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Stability on a cone in terms of two measures for impulsive differential equations with "supremum"[☆]

Snezhana G. Hristova

Department of Applied Mathematics and Modeling, Plovdiv University, 4000 Plovdiv, Bulgaria

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ABSTRACT

The stability of nonlinear impulsive differential equations with "supremum" is studied. A special type of stability, combining two different measures and a dot product, is defined. The definition is a generalization of several types of stability known in the literature. Razumikhin's method as well as a comparison method for scalar impulsive ordinary differential equations have been employed.

1. Introduction

The problem of stability of solutions of differential equations via the Lyapunov method has been successfully investigated in the past. One type of stability, very useful in real world problems, deals with two different measures. Stability in terms of two measures has been studied by means of various types of Lyapunov function ([1], the book [2] and references cited therein).

The objects of investigation of the paper are impulsive differential equations with "supremum" (IDES). These equations are adequate models of real processes in which the present state depends significantly on the maximal value of the state in a past time interval and at the same time it has instantaneous changes at certain moments. Note that IDES are used, for example, in the theory of automatic control of technical systems, when the law of regulation depends on maximum values of some regulated state parameters over certain time intervals [3]. Some results for IDES are obtained in [4,5].

In the present paper a new type of stability is defined. The definition for stability combines the application of two different measures and a dot product over a cone. Sufficient conditions for the existence of the defined type of stability of IDES are obtained. These conditions are based on the application of cone valued Lyapunov functions and comparison results. The dot product in the definition gives us the possibility to assign different weights to the components of the solution. The dot product allows us to use a scalar differential equation as a comparison equation.

2. Preliminary notes and definitions

Let $\mathbb{R}_+ = [0, \infty)$, $r \geq 0$ be a fixed number, and $\{\tau_k\}_1^\infty$ be an increasing sequence of points in $\mathbb{R}_+$ such that $\lim_{k\to\infty} \tau_k = \infty$.

Denote by $PC(X, Y)$ $(X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $\tau_k \in X$ and $u(\tau_k) = u(\tau_k - 0)$. Denote by $PC^1(X, Y)$ the set of all functions $u \in PC(X, Y)$ which are continuously differentiable for $t \in X$, $t \neq \tau_k$.

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E-mail address: snehri@uni-plovdiv.bg.

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Consider the impulsive differential equations with "supremum" (IDES)

$$x' = F(t, x(t), \sup_{s\in[t-r,t]} x(s)) \quad \text{for } t \geq 0,\ t \neq \tau_k, \tag{1}$$

$$x(\tau_k + 0) = I_k(x(\tau_k - 0)) \quad \text{for } k = 1, 2, \dots, \tag{2}$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, 3, \dots$.

Let $t_0 \geq 0$ be a given point and $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$. We denote by $x(t; t_0, \phi)$ the solution of the system IDES (1), (2) with initial conditions

$$x(t; t_0, \phi) = \phi(t), \quad t \in [t_0 - r, t_0], \qquad x(t_0 + 0; t_0, \phi) = \phi(t_0). \tag{3}$$

Assume the solution of the initial value problem (1), (2), (3) exists on $[t_0 - r, \infty)$ for any function $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$.

Let $x, y \in \mathbb{R}^n$. Denote by $(x \cdot y)$ the dot product of vectors x and y.

Let $\mathcal{K} \subset \mathbb{R}^n$ be a cone and consider the set

$$\mathcal{K}^* = \{\varphi \in \mathbb{R}^n : (\varphi \cdot x) \geq 0,\ \text{for all } x \in \mathcal{K}\}.$$

Consider following sets

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(s) \text{ is strictly increasing and } a(0) = 0\};$$
$$CK = \{b \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+] : b(t, \cdot) \in K \text{ for any fixed } t \in \mathbb{R}_+\};$$
$$\mathcal{G} = \{h \in C[[-r, \infty) \times \mathbb{R}^n, \mathcal{K}] : \inf_{x\in\mathbb{R}^n} h(t, x) = 0 \text{ for each } t \geq -r\};$$
$$\tilde{\mathcal{S}}(h, \rho, \varphi_0) = \{(t, x) \in [0, \infty) \times \mathbb{R}^n : (\varphi_0 \cdot h(t, x)) < \rho\},$$

where ρ be a positive constant, $\varphi_0 \in \mathcal{K}^*$, $h \in \mathcal{G}$.

Let $h_0 \in \mathcal{G}$, $\varphi_0 \in \mathcal{K}^*$. For any $t \in \mathbb{R}_+$ and $\phi \in PC([t - r, t], \mathbb{R}^n)$ we define

$$H_0(t, \phi, \varphi_0) = \sup\{(\varphi_0 \cdot h_0(t + s, \phi(t + s))) : s \in [-r, 0]\}. \tag{4}$$

We define a new type of stability of IDES, that combines the ideas of stability in terms of two measures [2] and the application of a dot product.

Definition 1. Let $\varphi_0 \in \mathcal{K}^*$, $h, h_0 \in \mathcal{G}$. The system of IDES (1), (2) is said to be

(S1) φ_0-*stable* in terms of measures h_0 and h if for every $\epsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that for any $\phi \in PC([t_0 - r, t_0], \mathbb{R}^n)$ inequality $H_0(t_0, \phi, \varphi_0) < \delta$ implies $(\varphi_0 \cdot h(t, x(t; t_0, \phi))) < \epsilon$ for $t \geq t_0$, where the function H_0 is defined by (4), $x(t; t_0, \phi)$ is a solution of IDES (1), (2) with initial condition (3);

(S2) *uniformly* φ_0-*stable* in terms of measures h_0 and h if (S1) is satisfied, where δ is independent on t_0.

We consider the set $\mathcal{L}$ of cone-valued functions $V(t, x) \in PC^1([-r, \infty) \times \mathbb{R}^n, \mathcal{K})$, $V = (V_1, V_2, \dots, V_n)$, that are component wise Lipschitz in x.

Let $t \in \mathbb{R}_+$, $t \neq \tau_k$, $(k = 1, 2, \dots)$, $x \in \mathbb{R}^n$, $V(t, x) \in \mathcal{L}$, $V = (V_1, V_2, \dots, V_n)$, and $\phi \in PC([t - r, t], \mathbb{R}^n)$. We define a derivative $\mathcal{D}V(t, x)$ of the function V along the system of IDES (1), (2) by the equalities

$$\mathcal{D}V_i(t, \phi(t)) = \frac{\partial V_i(t, \phi(t))}{\partial t} + \sum_{j=1}^{n} \frac{\partial V_i(t, \phi(t))}{\partial x_j} F_j(t, \phi(t), \sup_{s\in[-r,0]} \phi(t + s)), \quad i = 1, 2, \dots, n,$$

where $\mathcal{D}V(t, x) = (\mathcal{D}V_1(t, x), \mathcal{D}V_2(t, x), \dots, \mathcal{D}V_n(t, x))$.

In the proof of the main results we use the following comparison scalar impulsive ordinary differential equation:

$$u' = g(t, u), \quad t \neq \tau_k, \qquad u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \dots, \tag{5}$$

where $u \in \mathbb{R}$, $g(t, 0) \equiv 0$, $\xi_k(0) = 0$ for $k = 1, 2, \dots$.

Lemma 1. *Let the vector $\varphi_0 \in \mathcal{K}^*$ and the function $V \in \mathcal{L}$ be such that*

(i) *for any number $t \in [0, T]$, $t \neq \tau_k$, $k = 1, 2, \dots$ and for any function $\psi \in PC([t - r, t], \mathbb{R}^n)$ such that $(\varphi_0 \cdot V(t, \psi(t))) \geq (\varphi_0 \cdot V(t + s, \psi(t + s)))$ for $s \in [-r, 0)$ the inequality $(\varphi_0 \cdot \mathcal{D}V(t, \psi(t))) \leq g(t, (\varphi_0 \cdot V(t, \psi(t))))$ holds, where $g \in C([0, T] \times \mathbb{R}_+, \mathbb{R})$;*

(ii) *$(\varphi_0 \cdot V(\tau_k + 0, I_k(x))) \leq \xi_k(\varphi_0 \cdot V(\tau_k, x))$, for $k : \tau_k \in [0, T]$, where the functions $\xi_k \in K$.*

Then the inequality $\sup_{s\in[t_0-r,t_0]}(\varphi_0 \cdot V(s, \varphi(s))) \leq u_0$ implies the validity of the inequality $(\varphi_0 \cdot V(t, x(t; t_0, \phi))) \leq u^*(t)$ for $t \in [t_0, T]$, where $u^*(t) = u^*(t; t_0, u_0)$ is the maximal solution of (5) with initial condition $u^*(t_0) = u_0$.

Note that the inequality $(\varphi_0 \cdot V(t_0, \varphi(t_0))) \leq u_0$, that is satisfied only at the initial point, is not enough for the validity of the claim of Lemma 1.

Definition 2. Let $h, h_0 \in \mathcal{G}$. Function h_0 is *uniformly* φ_0-*finer* than h with a constant $\delta > 0$ and a function $p \in K$ if for any point $(t, x) \in [-r, \infty) \times \mathbb{R}^n$ such that $(\varphi_0 \cdot h_0(t, x)) < \delta$ the inequality $(\varphi_0 \cdot h(t, x)) \leq p((\varphi_0 \cdot h_0(t, x)))$ holds.

3. Main results

We will give sufficient conditions for φ_0-stability in terms of two measures of systems of impulsive differential equations with "supremum".

Theorem 1. *Let following conditions be fulfilled:*

1. *The functions $F \in PC[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $I_k \in C[\mathbb{R}^n, \mathbb{R}^n]$, $k = 1, 2, \dots$.*
2. *The vector $\varphi_0 \in \mathcal{K}^*$.*
3. *The functions $h_0, h \in \mathcal{G}$, h_0 is uniformly φ_0-finer than h.*
4. *There exists a function $V \in \mathcal{L}$ such that*
   (i) *$b(\varphi_0 \cdot h(t,x)) \le (\varphi_0 \cdot V(t,x)) \le a((\varphi_0 \cdot h_0(t,x)))$, $(t,x) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$ where $a, b \in K$;*
   (ii) *for any number $t \ge 0$, $t \ne \tau_k$, $k = 1, 2, \dots$ and for any function $\psi \in PC([t-r,t], \mathbb{R}^n)$ such that $(\varphi_0 \cdot V(t, \psi(t))) \ge (\varphi_0 \cdot V(t+s, \psi(t+s)))$ for $s \in [-r, 0)$ and $(t, \psi(t)) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$ the inequality $(\varphi_0 \cdot \mathcal{D}V(t, \psi(t))) \le g(t, (\varphi_0 \cdot V(t, \psi(t))))$ holds, where $g \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R})$, $g(t,0) \equiv 0$, $\rho > 0$ is a constant.*
   (iii) *$(\varphi_0 \cdot V(\tau_k + 0, I_k(x))) \le \xi_k(\varphi_0 \cdot V(\tau_k, x))$, for $(\tau_k, x) \in \tilde{\mathcal{S}}(h, \rho, \varphi_0)$ where the functions $\xi_k \in K$.*
5. *The zero solution of the scalar impulsive differential equation (5) is equi stable.*

Then system of IDES (1), (2) is φ_0-stable in terms of two measures h and h_0.

The proof of Theorem 1 is based on the application of a modified Razumikhin method and Lemma 1.
In the following theorem sufficient conditions for uniform φ_0-stability in terms of two measures are given.

Theorem 2. *Let the conditions* 1, 2, 3, 4 *of* Theorem 1 *be satisfied and the zero solution of scalar impulsive differential equation (5) be uniformly stable.*
Then system of IDES (1), (2) is uniformly φ_0-stable in terms of measures h_0 and h.

Now we will illustrate the application of the defined φ_0-stability in terms of two measures on an example.

Example 1. Consider the system of differential equations with "supremum"

$$x'(t) = -x(t) + 4y(t) + \frac{1}{2} \max_{s \in [t-r,t]} x(s), \quad t \ne k, \qquad x(k+0) = \frac{1}{2} x(k), \tag{6}$$

$$y'(t) = -x(t) - y(t) + \frac{1}{2} \max_{s \in [t-r,t]} y(s), \quad t \ne k, \qquad y(k+0) = \frac{1}{2} y(k), \tag{7}$$

with initial conditions

$$x(t) = \phi_1(t - t_0), \qquad y(t) = \phi_2(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{8}$$

where $x, y \in \mathbb{R}$, $r > 0$, $t_0 \ge 0$, $\phi_i \in PC([t_0 - r, t_0], \mathbb{R})$, $i = 1, 2$.

Let $h_0(t,x,y) = (|x|, |y|)$, $h(t,x,y) = (x^2, y^2)$, $h, h_0 \in \mathcal{G}$.
Consider the function $V : \mathbb{R}^2 \to \mathcal{K}$, $V = (V_1, V_2)$, $V_1(x,y) = \frac{1}{2}x^2$, $V_2(x,y) = \frac{1}{2}y^2$, where $\mathcal{K} = \{(x,y) : x \ge 0, y \ge 0\} \subset \mathbb{R}^2$ is a cone. We note that $V \in \mathcal{L}$.
Now, let us consider the vector $\varphi_0 = (1, 4)$.
It is easy to check the validity of conditions of Theorem 1 for functions $b(s) = \frac{1}{2}s \in K$, $g(t,s) = -s$ and $\xi_k(s) = \frac{1}{2}s$. Consider the scalar comparison impulsive equation $u' = -u$, $t \ne k$, $u(k+0) = \frac{1}{2}u(k)$ which solution $u(t) = u_0 \frac{1}{2^k} e^{-t+t_0}$, $t \in (k, k+1]$ is equi stable. According to Theorem 1 the system of IDES (6), (7) is φ_0-stable in terms of both measures h_0, h, i.e. for every $\epsilon > 0$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that inequality $\max_{s \in [-r,0]}(|\phi_1(t_0+s)| + 4|\phi_2(t_0+s))| < \delta$ implies $x^2(t) + 4y^2(t) < \epsilon$ for $t \ge t_0$, where $x(t), y(t)$ is the solution of the initial value problem (6)–(8).
Note that if the vector φ_0 is different than the above considered, for example, if $\varphi_0 = (1, 1)$ then the conditions of Theorem 1 are not satisfied. Therefore the stability depends significantly on the choice of the vector φ_0.

Partial cases. Note that as partial cases of the above results we obtain stability results for various types of differential equation, some of them are known in the literature:
Case 1. Let $I_k(x) \equiv x$. Then we obtain φ_0-stability in terms of two measures of differential equations with "maxima".
Case 2. Let $r = 0$. Then we obtain φ_0-stability in terms of two measures of impulsive differential equations.
Case 3. Let $r = 0$ and $h(t,y) = h_0(t,y) = (|y_1|, |y_2|, \dots, |y_n|)$. Then the above results reduce to results for φ_0-stability of impulsive differential equations [6].
Case 4. Let $I_k(x) \equiv x$ and φ_0 be the unit vector. Then the results reduce to results for stability in terms of two measures of differential equations with "maxima".
Case 5. Let $r = 0$ and φ_0 be the unit vector. Then the results reduce to results for stability in terms of two measures of impulsive differential equations [7].

Case 6. Let $r = 0$, $I_k(x) \equiv x$ and φ_0 be the unit vector. Then we obtain stability in terms of two measures of ordinary differential equations [2].

Case 7. Let $r = 0$, $I_k(x) \equiv x$ and $h(t, y) = h_0(t, y) = (|y_1|, |y_2|, \dots, |y_n|)$. Then we obtain φ_0-stability in terms of two measures for differential equations [8].

Case 8. Let $r = 0$, $h(t, y) = h_0(t, y) = (|y_1|, |y_2|, \dots, |y_n|)$ and φ_0 be the unit vector. Then we obtain the stability of impulsive differential equations [9].

Case 9. Let $r = 0$, $I_k(x) \equiv x$, $h(t, y) = h_0(t, y) = (|y_1|, |y_2|, \dots, |y_n|)$ and vector φ_0 consists of 0s and 1s. Then we obtain partial stability.

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