On the Cauchy problem for a two-component Degasperis–Procesi system

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A R T I C L E   I N F O

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This paper is concerned with the Cauchy problem for a two-component Degasperis–Procesi system. Firstly, the local well-posedness for this system in the nonhomogeneous Besov spaces is established. Then the precise blow-up scenario for strong solutions to the system is derived. Finally, two new blow-up criterions and the exact blow-up rate of strong solutions to the system are presented.

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1. Introduction

In this paper we consider the Cauchy problem of the following two-component Degasperis–Procesi system:

$$\begin{cases} m_t + 3mu_x + m_x u + k_3 \rho \rho_x = 0, & t > 0, \ x \in \mathbb{R}, \\ \rho_t + k_2 u \rho_x + (k_1 + k_2) u_x \rho = 0, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$, while $(k_1, k_2, k_3) = (1, 1, c)$ or $(c, 1, 0)$ and c takes an arbitrary value.

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Using the Green function $p(x) \triangleq \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ and the identity $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$, we can rewrite System (1.1) as follows:

$$\begin{cases} u_t + uu_x = P(D)\left(\dfrac{3}{2}u^2 + \dfrac{k_3}{2}\rho^2\right), & t > 0,\ x \in \mathbb{R}, \\ \rho_t + k_2 u\rho_x = -(k_1 + k_2)u_x\rho, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \qquad (1.2)$$

where the operator $P(D) \triangleq -\partial_x(1 - \partial_x^2)^{-1}$.

System (1.1) as the Hamiltonian extension of the Degasperis–Procesi equation was firstly proposed in [43]. In particular, for $(k_1, k_2, k_3) = (c, 1, 0)$, System (1.1) is no more coupled and the equation on ρ becomes linear. Therefore, we only consider the case $(k_1, k_2, k_3) = (1, 1, c)$ in the present paper. That is

$$\begin{cases} u_t + uu_x = P(D)\left(\dfrac{3}{2}u^2 + \dfrac{c}{2}\rho^2\right), & t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\rho_x = -2u_x\rho, & t > 0,\ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0, x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \qquad (1.3)$$

For $\rho \equiv 0$, System (1.1) becomes the Degasperis–Procesi equation [22]. It was proved formally integrable by constructing a Lax pair [21] and the direct and inverse scattering approach to pursue it can be seen in [12]. Moreover, they also presented [21] that the DP equation has a bi-Hamiltonian structure and an infinite number of conservation laws, and admits exact peakon solutions which are analogous to the Camassa–Holm peakons [1,16,17]. It is worth pointing out that solutions of this type are not mere abstractizations: the peakons replicate a feature that is characteristic for the waves of great height—waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves cf. the papers [6,9,44]. The DP equation is a model for nonlinear shallow water dynamics cf. the discussion in [14]. The numerical stability of solitons and peakons, the multi-soliton solutions and their peakon limits, together with an inverse scattering method to compute n-peakon solutions to DP equation have been investigated respectively in [34,40,41]. Furthermore, the traveling wave solutions and the classification of all weak traveling wave solutions to DP equation were presented in [37,45]. After the DP equation appeared, it has been studied in many works [13,27, 33,37,39,41,47,50]. For example, the author established the local well-posedness to DP equation with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ on the line [50] and on the circle [47], and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to DP equation were shown in [51,52]. Similar to the Camassa–Holm equation [5,7,8,15,48,49], the DP equation has not only global strong solutions [38,51] but also blow-up solutions [25,26,38,51]. Apart from these, it has global entropy weak solutions in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$, cf. [4].

Although the DP equation is very similar to the Camassa–Holm equation in many aspects, especially in the structure of equation, there are some essential differences between the two equations. One of the famous features of DP equation is that it has not only peakon solutions $u_c(t, x) = ce^{-|x-ct|}$ with $c > 0$ [21] and periodic peakon solutions [52], but also shock peakons [39] and the periodic shock waves [26]. Besides, the Camassa–Holm equation is a re-expression of geodesic flow on the diffeomorphism group [13] or on the Bott–Virasoro group [42], while the DP equation can be regarded as a non-metric Euler equation [23].

Recently, a large amount of literature was devoted to the two-component Camassa–Holm system [3,11,24,28–32,35,46]. It is noted that the authors in [46] studied the analytic solutions of the Cauchy problem for two-component Camassa–Holm shallow water systems, which were proved in both variables, globally in space and locally in time. The used approach in [46] depends strongly on

the structure of the given system, so we can obtain the same analyticity results of System (1.3) as those in two-component Camassa–Holm system. It is worth pointing out that analytic regularity results hold also for the governing equations cf. the recent paper [10]. This is a further indication that in finding these approximations, structural properties are preserved. However, the Cauchy problem of System (1.3) in Besov spaces has not been discussed yet. The goal of this paper is to establish the local well-posedness of System (1.3) in the nonhomogeneous Besov spaces, derive the precise blow-up scenario of strong solutions to the system, and give the new blow-up criterions with respect to the initial data and the exact blow-up rate of strong solutions to the system. Most of our results can be carried out to the periodic case and to homogeneous Besov spaces.

To solve the problem, we mainly use the ideas of [18,31,32,38]. One of the difficulties is the treatment of critical index in proving local well-posedness of System (1.3), which has been overcome by the interpolation method in some sense. On the other hand, the $H^1 \times L^2$-norm conserved quantity plays a key role in studying the blow-up phenomenon of the two-component Camassa–Holm system [31,32]. Unfortunately, one cannot find this similar conservation law of System (1.3). This difficulty has been dealt with in some sense by obtaining a priori estimate L^∞-norm of the first component of the solutions to System (1.3) and making good use of the structure of the system itself.

We now conclude this introduction by outlining the rest of the paper. In Section 2, we will recall some facts on the Littlewood–Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties, and the transport equation theory. In Section 3, we establish the local well-posedness of the system. In Section 4, we derive the precise blow-up scenario for strong solutions to the system. Section 5 is devoted to some new blow-up results and the exact blow-up rate of strong solutions to the system.

2. Preliminaries

In this section, we will recall some facts on the Littlewood–Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties, and the transport equation theory, which will be used in the sequel.

Proposition 2.1 (*Littlewood–Paley decomposition*). (*See [20].*) *There exists a couple of smooth functions* (χ, φ) *valued in* $[0, 1]$, *such that* χ *is supported in the ball* $B \triangleq \{\xi \in \mathbb{R}^n: |\xi| \leqslant \frac{4}{3}\}$, *and* φ *is supported in the ring* $C \triangleq \{\xi \in \mathbb{R}^n: \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. *Moreover,*

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1,$$

and

$$\mathrm{supp}\, \varphi(2^{-q}\cdot) \cap \mathrm{supp}\, \varphi(2^{-q'}\cdot) = \emptyset, \quad if\ |q - q'| \geqslant 2,$$
$$\mathrm{supp}\, \chi(\cdot) \cap \mathrm{supp}\, \varphi(2^{-q}\cdot) = \emptyset, \quad if\ q \geqslant 1.$$

Then for all $u \in \mathcal{S}'$, we can define the nonhomogeneous dyadic blocks as follows. Let

$$\Delta_q u \triangleq 0, \quad \text{if } q \leqslant -2,$$
$$\Delta_{-1} u \triangleq \chi(D)u = \mathcal{F}^{-1}\chi \mathcal{F}u,$$
$$\Delta_q u \triangleq \varphi(2^{-q}D)u = \mathcal{F}^{-1}\varphi(2^{-q}\xi)\mathcal{F}u, \quad \text{if } q \geqslant 0.$$

Hence,

$$u = \sum_{q \in \mathbb{Z}} \Delta_q u \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where the right-hand side is called the nonhomogeneous Littlewood–Paley decomposition of u.

Remark 2.1.

(1) The low frequency cut-off S_q is defined by

$$S_q u \triangleq \sum_{p=-1}^{q-1} \Delta_p u = \chi\big(2^{-q}D\big)u = \mathcal{F}^{-1}\chi\big(2^{-q}\xi\big)\mathcal{F}u, \quad \forall q \in \mathbb{N}.$$

(2) The Littlewood–Paley decomposition is quasi-orthogonal in L^2 in the following sense:

$$\Delta_p \Delta_q u \equiv 0, \quad \text{if } |p - q| \geqslant 2,$$

$$\Delta_q (S_{p-1}u \Delta_p v) \equiv 0, \quad \text{if } |p - q| \geqslant 5,$$

for all $u, v \in \mathcal{S}'(\mathbb{R}^n)$.
(3) Thanks to Young's inequality, we get

$$\|\Delta_q u\|_{L^p}, \qquad \|S_q u\|_{L^p} \leqslant C \|u\|_{L^p}, \quad \forall 1 \leqslant p \leqslant \infty,$$

where C is a positive constant independent of q.

Definition 2.1 *(Besov spaces).* (See [20].) Let $s \in \mathbb{R}, 1 \leqslant p, r \leqslant \infty$. The nonhomogeneous Besov space $B_{p,r}^s(\mathbb{R}^n)$ ($B_{p,r}^s$ for short) is defined by

$$B_{p,r}^s\big(\mathbb{R}^n\big) \triangleq \big\{ f \in \mathcal{S}'\big(\mathbb{R}^n\big): \|f\|_{B_{p,r}^s} < \infty \big\},$$

where

$$\|f\|_{B_{p,r}^s} \triangleq \big\| 2^{qs} \Delta_q f \big\|_{l^r(L^p)} = \big\| \big(2^{qs}\|\Delta_q f\|_{L^p}\big)_{q \geqslant -1} \big\|_{l^r}.$$

If $s = \infty$, $B_{p,r}^\infty \triangleq \bigcap_{s \in \mathbb{R}} B_{p,r}^s$.

Definition 2.2. Let $T > 0$, $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$. Set

$$E_{p,r}^s(T) \triangleq C\big([0, T]; B_{p,r}^s\big) \cap C^1\big([0, T]; B_{p,r}^{s-1}\big), \quad \text{if } r < \infty,$$

$$E_{p,\infty}^s(T) \triangleq L^\infty\big([0, T]; B_{p,\infty}^s\big) \cap \text{Lip}\big([0, T]; B_{p,\infty}^{s-1}\big)$$

and

$$E_{p,r}^s \triangleq \bigcap_{T>0} E_{p,r}^s(T).$$

Remark 2.2. By Definition 2.1 and Remark 2.1(3), we can deduce that

$$\|\Delta_q u\|_{B_{p,r}^s}, \|S_q u\|_{B_{p,r}^s} \leqslant C \|u\|_{B_{p,r}^s},$$

where C is a positive constant independent of q.

In the following proposition, we list some important properties of Besov spaces.

Proposition 2.2. *(See [2,20,32].) Suppose that $s \in \mathbb{R}$, $1 \leqslant p, r, p_i, r_i \leqslant \infty$, $i = 1, 2$. We have*

(1) *Topological properties*: $B_{p,r}^s$ *is a Banach space which is continuously embedded in* $\mathcal{S}'$.
(2) *Density*: $\mathcal{C}_c^\infty$ *is dense in* $B_{p,r}^s \Leftrightarrow 1 \leqslant p, r < \infty$.
(3) *Embedding*: $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-n(\frac{1}{p_1} - \frac{1}{p_2})}$, *if $p_1 \leqslant p_2$ and $r_1 \leqslant r_2$,*

$$B_{p,r_2}^{s_2} \hookrightarrow B_{p,r_1}^{s_1} \quad \textit{locally compact}, \quad \textit{if } s_1 < s_2.$$

(4) *Algebraic properties*: $\forall s > 0$, $B_{p,r}^s \bigcap L^\infty$ *is an algebra. Moreover, $B_{p,r}^s$ is an algebra, provided that $s > \frac{n}{p}$ or $s \geqslant \frac{n}{p}$ and $r = 1$.*
(5) *1-D Morse-type estimates*:
    (i) *For $s > 0$,*

$$\|fg\|_{B_{p,r}^s(\mathbb{R})} \leqslant C \big(\|f\|_{B_{p,r}^s(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|g\|_{B_{p,r}^s(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R})} \big). \tag{2.1}$$

   (ii) *$\forall s_1 \leqslant \frac{1}{p} < s_2$ ($s_2 \geqslant \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, we have*

$$\|fg\|_{B_{p,r}^{s_1}(\mathbb{R})} \leqslant C \|f\|_{B_{p,r}^{s_1}(\mathbb{R})} \|g\|_{B_{p,r}^{s_2}(\mathbb{R})}. \tag{2.2}$$

   (iii) *In Sobolev spaces $H^s(\mathbb{R}) = B_{2,2}^s(\mathbb{R})$, we have for $s > 0$,*

$$\|f \partial_x g\|_{H^s(\mathbb{R})} \leqslant C \big(\|f\|_{H^{s+1}(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} + \|f\|_{L^\infty(\mathbb{R})} \|\partial_x g\|_{H^s(\mathbb{R})} \big), \tag{2.3}$$

   where C is a positive constant independent of f and g.
(6) *Complex interpolation*:

$$\|f\|_{B_{p,r}^{\theta s_1 + (1-\theta) s_2}} \leqslant \|f\|_{B_{p,r}^{s_1}}^\theta \|f\|_{B_{p,r}^{s_2}}^{1-\theta}, \quad \forall u \in B_{p,r}^{s_1} \cap B_{p,r}^{s_1}, \quad \forall \theta \in [0, 1]. \tag{2.4}$$

(7) *Fatou lemma*: *if $(u_n)_{n \in \mathbb{N}}$ is bounded in $B_{p,r}^s$ and $u_n \to u$ in $\mathcal{S}'$, then $u \in B_{p,r}^s$ and*

$$\|u\|_{B_{p,r}^s} \leqslant \liminf_{n \to \infty} \|u_n\|_{B_{p,r}^s}.$$

(8) *Let $m \in \mathbb{R}$ and f be an S^m-multiplier (i.e., $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}^n$, $\exists$ a constant C_α, s.t. $|\partial^\alpha f(\xi)| \leqslant C_\alpha (1 + |\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbb{R}^n$). Then the operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.*

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

Lemma 2.1 *(A priori estimates in Besov spaces). (See [18,20].) Let $1 \leqslant p, r \leqslant \infty$ and $s > -\min(\frac{1}{p}, 1 - \frac{1}{p})$. Assume that $f_0 \in B_{p,r}^s$, $F \in L^1(0, T; B_{p,r}^s)$, and $\partial_x v$ belongs to $L^1(0, T; B_{p,r}^{s-1})$ if $s > 1 + \frac{1}{p}$ or to $L^1(0, T; B_{p,r}^{\frac{1}{p}} \cap L^\infty)$ otherwise. If $f \in L^\infty(0, T; B_{p,r}^s) \bigcap C([0, T]; \mathcal{S}')$ solves the following 1-D linear transport equation:*

$$(T) \quad \begin{cases} \partial_t f + v \, \partial_x f = F, \\ f|_{t=0} = f_0, \end{cases}$$

then there exists a constant C depending only on s, p and r, and such that the following statements hold:

(1) *If* $r = 1$ *or* $s \neq 1 + \frac{1}{p}$,

$$\|f(t)\|_{B^s_{p,r}} \leqslant \|f_0\|_{B^s_{p,r}} + \int_0^t \|F(\tau)\|_{B^s_{p,r}} \, d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{B^s_{p,r}} \, d\tau$$

or hence,

$$\|f(t)\|_{B^s_{p,r}} \leqslant e^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B^s_{p,r}} \, d\tau \right)$$

with $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{\frac{1}{p}}_{p,r} \cap L^\infty} \, d\tau$ *if* $s < 1 + \frac{1}{p}$ *and* $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{s-1}_{p,r}} \, d\tau$ *else.*

(2) *If* $s \leqslant 1 + \frac{1}{p}$, *and* $f_0' \in L^\infty$, $\partial_x f \in L^\infty((0, T]) \times \mathbb{R})$ *and* $\partial_x F \in L^1(0, T; L^\infty)$, *then*

$$\|f(t)\|_{B^s_{p,r}} + \|\partial_x f(t)\|_{L^\infty}$$

$$\leqslant e^{CV(t)} \left(\|f_0\|_{B^s_{p,r}} + \|\partial_x f_0\|_{L^\infty} + \int_0^t e^{-CV(\tau)} \left(\|F(\tau)\|_{B^s_{p,r}} + \|\partial_x F(\tau)\|_{L^\infty} \right) d\tau \right),$$

with $V(t) = \int_0^t \|\partial_x v(\tau)\|_{B^{\frac{1}{p}}_{p,r} \cap L^\infty} \, d\tau$.

(3) *If* $f = v$, *then for all* $s > 0$, (1) *holds true with* $V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty} \, d\tau$.

(4) *If* $r < \infty$, *then* $f \in C([0, T]; B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0, T]; B^{s'}_{p,1})$ *for all* $s' < s$.

Lemma 2.2 *(Existence and uniqueness).* *(See [20].)* *Let* p, r, s, f_0 *and* F *be as in the statement of Lemma* 2.1. *Assume that* $v \in L^\rho(0, T; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$, *and* $\partial_x v \in L^1(0, T; B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$, *and* $\partial_x v \in L^1(0, T; B^{\frac{1}{p}}_{p,\infty} \cap L^\infty)$ *if* $s < 1 + \frac{1}{p}$. *Then* (T) *has a unique solution* $f \in L^\infty(0, T; B^s_{p,r}) \bigcap (\bigcap_{s' < s} C([0, T]; B^{s'}_{p,1}))$ *and the inequalities of Lemma* 2.1 *can hold true. Moreover, if* $r < \infty$, *then* $f \in C([0, T]; B^s_{p,r})$.

Lemma 2.3 *(A priori estimate in Sobolev spaces).* *(See [32].)* *Let* $0 < \sigma < 1$. *Assume that* $f_0 \in H^\sigma$, $F \in L^1(0, T; H^\sigma)$, *and* $v, \partial_x v \in L^1(0, T; L^\infty)$. *If* $f \in L^\infty(0, T; H^\sigma) \bigcap C([0, T]; \mathcal{S}')$ *solves* (T), *then* $f \in C([0, T]; H^\sigma)$, *and there exists a constant* C *depending only on* σ *such that the following statement holds:*

$$\|f(t)\|_{H^\sigma} \leqslant \|f_0\|_{H^\sigma} + C \int_0^t \|F(\tau)\|_{H^\sigma} \, d\tau + C \int_0^t V'(\tau) \|f(\tau)\|_{H^\sigma} \, d\tau$$

or hence,

$$\|f(t)\|_{H^\sigma} \leqslant e^{CV(t)} \left(\|f_0\|_{H^\sigma} + \int_0^t \|F(\tau)\|_{H^\sigma} \, d\tau \right)$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty}) \, d\tau$.

3. Local well-posedness

In this section, we will establish the local well-posedness of System (1.3) in the nonhomogeneous Besov spaces.

For completeness, we firstly apply the classical Kato's semigroup theory [36] to obtain the local well-posedness of System (1.3) in Sobolev spaces. More precisely, we have

Theorem 3.1. *Suppose that* $z_0 \triangleq \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$, *and* $s \geqslant 2$. *There exists a maximal existence time* $T = T(\|z_0\|_{H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})}) > 0$, *and a unique solution* $z \triangleq \binom{u}{\rho}$ *to System* (1.3) *such that*

$$z = z(\cdot, z_0) \in C\big([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\big) \cap C^1\big([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})\big).$$

Moreover, the solution depends continuously on the initial data, that is, the mapping $z_0 \mapsto z(\cdot, z_0)$:

$$H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \to C\big([0, T]; H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\big) \cap C^1\big([0, T]; H^{s-1}(\mathbb{R}) \times H^{s-2}(\mathbb{R})\big)$$

is continuous.

Proof. The proof is very similar to that in [24], so we omit it here. □

Now we pay attention to the case in the nonhomogeneous Besov spaces. Uniqueness and continuity with respect to the initial data in some sense can be obtained by the following a priori estimates.

Lemma 3.1. *Let* $1 \leqslant p, r \leqslant \infty$ *and* $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$. *Suppose that we are given* $\binom{u^i}{\rho^i} \in L^\infty(0, T; B_{p,r}^s) \cap C([0, T]; \mathcal{S}') \times L^\infty(0, T; B_{p,r}^{s-1}) \cap C([0, T]; \mathcal{S}')$ $(i = 1, 2)$ *two solutions of System* (1.3) *with the initial data* $\binom{u_0^i}{\rho_0^i} \in B_{p,r}^s \times B_{p,r}^{s-1}$ $(i = 1, 2)$ *and let* $u^{12} \triangleq u^2 - u^1$ *and* $\rho^{12} \triangleq \rho^2 - \rho^1$. *Then for all* $t \in [0, T]$, *we have*

(1) *if* $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$, *but* $s \neq 2 + \frac{1}{p}, 3 + \frac{1}{p}$, *then*

$$\begin{aligned} &\big\|u^{12}(t)\big\|_{B_{p,r}^{s-1}} + \big\|\rho^{12}(t)\big\|_{B_{p,r}^{s-2}} \\ &\leqslant \big(\big\|u_0^{12}\big\|_{B_{p,r}^{s-1}} + \big\|\rho_0^{12}\big\|_{B_{p,r}^{s-2}}\big) e^{C \int_0^t (\|u^1(\tau)\|_{B_{p,r}^s} + \|u^2(\tau)\|_{B_{p,r}^s} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}}) d\tau}; \end{aligned} \quad (3.1)$$

(2) *if* $s = 2 + \frac{1}{p}$, *then*

$$\begin{aligned} &\big\|u^{12}(t)\big\|_{B_{p,r}^{s-1}} + \big\|\rho^{12}(t)\big\|_{B_{p,r}^{s-2}} \\ &\leqslant C\big(\big\|u_0^{12}\big\|_{B_{p,r}^{s-1}} + \big\|\rho_0^{12}\big\|_{B_{p,r}^{s-2}}\big)^\theta \times \big(\big\|u^1(t)\big\|_{B_{p,r}^s} + \big\|u^2(t)\big\|_{B_{p,r}^s}\big)^{1-\theta} \\ &\quad \times e^{\theta C \int_0^t (\|u^1(\tau)\|_{B_{p,r}^s} + \|u^2(\tau)\|_{B_{p,r}^s} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}}) d\tau} + \big(\big\|u_0^{12}\big\|_{B_{p,r}^{s-1}} + \big\|\rho_0^{12}\big\|_{B_{p,r}^{s-2}}\big) \\ &\quad \times e^{C \int_0^t (\|u^1(\tau)\|_{B_{p,r}^s} + \|u^2(\tau)\|_{B_{p,r}^s} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}}) d\tau}; \end{aligned}$$

(3) *if* $s = 3 + \frac{1}{p}$, *then*

$$\|u^{12}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(t)\|_{B_{p,r}^{s-2}}$$

$$\leqslant C\big(\|u_0^{12}\|_{B_{p,r}^{s-1}} + \|\rho_0^{12}\|_{B_{p,r}^{s-2}}\big)^{\theta} \times \big(\|\rho^1(t)\|_{B_{p,r}^{s-1}} + \|\rho^2(t)\|_{B_{p,r}^{s-1}}\big)^{1-\theta}$$

$$\times e^{\theta C \int_0^t (\|u^1(\tau)\|_{B_{p,r}^{s}} + \|u^2(\tau)\|_{B_{p,r}^{s}} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}}) \, d\tau} + \big(\|u_0^{12}\|_{B_{p,r}^{s-1}} + \|\rho_0^{12}\|_{B_{p,r}^{s-2}}\big)$$

$$\times e^{C \int_0^t (\|u^1(\tau)\|_{B_{p,r}^{s}} + \|u^2(\tau)\|_{B_{p,r}^{s}} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}}) \, d\tau},$$

where $\theta \in (0, 1)$.

Proof. It is obvious that $\binom{u^{12}}{\rho^{12}} \in L^{\infty}(0, T; B_{p,r}^{s}) \cap C([0, T]; \mathcal{S}') \times L^{\infty}(0, T; B_{p,r}^{s-1}) \cap C([0, T]; \mathcal{S}')$ solves the following Cauchy problem of the transport equations:

$$\begin{cases} \partial_t u^{12} + u^1 \partial_x u^{12} = F(t, x), \\ \partial_t \rho^{12} + u^1 \partial_x \rho^{12} = G(t, x), \\ u^{12}|_{t=0} = u_0^{12} \triangleq u_0^2 - u_0^1, \\ \rho^{12}|_{t=0} = \rho_0^{12} \triangleq \rho_0^2 - \rho_0^1, \end{cases} \tag{3.2}$$

where $F(t, x) \triangleq -u^{12} \partial_x u^2 + P(D)(\frac{3}{2} u^{12}(u^1 + u^2) + \frac{c}{2} \rho^{12}(\rho^1 + \rho^2))$ and $G(t, x) \triangleq -u^{12} \partial_x \rho^2 - 2(\rho^{12} \partial_x u^1 + \rho^2 \partial_x u^{12})$.

Claim. For all $s > \max(1 + \frac{1}{p}, \frac{3}{2})$ and $t \in [0, T]$, we have

$$\|F(t)\|_{B_{p,r}^{s-1}}, \|G(t)\|_{B_{p,r}^{s-2}} \leqslant C\big(\|u^{12}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(t)\|_{s-2}^{s-2}\big)$$

$$\times \big(\|u^1(t)\|_{B_{p,r}^{s}} + \|u^2(t)\|_{B_{p,r}^{s}} + \|\rho^1(t)\|_{B_{p,r}^{s-1}} + \|\rho^2(t)\|_{B_{p,r}^{s-1}}\big),$$

where $C = C(s, p, r, c)$ is a positive constant.

Indeed, for $s > 1 + \frac{1}{p}$, $B_{p,r}^{s-1}$ is an algebra, by Proposition 2.2(4), we have

$$\|u^{12} \partial_x u^2\|_{B_{p,r}^{s-1}} \leqslant C \|u^{12}\|_{B_{p,r}^{s-1}} \|\partial_x u^2\|_{B_{p,r}^{s-1}} \leqslant C \|u^{12}\|_{B_{p,r}^{s-1}} \|u^2\|_{B_{p,r}^{s}}.$$

Note that $P(D) \in Op(S^{-1})$. According to Proposition 2.2(8) and (2.2), we obtain

$$\left\| P(D)\left(\frac{3}{2} u^{12}(u^1 + u^2) \right) \right\|_{B_{p,r}^{s-1}} \leqslant C \|u^{12}\|_{B_{p,r}^{s-1}} \big(\|u^1\|_{B_{p,r}^{s-1}} + \|u^2\|_{B_{p,r}^{s-1}} \big)$$

and

$$\left\| P(D)\left(\frac{c}{2} \rho^{12}(\rho^1 + \rho^2) \right) \right\|_{B_{p,r}^{s-1}} \leqslant C \|\rho^{12}\|_{B_{p,r}^{s-2}} \big(\|\rho^1\|_{B_{p,r}^{s-1}} + \|\rho^2\|_{B_{p,r}^{s-1}} \big),$$

if $\max(1 + \frac{1}{p}, \frac{3}{2}) < s \leqslant 2 + \frac{1}{p}$.

Otherwise, these inequalities can also hold true in view of the fact $B_{p,r}^{s-2}$ is an algebra as $s > 2 + \frac{1}{p}$. Therefore,

$$\|F(t)\|_{B_{p,r}^{s-1}} \leqslant C \big(\|u^{12}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(t)\|_{s-2}^{s-2} \big)$$
$$\times \big(\|u^1(t)\|_{B_{p,r}^s} + \|u^2(t)\|_{B_{p,r}^s} + \|\rho^1(t)\|_{B_{p,r}^{s-1}} + \|\rho^2(t)\|_{B_{p,r}^{s-1}} \big).$$

On the other hand, thanks to (2.2), we get

$$\|u^{12}\partial_x\rho^2\|_{B_{p,r}^{s-2}} \leqslant C\|u^{12}\|_{B_{p,r}^{s-1}}\|\partial_x\rho^2\|_{B_{p,r}^{s-2}},$$
$$\|\rho^{12}\partial_x u^1\|_{B_{p,r}^{s-2}} \leqslant C\|\rho^{12}\|_{B_{p,r}^{s-2}}\|\partial_x u^1\|_{B_{p,r}^{s-1}}$$

and

$$\|\rho^2\partial_x u^{12}\|_{B_{p,r}^{s-2}} \leqslant C\|\rho^2\|_{B_{p,r}^{s-1}}\|\partial_x u^{12}\|_{B_{p,r}^{s-2}},$$

if $\max(1 + \frac{1}{p}, \frac{3}{2}) < s \leqslant 2 + \frac{1}{p}$.

For $s > 2 + \frac{1}{p}$, we can handle it in a similar way. Therefore,

$$\|G(t)\|_{B_{p,r}^{s-2}} \leqslant C \big(\|u^{12}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(t)\|_{s-2}^{s-2} \big)$$
$$\times \big(\|u^1(t)\|_{B_{p,r}^s} + \|u^2(t)\|_{B_{p,r}^s} + \|\rho^1(t)\|_{B_{p,r}^{s-1}} + \|\rho^2(t)\|_{B_{p,r}^{s-1}} \big).$$

This proves our Claim.

Applying Lemma 2.1(1) and the fact that $\|\partial_x w(t)\|_{B_{p,r}^{\frac{1}{p}} \cap L^\infty} \leqslant C\|w(t)\|_{B_{p,r}^s}$, $\|\partial_x w(t)\|_{B_{p,r}^{s-3}} \leqslant C\|\partial_x w(t)\|_{B_{p,r}^{s-2}} \leqslant C\|w(t)\|_{B_{p,r}^s}$, if $w \in B_{p,r}^s$ with $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$, we can obtain, for case (1),

$$\|u^{12}(t)\|_{B_{p,r}^{s-1}} \leqslant \|u_0^{12}\|_{B_{p,r}^{s-1}} + \int_0^t \|F(\tau)\|_{B_{p,r}^{s-1}} \, d\tau + C \int_0^t \|u^1(\tau)\|_{B_{p,r}^s} \|u^{12}(\tau)\|_{B_{p,r}^{s-1}} \, d\tau$$

and

$$\|\rho^{12}(t)\|_{B_{p,r}^{s-2}} \leqslant \|\rho_0^{12}\|_{B_{p,r}^{s-2}} + \int_0^t \|G(\tau)\|_{B_{p,r}^{s-2}} \, d\tau + C \int_0^t \|u^1(\tau)\|_{B_{p,r}^s} \|\rho^{12}(\tau)\|_{B_{p,r}^{s-2}} \, d\tau,$$

which together with the Claim yield

$$\|u^{12}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(t)\|_{B_{p,r}^{s-2}}$$

$$\leqslant \|u_0^{12}\|_{B_{p,r}^{s-1}} + \|\rho_0^{12}\|_{B_{p,r}^{s-2}} + C \int_0^t \big(\|u^{12}(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^{12}(\tau)\|_{s-2}^{s-2} \big)$$

$$\times \big(\|u^1(\tau)\|_{B_{p,r}^s} + \|u^2(\tau)\|_{B_{p,r}^s} + \|\rho^1(\tau)\|_{B_{p,r}^{s-1}} + \|\rho^2(\tau)\|_{B_{p,r}^{s-1}} \big) \, d\tau.$$

Taking advantage of Gronwall's inequality, we get (3.1).

For the critical case (2) $s = 2 + \frac{1}{p}$, we here use the interpolation method to deal with it. Indeed, if we choose $s_1 \in (\max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2}) - 1, s - 1)$, $s_2 \in (s - 1, s)$ and $\theta = \frac{s_2 - (s-1)}{s_2 - s_1} \in (0, 1)$, then $s - 1 = \theta s_1 + (1 - \theta)s_2$. According to Proposition 2.2(6) and the consequence of case (1), we have

$$
\begin{aligned}
\left\| u^{12}(t) \right\|_{B_{p,r}^{s-1}} &\leqslant \left\| u^{12}(t) \right\|_{B_{p,r}^{s_1}}^{\theta} \left\| u^{12}(t) \right\|_{B_{p,r}^{s_2}}^{1-\theta} \\
&\leqslant \left(\left\| u^1(t) \right\|_{B_{p,r}^{s_2}} + \left\| u^2(t) \right\|_{B_{p,r}^{s_2}} \right)^{1-\theta} \left(\left\| u_0^{12} \right\|_{B_{p,r}^{s_1}} + \left\| \rho_0^{12} \right\|_{B_{p,r}^{s_1-1}} \right)^{\theta} \\
&\qquad \times e^{\theta C \int_0^t (\| u^1(\tau) \|_{B_{p,r}^{s_1+1}} + \| u^2(\tau) \|_{B_{p,r}^{s_1+1}} + \| \rho^1(\tau) \|_{B_{p,r}^{s_1}} + \| \rho^2(\tau) \|_{B_{p,r}^{s_1}}) d\tau} \\
&\leqslant C \left(\left\| u_0^{12} \right\|_{B_{p,r}^{s-1}} + \left\| \rho_0^{12} \right\|_{B_{p,r}^{s-2}} \right)^{\theta} \left(\left\| u^1(t) \right\|_{B_{p,r}^{s}} + \left\| u^2(t) \right\|_{B_{p,r}^{s}} \right)^{1-\theta} \\
&\qquad \times e^{\theta C \int_0^t (\| u^1(\tau) \|_{B_{p,r}^{s}} + \| u^2(\tau) \|_{B_{p,r}^{s}} + \| \rho^1(\tau) \|_{B_{p,r}^{s-1}} + \| \rho^2(\tau) \|_{B_{p,r}^{s-1}}) d\tau}.
\end{aligned}
$$

On the other hand, thanks to $s - 2 = \frac{1}{p} < 1 + \frac{1}{p}$, then the estimate for $\| v^{12}(t) \|_{B_{p,r}^{s-2}}$ in case (1) can also hold true. Hence, we can get the desired result.

For the critical case (3) $s = 3 + \frac{1}{p}$, its proof is very similar to that of case (2). Therefore, we complete our proof of Lemma 3.1. □

We next construct the approximation solutions to System (1.3) as follows.

Lemma 3.2. *Let p and r be as in the statement of Lemma 3.1. Assume that $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$ and $s \neq 2 + \frac{1}{p}$, $z_0 \triangleq \binom{u_0}{\rho_0} \in B_{p,r}^s \times B_{p,r}^{s-1}$ and $z^0 \triangleq \binom{u^0}{\rho^0} = \binom{0}{0}$. Then*

(1) *there exists a sequence of smooth functions $(z^n)_{n \in \mathbb{N}} \triangleq \binom{u^n}{\rho^n}_{n \in \mathbb{N}}$ belonging to $(C(\mathbb{R}^+; B_{p,r}^\infty))^2$ and solving the following linear transport equations by induction:*

$$
(T_n) \quad \begin{cases}
\partial_t u^{n+1} + u^n \partial_x u^{n+1} = P(D)\left(\dfrac{3}{2}(u^n)^2 + \dfrac{c}{2}(\rho^n)^2 \right), \\[2mm]
\partial_t \rho^{n+1} + u^n \partial_x \rho^{n+1} = -2\rho^n \partial_x u^n, \\[2mm]
u^{n+1}|_{t=0} \triangleq u_0^{n+1}(x) = S_{n+1} u_0, \\[2mm]
\rho^{n+1}|_{t=0} \triangleq \rho_0^{n+1}(x) = S_{n+1} \rho_0,
\end{cases}
$$

(2) *there exists $T > 0$ such that the solution $(z^n)_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)$ and a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1}) \times C([0, T]; B_{p,r}^{s-2})$, whence it converges to some limit $z \triangleq \binom{u}{\rho} \in C([0, T]; B_{p,r}^{s-1}) \times C([0, T]; B_{p,r}^{s-2})$.*

Proof. Since all the data $S_{n+1} u_0, S_{n+1} \rho_0 \in B_{p,r}^\infty$, it then follows from Lemma 2.2 and by induction with respect to the index n that (1) holds.

To prove (2), applying Remark (2.2) and simulating the proof of Lemma 3.1(1), we obtain that for $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$ and $s \neq 2 + \frac{1}{p}$,

$$
a_{n+1}(t) \leqslant C e^{C U^n(t)} \left(A + \int_0^t e^{-C U^n(\tau)} a_n^2(\tau) \, d\tau \right), \tag{3.3}
$$

where $a_n(t) \triangleq \| u^n(t) \|_{B_{p,r}^s} + \| \rho^n(t) \|_{B_{p,r}^{s-1}}$, $A \triangleq \| u_0 \|_{B_{p,r}^s} + \| \rho_0 \|_{B_{p,r}^{s-1}}$ and $U^n(t) \triangleq \int_0^t \| u^n(\tau) \|_{B_{p,r}^s} \, d\tau$.

Choose $0 < T < \frac{1}{2C^2A}$ and suppose that

$$a_n(t) \leqslant \frac{CA}{1 - 2C^2At}, \quad \forall t \in [0, T]. \tag{3.4}$$

Noting that $e^{C(U^n(t) - U^n(\tau))} \leqslant \sqrt{\frac{1 - 2C^2A\tau}{1 - 2C^2At}}$ and substituting (3.4) into (3.3) yields

$$
\begin{aligned}
a_{n+1}(t) &\leqslant \frac{CA}{\sqrt{1 - 2C^2At}} + \frac{C}{\sqrt{1 - 2C^2At}} \int_0^t \frac{C^2A^2}{(1 - 2C^2A\tau)^{\frac{3}{2}}} \, d\tau \\
&= \frac{CA}{\sqrt{1 - 2C^2At}} + \frac{C}{\sqrt{1 - 2C^2At}} \left(\frac{A}{\sqrt{1 - 2C^2At}} - A \right) \\
&\leqslant \frac{CA}{1 - 2C^2At},
\end{aligned}
$$

which implies that

$$\left(z^n\right)_{n \in \mathbb{N}} \quad \text{is uniformly bounded in } C\left([0, T]; B_{p,r}^s\right) \times C\left([0, T]; B_{p,r}^{s-1}\right).$$

Using the equations (T_n) and the similar argument in the proof of Lemma 3.1(1), one can easily prove that

$$\begin{pmatrix} \partial_t u^{n+1} \\ \partial_t \rho^{n+1} \end{pmatrix}_{n \in \mathbb{N}} \quad \text{is uniformly bounded in } C\left([0, T]; B_{p,r}^{s-1}\right) \times C\left([0, T]; B_{p,r}^{s-2}\right).$$

Hence,

$$\left(z^n\right)_{n \in \mathbb{N}} \quad \text{is uniformly bounded in } E_{p,r}^s(T) \times E_{p,r}^{s-1}(T).$$

Now it suffices to show that $(z^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1}) \times C([0, T]; B_{p,r}^{s-2})$. Indeed, for all $m, n \in \mathbb{N}$, from (T_n), we have

$$
\begin{aligned}
&\partial_t \left(u^{n+m+1} - u^{n+1}\right) + u^{n+m} \partial_x \left(u^{n+m+1} - u^{n+1}\right) \\
&= P(D) \left(\frac{3}{2} \left(u^{n+m} - u^n\right)\left(u^{n+m} + u^n\right) + \frac{c}{2} \left(\rho^{n+m} - \rho^n\right)\left(\rho^{n+m} + \rho^n\right) \right) + \left(u^n - u^{n+m}\right) \partial_x u^{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\partial_t \left(\rho^{n+m+1} - \rho^{n+1}\right) + u^{n+m} \partial_x \left(\rho^{n+m+1} - \rho^{n+1}\right) \\
&= -2\left(\left(\rho^{n+m} - \rho^n\right)\partial_x u^n + \rho^{n+m} \partial_x \left(u^{n+m} - u^n\right)\right) + \left(u^n - u^{n+m}\right)\partial_x \rho^{n+1}.
\end{aligned}
$$

Similar to the proof of Lemma 3.1(1), for $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$ and $s \neq 2 + \frac{1}{p}, 3 + \frac{1}{p}$, we can obtain that

$$b_{n+1}^m(t) \leqslant Ce^{CU^{n+m}(t)} \left(b_{n+1}^m(0) + \int_0^t e^{-CU^{n+m}(\tau)} b_n^m(\tau) d_n^m(\tau) \, d\tau \right),$$

where $b_n^m(t) \triangleq \|(u^{n+m} - u^n)(t)\|_{B_{p,r}^{s-1}} + \|(\rho^{n+m} - \rho^n)(t)\|_{B_{p,r}^{s-2}}$, $U^{n+m}(t) \triangleq \int_0^t \|u^{n+m}(\tau)\|_{B_{p,r}^s} d\tau$, and
$d_n^m(t) \triangleq \|u^n(t)\|_{B_{p,r}^s} + \|u^{n+1}(t)\|_{B_{p,r}^s} + \|u^{n+m}(t)\|_{B_{p,r}^s} + \|\rho^n(t)\|_{B_{p,r}^{s-1}} + \|\rho^{n+1}(t)\|_{B_{p,r}^{s-1}} + \|\rho^{n+m}(t)\|_{B_{p,r}^{s-1}}$.

Thanks to Remark 2.1, we have

$$
\left\| \sum_{q=n+1}^{n+m} \Delta_q u_0 \right\|_{B_{p,r}^{s-1}} = \left(\sum_{k \geqslant -1} 2^{k(s-1)r} \left\| \Delta_k \left(\sum_{q=n+1}^{n+m} \Delta_q u_0 \right) \right\|_{L^p}^r \right)^{\frac{1}{r}}
$$
$$
\leqslant C \left(\sum_{k=n}^{n+m+1} 2^{-kr} 2^{ksr} \|\Delta_k u_0\|_{L^p}^r \right)^{\frac{1}{r}}
$$
$$
\leqslant C 2^{-n} \|u_0\|_{B_{p,r}^s}.
$$

Similarly,

$$
\left\| \sum_{q=n+1}^{n+m} \Delta_q \rho_0 \right\|_{B_{p,r}^{s-2}} \leqslant C 2^{-n} \|\rho_0\|_{B_{p,r}^{s-1}}.
$$

Hence, we obtain

$$
b_{n+1}^m(0) \leqslant C 2^{-n} \left(\|u_0\|_{B_{p,r}^s} + \|\rho_0\|_{B_{p,r}^{s-1}} \right).
$$

According to the fact that $(z^n)_{n \in \mathbb{N}}$ is uniformly bounded in $E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)$, we can find a positive constant C_T independent of n, m such that

$$
b_{n+1}^m(t) \leqslant C_T \left(2^{-n} + \int_0^t b_n^m(\tau) d\tau \right), \quad \forall t \in [0, T].
$$

Arguing by induction with respect to the index n, we can obtain

$$
b_{n+1}^m(t) \leqslant C_T \left(2^{-n} \sum_{k=0}^n \frac{(2TC_T)^k}{k!} + C_T^{n+1} \int_0^t \frac{(t-\tau)^n}{n!} d\tau \right)
$$
$$
\leqslant \left(C_T \sum_{k=0}^n \frac{(2TC_T)^k}{k!} \right) 2^{-n} + C_T \frac{(TC_T)^{n+1}}{(n+1)!},
$$

which implies the desired result.

On the other hand, for the critical point $3 + \frac{1}{p}$, we can apply the interpolation method which has been used in the proof of Lemma 3.1 to show that $(z^n)_{n \in \mathbb{N}}$ is also a Cauchy sequence in $C([0, T]; B_{p,r}^{s-1}) \times C([0, T]; B_{p,r}^{s-2})$ for this critical case. Therefore, we have completed the proof of Lemma 3.2. □

Now we are in the position to prove the main theorem of this section.

Theorem 3.2. *Assume that $1 \leqslant p, r \leqslant \infty$ and $s > \max(2 - \frac{1}{p}, 1 + \frac{1}{p}, \frac{3}{2})$ with $s \neq 2 + \frac{1}{p}$. Let $z_0 \triangleq \binom{u_0}{\rho_0} \in B_{p,r}^s \times B_{p,r}^{s-1}$ and $z \triangleq \binom{u}{\rho}$ be the obtained limit in Lemma 3.2. Then there exists a time $T > 0$ such that $z \in$*

$E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)$ is the unique solution to System (3.1), and the mapping $z_0 \mapsto z$: is continuous from $B_{p,r}^s \times B_{p,r}^{s-1}$ into

$$C\big([0,T]; B_{p,r}^{s'}\big) \cap C^1\big([0,T]; B_{p,r}^{s'-1}\big) \times C\big([0,T]; B_{p,r}^{s'-1}\big) \cap C^1\big([0,T]; B_{p,r}^{s'-2}\big)$$

for all $s' < s$ if $r = \infty$ and $s' = s$ otherwise.

Proof. We first claim that $z \in E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)$ solves System (3.1).
   In fact, according to Lemma 3.2(2) and Proposition 2.2(7), one can get

$$z \in L^\infty\big([0,T]; B_{p,r}^s\big) \times L^\infty\big([0,T]; B_{p,r}^{s-1}\big).$$

For all $s' < s$, Lemma 3.2(2) applied again, together with an interpolation argument yields

$$z^n \to z, \quad \text{as } n \to \infty, \quad \text{in } C\big([0,T]; B_{p,r}^{s'}\big) \times C\big([0,T]; B_{p,r}^{s'-1}\big).$$

Taking limit in (T_n), we can see that z solves System (3.1) in the sense of $C([0,T]; B_{p,r}^{s'-1}) \times C([0,T]; B_{p,r}^{s'-2})$ for all $s' < s$.
   Making use of the equations in System (1.3) twice and the similar proof in the Claim of Lemma 3.1, together with Lemma 2.1(4) and Lemma 2.2 yields $z \in E_{p,r}^s(T) \times E_{p,r}^{s-1}(T)$.
   On the other hand, the continuity with respect to the initial data in

$$C\big([0,T]; B_{p,r}^{s'}\big) \cap C^1\big([0,T]; B_{p,r}^{s'-1}\big) \times C\big([0,T]; B_{p,r}^{s'-1}\big) \cap C^1\big([0,T]; B_{p,r}^{s'-2}\big) \quad \big(\forall s' < s\big)$$

can be obtained by Lemma 3.1 and a simple interpolation argument. While the continuity in $C([0,T]; B_{p,r}^s) \cap C^1([0,T]; B_{p,r}^{s-1}) \times C([0,T]; B_{p,r}^{s-1}) \cap C^1([0,T]; B_{p,r}^{s-2})$ when $r < \infty$ can be proved through the use of a sequence of viscosity approximation solutions $\big(\begin{smallmatrix} u_\varepsilon \\ \rho_\varepsilon \end{smallmatrix}\big)_{\varepsilon > 0}$ for System (1.3) which converges uniformly in $C([0,T]; B_{p,r}^s) \cap C^1([0,T]; B_{p,r}^{s-1}) \times C([0,T]; B_{p,r}^{s-1}) \cap C^1([0,T]; B_{p,r}^{s-2})$. This completes the proof of Theorem 3.1. $\square$

Remark 3.1.

(1) Note that for every $s \in \mathbb{R}$, $B_{2,2}^s = H^s$. Theorem 3.2 holds true in the corresponding Sobolev spaces with $\frac{3}{2} < s \neq \frac{5}{2}$, which almost improves the result of Theorem 3.1 proved by Kato's theory, where $s \geqslant 2$ is required. Therefore, Theorem 3.2 together with Theorem 3.1 implies that the conclusion of Theorem 3.1 holds true for all $s > \frac{3}{2}$.

(2) As we know, $u_c(t,x) = ce^{-|x-ct|}$ with $c \in \mathbb{R}$ is the solitary wave solution to DP equation [21], then the index $s = \frac{3}{2}$ is critical in Besov spaces $B_{2,r}^s$ in the following sense [19]: System (1.3) is not local well-posedness in $B_{2,\infty}^{\frac{3}{2}}$. More precisely, there exists a global solution $u_1 \in L^\infty(\mathbb{R}^+; B_{2,\infty}^{\frac{3}{2}})$ and $v \equiv 0$ to System (1.3) such that for any $T > 0$ and $\varepsilon > 0$, there exists a solution $u_2 \in L^\infty(0,T; B_{2,\infty}^{\frac{3}{2}})$ and $v \equiv 0$ to System (1.3) with

$$\big\| u_1(0) - u_2(0) \big\|_{B_{2,\infty}^{\frac{3}{2}}} \leqslant \varepsilon \quad \text{but} \quad \| u_1 - u_2 \|_{L^\infty(0,T; B_{2,\infty}^{\frac{3}{2}})} \geqslant 1.$$

4. The precise blow-up scenario

In this section, we will derive the precise blow-up scenario of strong solutions to System (1.3). Firstly, let us consider the following differential equation:

$$\begin{cases} q_t = u(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

where u denotes the first component of the solution z to System (1.3).

The following lemmas are very crucial to study the blow-up phenomena of strong solutions to System (1.3).

Lemma 4.1. *(See [5].) Let* $u \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$, $s \geqslant 2$. *Then Eq. (4.1) has a unique solution* $q \in C^1([0, T) \times \mathbb{R}; \mathbb{R})$. *Moreover, the map* $q(t, \cdot)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$q_x(t, x) = \exp\left(\int_0^t u_x\big(s, q(s, x)\big) \, ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 4.2. *Let* $z_0 \triangleq \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s > \frac{3}{2}$ *and* $T > 0$ *be the maximal existence time of the corresponding solution* $z \triangleq \binom{u}{\rho}$ *to System* (1.3), *which is guaranteed by Remark* 3.1(1). *Then we have*

$$\rho\big(t, q(t, x)\big) q_x^2(t, x) = \rho_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{4.2}$$

Moreover, if there exists an $M > 0$ *such that* $u_x(t, x) \geqslant -M$ *for all* $(t, x) \in [0, T) \times \mathbb{R}$, *then*

$$\big\| \rho(t, \cdot) \big\|_{L^\infty}, \big\| \rho(t, \cdot) \big\|_{L^2} \leqslant e^{2Mt} \| \rho_0 \|_{H^{s-1}}, \quad \forall t \in [0, T).$$

Proof. Differentiating the left-hand side of Eq. (4.2) with respect to t and making use of (4.1) and System (1.3), we obtain

$$\begin{aligned} \frac{d}{dt} \big(\rho\big(t, q(t, x)\big) q_x^2(t, x) \big) &= \big(\rho_t(t, q) + \rho_x(t, q) q_t(t, x) \big) q_x^2(t, x) + 2\rho(t, q) q_x(t, x) q_{xt}(t, x) \\ &= \big(\rho_t(t, q) + \rho_x(t, q) u(t, q) + 2\rho(t, q) u_x(t, q) \big) q_x^2(t, x) \\ &= 0. \end{aligned}$$

This proves (4.2). By Lemma 4.1, in view of (4.2) and the assumption of the lemma, we obtain for all $t \in [0, T)$

$$\begin{aligned} \big\| \rho(t, \cdot) \big\|_{L^\infty} &= \big\| \rho\big(t, q(t, \cdot)\big) \big\|_{L^\infty} \\ &= \big\| e^{-2 \int_0^t u_x(s, \cdot) \, ds} \rho_0(\cdot) \big\|_{L^\infty} \\ &\leqslant e^{2Mt} \| \rho_0(\cdot) \|_{L^\infty}. \end{aligned}$$

By (4.2) and Lemma 4.1, we get

$$\int_{\mathbb{R}} \left| \rho(t,x) \right|^2 dx = \int_{\mathbb{R}} \left| \rho\big(t, q(t,x)\big) \right|^2 q_x(t,x)\, dx = \int_{\mathbb{R}} \left| \rho_0(x) \right|^2 q_x^{-3}(t,x)\, dx$$

$$\leqslant e^{3Mt} \int_{R} \left| \rho_0(x) \right|^2 dx, \quad \forall t \in [0,T).$$

This completes the proof of the lemma. $\square$

As mentioned in the Introduction, the H^1-norm of the solutions to DP equation is not conserved. However, what saves the game in some sense is to establish a priori estimate for the L^∞-norm of the first component u of the strong solutions to System (1.3).

Lemma 4.3. *Let* $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ *with* $s > \frac{3}{2}$ *and* T *be the maximal existence time of the solution* $z = \binom{u}{\rho}$ *to System* (1.3), *which is guaranteed by Remark* 3.1(1). *Assume that there is an* $M > 0$ *such that* $\|\rho(t, \cdot)\|_{L^\infty}, \|\rho(t, \cdot)\|_{L^2} \leqslant e^{2Mt} \|\rho_0\|_{H^{s-1}}$ *for all* $t \in [0,T)$. *Then for all* $t \in [0,T)$, *we have*

$$\left\| u(t) \right\|_{L^2}^2 \leqslant 2\, e^{2|c|t} \Big(2\|u_0\|_{L^2}^2 + |c|t(1 + 8Mt)\big(e^{2Mt} \|\rho_0\|_{H^{s-1}} \big)^4 \Big) \tag{4.3}$$

and

$$\left\| u(t) \right\|_{L^\infty} \leqslant \frac{3}{2} t e^{2|c|t} \Big(2\|u_0\|_{L^2}^2 + |c|t(1 + 8Mt)\big(e^{2Mt} \|\rho_0\|_{H^{s-1}} \big)^4 + \frac{|c|}{4}\big(e^{2Mt} \|\rho_0\|_{H^{s-1}} \big)^2 \Big) + \|u_0\|_{L^\infty}$$

$$\triangleq J(t). \tag{4.4}$$

Proof. By a standard density argument, here we may assume $s \geqslant 3$ to prove the lemma. Set $w \triangleq (4 - \partial_x^2)^{-1} u$. By the first equation of System (1.1) and the fact that $(\hat{m}_t, \hat{w}) = (\hat{m}, \hat{w}_t)$ or $\int_{\mathbb{R}} m_t w\, dx = \int_{\mathbb{R}} m w_t\, dx$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} mw\, dx = \frac{1}{2}\int_{\mathbb{R}} m_t w\, dx + \frac{1}{2}\int_{\mathbb{R}} m w_t\, dx = \int_{\mathbb{R}} m_t w\, dx$$

$$= -3\int_{\mathbb{R}} wmu_x\, dx - \int_{\mathbb{R}} wm_x u\, dx - c\int_{\mathbb{R}} w\rho\rho_x\, dx$$

$$= -\int_{\mathbb{R}} w(mu)_x\, dx - 2\int_{\mathbb{R}} wmu_x\, dx + \frac{c}{2}\int_{\mathbb{R}} w_x\rho^2\, dx.$$

While

$$\int_{\mathbb{R}} w(mu)_x\, dx = -\int_{\mathbb{R}} w_x mu\, dx = \int_{\mathbb{R}} w_x u^2\, dx - \int_{\mathbb{R}} w_x u_x^2\, dx,$$

and

$$2\int_{\mathbb{R}} wmu_x\, dx = -\int_{\mathbb{R}} w_x u^2 + \int_{\mathbb{R}} w_x u_x^2\, dx.$$

Combining the above three equalities, we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} mw\,dx = c\int_{\mathbb{R}} w_x\rho^2\,dx.$$

Integrating from 0 to t on both sides of the above equality, we have

$$\int_{\mathbb{R}} mw\,dx = \int_{\mathbb{R}} m_0 w_0\,dx + c\int_0^t\int_{\mathbb{R}} w_x\rho^2\,dx\,ds,$$

which implies

$$\|u(t)\|_{L^2}^2 = \|\hat{u}(t)\|_{L^2}^2 \leqslant 4\int_{\mathbb{R}}\frac{1+\xi^2}{4+\xi^2}|\hat{u}(t,\xi)|^2\,d\xi = 4(\hat{m}(t),\hat{w}(t))$$

$$= 4\big(m(t),w(t)\big) = 4(m_0,w_0) + 4c\int_0^t\int_{\mathbb{R}} w_x\rho^2\,dx\,ds$$

$$\leqslant 4\|u_0\|_{L^2}^2 + 4c\int_0^t\int_{\mathbb{R}} w_x\rho^2\,dx\,ds.$$

Note that

$$\big\|w_x(t)\big\|_{L^2}^2 = \big\|\partial_x\big(4-\partial_x^2\big)^{-1}u(t)\big\|_{L^2}^2$$

$$\leqslant \big\|u(t)\big\|_{L^2}^2.$$

Besides, by the assumption of the lemma, we have

$$\big\|\rho(t,\cdot)\big\|_{L^4}^4 \leqslant \big\|\rho(t,\cdot)\big\|_{L^\infty}^2\big\|\rho(t,\cdot)\big\|_{L^2}^2 \leqslant \big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^4, \quad \forall t\in[0,T).$$

Hence,

$$\big\|u(t)\big\|_{L^2}^2 \leqslant 4\|u_0\|_{L^2}^2 + 2c\int_0^t\big(\big\|w_x(s,\cdot)\big\|_{L^2}^2 + \big\|\rho(s,\cdot)\big\|_{L^4}^4\big)\,ds$$

$$\leqslant 4\|u_0\|_{L^2}^2 + 2|c|t\big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^4 + 2|c|\int_0^t\big\|u(s)\big\|_{L^2}^2\,ds.$$

By Gronwall's inequality, we can reach (4.3).

Next we prove (4.4). Indeed, by the first equation in System (1.3), we have

$$u_t + uu_x = -\partial_x p * \left(\frac{3}{2}u^2 + \frac{c}{2}\rho^2\right).$$

Applying Young's inequality and noting that $\|\partial_x p\|_{L^\infty} \leqslant \frac{1}{2}$, we have

$$\left\| -\partial_x p * \left(\frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right) \right\|_{L^\infty} \leqslant \|\partial_x p\|_{L^\infty} \left\| \frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right\|_{L^1}$$

$$\leqslant \frac{3}{4} \|u\|_{L^2}^2 + \frac{|c|}{4} \|\rho\|_{L^2}^2$$

$$\leqslant \frac{3}{4} \|u\|_{L^2}^2 + \frac{|c|}{4} \left(e^{2Mt} \|\rho_0\|_{H^{s-1}} \right)^2.$$

Besides, in view of (4.1), we have

$$\frac{du(t, q(t,x))}{dt} = u_t\big(t, q(t,x)\big) + u_x\big(t, q(t,x)\big) q_t(t,x) = (u_t + u u_x)\big(t, q(t,x)\big).$$

Thanks to (4.3) and the facts above, we deduce

$$-P(t) \leqslant \frac{du(t, q(t,x))}{dt} \leqslant P(t),$$

where

$$P(t) \triangleq \frac{3}{2} e^{2|c|t} \left(2\|u_0\|_{L^2}^2 + |c|t(1 + 8Mt)\left(e^{2Mt} \|\rho_0\|_{H^{s-1}} \right)^4 + \frac{|c|}{4} \left(e^{2Mt} \|\rho_0\|_{H^{s-1}} \right)^2 \right).$$

Integrating the above inequalities with respect to $t < T$ on $[0, t]$ yields

$$-tP(t) + u_0(x) \leqslant u\big(t, q(t,x)\big) \leqslant tP(t) + u_0(x).$$

Therefore, in view of Lemma 4.1, we get the desired result. This completes the proof of the lemma. □

Corollary 4.1. *Let $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and T be the maximal existence time of the solution $z = \binom{u}{\rho}$ to System (1.3), which is guaranteed by Remark 3.1(1). If $\partial_x u \in L^1(0, T; L^\infty)$, then for all $t \in [0, T)$, we have*

$$\|u(t)\|_{L^\infty} \leqslant \frac{3}{2} t e^{2|c|t} \left(2\|u_0\|_{L^2}^2 + |c|t \left(1 + 8 \int\limits_0^t \|\partial_x u(\tau)\|_{L^\infty} \, d\tau \right) \left(e^{2 \int_0^t \|\partial_x u(\tau)\|_{L^\infty} \, d\tau} \|\rho_0\|_{H^{s-1}} \right)^4 \right.$$

$$\left. + \frac{|c|}{4} \left(e^{2 \int_0^t \|\partial_x u(\tau)\|_{L^\infty} \, d\tau} \|\rho_0\|_{H^{s-1}} \right)^2 \right) + \|u_0\|_{L^\infty}$$

$$\triangleq L(t).$$

Proof. By the proof of Lemma 4.2, we also have

$$\|\rho(t, \cdot)\|_{L^\infty}, \|\rho(t, \cdot)\|_{L^2} \leqslant e^{2 \int_0^t \|\partial_x u(\tau)\|_{L^\infty} \, d\tau} \|\rho_0\|_{H^{s-1}}, \quad \forall t \in [0, T).$$

It is then easy to prove the corollary by a similar argument as in the proof of Lemma 4.3. □

Theorem 4.1. *Let $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and T be the maximal existence time of the solution $z = \binom{u}{\rho}$ to System* (1.3), *which is guaranteed by Remark* 3.1(1). *If $T < \infty$, then*

$$\int_0^T \left\| \partial_x u(\tau) \right\|_{L^\infty} d\tau = \infty.$$

Proof. We will prove the theorem by induction with respect to the regular index s $(s > \frac{3}{2})$ as follows.

Step 1. For $s \in (\frac{3}{2}, 2)$, by Lemma 2.3 and the second equation of System (1.3), we have

$$\left\| \rho(t) \right\|_{H^{s-1}} \leqslant \left\| \rho_0 \right\|_{H^{s-1}} + C \int_0^t \left\| \partial_x u(\tau) \rho(\tau) \right\|_{H^{s-1}} d\tau$$

$$+ C \int_0^t \left\| \rho(\tau) \right\|_{H^{s-1}} \left(\left\| u(\tau) \right\|_{L^\infty} + \left\| \partial_x u(\tau) \right\|_{L^\infty} \right) d\tau.$$

Applying (2.1), we get

$$\left\| \partial_x u \rho \right\|_{H^{s-1}} \leqslant C \left(\left\| \partial_x u \right\|_{H^{s-1}} \|\rho\|_{L^\infty} + \left\| \partial_x u \right\|_{L^\infty} \|\rho\|_{H^{s-1}} \right). \tag{4.5}$$

Thus,

$$\left\| \rho(t) \right\|_{H^{s-1}} \leqslant \left\| \rho_0 \right\|_{H^{s-1}} + C \int_0^t \left\| \partial_x u(\tau) \right\|_{H^{s-1}} \left\| \rho(\tau) \right\|_{L^\infty} d\tau$$

$$+ C \int_0^t \left\| \rho(\tau) \right\|_{H^{s-1}} \left(\left\| u(\tau) \right\|_{L^\infty} + \left\| \partial_x u(\tau) \right\|_{L^\infty} \right) d\tau. \tag{4.6}$$

On the other hand, thanks to Lemma 2.1(3) and the first equation of System (1.3), we have ($\forall s > 1$, indeed)

$$\left\| u(t) \right\|_{H^s} \leqslant \|u_0\|_{H^s} + C \int_0^t \left\| P(D) \left(\frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right)(\tau) \right\|_{H^s} d\tau$$

$$+ C \int_0^t \left\| u(\tau) \right\|_{H^s} \left\| \partial_x u(\tau) \right\|_{L^\infty} d\tau.$$

By Proposition 2.2(8) and (2.1), we have

$$\left\| P(D) \left(\frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right) \right\|_{H^s} \leqslant C \left\| \frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right\|_{H^{s-1}}$$

$$\leqslant C \left(\|u\|_{H^{s-1}} \|u\|_{L^\infty} + \|\rho\|_{H^{s-1}} \|\rho\|_{L^\infty} \right).$$

Hence,

$$\left\|u(t)\right\|_{H^s} \leqslant \|u_0\|_{H^s} + C \int_0^t \left\|u(\tau)\right\|_{H^s} \left(\left\|u(\tau)\right\|_{L^\infty} + \left\|\partial_x u(\tau)\right\|_{L^\infty}\right) d\tau$$

$$+ C \int_0^t \left\|\rho(\tau)\right\|_{H^{s-1}} \left\|\rho(\tau)\right\|_{L^\infty} d\tau. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \leqslant \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C \int_0^t \left(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\right)$$

$$\times \left(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|\rho\|_{L^\infty}\right) d\tau.$$

Thanks to Gronwall's inequality, we have

$$\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}$$

$$\leqslant \left(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\right) e^{C \int_0^t (\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\rho(\tau)\|_{L^\infty}) d\tau}. \tag{4.8}$$

Therefore, if $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau < \infty$, then we deduce from (4.8), Corollary 4.1 and the fact $\|\rho(t, \cdot)\|_{L^\infty} \leqslant e^{2 \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau} \|\rho_0\|_{H^{s-1}}$ that

$$\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}$$

$$\leqslant \left(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\right) e^{Ct(e^{2 \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau} \|\rho_0\|_{H^{s-1}} + L(t)) + C \int_0^t \|\partial_x u(\tau)\|_{L^\infty} d\tau}. \tag{4.9}$$

Hence,

$$\limsup_{t \to T} \left(\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\right) < \infty, \tag{4.10}$$

which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s \in (\frac{3}{2}, 2)$.

Step 2. For $s \in [2, \frac{5}{2})$, Lemma 2.1(1) applied to the second equation of System (1.3), we get

$$\left\|\rho(t)\right\|_{H^{s-1}} \leqslant \|\rho_0\|_{H^{s-1}} + C \int_0^t \left\|\partial_x u(\tau) \rho(\tau)\right\|_{H^{s-1}} d\tau$$

$$+ C \int_0^t \left\|\rho(\tau)\right\|_{H^{s-1}} \left\|\partial_x u(\tau)\right\|_{H^{\frac{1}{2}} \cap L^\infty} d\tau,$$

together with (4.5) implies that

$$\left\| \rho(t) \right\|_{H^{s-1}} \leqslant \left\| \rho_0 \right\|_{H^{s-1}} + C \int_0^t \left\| \partial_x u(\tau) \right\|_{H^{s-1}} \left\| \rho(\tau) \right\|_{L^\infty} d\tau$$

$$+ C \int_0^t \left\| \rho(\tau) \right\|_{H^{s-1}} \left\| \partial_x u(\tau) \right\|_{H^{\frac{1}{2}} \cap L^\infty} d\tau. \tag{4.11}$$

By (4.7) and (4.11), we have

$$\left\| u(t) \right\|_{H^s} + \left\| \rho(t) \right\|_{H^{s-1}} \leqslant \left\| u_0 \right\|_{H^s} + \left\| \rho_0 \right\|_{H^{s-1}} + C \int_0^t \left(\left\| u \right\|_{H^s} + \left\| \rho \right\|_{H^{s-1}} \right) \left(\left\| u \right\|_{H^{\frac{3}{2}+\varepsilon_0}} + \left\| \rho \right\|_{L^\infty} \right) d\tau,$$

where $\varepsilon_0 \in (0, \frac{1}{2})$ and we used the fact that $H^{\frac{1}{2}+\varepsilon_0} \hookrightarrow H^{\frac{1}{2}} \cap L^\infty$. Thanks to Gronwall's inequality again, we have

$$\left\| u(t) \right\|_{H^s} + \left\| \rho(t) \right\|_{H^{s-1}}$$

$$\leqslant \left(\left\| u_0 \right\|_{H^s} + \left\| \rho_0 \right\|_{H^{s-1}} \right) e^{C \int_0^t \left(\left\| u(\tau) \right\|_{H^{\frac{3}{2}+\varepsilon_0}} + \left\| \rho(\tau) \right\|_{L^\infty} \right) d\tau}. \tag{4.12}$$

Therefore if $T < \infty$ satisfies $\int_0^T \left\| \partial_x u(\tau) \right\|_{L^\infty} d\tau < \infty$, then we deduce from (4.12), (4.10) with $\frac{3}{2} + \varepsilon_0 \in (\frac{3}{2}, 2)$ and the fact $\left\| \rho(t, \cdot) \right\|_{L^\infty} \leqslant e^{2 \int_0^t \left\| \partial_x u(\tau) \right\|_{L^\infty} d\tau} \left\| \rho_0 \right\|_{H^{s-1}}$ that

$$\limsup_{t \to T} \left(\left\| u(t) \right\|_{H^s} + \left\| \rho(t) \right\|_{H^{s-1}} \right) < \infty, \tag{4.13}$$

which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s \in [2, \frac{5}{2})$.

Step 3. For $s \in (2, 3)$, by differentiating the second equation of System (1.3) with respect to x, we have

$$\partial_t \rho_x + u \, \partial_x \rho_x + 3 u_x \rho_x + 2 u_{xx} \rho = 0.$$

By Lemma 2.3, we get

$$\left\| \partial_x \rho(t) \right\|_{H^{s-2}} \leqslant \left\| \partial_x \rho_0 \right\|_{H^{s-2}} + C \int_0^t \left\| (3 u_x \rho_x + 2 \rho u_{xx})(\tau) \right\|_{H^{s-2}} d\tau$$

$$+ C \int_0^t \left\| \partial_x \rho(\tau) \right\|_{H^{s-2}} \left(\left\| u(\tau) \right\|_{L^\infty} + \left\| \partial_x u(\tau) \right\|_{L^\infty} \right) d\tau.$$

Thanks to (2.3), we have

$$\left\| u_x \rho_x \right\|_{H^{s-2}} \leqslant C \left(\left\| \partial_x u \right\|_{H^{s-1}} \left\| \rho \right\|_{L^\infty} + \left\| \partial_x u \right\|_{L^\infty} \left\| \partial_x \rho \right\|_{H^{s-2}} \right)$$

and

$$\|\rho u_{xx}\|_{H^{s-2}} \leqslant C\big(\|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty} + \|\rho\|_{L^\infty}\|u_{xx}\|_{H^{s-2}}\big).$$

Hence,

$$\big\|\partial_x \rho(t)\big\|_{H^{s-2}} \leqslant \|\partial_x \rho_0\|_{H^{s-2}} + C\int_0^t \big(\|u(\tau)\|_{H^s} + \|\rho(\tau)\|_{H^{s-1}}\big)$$

$$\times \big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty} + \|\rho(\tau)\|_{L^\infty}\big)\,d\tau,$$

which together with (4.7) and (4.6) with $s-2$ instead of $s-1$, yields that

$$\big\|u(t)\big\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}} \leqslant \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)$$

$$\times \big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty} + \|\rho\|_{L^\infty}\big)\,d\tau.$$

Similar to Step 1, we can easily prove the theorem for $s \in (2,3)$.

Step 4. For $s = k \in \mathbb{N}$ and $k \geqslant 3$, by differentiating the second equation of System (1.3) $k-2$ times with respect to x, we get

$$(\partial_t + u\partial_x)\partial_x^{k-2}\rho + \sum_{l_1+l_2=k-3,\, l_1,l_2 \geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1} u\,\partial_x^{l_2+1}\rho + 2\rho\,\partial_x^{k-1} u = 0,$$

which together with Lemma 2.1(1), implies that

$$\big\|\partial_x^{k-2}\rho(t)\big\|_{H^1} \leqslant \big\|\partial_x^{k-2}\rho_0\big\|_{H^1} + C\int_0^t \big\|\partial_x^{k-2}\rho(\tau)\big\|_{H^1}\big\|\partial_x u(\tau)\big\|_{H^{\frac{1}{2}}\cap L^\infty}\,d\tau$$

$$+ C\int_0^t \bigg\| \sum_{l_1+l_2=k-3,\, l_1,l_2 \geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1} u\,\partial_x^{l_2+1}\rho + 2\rho\partial_x^{k-1} u \bigg\|_{H^1}\,d\tau.$$

Since H^1 is an algebra, it follows that

$$\big\|\rho\partial_x^{k-1} u\big\|_{H^1} \leqslant C\|\rho\|_{H^1}\big\|\partial_x^{k-1} u\big\|_{H^1} \leqslant C\|\rho\|_{H^1}\|u\|_{H^s}$$

and

$$\bigg\| \sum_{l_1+l_2=k-3,\, l_1,l_2 \geqslant 0} C_{l_1,l_2}\partial_x^{l_1+1} u\,\partial_x^{l_2+1}\rho \bigg\|_{H^1} \leqslant C\|u\|_{H^{s-1}}\|\rho\|_{H^{s-1}}.$$

Then, we have

$$\big\|\partial_x^{k-2}\rho(t)\big\|_{H^1} \leqslant \big\|\partial_x^{k-2}\rho_0\big\|_{H^1} + C\int_0^t \big(\|u(\tau)\|_{H^s} + \|\rho(\tau)\|_{H^{s-1}}\big)$$

$$\times \big(\|u(\tau)\|_{H^{s-1}} + \|\rho(\tau)\|_{H^1}\big)\,d\tau. \tag{4.14}$$

By the classical Gagliardo–Nirenberg inequality, we have for $\sigma \in (0, 1)$,

$$\|\rho(t)\|_{H^{s-1}} \leqslant C\big(\|\rho(t)\|_{H^{\sigma}} + \|\partial_x^{k-2}\rho(t)\|_{H^1}\big),$$

which together with (4.14), (4.7) and (4.6) with σ instead of $s-1$, yields that

$$\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \leqslant C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) + C\int\limits_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)$$

$$\times \big(\|u\|_{H^{s-1}} + \|\rho\|_{H^1}\big)\, d\tau.$$

By Gronwall's inequality, we obtain

$$\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}$$
$$\leqslant C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big)e^{C\int_0^t(\|u(\tau)\|_{H^{s-1}} + \|\rho(\tau)\|_{H^1})\, d\tau}. \qquad (4.15)$$

If $T < \infty$ satisfies $\int_0^T \|\partial_x u(\tau)\|_{L^\infty}\, d\tau < \infty$, applying Step 3 and arguing by induction assumption, we can obtain that $\|u(t)\|_{H^{s-1}} + \|\rho(t)\|_{H^1}$ is uniformly bounded. Thanks to (4.15), we get

$$\limsup_{t\to T}\big(\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\big) < \infty, \qquad (4.16)$$

which contradicts the assumption that $T < \infty$ is the maximal existence time. This completes the proof of the theorem for $s = k \in \mathbb{N}$ and $k \geqslant 3$.

Step 5. For $s \in (k, k+1)$, $k \in \mathbb{N}$ and $k \geqslant 3$, by differentiating the second equation of System (1.3) $k - 1$ times with respect to x, we get

$$(\partial_t + u\partial_x)\partial_x^{k-1}\rho + \sum_{l_1+l_2=k-2,\, l_1, l_2 \geqslant 0} C_{l_1, l_2} \partial_x^{l_1+1} u\, \partial_x^{l_2+1}\rho + 2\rho\, \partial_x^k u = 0.$$

Applying Lemma 2.3 with $s - k \in (0, 1)$, we have

$$\big\|\partial_x^{k-1}\rho(t)\big\|_{H^{s-k}} \leqslant \big\|\partial_x^{k-1}\rho_0\big\|_{H^{s-k}} + C\int\limits_0^t \big\|\partial_x^{k-1}\rho(\tau)\big\|_{H^{s-k}}\big(\|u(\tau)\|_{L^\infty} + \|\partial_x u(\tau)\|_{L^\infty}\big)\, d\tau$$

$$+ C\int\limits_0^t \bigg\| \sum_{l_1+l_2=k-2,\, l_1, l_2 \geqslant 0} C_{l_1, l_2} \partial_x^{l_1+1} u(\tau)\partial_x^{l_2+1}\rho(\tau) + 2\rho(\tau)\, \partial_x^k u(\tau) \bigg\|_{H^1}\, d\tau. \qquad (4.17)$$

For each $\varepsilon_0 \in (0, \frac{1}{2})$, using (2.3) and the fact that $H^{\frac{1}{2}+\varepsilon_0} \hookrightarrow L^\infty$, we have

$$\big\|\rho\, \partial_x^k u\big\|_{H^{s-k}} \leqslant C\big(\big\|\partial_x^k u\big\|_{H^{s-k}}\|\rho\|_{L^\infty} + \big\|\partial_x^{k-1} u\big\|_{L^\infty}\|\rho\|_{H^{s-k+1}}\big)$$
$$\leqslant C\big(\|u\|_{H^s}\|\rho\|_{L^\infty} + \|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}}\|\rho\|_{H^{s-k+1}}\big) \qquad (4.18)$$

and

$$\left\| \sum_{l_1+l_2=k-2,\, l_1,l_2 \geqslant 0} C_{l_1,l_2} \partial_x^{l_1+1} u \partial_x^{l_2+1} \rho \right\|_{H^{s-k}}$$

$$\leqslant C \sum_{l_1+l_2=k-2,\, l_1,l_2 \geqslant 0} C_{l_1,l_2} \big(\big\| \partial_x^{l_1+1} u \big\|_{L^\infty} \big\| \partial_x^{l_2+1} \rho \big\|_{H^{s-k}} + \big\| \partial_x^{l_1+1} u \big\|_{H^{s-k+1}} \big\| \partial_x^{l_2} \rho \big\|_{L^\infty} \big)$$

$$\leqslant C \big(\|u\|_{H^s} \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon_0}} + \|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}} \|\rho\|_{H^{s-1}} \big). \tag{4.19}$$

Combining (4.17), (4.18) and (4.19), we can get

$$\big\| \partial_x^{k-1} \rho(t) \big\|_{H^{s-k}} \leqslant \big\| \partial_x^{k-1} \rho_0 \big\|_{H^{s-k}} + C \int_0^t \big(\|u(\tau)\|_{H^s} + \|\rho(\tau)\|_{H^{s-1}} \big)$$

$$\times \big(\|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon_0}} \big) \, d\tau,$$

which together with (4.7) and (4.6) with $s-k \in (0,1)$ instead of $s-1$, yields that

$$\big\| u(t) \big\|_{H^s} + \big\| \rho(t) \big\|_{H^{s-1}} \leqslant C \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} \big) + C \int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}} \big)$$

$$\times \big(\|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon_0}} \big) \, d\tau.$$

Thanks to Gronwall's inequality again, we obtain

$$\big\| u(t) \big\|_{H^s} + \big\| \rho(t) \big\|_{H^{s-1}}$$

$$\leqslant C \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} \big) e^{C \int_0^t (\|u\|_{H^{k-\frac{1}{2}+\varepsilon_0}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon_0}}) \, d\tau}.$$

Noting that $k-\frac{1}{2}+\varepsilon_0 < k$, $k-\frac{3}{2}+\varepsilon_0 < k-1$ and $k \geqslant 3$, and applying Step 3 and the similar argument by induction as in Step 4, we can easily get the desired result.

Consequently, we have completed the proof of the theorem from Step 1 to Step 5. □

The following main theorem of this section shows the precise blow-up scenario for sufficiently regular solutions to System (1.3).

Theorem 4.2. *Let $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and $T > 0$ be the maximal existence time of the corresponding solution $z = \binom{u}{\rho}$ to System (1.3), which is guaranteed by Remark 3.1(1). Then the corresponding solution z blows up in finite time if and only if*

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}} \big\{ u_x(t,x) \big\} = -\infty.$$

Proof. As mentioned earlier, we only need to prove the theorem for $s \geqslant 3$. Assume that the solution z blows up in finite time ($T < \infty$) and there exists an $M > 0$ such that

$$u_x(t,x) \geqslant -M, \quad \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{4.20}$$

By Lemma 4.2, we have

$$\left\|\rho(t,\cdot)\right\|_{L^\infty}, \left\|\rho(t,\cdot)\right\|_{L^2} \leqslant e^{2Mt}\|\rho_0\|_{H^{s-1}}, \quad \forall t \in [0, T).$$

Differentiating the first equation in System (1.3) with respect to x and noting that $\partial_x^2 p * f = p * f - f$, we have

$$u_{tx} = -u_x^2 - uu_{xx} - p * \left(\frac{3}{2}u^2 + \frac{c}{2}\rho^2\right) + \frac{3}{2}u^2 + \frac{c}{2}\rho^2. \tag{4.21}$$

Note that

$$\frac{du_x(t, q(t, x))}{dt} = u_{xt}\big(t, q(t, x)\big) + u_{xx}\big(t, q(t, x)\big)q_t(t, x)$$

$$= (u_{tx} + uu_{xx})\big(t, q(t, x)\big). \tag{4.22}$$

By (4.21) and (4.22), in view of $u_x^2 \geqslant 0$, $p * u^2 \geqslant 0$, $\|p * \rho^2\|_{L^\infty} \leqslant \|p\|_{L^1}\|\rho\|_{L^\infty}^2 \leqslant (e^{2Mt}\|\rho_0\|_{H^{s-1}})^2$ and (4.4), we obtain

$$\frac{du_x(t, q(t, x))}{dt} = -u_x^2\big(t, q(t, x)\big) - p * \left(\frac{3}{2}u^2 + \frac{c}{2}\rho^2\right)\big(t, q(t, x)\big) + \left(\frac{3}{2}u^2 + \frac{c}{2}\rho^2\right)\big(t, q(t, x)\big)$$

$$\leqslant |c|\big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^2 + \frac{3}{2}J^2(t).$$

Integrating the above inequality with respect to $t < T$ on $[0, t]$ yields that

$$u_x\big(t, q(t, x)\big) \leqslant u_x(0) + |c|t\big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^2 + \frac{3}{2}tJ^2(t), \quad \forall t \in [0, T).$$

Then for all $t \in [0, T)$, we have

$$\sup_{x \in \mathbb{R}} u_x(t, x) \leqslant \|\partial_x u_0\|_{L^\infty} + |c|t\big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^2 + \frac{3}{2}tJ^2(t)$$

$$\leqslant \|u_0\|_{H^s} + |c|t\big(e^{2Mt}\|\rho_0\|_{H^{s-1}}\big)^2 + \frac{3}{2}tJ^2(t),$$

which together with (4.20) and $T < \infty$, implies that

$$\int_0^T \left\|\partial_x u(\tau)\right\|_{L^\infty} d\tau < \infty.$$

This contradicts Theorem 4.1.

On the other hand, by Sobolev's imbedding theorem, we can see that if

$$\liminf_{t \to T} \inf_{x \in \mathbb{R}}\left\{u_x(t, x)\right\} = -\infty,$$

then the solution z will blow up in finite time. This completes the proof of the theorem. $\square$

Remark 4.1. Theorem 4.2 implies that the blow-up phenomena of the solution z to System (1.3) depends only on the slope of the first component u. That is, the first component u must blow up before the second component ρ in finite time.

5. Blow-up

In this section, we will state two new blow-up criterions with respect to the initial data and the exact blow-up rate of strong solutions to System (1.3).

Remark 4.1 and Lemma 4.2 imply that if we want to study the fine structure of finite time singularities, one should assume in the following that there is an $M > 0$ such that $\|\rho(t, \cdot)\|_{L^\infty}, \|\rho(t, \cdot)\|_{L^2} \leqslant e^{2Mt}\|\rho_0\|_{H^{s-1}}$ for all $t \in [0, T)$. Next we will apply Lemma 4.3 to establish our first blow-up result with respect to the initial data.

Theorem 5.1. Let $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and T be the maximal existence time of the solution $z = \binom{u}{\rho}$ to System (1.3), which is guaranteed by Remark 3.1(1). Assume that there is an $M > 0$ such that $\|\rho(t, \cdot)\|_{L^\infty}, \|\rho(t, \cdot)\|_{L^2} \leqslant e^{2Mt}\|\rho_0\|_{H^{s-1}}$ for all $t \in [0, T)$. Let $\varepsilon > 0$ and $T^\star \triangleq \frac{\ln(1+\frac{2}{\varepsilon})}{\sqrt{|c|+1}(\|\rho_0\|_{H^{s-1}}+1)} > 0$. If there is a point $x_0 \in \mathbb{R}$ such that

$$u_0'(x_0) < -(1+\varepsilon)K(T^\star),$$

where $K(T^\star) \triangleq (\frac{3(|c|+1)}{4}(e^{2MT^\star}(\|\rho_0\|_{H^{s-1}}+1))^2 + \frac{3}{2}J^2(T^\star))^{\frac{1}{2}} > 0$, then $T < T^\star$. In other words, the corresponding solution to System (1.3) blows up in finite time.

Proof. As mentioned earlier, we may assume $s = 3$ here. By (4.21) and (4.22), in view of $p * (\frac{3}{2}u^2) \geqslant 0$ and $\|p * \rho^2\|_{L^\infty} \leqslant \|p\|_{L^\infty}\|\rho\|_{L^2}^2 \leqslant \frac{1}{2}(e^{2Mt}\|\rho_0\|_{H^{s-1}})^2$, we obtain

$$\frac{du_x(t, q(t, x))}{dt} \leqslant -u_x^2(t, q(t, x)) - p * \left(\frac{c}{2}\rho^2(t, q(t, x))\right) + \frac{3}{2}u^2(t, q(t, x)) + \frac{c}{2}\rho^2(t, q(t, x))$$

$$\leqslant -u_x^2(t, q(t, x)) + \frac{3|c|}{4}(e^{2Mt}\|\rho_0\|_{H^{s-1}})^2 + \frac{3}{2}\|u(t)\|_{L^\infty}^2. \tag{5.1}$$

Set $m(t) \triangleq u_x(t, q(t, x_0))$ and fix $\varepsilon > 0$. From (5.1) and (4.4), we have

$$\frac{dm(t)}{dt} \leqslant -m^2(t) + K^2(T^\star), \quad \forall t \in [0, T^\star] \cap [0, T). \tag{5.2}$$

Since $m(0) < -(1+\varepsilon)K(T^\star) < -K(T^\star)$, it then follows that

$$m(t) \leqslant -K(T^\star), \quad \forall t \in [0, T^\star] \cap [0, T).$$

By solving the inequality (5.2), we get

$$\frac{m(0) + K(T^\star)}{m(0) - K(T^\star)}e^{2K(T^\star)t} - 1 \leqslant \frac{2K(T^\star)}{m(t) - K(T^\star)} \leqslant 0. \tag{5.3}$$

Noting that $m(0) < -(1+\varepsilon)K(T^\star)$ and $2K(T^\star)T^\star \geqslant \ln(1+\frac{2}{\varepsilon})$, we deduce that

$$\ln\frac{m(0) - K(T^\star)}{m(0) + K(T^\star)} \leqslant 2K(T^\star)T^\star. \tag{5.4}$$

By (5.3), (5.4) and the fact $0 < \frac{m(0)+K(T^{\star})}{m(0)-K(T^{\star})} < 1$, there exists

$$0 < T < \frac{1}{2K(T^{\star})} \ln \frac{m(0) - K(T^{\star})}{m(0) + K(T^{\star})} \leqslant T^{\star},$$

such that $\lim_{t \to T} m(t) = -\infty$. This completes the proof of the theorem. $\quad\square$

In order to establish the second blow-up result, we need the following useful lemma.

Lemma 5.1. *(See [8].) Let $T > 0$ and $u \in C^1([0, T); H^2)$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) \triangleq \inf_{x \in \mathbb{R}} \big(u_x(t, x)\big) = u_x\big(t, \xi(t)\big).$$

The function $m(t)$ is absolutely continuous on $(0, T)$ with

$$\frac{dm}{dt} = u_{tx}\big(t, \xi(t)\big) \quad a.e. \ on \ (0, T).$$

Theorem 5.2. *Let $z_0 = \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and T be the maximal existence time of the solution $z = \binom{u}{\rho}$ to System (1.3), which is guaranteed by Remark 3.1(1). Assume that $c \geqslant 0$ and the initial data satisfies that u_0 is odd, ρ_0 is even, $u_0'(0) < 0$ and $\rho_0(0) = 0$. Then $T \leqslant -\frac{1}{u_0'(0)} \triangleq T_0$ and $\lim_{t \to T_0} u_x(t, 0) = -\infty$. Moreover, if there is some $x_0 \in \mathbb{R}$ such that $u_0'(x_0) = \inf_{x \in \mathbb{R}} u_0'(x)$ and $\rho_0'(x_0) \neq 0$, then there exists a $T_1 \in (0, -\frac{1}{u_0'(0)}]$, such that $\limsup_{t \to T_1} (\sup_{x \in \mathbb{R}} \rho_x(t, x)) = +\infty$, if $\rho_0'(x_0) > 0$ and $\liminf_{t \to T_1} (\inf_{x \in \mathbb{R}} \rho_x(t, x)) = -\infty$ otherwise.*

Proof. We may assume $s = 3$ here. By the assumption u_0 is odd, ρ_0 is even, and the structure of System (1.3), we have $u(t, x)$ is odd and $\rho(t, x)$ is even with respect to x for $t \in (0, T)$. Thus, $u(t, 0) = 0$ and $\rho_x(t, 0) = 0$.

Since $\rho_0(0) = 0$ and the second equation of System (1.3), it follows that

$$\rho(t, 0) = \rho_0(0)e^{-2\int_0^t u_x(s, 0)\,ds} = 0.$$

Set $M(t) \triangleq u_x(t, 0)$. By (4.21), (4.22) and in view of $c \geqslant 0$, $p * u^2 \geqslant 0$ and $p * \rho^2 \geqslant 0$, we have

$$\frac{dM(t)}{dt} = -M^2(t) - p * \left(\frac{3}{2}u^2(t, 0) + \frac{c}{2}\rho^2(t, 0) \right)$$

$$\leqslant -M^2(t). \tag{5.5}$$

Note that if $M(0) = u_0'(0) < 0$, then $M(t) \leqslant M(0) < 0$ for all $t \in (0, T]$. From (5.5), we obtain $T \leqslant -\frac{1}{u_0'(0)}$ and

$$u_x(t, 0) = M(t) \leqslant \frac{u_0'(0)}{1 + tu_0'(0)} \to -\infty, \tag{5.6}$$

as $t \to -\frac{1}{u_0'(0)}$.

On the other hand, applying Eq. (4.1) and differentiating the second equation in System (1.3) with respect to x, we get

$$\frac{d\rho_x(t, q(t,x))}{dt} = (-3u_x\rho_x - 2u_{xx}\rho)\big(t, q(t,x)\big). \tag{5.7}$$

By Lemma 5.1, there exists $\xi(t) \in \mathbb{R}$ such that

$$u_x\big(t, \xi(t)\big) = \inf_{x\in\mathbb{R}}\big(u_x(t,x)\big) \quad \forall t \in [0, T). \tag{5.8}$$

Hence,

$$u_{xx}\big(t, \xi(t)\big) = 0 \quad \text{a.e. } t \in [0, T). \tag{5.9}$$

By (5.7), (5.9) and Lemma 4.1, we have

$$\frac{d\rho_x(t, \xi(t))}{dt} = -3u_x\big(t, \xi(t)\big)\rho_x\big(t, \xi(t)\big),$$

together with the assumption $u_0'(x_0) = \inf_{x\in\mathbb{R}} u_0'(x)$ and (5.8) yields $\xi(0) = x_0$,

$$\rho_x\big(t, \xi(t)\big) = \rho_0'(x_0)e^{-3\int_0^t u_x(s, \xi(s))\,ds} = \rho_0'(x_0)e^{-3\int_0^t \inf_{x\in\mathbb{R}} u_x(s,x)\,ds}. \tag{5.10}$$

Thanks to (5.6) again, we have for all $t \in [0, T)$,

$$e^{-3\int_0^t \inf_{x\in\mathbb{R}} u_x(s,x)\,ds} \geqslant e^{-3\int_0^t \frac{u_0'(0)}{1+su_0'(0)}\,ds} = \frac{1}{(1 + u_0'(0)t)^3} \to +\infty,$$

as $t \to -\frac{1}{u_0'(0)}$. This implies the desired result and we have completed the proof of the theorem. $\quad\square$

We conclude this section with the exact blow-up rate for blowing-up solutions to System (1.3).

Theorem 5.3. *Let $z_0 \triangleq \binom{u_0}{\rho_0} \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ with $s > \frac{3}{2}$ and $T < \infty$ be the blow-up time of the corresponding solution $z \triangleq \binom{u}{\rho}$ to System (1.3). Assume that there is an $M > 0$ such that $\|\rho(t, \cdot)\|_{L^\infty}, \|\rho(t, \cdot)\|_{L^2} \leqslant e^{2Mt}\|\rho_0\|_{H^{s-1}}$ for all $t \in [0, T)$. Then*

$$\lim_{t\to T}\Big(\inf_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\Big) = -1. \tag{5.11}$$

Proof. We may assume $s = 3$ here. By the assumptions of the theorem and (4.4), we can find an $M_0 > 0$, such that

$$\|\rho(t, \cdot)\|_{L^\infty}, \|u(t, \cdot)\|_{L^\infty} \leqslant M_0, \quad \forall t \in [0, T).$$

Hence,

$$\|p * \rho^2\|_{L^\infty} \leqslant \|p\|_{L^1}\|\rho\|_{L^\infty}^2 \leqslant M_0^2 \quad \text{and} \quad \|p * u^2\|_{L^\infty} \leqslant \|p\|_{L^1}\|u\|_{L^\infty}^2 \leqslant M_0^2,$$

which together with (4.21) and (4.22), implies that

$$\left|\frac{dm(t)}{dt} + m^2(t)\right| \leqslant \widetilde{K}, \tag{5.12}$$

where $\widetilde{K} = \widetilde{K}(|c|, M_0)$ is a positive constant.

For every $\varepsilon \in (0, \frac{1}{2})$, using the fact $m(t) \leqslant u_x(t, 0)$ and (5.6), we can find a $t_0 \in (0, T)$ such that

$$m(t_0) < -\sqrt{\widetilde{K} + \frac{\widetilde{K}}{\varepsilon}} < -\sqrt{\widetilde{K}}.$$

Thanks to (5.12) again, we have

$$m(t) \leqslant -\sqrt{\widetilde{K}}.$$

This implies that $m(t)$ is decreasing on $[t_0, T)$, hence,

$$m(t) < -\sqrt{\widetilde{K} + \frac{\widetilde{K}}{\varepsilon}} < -\sqrt{\frac{\widetilde{K}}{\varepsilon}}, \quad \forall t \in [t_0, T).$$

Noting that $-m^2(t) - \widetilde{K} \leqslant \frac{dm(t)}{dt} \leqslant -m^2(t) + \widetilde{K}$ a.e. $t \in (t_0, T)$, we get

$$1 - \varepsilon \leqslant \frac{d}{dt}\left(\frac{1}{m(t)}\right) \leqslant 1 + \varepsilon \quad \text{a.e. } t \in (t_0, T). \tag{5.13}$$

Integrating (5.13) with respect to $t \in [t_0, T)$ on (t, T) and applying $\lim_{t \to T} m(t) = -\infty$ again, we deduce that

$$(1 - \varepsilon)(T - t) \leqslant -\frac{1}{m(t)} \leqslant (1 + \varepsilon)(T - t). \tag{5.14}$$

Since $\varepsilon \in (0, \frac{1}{2})$ is arbitrary, it then follows from (5.14) that (5.11) holds. This completes the proof of the theorem. □

Acknowledgments

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