_4(t_1) \right) E \| X(t_1 - \delta(t_1)) \|_H^2.
\end{split}
\end{equation}

(3.17)

Next, we split the following cases to derive the desired assertion.

**Case 1:** Let $t_1 - \rho(t_1) \geq 0$, $t_1 - r(t_1) \geq 0$, $t_1 - \delta(t_1) \geq 0$. By the definition of $K(t)$, we then have from (3.9) and (3.15) that

\begin{equation}
\begin{split}
\frac{dK(t_1)}{dt} &\leq \left[ \sigma - \theta(t_1) + \alpha_1(t_1) - a + \gamma_2 \right] K(t_1) \\
+ &e^{\sigma t} \exp \left( - \int_{t_1 - \rho(t_1)}^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \frac{\delta_2 + \alpha_2(t_1)}{\gamma_2} K(t_1 - \rho(t_1)) \\
+ &e^{\sigma t} \exp \left( - \int_{t_1 - r(t_1)}^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_3 + \alpha_3(t_1) \right) K(t_1 - r(t_1)) \\
+ &e^{\sigma t} \exp \left( - \int_{t_1 - \delta(t_1)}^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_4 + \alpha_4(t_1) \right) K(t_1 - \delta(t_1)) \\
\leq &\left[ \sigma - a + \gamma_2 + \frac{e^{\sigma t} \delta_2}{\gamma_2} + e^{\sigma t} \delta_3 + e^{\sigma t} \delta_4 \right] \bar{M} \\
< &0,
\end{split}
\end{equation}

which contradicts with (3.16). That is, the desired assertion (3.14) must hold.
Then there exists $T$ such that for all $t \geq 0$, $t_1 - r(t_1) \leq 0, t_1 - \delta(t_1) \leq 0$. Taking into account the definition of $K(t)$, it follows that

$$
\frac{dK(t_1)}{dt} \leq \left[ \sigma - \theta(t_1) + \alpha_1(t_1) - a + \gamma_2 \right] K(t_1) + e^{\sigma \theta(t_1)} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \frac{\delta_2 + \alpha_2(t_1)}{\gamma_2} K(t_1 - \rho(t_1)) + e^{\sigma \tau(t_1)} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_3 + \alpha_3(t_1) \right) K(t_1 - r(t_1)) + e^{\sigma \delta(t_1)} \exp \left( - \int_0^{t_1} \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right) \left( \delta_4 + \alpha_4(t_1) \right) K(t_1 - \delta(t_1)).
$$

(3.18)

Putting (3.9), (3.14) and (3.15) into (3.18) gets

$$
\frac{dK(t_1)}{dt} \leq \left[ \sigma - a + \gamma_2 + e^{\sigma \tau} \frac{\delta_2}{\gamma_2} + e^{\sigma \tau} \delta_3 + e^{\sigma \tau} \delta_4 \right] \tilde{M} < 0.
$$

By contradiction, we must have (3.14) holds for any $t \geq 0$. For other cases, in the same way as Cases 1 and 2 were done, we can show (3.14). Thus we obtain from (3.14) that

$$
\mathbb{E} \left\| X(t) \right\|_H^2 \leq \tilde{M} e^{-\sigma t} \exp \left( \int_0^t \left[ \theta(s) + e^{\sigma s} \beta(s) \right] ds \right).
$$

which, combining (3.11), immediately implies that with $B = e^{R_1 + R_1}$

$$
\mathbb{E} \left\| X(t) \right\|_H^2 \leq \tilde{M} B e^{-\sigma t}.
$$

Consequently, the exponential stability in mean square of energy solution to (3.1) follows. 

\section{4. Almost sure exponential stability}

In what follows, the almost sure exponential stability of energy solution to (3.1) is considered. First of all, we recall the following result [12] which lays the foundation for almost sure exponential stability of energy solution to (3.1).

\textbf{Lemma 4.1.} For any $t \geq 0$, there exists a constant $C > 0$ such that

$$
\mathbb{E} \sup_{0 \leq s \leq t} \left\| \int_0^s \left[ k(l, X(l - \delta(l)), u) \right]_H^2 + 2 \left\| X(l - \delta(l)) \right\|_H \tilde{N}(dl, du) \right\| \leq C \mathbb{E} \int_0^t \left\| k(s, X(s - \delta(s)), u) \right\|_H^2 \lambda(du) ds + \frac{1}{4} \mathbb{E} \sup_{0 \leq s \leq t} \left\| X(s) \right\|_H^2.
$$

(4.1)

\textbf{Theorem 4.1.} Assume that all the conditions of Theorem 3.1 are satisfied. Furthermore, we impose the following assumption:

(g) Both $\alpha_i(t)$ and $e^{\sigma \theta} \beta_i(t)$ $(i = 1, 2, 3, 4)$ are bounded functions.

Then there exists $T(\omega) > 0$ such that for all $t > T(\omega)$, with probability one

$$
\left\| X(t) \right\|_H^2 \leq e^{\sigma / 2} e^{-\sigma t / 2}.
$$

(4.2)

In other words, the energy solution $X(t)$ to (3.1) is almost surely exponentially stable.

\textbf{Proof.} Let $N_1, N_2$ and $N_3$ be positive integers such that

$$
N_1 - \rho(N_1) \geq N_1 - \tau \geq 1, \quad N_2 - r(N_2) \geq N_2 - \tau \geq 1, \quad N_3 - r(N_3) \geq N_3 - \tau \geq 1.
$$
Set $N = \max\{N_1, N_2, N_3\}$ and $I_N = [N, N + 1]$. Furthermore, denote
\[
\alpha(t) = \alpha_1(t) + \gamma_2 + \frac{\delta_2 + \alpha_2(t)}{\gamma_2}e^{\sigma \tau} + 33(\delta_3 + \alpha_3(t))e^{\sigma \tau} + (C + 1)(\delta_4 + \alpha_4(t))e^{\sigma \tau},
\]
\[
\beta(t) = \beta_1(t) + \frac{\beta_2(t)}{\gamma_2} + 33\beta_3(t) + (C + 1)\beta_4(t),
\]
where $C$ is the given constant in Lemma 4.1. Since $X(t)$ is the energy solution to (3.1), it follows that for any $t \in [N, N + 1]$
\[
\|X(t)\|_H^2 = \|X(N)\|_H^2 + 2 \int_N^t \langle X(s), A(s, X(s)) + f(s) \rangle ds + 2 \int_N^t \langle X(s), h(s, X(s - \rho(s))) \rangle ds
\]
\[+
\int_N^t \|g(s, X(s - \rho(s)))\|_{L_Q}^2 ds + 2 \int_N^t \langle X(s), g(s, X(s - \rho(s))) \rangle dW(s)
\]
\[+
\int_N^t \int_X \|k(s, X(s - \delta(s)), u)\|^2_H ds \lambda(du)
\]
\[+
\int_N^t \int_X [\|k(s, X(s - \delta(s)), u)\|^2_H + 2\|X_{s-}, k(s, X(s - \delta(s)), u)\|_H] \tilde{N}(ds, du).
\]
This implies that
\[
\mathbb{E} \sup_{t \in I_N} \|X(t)\|_H^2 \leq \mathbb{E} \|X(N)\|_H^2 + 2 \mathbb{E} \sup_{t \in I_N} \int_N^t \langle X(s), A(s, X(s)) + f(s) \rangle ds
\]
\[+
2 \mathbb{E} \sup_{t \in I_N} \int_N^t \langle X(s), h(s, X(s - \rho(s))) \rangle ds + \int_N^{N+1} \mathbb{E} \|g(s, X(s - \rho(s)))\|_{L_Q}^2 ds
\]
\[+
2 \mathbb{E} \sup_{t \in I_N} \int_N^t \langle X(s), g(s, X(s - \rho(s))) \rangle dW(s)
\]
\[+
\int_N^{N+1} \int_X \mathbb{E} \|k(s, X(s - \delta(s)), u)\|^2_H \lambda(du) ds
\]
\[+
\mathbb{E} \sup_{t \in I_N} \int_N^t \int_X [\|k(s, X(s - \delta(s)), u)\|^2_H + 2\|X_{s-}, k(s, X(s - \delta(s)), u)\|_H] \tilde{N}(ds, du).
\] (4.3)
From (3.10), it is easy to show that
\[
2 \mathbb{E} \sup_{t \in I_N} \int_N^t \langle X(s), A(s, X(s)) + f(s) \rangle ds \leq \int_N^{N+1} [\alpha_1(s)\mathbb{E} \|X(s)\|_H^2 + \beta_1(s)] ds.
\] (4.4)
On the other hand, derive by virtue of (3.3)
\[
2 \mathbb{E} \sup_{t \in I_N} \int_N^t \langle X(s), h(s, X(s - \rho(s))) \rangle ds
\]
\[\leq \int_N^{N+1} \left[ \gamma_2 \mathbb{E} \|X(s)\|_H^2 + \frac{E \|h(s, X(s - \rho(s)))\|_{L_Q}^2}{\gamma_2} \right] ds
\]
\[\leq \int_N^{N+1} \left[ \gamma_2 \mathbb{E} \|X(s)\|_H^2 + \frac{\delta_2 + \alpha_2(s)}{\gamma_2} \mathbb{E} \|X(s - \rho(s))\|_H^2 + \frac{\beta_2(s)}{\gamma_2} \right] ds.
\] (4.5)
By the Burkholder–Davis–Gundy inequality [10], the following inequality holds:

\[
2\mathbb{E} \sup_{t \in I} \int_0^t \left( X(s), g(s, X(s - r(s))) \right) dW(s) \leq \frac{1}{2} \mathbb{E} \sup_{t \in I} \|X(t)\|_H^2 + 32 \int \mathbb{E} \|g(s, X(s - r(s)))\|_2^2 \, ds.
\]  

(4.6)

Applying Lemma 4.1, for any \( t > 0 \) and certain positive constant \( C \) we can yield that

\[
\mathbb{E} \sup_{t \in I} \int_0^t \left[ \|k(s, X(s - \delta(s)), u)\|_H^2 + 2\|X_s - k(s, X(s - \delta(s)), u)\|_H \right] \tilde{N}(du, ds)
\leq C \int_0^t \|k(s, X(s - \delta(s)), u)\|_H^2 \lambda(du) \, ds + \frac{1}{4} \mathbb{E} \sup_{t \in I} \|X(t)\|_H^2.
\]  

(4.7)

Substituting (4.4)–(4.7) into (4.3) and combining (3.3) and (3.5), we must have

\[
\begin{align*}
\mathbb{E} \sup_{t \in I} \|X(t)\|_H^2 &\leq 4\mathbb{E} \|X(N)\|_H^2 + 4 \int \left[ \alpha_1(s)\mathbb{E} \|X(s)\|_H^2 + \beta_1(s) \right] \, ds \\
&\quad+ 4 \int \left[ \gamma_2 \mathbb{E} \|X(s)\|_H^2 + \frac{\delta_2 + \alpha_2(s)}{\gamma_2} \mathbb{E} \|X(s - \rho(s))\|_H^2 + \frac{\beta_2(s)}{\gamma_2} \right] \, ds \\
&\quad+ 132 \int \left[ (\delta_3 + \alpha_3(s))\mathbb{E} \|X(s - r(s))\|_H^2 + \beta_3(s) \right] \, ds \\
&\quad+ 4(C + 1) \int \left[ (\delta_4 + \alpha_4(s))\mathbb{E} \|X(s - \delta(s))\|_H^2 + \beta_4(s) \right] \, ds \\
&\leq 4Me^{-\sigma N} + 4 \int Me^{-\sigma s} [\alpha(s) + \beta(s)e^{\sigma s}] \, ds.
\end{align*}
\]  

(4.8)

In view of assumption (g), there exists a certain constant \( B_1 > 0 \) satisfying

\[
\alpha(t) + e^{\sigma t}\beta(s) \leq B_1.
\]

This, in addition to (4.8), yields that

\[
\mathbb{E} \sup_{t \in I} \|X(t)\|_H^2 \leq 4Me^{-\sigma N} \left( 1 + \frac{B_1}{\sigma} \right).
\]

By the Chebyshev inequality, for any fixed positive real number \( \epsilon_N \), then we get that

\[
P \left\{ \sup_{t \in I} \|X(t)\|_H^2 > \epsilon_N^2 \right\} \leq \frac{\mathbb{E} \sup_{t \in I} \|X(t)\|_H^2}{\epsilon_N^2} \leq \frac{4Me^{-\sigma N}(1 + B_1/\sigma)}{\epsilon_N^2}.
\]

Since \( \epsilon_N \) is arbitrary, letting \( \epsilon_N^2 = e^{-\sigma N/2} \), obtain

\[
P \left\{ \sup_{t \in I} \|X(t)\|_H^2 > \epsilon_N^2 \right\} \leq 4Me^{-\sigma N/2} \left( 1 + \frac{B_1}{\sigma} \right).
\]

In the conclusion, from the Borel–Cantelli lemma, it follows that there exists \( T(\omega) > 0 \) such that for all \( t > T(\omega) \) almost surely

\[
\|X(t)\|_H^2 \leq e^{\sigma/2}e^{-\sigma t/2}.
\]
5. An illustrative example

In this section, we construct one example to demonstrate the effectiveness of this theory.

Assume $B(t)$, $t \geq 0$, is a real standard Brownian motion and $\tilde{N}(\cdot, \cdot)$ is a compensated Poisson random measure on $[1, \infty]$ with parameter $\lambda(dy)dt$ such that $\int_1^\infty y^2 \lambda(dy) < \infty$. Let $b_1, b_2, b_3, k$ be positive constants, and $k_1, k_2, k_3 : [0, \infty) \to \mathbb{R}^+$ be continuous functions and $k_1^2, k_2^2, k_3^2$ be decreasing, bounded and integrable functions. Furthermore, set

$$\rho(t) = \frac{1}{1 + |\sin t|}, \quad r(t) = \frac{1}{1 + |\cos t|}, \quad \delta(t) = |\sin t|.$$ 

Consider stochastic partial equation with finite delays and jumps in the following form:

$$dZ(t, x) = \mu \frac{\partial^2}{\partial x^2} Z(t, x) dt + \left[(b_1 + k_1(t))Z(t - \rho(t), x) + e^{-kt}p\right] dt + \left[(b_2 + k_2(t))Z(t - r(t), x)\right] dB(t)$$
$$+ \int_1^\infty \left[(b_3 + k_3(t))Z(t - \delta(t), x)\right] y \tilde{N}(dt, dy), \quad t \geq 0, \mu > 0, 0 < x < \pi,$$

$$Z(t, 0) = Z(t, \pi) = 0, \quad t \geq 0; \quad Z(\theta, x) = \varphi(\theta, x), \quad 0 \leq x \leq \pi, \theta \in [-1, 0],$$

$$\varphi(\theta, \cdot) \in H = L^2(0, \pi), \quad \varphi(\cdot, x) \in C([-1, 0]; \mathbb{R}). \tag{5.1}$$

In this example, we take $H = L^2(0, \pi)$ and $A = \mu \frac{\partial^2}{\partial x^2}$ with domain $\mathcal{D}(A) = H^1_0(0, \pi) \cap H^2(0, \pi)$, where

$$H^1_0(0, \pi) = \left\{ u \in L^2(0, \pi) : \frac{\partial u}{\partial x} \in L^2(0, \pi), u(0) = u(\pi) = 0 \right\},$$

and

$$H^2(0, \pi) = \left\{ u \in L^2(0, \pi) : \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} \in L^2(0, \pi) \right\}.$$ 

The norms of $H$ and $H^1_0(0, \pi)$ are defined by $\|u\|_H^2 = \int_0^\pi \xi^2(x) dx$ for any $\xi \in H$ and $\|u\|^2 = \int_0^\pi (\frac{\partial u}{\partial x})^2 dx$ for any $u \in H^1_0(0, \pi)$. It is easy to show that for arbitrary $u \in H^1_0(0, \pi)$

$$\langle u, Au \rangle \leq -\mu \|u\|_H^2. \tag{5.2}$$

We can also show for $p \in H$ with $\|p\|_H < \infty$ that

$$\| (b_1 + k_1(t))Z(t - \rho(t), x) + e^{-kt}p \|_H^2 \leq 4(b_1^2 + k_1^2(t)) \| Z(t - \rho(t), x) \|_H^2 + 2e^{-2kt} \| p \|_H^2,$$

$$\| (b_2 + k_2(t))Z(t - r(t), x) \|_H^2 \leq 4(b_2^2 + k_2^2(t)) \| Z(t - r(t), x) \|_H^2,$$

$$\int_1^\infty \| (b_3 + k_3(t))Z(t - \delta(t), x) \|_H^2 \lambda(dy) \leq 2(b_3^2 + k_3^2(t)) \int_1^\infty y^2 \xi^2(dy) \| Z(t - \delta(t), x) \|_H^2.$$ 

Note that, $\lambda_1 = 1$, $\delta_1 = 2\mu$, $\delta_2 = 4b_1^2$, $\delta_3 = 4b_2^2$, $\delta_4 = 2b_3^2$, $\alpha_2(t) = 4k_1^2(t)$, $\alpha_3(t) = 4k_2^2(t)$, $\alpha_4(t) = 2 \int_1^\infty y^2 \xi^2(dy) k_3^2(t)$, $\beta_2(t) = 2e^{-2kt} \| p \|_H^2$, $\beta_3(t) = 0$, $\beta_4(t) = 0$. Let

$$\mu > 2b_1 + 2b_2^2 + b_3^2 \int_1^\infty y^2 \xi^2(dy),$$

and take $\alpha_1(t) = 1, \sigma_1 = k$. By Theorems 3.1 and 4.1, the energy solution to (5.1) is exponentially stable and almost surely exponentially stable.

Remark 5.1. Unlike earlier studies, ours do not make use of general methods such as Lyapunov methods, Itô formula methods and so forth. As we know, in general, it is impossible to construct suitable Lyapunov’s function (functional) for stochastic partial differential equations with retarded arguments, even for constant delays to deal with stability. In this work, we use the estimate of the coefficients functions of stochastic partial differential equations with variable delays and jumps to derive the sufficient conditions for stability.

Remark 5.2. For $k = 0$ or $\lambda = 0$, (3.1) has been studied by Taniguchi [13] and Wan [14]. Therefore, we improve by the energy inequality the existing results to cover a class of more general stochastic partial differential equations with variable delays and jumps. Moreover, unlike Taniguchi [13], we need not require the functions $\rho(t), r(t), \delta(t)$ to be differentiable.
References