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ScienceDirect
Journal of Multivariate Analysis 99 (2008) 2096–2107
Journal of Multivariate Analysis
www.elsevier.com/locate/jmva

Characterizations of multivariate life distributions

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Received 26 March 2007
Available online 8 February 2008

Abstract

Characterizations of multivariate distributions has been a topic of great interest in applied statistics literature for the last three decades. In this paper, we develop characterizations of multivariate lifetime distributions by relationship between multivariate failure rates (reversed failure rates) and the left (right) truncated expectations of functions of random variables. We, then, discuss the application of the results to derive a multivariate Stein type identity.
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AMS subject classifications: 62H05; 62N05

Keywords: Multivariate distributions; Multivariate failure rates; Multivariate reversed failure rates; Conditional expectation; Stein's identity; Pearson family

1. Introduction

Characterization of lifetime distributions by relationship between failure rate (reversed failure rate) and the left (right) truncated expectations of functions of random variables has been a fertile area of research during the last two decades. The various results in this connection in the univariate case deal with specific distributions like normal [15], gamma and negative binomial [26], beta, binomial and Poisson [3], gamma [13], mixture of exponentials [23], gamma and chi-square [2], mixtures of exponential, Lomax and beta [1], discrete models [18], families of distributions like Pearson system [19] and its extension [32], exponential family [8], exponential type [9] and some general classes of distributions that include most of the lifetime models used in practice [29,24,4,10,20]. The general form of the relationships employed in most of these papers

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doi:10.1016/j.jmva.2008.02.003

is of the form

$$E[B(X)|X > x] - \mu = h(x)g(x), \tag{1}$$

where $B(X)$ is a suitably chosen function of a continuous random variable X, $\mu = E(B(X))$, $h(x) = f(x)/\overline{F}(x)$, the failure rate of X and $g(x)$ a differentiable non-negative real-valued function. One can write an equivalent form of Eq. (1) in terms of reversed failure rate $\lambda(x) = f(x)/F(x)$ as

$$\mu - E[B(X)|X \le x] = \lambda(x)\, g(x). \tag{2}$$

When (1) or (2) is satisfied for all x, the density $f(x)$ of X is uniquely determined as

$$\frac{f'(x)}{f(x)} = \frac{\mu - B(x) - g'(x)}{g(x)}, \tag{3}$$

where prime denotes the derivative, and hence for a fixed $B(x)$, the $g(x)$ function characterizes the distribution of X. Apart from identifying the distribution through (1) or (2), these relationships form necessary and sufficient conditions for deriving lower bounds to the variance of any absolutely continuous functions $C(X)$ of X. Moreover, such lower bounds compare favourably with their counterparts like the Cramer–Rao and Chapman–Robbin inequalities in estimation theory, under conditions of validity of the latter [21,22]. While the search for identities of the form Eqs. (1) and (2) is well documented in the univariate case, there are no comparable developments to augment these in the multivariate case other than the pioneering works by Ruiz and Navarro [30] and Kotz et al. [14]. They point out that there exist multivariate distributions which do not satisfy their general relationship and further that the matrix function involved there can be chosen in different ways. With the advancement of complex equipments and systems consisting of several components with dependent lifetimes, multivariate failure rates and mean residual life and the analysis of their behaviour becomes essential to understand equipment reliability and to model lifetime distributions. Motivated by these considerations, the present paper attempts to propose a general framework through which relationships between multivariate failure rates and conditional expectations of functions of random variables can be explored. Though it is possible to have several types of extensions to the multivariate case, the focus here is to discuss those generalizations that reduce to (1) for univariate models in view of the considerable interest generated in the latter formulation.

Since the concept of failure rate in the univariate case lends itself to different definitions in higher dimensions, the characteristic properties (1) and (2) can be generalized to varying forms corresponding to the definition used. Accordingly in Section 2 we use the vector-valued multivariate failure rate of Johnson and Kotz [12] and in Section 3, the scalar failure rate is employed to derive the relevant results. In Section 4 we discuss the applications of our results to derive a multivariate Stein type identity.

2. Relationships for vector failure rates

Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ be a random vector in $\mathbb{R}^p$ supported by the p-dimensional rectangle $S_p = \{\boldsymbol{x}| -\infty \le a_i < x_i < b_i \le \infty, i = 1, 2, \ldots, p\}$, with absolutely continuous survival function $\overline{F}(\boldsymbol{x}) = P(\boldsymbol{X} > \boldsymbol{x})$, distribution function $F(\boldsymbol{x})$ and density function $f(\boldsymbol{x})$, where the ordering implied in $\boldsymbol{X} > \boldsymbol{x}$ is lexicographic. The vector-valued multivariate failure rate of Johnson and Kotz [12] is defined as

$$\boldsymbol{h}(\boldsymbol{x}) = -\nabla \log \overline{F}(\boldsymbol{x}), \tag{4}$$

where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p})$ is the p-dimensional gradient operator. Analogously, the multivariate reversed failure rate has the definition

$$\lambda(\boldsymbol{x}) = \nabla \log F(\boldsymbol{x}). \tag{5}$$

Notice that the ith components of the vectors $\boldsymbol{h}(\boldsymbol{x})$ and $\lambda(\boldsymbol{x})$ are respectively obtained as

$$h_i(\boldsymbol{x}) = -\frac{\partial \log \overline{F}}{\partial x_i} \quad \text{and} \quad \lambda_i(\boldsymbol{x}) = \frac{\partial \log F}{\partial x_i}. \tag{6}$$

Further assume that $B_i(X_i)$, $i = 1, 2, \ldots, p$ are real-valued non-constant functions satisfying $E(B_i(X_i)) < \infty$. Write

$$m_i(\boldsymbol{x}) = E(B_i(X_i)|\boldsymbol{X} > \boldsymbol{x}) \tag{7}$$

and

$$r_i(\boldsymbol{x}) = E(B_i(X_i)|\boldsymbol{X} \le \boldsymbol{x}). \tag{8}$$

Then the multivariate mean residual life $E(\boldsymbol{X} - \boldsymbol{x}|\boldsymbol{X} > \boldsymbol{x})$ has its ith component derived from (7) by substituting $B_i(X_i) = X_i - x_i$ in (7) and similarly the ith component of the reversed mean residual life $E(\boldsymbol{x} - \boldsymbol{X}|\boldsymbol{X} \le \boldsymbol{x})$ is the special case of (8) when $B_i(X_i) = x_i - X_i$.

With these definitions and notations, we establish characterizations of multivariate distributions by relationship between failure rates and conditional expectations.

Theorem 2.1. *A necessary and sufficient condition for the p-dimensional random variable $\boldsymbol{X}$ defined above to satisfy the relationship*

$$g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = m_i(\boldsymbol{x}) - \mu_i, \quad i = 1, 2, \ldots, p \tag{9}$$

for all $\boldsymbol{x}$ in S_p, some positive real-valued function $g_i(\boldsymbol{x})$ defined on S_p and differentiable in each of the arguments and $\mu_i = E(B_i(X_i)|\boldsymbol{X}_i^* > \boldsymbol{x}_i^*)$, is that the distribution of $\boldsymbol{X}$ is determined from

$$\frac{\partial \overline{F}}{\partial x_i} = -C_i(\boldsymbol{x}_i^*) \exp\left[\int_{a_i}^{x_i} \left(\frac{\mu_i - B_i(t_i) - \frac{\partial g_i(\boldsymbol{x}_i, t_i)}{\partial t_i}}{g(\boldsymbol{x}_i, t_i)}\right) \mathrm{d}t_i\right], \quad i = 1, 2, \ldots, p, \tag{10}$$

where $C_i(\boldsymbol{x}_i^*)$ is determined such that $F(\boldsymbol{x})$ is a distribution function, $\boldsymbol{x}_i^* = (x_1 \ldots x_{i-1}\ x_{i+1} \ldots x_p)$ and $(\boldsymbol{x}_i, t_i)$ is the vector $\boldsymbol{x}$ in which the ith component x_i is replaced by t_i.

Proof. To prove the necessary part, we assume (9) and observe that

$$\begin{aligned} m_i(\boldsymbol{x}) = E(B_i(X_i)|\boldsymbol{X} > \boldsymbol{x}) &= \left[\overline{F}(\boldsymbol{x})\right]^{-1} \int_{x_1}^{b_1} \ldots, \int_{x_p}^{b_p} B_i(t_i)(-1)^p \frac{\partial^p \overline{F}}{\partial t_1 \ldots, \partial t_p} \mathrm{d}t_1 \ldots, \mathrm{d}t_p \\ &= -\left[\overline{F}(\boldsymbol{x})\right]^{-1} \int_{x_i}^{b_i} B_i(t_i) \frac{\partial \overline{F}(\boldsymbol{x}_i, t_i)}{\partial t_i} \mathrm{d}t_i. \end{aligned}$$

Hence condition (9) is equivalent to

$$g_i(\boldsymbol{x}) \frac{\partial \overline{F}}{\partial x_i} = \int_{x_i}^{b_i} (B_i(t_i) - \mu_i) \frac{\partial \overline{F}(\boldsymbol{x}_i, t_i)}{\partial t_i} \mathrm{d}t_i$$

which on differentiation with respect to x_i yields

$$g_i(\boldsymbol{x})\frac{\partial^2 \overline{F}(\boldsymbol{x})}{\partial x_i^2} + \frac{\partial \overline{F}(\boldsymbol{x})}{\partial x_i}\frac{\partial g_i(\boldsymbol{x})}{\partial x_i} = (\mu_i - B_i(x_i))\frac{\partial \overline{F}(\boldsymbol{x})}{\partial x_i}. \tag{11}$$

One solution of the second-order partial differential equation (11) is $\overline{F}$ as a function independent of x_i, which is inadmissible for our purpose. For the other solution, set $G_i(\boldsymbol{x}) = -\frac{\partial \overline{F}}{\partial x_i}$ to write

$$\frac{\partial \log G_i(\boldsymbol{x})}{\partial x_i} = [g_i(\boldsymbol{x})]^{-1}\left(\mu_i - B_i(x_i) - \frac{\partial g_i(\boldsymbol{x})}{\partial x_i}\right).$$

Integration over (a_i, x_i) leads to (10). By retracing the above steps, the sufficiency part is established and this completes the proof. □

Remark 1. The distribution of $\boldsymbol{X}$ is found by solving the p simultaneous equations (10). As seen from (10), for a fixed $B_i(X_i)$, the functional form of $g_i(\boldsymbol{x})$ characterizes the distribution

Examples

1. The multivariate Weibull distribution with

$$\overline{F}(\boldsymbol{x}) = \exp\left[-\sum_{i=1}^{p}\lambda_i x_i^{\alpha_i} - \sum_{i<j}\lambda_{ij}x_i^{\alpha_i}x_j^{\alpha_j} - \cdots - \lambda_{12\ldots,p}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_p^{\alpha_p}\right],$$
$$x_i, \alpha_i, \lambda > 0$$

the choice, $B_i(X_i) = X_i^{\alpha_i}$ leads to

$$g_i(\boldsymbol{x}) = x_i/\alpha_i\left[\lambda_i + \sum_j \lambda_{ij}x_j^{\alpha_j} + \cdots + \lambda_{12\ldots,p}x_j^{\alpha_j}\ldots, x_p^{\alpha_p}\right], \quad i, j = 1, 2\ldots p,\ i < j.$$

When the α's are taken to be unity the above result reduces to that of Gumbel's multivariate exponential distribution with $h_i(\boldsymbol{x})$ independent of x_i and $\alpha_i = 2$ gives multivariate Rayleigh model with marginal linear failure rates.

2. The Lomax law

$$\overline{F}(\boldsymbol{x}) = \left(1 + \sum_{i=1}^{p}\alpha_i x_i\right)^{-\beta}, \quad x_i, \alpha_i, \beta > 0$$

has a reciprocal linear failure rate that increases with each x_i and a decreasing linear mean residual life. In this case, when $B_i(X_i) = X_i$,

$$g_i(\boldsymbol{x}) = \left(1 + \sum \alpha_i x_i\right)[\alpha_i(\beta - 1)]^{-1}x_i, \quad \beta > 1.$$

3. Multivariate beta distribution

$$\overline{F}(\boldsymbol{x}) = \left(1 - \sum \alpha_i x_i\right)^{\beta}, \quad 0 < x_1 < \frac{1}{\alpha_1}, \ldots, x_p < \frac{1 - \sum_{i=1}^{p-1}\alpha_i x_i}{\alpha_p}, \alpha_i,\ \beta > 0$$

with decreasing reciprocal linear failure rate and increasing linear mean residual life satisfies

$$g_i(\boldsymbol{x}) = \left(1 - \sum \alpha_i x_i\right) [\alpha_i(\beta + 1)]^{-1} x_i$$

when $B_i(X_i) = X_i$.

Remark 2. The identity connecting the mean residual life and the failure rate is

$$E(X_i - x_i | \boldsymbol{X} > \boldsymbol{x}) = g_i(\boldsymbol{x}) h_i(\boldsymbol{x}) - x_i + E(X_i | \boldsymbol{X}_i^* > \boldsymbol{x}_i^*), \quad i = 1, 2, \ldots, p.$$

By appropriately choosing $g_i(\boldsymbol{x})$, the characterizations of Roy [28] and improved version of these by Asadi [6] of the bivariate exponential, Lomax and beta models arise as particular cases of the above expression.

Remark 3. When $X_1, X_2, \ldots, X_p$ are independent random variables, Eq. (10) reduces to the marginal distributions of each X_i, with $g_i(x_i)$ replacing $g_i(\boldsymbol{x})$. The resulting expression is evidently of the form Eq. (1) and subsumes most of the univariate results, see [20] for details.

Theorem 2.2. *The relationship*

$$q_i(\boldsymbol{x}) \lambda_i(\boldsymbol{x}) = \mu_i^* - r_i(\boldsymbol{x}), \quad i = 1, 2, \ldots, p \tag{12}$$

is satisfied for all $\boldsymbol{x}$ *in* S_p *and some positive real-valued function* $q_i(\boldsymbol{x})$ *defined on* S_p *and differentiable in each* x_i *if and only if*

$$\frac{\partial F}{\partial x_i} = K_i(\boldsymbol{x}_i^*) \exp\left[\int_{x_i}^{b_i} \left(\frac{\mu_i^* - B_i(t_i) - \frac{\partial q_i(\boldsymbol{x}_i, t_i)}{\partial t_i}}{q_i(\boldsymbol{x}_i, t_i)}\right) \mathrm{d}t_i\right], \quad i = 1, 2, \ldots, p, \tag{13}$$

where $\mu_i^* = E(X_i | \boldsymbol{X}_i^* \le \boldsymbol{x}_i^*)$.

The proof is similar to that of Theorem 2.2 and therefore omitted.

Remark 4. For $p = 1$, it is seen from (9) and (12) that $q_1(x_1) = g_1(x_1)$, but in higher dimensions q_i and g_i need not be the same.

3. Scalar failure rates

In this section, we consider functions $B_i(X_i)$ defined in Section 2 and functions $u_i(\boldsymbol{x})$ defined on S_p, positive, real-valued and differentiable. Instead of vector failure rates, we look at scalar conditional failure rates

$$a_i(x_i | \boldsymbol{x}_i^*) = f_i(x_i | \boldsymbol{x}_i^*) / \overline{F}_i(x_i | \boldsymbol{x}_i^*),$$

where

$$\overline{F}_i(\boldsymbol{x}_i | x_i^*) = P(X_i > x_i | \boldsymbol{X}_i^* = \boldsymbol{x}_i^*), \quad i = 1, 2, \ldots, p.$$

Theorem 3.1. *A necessary and sufficient condition that a p-dimensional random vector with support* S_p *and conditional failure rates* $a_i(x_i | \boldsymbol{x}_i^*)$ *satisfies*

$$a_i(x_i | \boldsymbol{x}_i^*) u_i(\boldsymbol{x}) = E(B_i(X_i) | X_i > x_i, \boldsymbol{X}_i^* = \boldsymbol{x}_i^*) - e_i, \quad i = 1, 2, \ldots, p, \tag{14}$$

where $e_i = E(B_i(X_i) | \boldsymbol{X}_i^* = \boldsymbol{x}_i^*)$ *is that the density function* $f(\boldsymbol{x})$ *has the form*

$$f(\boldsymbol{x}) = c_i(\boldsymbol{x}_i^*) \exp\left[\int_{a_i}^{x_i} \left(\frac{e_i - B_i(t_i) - \frac{\partial u_i(\boldsymbol{x}_i, t_i)}{\partial t}}{u_i(\boldsymbol{x}_i, t_i)}\right) \mathrm{d}t_i\right], \quad i = 1, 2, \ldots, p. \tag{15}$$

Proof. Assume that (14) is satisfied for all $\boldsymbol{x}$ in S_p. Then

$$a_i(x_i|\boldsymbol{x}_i^*)u_i(\boldsymbol{x}) = \left[\overline{F}(x_i|\boldsymbol{x}_i^*)\right]^{-1}\int_{x_i}^{b_i} B_i(t_i)f(t_i|\boldsymbol{x}_i^*)\mathrm{d}t_i - e_i$$

$$f(x_i|\boldsymbol{x}_i^*)u_i(\boldsymbol{x}) = \int_{x_i}^{b_i}(B_i(t_i)-e_i)f_i(t_i|\boldsymbol{x}_i^*)\mathrm{d}t_i$$

$$\text{or}\quad f(\boldsymbol{x})u_i(\boldsymbol{x}) = \int_{x_i}^{b_i}(B_i(t_i)-e_i)f(\boldsymbol{x}_i,t_i)\mathrm{d}t_i. \tag{16}$$

Differentiating (16) with respect to x_i,

$$\frac{\partial f}{\partial x_i}u_i + f\frac{\partial u_i}{\partial x_i} = [e_i - B_i(x_i)]f(\boldsymbol{x})$$

or

$$\frac{\partial \log f}{\partial x_i} = \left(e_i - B_i(x_i) - \frac{\partial u_i}{\partial x_i}\right)\Big/ u_i(\boldsymbol{x}).$$

Integrating over (a_i, x_i) the form (15) is obtained. Converse part follows by retracing the above steps. □

Remark 5. When $X_1,\ldots,X_n$ are independent, $a_i(x_i|\boldsymbol{x}_i^*) = a_i(x_i)$, the univariate failure rate of X_i and $u_i(\boldsymbol{x}) = u_i(x_i)$, giving the univariate result identical with that mentioned in Remark 3.

Remark 6. Taking $B_i(X_i) = X_i$, $E(X_i|X_i > x_i, \boldsymbol{X}_i^* = \boldsymbol{x}_i^*) = e(\boldsymbol{x}) + x_i$ where $e(\boldsymbol{x})$ is the conditional mean residual life studied in [31].

We now give examples of $u_i(\boldsymbol{x})$ that characterize some standard distributions and families. The bivariate Pearson family discussed in [34] is defined for a random vector (X_1, X_2) by the equations

$$\frac{\partial \log f}{\partial x_i} = \frac{L_i(x_1,x_2)}{Q_i(x_1,x_2)},\quad -\infty \le a_i < x_i < b_i \le \infty, i = 1,2, \tag{17}$$

where L_i and Q_i are respectively linear and quadratic functions in x_1 and x_2 given by

$$L_i = l_i x_1 + m_i x_2 + n_i$$

and

$$Q_i = A_i x_1^2 + 2H_i x_1 x_2 + D_i x_2^2 + 2G_i x_1 + 2K_i x_2 + C_i,\quad i = 1,2$$

such that L_i/Q_i is irreducible and satisfies

$$\frac{\partial}{\partial x_2}\frac{L_1}{Q_1} = \frac{\partial}{\partial x_1}\frac{L_2}{Q_2}.$$

Specialising to $B_i(X_i) = X_i$, we have from Eq. (14) that

$$a_i(x_i|x_j)u_i(x_1,x_2) = E(X_i|X_i > x_i, X_j = x_j) - E(X_i|X_j = x_j),\\ i,j = 1,2,\ i \ne j. \tag{18}$$

Theorem 3.2. *The random vector* (X_1, X_2) *is distributed according to the bivariate Pearson family Eq.* (17) *if and only if Eq.* (18) *holds with* $u_i(x_1, x_2)$ *as a quadratic function in* (x_1, x_2).

Proof. Assume that (X_1, X_2) has a density satisfying (17). Then with $B_i(X_i) = X_i$, in Theorem 3.1, we have the relation

$$\frac{\partial \log f_i}{\partial x_i} = \frac{L_i}{Q_i} = \left(e_i - x_i - \frac{\partial u_i}{\partial x_i}\right) \Big/ u_i$$

or

$$-u_i(x_1, x_2) L_i(x_1, x_2) = Q_i \frac{\partial u_i}{\partial x_i} - (e_i - x_i) Q_i. \tag{19}$$

Further from equation (17) and the expression for L_i and Q_i, we obtain

$$\int_{a_i}^{b_i} Q_i \frac{\partial f}{\partial x_i} \mathrm{d}x_i = \int_{a_i}^{b_i} L_i f \, \mathrm{d}x_i, \quad i = 1, 2.$$

Integrating the left-hand side by parts and assuming that $Q_i f$ tends to zero at the boundary points of the probability domain,

$$\int_{a_i}^{b_i} \left(L_i + \frac{\partial Q_i}{\partial x_i}\right) f \mathrm{d}x_i = 0$$

which gives for $i = 1$,

$$\int_{a_1}^{b_1} [(l_1 x_1 + m_1 x_2 + n_1) + (2A_1 x_1 + 2H_1 x_2 + 2G_1)] f(x_1, x_2) \mathrm{d}x_1 = 0$$

that simplifies to

$$e_1(x_2) = -\frac{1}{(l_1 + 2A_1)} [(m_1 + 2H_1) x_2 + (n_1 + 2G_1)]. \tag{20}$$

Similarly

$$e_2(x_1) = -\frac{1}{(m_2 + 2D_2)} [(l_2 + 2H_2) x_1 + (n_2 + 2K_2)]. \tag{21}$$

The right-hand side of Eq. (20) being a cubic function in x_i, the left-hand side also must be a cubic leaving the only option that u_i has to be a quadratic. The exact expression for u_i is obtained in practice by comparing the coefficients of like powers on both sides of (19). Conversely if (19) is true with u_i as quadratic functions, then $\frac{\partial \log f}{\partial x_i}$ is of the form L_i / Q_i and the proof is complete. □

Special cases

1. If $\boldsymbol{X}$ is bivariate normal with

$$f(\boldsymbol{x}) = \exp\left[-\frac{1}{2(1-\rho^2)}(x_1^2 - 2\rho x_1 x_2 + x_2^2)\right], \quad -\infty < x_i < \infty$$

then $e_i(x_j) = \rho x_j$ and $u_i(\boldsymbol{x}) = 1 - \rho^2$. Thus $u_i(\boldsymbol{x})$ is constant (independent of x_1 and x_2) characterizes the bivariate normal law.

2. The bivariate gamma

$$f(x_1, x_2) = \frac{c^{a+b}}{\Gamma(a)\Gamma(b)} x_1^{a-1}(x_2 - x_1)^{b-1}\bar{e}^{cx_2}, \quad 0 < x_1 < x_2,\ a, b, c > 0$$

is characterized by

$$u_1(\boldsymbol{x}) = \frac{x_1}{a+b}(x_2 - x_1)$$

and

$$u_2(\boldsymbol{x}) = \frac{x_2}{c} + \frac{2x_1x_2}{b} - \frac{b-2}{b(b-1)}x_1^2.$$

3. For the bivariate Type II b,

$$f(x_1, x_2) = \frac{x_1^{n_1-1}x_2^{n_2-1}}{\Gamma(n_1)\Gamma(-n_1-n_2)} \exp\left(-\frac{x_1+1}{x_2}\right), \quad x_1 > x_2 > 0,\ n_1 + n_2 < 0$$

$$u_1(\boldsymbol{x}) = x_1x_2 \quad \text{and} \quad u_2(\boldsymbol{x}) = -(n_2+2)^{-1}x_2^2, n_2 < -2.$$

Another example of characterization using Theorem 3.1 is that of the bivariate exponential conditional of Arnold and Strauss [5],

$$f(x_1, x_2) = C \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2]$$

by $u_i(x_1, x_2) = x_i/(\lambda_i + \theta x_j), i, j = 1, 2, i \neq j$.

Analogous result for concepts in reversed time is the following, which is proved using similar arguments as in Theorem 3.1.

Theorem 3.3. *Let $v_i(\boldsymbol{x})$ be positive real-valued and differentiable functions defined on S_p. Then a necessary and sufficient condition that $\boldsymbol{X}$ with reversed conditional failure rate*

$$b_i(x_i|\boldsymbol{x}_i^*) = f(x_i|\boldsymbol{x}_i^*)/F(x_i|\boldsymbol{x}_i^*)$$

satisfies

$$b_i(x_i|\boldsymbol{x}_i^*)v_i(\boldsymbol{x}) = E(B_i(X_i)|X_i \leq x_i, \boldsymbol{X}_i^* = \boldsymbol{x}_i^*) - m_i, \quad i = 1, 2, \ldots, p,$$

where $m_i = E(B_i(X_i)|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*)$ is that

$$f(\boldsymbol{x}) = C_i(\boldsymbol{x}_i^*) \exp\left[-\int_{x_i}^{b_i}\left(\frac{B_i(t_i) - m_i - \frac{\partial v_i}{\partial t_i}(\boldsymbol{x}_i, t_i)}{v_i(\boldsymbol{x}_i, t_i)}\right)\mathrm{d}t_i\right], \quad i = 1, 2, \ldots, p.$$

Example: The bivariate logistic model with

$$F(\boldsymbol{x}) = (1 + \bar{e}^{x_1} + \bar{e}^{x_2})^{-1}, \quad -\infty < x_i < \infty$$

is characterized by $v_i(\boldsymbol{x}) = 1 + \bar{e}^{x_1} + \bar{e}^{x_2}$, for a choice of $B_i(X_i) = e^{-X_i}$.

4. Stein type identities for truncated variables

In this section, we discuss the applications of Theorem 3.1 in developing Stein type identities involving truncated expectations. Stein [33] established that for a normal random variable with parameters μ and σ^2 and a differentiable function $g(x)$ satisfying $E|g'(X)| < \infty$,

$$E(g(X)(X-\mu)) = \sigma^2 g'(X).$$

Subsequently similar results, widely referred to as Stein type identities, were discovered for continuous and discrete variables and distributions belonging to the exponential family in the multivariate case. These results were applied to characterizations, probability theory, prediction and inference. We refer to [11,27,7,16,25] for details. We now establish a Stein type identity for truncated variables and give some examples. Apart from providing scope for applications mentioned above, our results give an alternative methodology to compute truncated moments, the importance of which is well documented in [17].

Theorem 4.1. *Assuming that*

$$E[(B_i(X_i)-e_i)g_i(\boldsymbol{X})|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*] \quad \textit{and} \quad E\left[u_i(\boldsymbol{X})\left|\frac{\partial g_i}{\partial x_i}\right|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*\right]$$

are finite, for a real-valued absolutely continuous function $g_i(\boldsymbol{x})$ defined on S_p with partial derivatives $\frac{\partial g_i}{\partial x_i}$, the following statements are equivalent

(i) $f(\boldsymbol{x}) = C_i(\boldsymbol{x}_i^*)\exp[\int_{a_i}^{x_i}(\frac{e_i - B_i(t_i) - \frac{\partial u_i}{\partial t_i}}{u_i(\boldsymbol{x}_i, t_i)})\mathrm{d}t_i]$
(ii) $u_i(\boldsymbol{x})a_i(x_i|\boldsymbol{x}_i^*) = E(B_i(X_i)|X_i > x_i, \boldsymbol{X}_i^* = \boldsymbol{x}_i^*) - e_i$
(iii) $E\left[(B_i(X_i)-e_i)g_i(\boldsymbol{X})|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*\right] = E[u_i(\boldsymbol{X})\frac{\partial g_i}{\partial X_i}|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*]$, $i = 1, 2, \ldots, p$.

Proof. The equivalence of (i) and (ii) is proved in Theorem 3.1. Hence we first consider the case when (ii) is true. Now

$$\begin{aligned} E[(B_i(X_i)-e_i)g_i(\boldsymbol{X})|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*] &= \int_{a_i}^{b_i}(B_i(x_i)-e_i)g_i(\boldsymbol{x})f(x_i|\boldsymbol{x}_i^*) \\ &= -\int_{a_i}^{b_i} g_i(\boldsymbol{x})\left[\frac{\partial}{\partial x_i}(f(x_i|\boldsymbol{x}_i^*)u_i(\boldsymbol{x}))\right]\mathrm{d}x_i \end{aligned}$$

on using (ii). Integrating by parts, the right-hand side becomes $E[u_i(\boldsymbol{X})\frac{\partial g_i}{\partial X_i}|\boldsymbol{x}_i^*]$, in view of the assumptions of the Theorem and hence (ii) implies (iii). Conversely assuming (iii), one can write it as

$$\begin{aligned} \int_{a_i}^{b_i} u_i(\boldsymbol{x})\frac{\partial g_i}{\partial x_i}f(x_i|\boldsymbol{x}_i^*)\mathrm{d}x_i &= \int_{a_i}^{b_i}(B_i(x_i)-e_i)g_i(\boldsymbol{x})f(x_i|\boldsymbol{x}_i^*)\mathrm{d}x_i \\ &= \int_{a_i}^{b_i}(B_i(x_i)-e_i)(g_i(\boldsymbol{x})-\alpha)f(x_i|\boldsymbol{x}_i^*)\mathrm{d}x_i, \end{aligned}$$

where $\alpha = E[g_i(X)|\boldsymbol{X}_i^* = \boldsymbol{x}_i^*]$. Then

$$\begin{aligned} \int_{a_i}^{b_i} u_i(\boldsymbol{x})\frac{\partial g_i}{\partial x_i}f(x_i|\boldsymbol{x}_i^*)\mathrm{d}x_i &= \int_{a_i}^{b_i}(B_i(x_i)-e_i)\left(\int_{a_i}^{x_i}\frac{\partial g_i(\boldsymbol{x}_i, t_i)}{\partial t_i}\mathrm{d}t_i\right)f(x_i|\boldsymbol{x}_i^*)\mathrm{d}x_i \\ &= \int_{a_i}^{b_i}\frac{\partial g_i}{\partial x_i}\int_{a_i}^{x_i}((B_i(t_i)-e_i)f(t_i|\boldsymbol{x}_i^*)\mathrm{d}t_i)\mathrm{d}x_i, \\ &\quad \text{(on changing the order of integration)} \\ &= \int_{a_i}^{b_i}\frac{\partial g_i}{\partial x_i}\int_{x_i}^{b_i}(e_i - B_i(t_i))f(t_i|\boldsymbol{x}_i^*)\mathrm{d}t_i, \end{aligned}$$

since $E[B_i(X_i)|\boldsymbol{x}_i^*] - e_i = 0$. Setting first $g_j(\boldsymbol{x}) = \cos\theta x_j$ and then $g_j(\boldsymbol{x}) = \sin\theta x_j$, where θ can be a function of $\boldsymbol{x}_j^*$, we can arrive at

$$\int_{a_j}^{b_j} \mathrm{e}^{\mathrm{i}\theta x_j} u_j(\boldsymbol{x}) f(x_j|\boldsymbol{x}_j^*)\mathrm{d}x_j = \int_{a_j}^{b_j} \mathrm{e}^{\mathrm{i}\theta x_j} \left(\int_{x_j}^{b_j} (e_j - B_j(t_j)) f(t_j|\boldsymbol{x}_j^*)\mathrm{d}t_j \right) \mathrm{d}x_j.$$

Hence by the uniqueness of the Fourier transforms,

$$u_j(\boldsymbol{x}) f(x_j|\boldsymbol{x}_j^*) = \int_{x_j}^{b_j} (e_j - B_j(t_j)) f(t_j|\boldsymbol{x}_j^*), \quad \text{for } j = 1, 2, \ldots, p$$

which is the same as (ii). Hence all the implications in the theorem are established. □

Remark 7. When X_i's are independent

$$E(B_i(X_i) - e_i)g_i(X_i) = E\left(u_i(X_i) \frac{\partial g_i(X_i)}{\partial X_i} \right)$$

which is a univariate Stein type identity.

Examples

1. In the bivariate normal case discussed above, for every absolutely continuous $g_i(\boldsymbol{x})$,

$$E\left[(X_i - m_i)g_i(\boldsymbol{X})|X_j = x_j\right] = (1-\rho^2)E\left(\frac{\partial g_i(\boldsymbol{X})}{\partial X_i} \right), \tag{22}$$

with $m_i = \rho x_j, i, j = 1, 2 i \neq j$. In particular, when X_1 and X_2 are independent

$$E((X_1 - \mu_1)g_1(X_1)) = E\frac{\partial g_1(X_1)}{\partial X_1}$$

which is the Stein's identity in the univariate $N(\mu, 1)$ case. Giving $g_1(X)$, the values X, X^2 ..., all the higher-order truncated moments can be calculated recursively. Hence Theorem 4.1 offers a simple method to calculate higher-order truncated moments other than the usual repeated integration using the density function.

2. In the bivariate exponential conditional distribution of Arnold and Strauss [5] in Section 3

$$E\left[(X_i - (\lambda_i + \theta x_j)^{-1})g_i(\boldsymbol{X})|X_j = x_j\right] = E\left(\frac{X_i}{\lambda_i + \theta x_j} \frac{\partial g_i(\boldsymbol{X})}{\partial X_i} \right), \quad i = 1, 2.$$

5. Discussion

The present paper proposes some characterizations of multivariate life distributions using relationship between conditional expectations and failure rates. The main application of these results appears to be in reliability analysis in identifying the appropriate models, based on relationship between the mean residual life (or other appropriate functions) and failure rate, either postulated or empirically seen to be approximately holding for the given lifetime data. Further the proposed Stein type identity encourages its use in multivariate analogues of applications in the univariate case, we have mentioned earlier. Multivariate versions of lower bounds to the variance of random vectors and its applications to estimation theory in the sense of [21] is being worked out and is expected to be presented in a future work.

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