The weight hierarchies and generalized weight spectra of the projective codes from degenerate quadrics

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Abstract

The weight hierarchies and generalized weight spectra of the projective codes from degenerate quadrics in projective spaces over finite fields are determined. These codes satisfy also the chain conditions.

1. Introduction

Motivated by applications in cryptography, Wei [11] introduced a generalization of the minimum Hamming weight of a linear code. Let \( \mathbb{F}_q \) be a finite field, where \( q \) is a prime power. For any code \( D \) of block length \( n \) over \( \mathbb{F}_q \), define the support \( \chi(D) \) by

\[
\chi(D) = \{ i \mid c_i \neq 0 \text{ for some } (c_1, c_2, \ldots, c_n) \in D \},
\]

and the support weight \( \omega_s(D) \) by

\[
\omega_s(D) = |\chi(D)|.
\]

Now let \( C \) be a linear \([n,k]\) code over \( \mathbb{F}_q \). For any \( r \), where \( 1 \leq r \leq k \), the \( r \)-th generalized Hamming weight of \( C \) is defined as

\[
d_r(C) = \min \{ \omega_s(D) \mid D \text{ is an } r\text{-dimensional subcode of } C \}.
\]

Obviously, the minimum Hamming weight of \( C \) is exactly \( d_1(C) \). The weight hierarchy of \( C \) is then defined to be the set of generalized Hamming weights

\[
\{ d_1(C), d_2(C), \ldots, d_k(C) \}.
\]

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In the terminology of Wei and Yang [12], we say that a linear \([n, k]\) code \(C\) satisfies the \textit{chain condition}, if there exist \(r\)-dimensional subcodes \(D_r\) of \(C\) for \(1 \leq r \leq k\) such that
\[
w_s(D_r) = d_r(C), \quad r = 1, 2, \ldots, k,
\]
and
\[
D_1 \subseteq D_2 \subseteq \cdots \subseteq D_k,
\]
Many applications of generalized Hamming weights are known. They are useful in cryptography [2, 11], in trellis coding [7, 13], in truncating a linear block code [5], etc.

The generalized Hamming weights have been determined for binary Hamming codes, MDS codes, Golay codes, Reed–Muller codes and their duals [11], Kasami codes [5], the projective codes from nondegenerate quadrics [10], the projective codes from nondegenerate Hermitian varieties [6], the projective codes from degenerate Hermitian varieties [16], and projective Reed–Muller codes of lower orders [15]. Some of these codes are proved to satisfy the chain condition [12].

Let \(C\) be a \([n, k]\) linear code. For \(r\), and \(i\), with \(1 \leq r \leq k\) and \(0 \leq i \leq n\), let \(A_{i}^{(r)}\) be the number of \(r\)-dimensional subcodes of \(C\) of support weight \(i\). The \(r\)-th generalized weight spectrum is the sequence
\[
A_{0}^{(r)}, A_{1}^{(r)}, \ldots, A_{n}^{(r)}.
\]
And the \(r\)-th generalized weight distribution function is the polynomial
\[
A^{(r)}(Z) = A_{0}^{(r)} + A_{1}^{(r)}Z + \cdots + A_{n}^{(r)}Z^n.
\]
We denote by \(A_i\), \(0 \leq i \leq n\), the number of codewords in \(C\) of weight \(i\). Then the sequence
\[
A_0, A_1, \ldots, A_n,
\]
is the \textit{weight spectrum} of \(C\). Obviously,
\[
A_0 = 1, \quad A_i = (q - 1)A_i^{(1)}, \quad i = 1, \ldots, n.
\]
The support weight distribution for irreducible cyclic codes was introduced by Helleseth et al. [3]. The generalized Hamming weights are actually the minimum support weights and the support weight distribution is exactly the same as the generalized weight spectrum.

The weight spectra of the projective codes from nondegenerate quadrics in projective spaces have been determined [10]. Some special cases of the projective codes from degenerate quadrics in projective space have obtained [1].

In the present paper the weight hierarchies and generalized weight spectra of the \(q\)-ary projective codes from degenerate quadrics in projective spaces are determined. It is also proved that these codes satisfy the chain condition.
2. Preliminaries

Let $C$ be a $q$-ary linear $[n,k]$ code. The following properties of the generalized Hamming weights are known:

(1) (Monotonicity) $1 \leq d_1(C) < d_2(C) < \cdots < d_k(C) \leq n$.

(2) (The generalized Singleton bound) $d_r(C) \leq n - k + r$ for $r = 1, 2, \ldots, k$.

(3) (Duality) Let $C^\perp$ be the dual code of $C$. Then

$$\{d_r(C^\perp) \mid 1 \leq r \leq n - k\} = \{1, 2, \ldots, n\} \setminus \{n + 1 - d_r(C) \mid 1 \leq r \leq k\}.$$

(4) (The Griesmer–Wei bound)

$$d_r(C) \geq \sum_{i=0}^{r-1} \left\lfloor \frac{d_i(C)}{q^i} \right\rfloor$$

for $r = 1, 2, \ldots, k$.

where $\lfloor x \rfloor$ denotes the smallest integer not less than $x$.

All these properties are proved for the case $q = 2$, for (1) cf. [3, 11], for (2)–(4) cf. [11]. When $q$ is a power of any prime, the proofs are the same.

As in [3] or [8], let $G$ be a generator matrix of $C$, and for any column vector $x \in \mathbb{F}_q^k$, the $k$-dimensional column vector space over $\mathbb{F}_q$, let $m_G(x)$, the number of occurrences of the vector $x$ as columns of $G$. Obviously, $w_s(C) = n - m_G(0^k)$, where $0^k$ is the zero vector of $\mathbb{F}_q^k$. Let $U$ be a subspace of $\mathbb{F}_q^k$ and define

$$m_G(U) = \sum_{x \in U} m_G(x).$$

If $M$ is an $r \times k$ matrix of rank $r$, then $MG$ generate an $r$-dimensional subcode of $C$, and any $r$-dimensional subcode is obtained in this way.

Let $D$ be an $r$-dimensional subcode of $C$ and $MG$ be a generator matrix of $D$, where $M$ is an $r \times k$ matrix of rank $r$. Define the dual of $D$ to be

$$D^\perp = \{x \in \mathbb{F}_q^k \mid Mx = 0\}.$$

Then $D^\perp$ is an $(k - r)$-dimensional subspace. And for any $r$, where $1 \leq r \leq k$, the map $D \mapsto D^\perp$ defined by (1) is a bijection from the set of $r$-dimensional subcodes of $C$ to the set of $(k - r)$-dimensional subspaces of $\mathbb{F}_q^k$. The following result can be found in [8].

**Lemma 1.** Let $D$ be a subcode of $C$. Then

$$w_s(D) = n - m_G(D^\perp).$$

3. The projective code from a degenerate quadric in $\text{PG}(k - 1, \mathbb{F}_q)$

A linear $[n,k]$ code $C$ is called a projective code if the columns of a generator matrix $G$ of $C$ can be regarded as distinct points of $\text{PG}(k - 1, \mathbb{F}_q)$, the $(k - 1)$-dimensional
projective space over \( \mathbb{F}_q \). So we have a point set in \( \text{PG}(k-1, \mathbb{F}_q) \), whose elements are the column vectors of \( G \), which will be denoted by \( \mathcal{S}_{C,G} \) and called the point set arising from \( C \) via \( G \). Different encoding matrices of \( C \) give rise to point sets which are projectively equivalent.

Two projective \([n,k]\) codes over \( \mathbb{F}_q \) are said to be equivalent, if one can be obtained from the other by permuting the coordinates of the codewords and multiplying them by non-zero elements of \( \mathbb{F}_q \).

Let \( G \) be a generator matrix of a projective \([n,k]\) code \( C \) and \( C' \) be a projective code equivalent to \( C \). Then the same transformation which transforms \( C \) to \( C' \) will transform the encoding matrix \( G \) of \( C \) to an encoding matrix \( G' \) of \( C' \). Clearly, \( \mathcal{S}_{C,G} = \mathcal{S}_{C',G'} \). It follows that in general, if \( G \) and \( G' \) are encoding matrices of two equivalent projective \([n,k]\) codes \( C \) and \( C' \), then \( \mathcal{S}_{C,G} \) and \( \mathcal{S}_{C',G'} \) are projectively equivalent. In the sequel, when we consider a fixed code, we denote \( \mathcal{S}_{C,G} \) by \( \mathcal{S}_G \).

**Theorem 2.** Let \( C \) be a \( q \)-ary projective \([n,k]\) code, \( G \) be a generator matrix of \( C \), and \( \mathcal{S}_G \) be the point set in \( \text{PG}(k-1, \mathbb{F}_q) \) arising from \( C \) via \( G \). For any \( r \), where \( 1 \leq r \leq k \), let \( D_r \) be an \( r \)-dimensional subcode, then there is an \((k - r - 1)\)-flat of \( \text{PG}(k-1, \mathbb{F}_q) \) \( P_{k-r-1} \) such that

\[
\omega_s(D_r) = n - |P_{k-r-1} \cap \mathcal{S}_G|,
\]

and

\[
d_r(C) = n - \max \{|P_{k-r-1} \cap \mathcal{S}_G|\},
\]

where \( P_{k-r-1} \) runs through all \((k - r - 1)\)-flats of \( \text{PG}(k-1, \mathbb{F}_q) \).

**Proof.** It follows from Lemma 1. Specializing Lemma of [4] and Lemma 1 of [8] to any \( q \)-ary projective \([n,k]\) code \( C \) we can also obtain the theorem. □

When \( q \) is odd, we choose a fixed nonsquare element \( z \) in \( \mathbb{F}_q \), and when \( q \) is even, we choose a fixed \( x \), where \( x \not\in \{x^2 + x \mid x \in \mathbb{F}_q \} \). Let \( k = 2v + \delta + l > 0 \), where \( \delta = 0, 1, \) or 2 and assume that \( v > 0, l \geq 0 \). Let \( Q_{2v+\delta+l} \) be the \((2v + \delta + l) \times (2v + \delta + l)\) matrix over \( \mathbb{F}_q \),

\[
Q_{2v+\delta+l} = \begin{pmatrix}
Q_{2v+\delta} & 0^{(l)} \\
0^{(v)} & I^{(v)}
\end{pmatrix},
\]

where \( Q_{2v+\delta} \) is defined as follows. When \( q \) is odd,

\[
Q_{2v} = \begin{pmatrix}
0^{(v)} & I^{(v)} \\
I^{(v)} & 0
\end{pmatrix}, \quad Q_{2v+1} = \begin{pmatrix}
0^{(v)} & I^{(v)} \\
I^{(v)} & 0
\end{pmatrix},
\]
and \( Q_{2v+2} = \begin{pmatrix} 0 & I^{(v)} \\ I^{(v)} & 0 \\ 1 & -1 \end{pmatrix} \).

When \( q \) is even,

\[
Q_{2v} = \begin{pmatrix} 0 & I^{(v)} \\ 0 & 0 \end{pmatrix}, \quad Q_{2v+1} = \begin{pmatrix} 0 & I^{(v)} \\ 0 & 0 \end{pmatrix},
\]

and

\[
Q_{2v+2} = \begin{pmatrix} 0 & I^{(v)} \\ I^{(v)} & 0 \\ 1 & \alpha \end{pmatrix}.
\]

The set of points \( \{x_1, x_2, \ldots, x_{2v+\delta+1}\} \) satisfying

\[
(x_1, x_2, \ldots, x_{2v+\delta+1})Q_{2v+\delta+1}^{-1}(x_1, x_2, \ldots, x_{2v+\delta+1}) = 0
\]

is a degenerate quadric in \( \text{PG}(2v+\delta+l-1, \mathbb{F}_q) \) in its normal form \[9\], which will also be denoted by \( Q_{2v+\delta+l} \). Let \( n = |Q_{2v+\delta+l}| \). Then from \[9\], we know

\[
n = \frac{(q^\delta - 1)(q^{2v+2\delta-1} + 1)}{q-1} q' + \frac{q' - 1}{q-1}
\]

\[
= \frac{q^{2v+2\delta+l-1} + q^{\delta+l} - q^{2v+2\delta+l-1} - 1}{q-1}.
\]

For each point of \( Q_{2v+\delta+l} \), choose a system of coordinates and regard it as a \( k \)-dimensional column vector. Arrange these \( n \) column vectors in any order into a \( k \times n \) matrix, denote it also by \( Q_{2v+\delta+l} \). It can be proved that \( Q_{2v+\delta+l} \) is of rank \( k \). Hence \( Q_{2v+\delta+l} \) can be regarded as a generator matrix of a \( q \)-ary projective \([n, k]\)-code, which will be denoted by \( C_{2v+\delta+l} \) and called the projective code from the degenerate quadric \( Q_{2v+\delta+l} \) in \( \text{PG}(k-1, \mathbb{F}_q) \). Obviously,

\[
Q_{2v+\delta+l} = Q_{2v+\delta+l}.
\]

4. Flats in \( \text{PG}(k - 1, \mathbb{F}_q) \) under the singular orthogonal group

Let \( A \) and \( B \) be two \( k \times k \) matrices over \( \mathbb{F}_q \). We introduce the notation

\[
A \equiv B,
\]

which means that \( A = B \) when \( q \) is odd, and that \( A + B \) is an alternate matrix when \( q \) is even. Moreover, \( A \) and \( B \) are said to be cogredient if there is a \( k \times k \) nonsingular matrix \( R \) such that \( RAR \equiv B \).
Denote by $GL_{2v+\delta+1}(\mathbb{F}_q)$ the general linear group of degree $2v + \delta + l$ over $\mathbb{F}_q$. Let

$$O_{2v+\delta+1}(\mathbb{F}_q) = \{ T \mid T \in GL_{2v+\delta+1}(\mathbb{F}_q) \text{ and } ^tTQ_{2v+\delta+1}T \equiv Q_{2v+\delta+1} \}.$$  

Clearly, $O_{2v+\delta+1}$ is a subgroup of $GL_{2v+\delta+1}(\mathbb{F}_q)$ and is called the singular orthogonal group of degree $2v + \delta + l$ over $\mathbb{F}_q$ with respect to $Q_{2v+\delta+1}$.

Define an action of $O_{2v+\delta+1}(\mathbb{F}_q)$ on the $(2v+3+l)$-dimensional column vector space $\mathbb{F}_q^{(2v+3+l)}$ as follows:

$$O_{2v+\delta+1}(\mathbb{F}_q) \times \mathbb{F}_q^{(2v+3+l)} \rightarrow \mathbb{F}_q^{(2v+3+l)}$$

$$(T, x_1, \ldots, x_{2v+3+l}) \mapsto T^t(x_1, \ldots, x_{2v+3+l}).$$

The action induces an action on $PG(k-1, \mathbb{F}_q)$, and the set of flats of $PG(k-1, \mathbb{F}_q)$ is subdivided into orbits under $O_{2v+\delta+1}(\mathbb{F}_q)$. Let $P$ be an $(m-1)$-flat, then it corresponds to an $m$-dimensional subspace of $\mathbb{F}_q^m$, and there is a $k \times m$ matrix whose $m$ columns form a basis of this subspace, if no ambiguity arises, we denote all of them by $P$.

Let $P$ be an $m$-dimensional subspace of $\mathbb{F}_q^m$. When $q$ is odd, the $m$-dimensional subspace $P$ and its corresponding $(m-1)$-flat are said to be type $(m, 2s + \gamma, s, \Gamma)$, where $\gamma = 0, 1, \text{ or } 2, \Gamma = 1 \text{ or } z \text{ when } \gamma = 1 \text{ and } \Gamma \text{ disappears when } \gamma = 0 \text{ or } 2$, if $^tPQ_{2v+\delta+1}P$ is cogredient respectively to

$$\begin{pmatrix}
0 & f(s) \\
f(s) & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}$$

or

$$\begin{pmatrix}
0 & f(s) \\
f(s) & 0 \\
z & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.$$

When $q$ is even, the subspace $P$ and its corresponding flat $P$ are said to be type $(m, 2s + \gamma, s)$, where $\gamma = 0, 1, \text{ or } 2$, if $^tPQ_{2v+\delta+1}P$ is cogredient to

$$\begin{pmatrix}
Q_{2s+\gamma} \\
0 & 0 & \cdots & 0
\end{pmatrix}.$$  

Moreover, when $\delta = 1$, subspaces and flats of type $(m, 2s + 1, s)$ are further subdivided into two types, namely, type $(m, 2s + 1, s, 1)$ and type $(m, 2s + 1, s, 0)$ according to $P$ not containing or containing a vector of the form

$$\begin{pmatrix}
0, \ldots, 0, 1, * \ldots, *
\end{pmatrix}_{2s}$$

respectively. We use the symbol $(m, 2s + \gamma, s, \Gamma)$, where $\Gamma = 0 \text{ or } 1 \text{ when } \gamma = 1$, and $\Gamma \text{ disappears when } \gamma = 0 \text{ or } 2$, to cover these four cases.
Let $E$ be the subspace of $\mathbb{F}_q^{(2v+\delta+1)}$ generated by $e_{2v+\delta+1}$, $e_{2v+\delta+2}, \ldots, e_{2v+\delta+l}$, where $e_j = \{0, \ldots, 0, 1, 0, \ldots, 0\}$, then $\dim E = l$. An $m$-dimensional subspace $P$ or an $(m-1)$-flat $P$ is said to be type $(m, 2s + \gamma, s, \Gamma, t)$ if:

1. $P$ is of type $(m, 2s + \gamma, s, \Gamma)$ and
2. $\dim(P \cap E) = t$.

Denote by $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)$ the set of subspaces or flats of type $(m, 2s + \gamma, s, \Gamma)$, and by $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l; t)$ the set of subspaces or flats of type $(m, 2s + \gamma, s, \Gamma, t)$ in $\mathbb{F}_q^{(2v+\delta+l)}$ or $\text{PG}(2v + \delta + l, \mathbb{F}_q)$, respectively, and let

$$N(m, 2s + \gamma, s, \Gamma; 2v + \delta + l) = |\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)|,$$

$$N(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l) = |\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)|.$$

The following results will be used in the sequel and can be found in [9].

Lemma 3. Let $q$ be odd. Then $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ is nonempty if and only if $t \leq l$ and

$$2s + \gamma \leq m - t \leq \begin{cases} v + s + \min\{\delta, \gamma\} & \text{if } \delta \neq 1, \text{ or } \gamma \neq 1, \\ or \gamma = \delta = 1 \text{ and } \Gamma = 1, \\ v + s & \text{if } \gamma = \delta = 1 \text{ and } \Gamma = z. \end{cases}$$

Moreover, if $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ is nonempty, then it is an orbit of subspaces or flats under $O_{2v+\delta+l}(\mathbb{F}_q)$.

Lemma 4. Let $q$ be odd. Then $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)$ is nonempty if and only if

$$\min\{l, m - 2s - \gamma\} \geq \begin{cases} \max\{0, m - v - s - \min\{\delta, \gamma\}\} & \text{if } \delta \neq 1, \text{ or } \gamma \neq 1, \\ or \gamma = \delta = 1 \text{ and } \Gamma = 1, \\ \max\{0, m - v - s\} & \text{if } \gamma = \delta = 1 \text{ and } \Gamma = z. \end{cases}$$

Moreover, $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)$ is the disjoint union of all the orbits $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ under $O_{2v+\delta+l}(\mathbb{F}_q)$, where $t$ satisfies (2).

Lemma 5. Let $q$ be even. Then $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ is nonempty if and only if $t \leq l$ and

$$2s + \gamma \leq m - t \leq \begin{cases} v + s + \min\{\delta, \gamma\} & \text{if } \delta \neq 1, \text{ or } \gamma \neq 1, \\ or \gamma = \delta = 1 \text{ and } \Gamma = 1, \\ v + s & \text{if } \gamma = \delta = 1 \text{ and } \Gamma = 0. \end{cases}$$

Moreover, if $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ is nonempty, then it is an orbit of subspaces or flats under $O_{2v+\delta+l}(\mathbb{F}_q)$. 
Lemma 6. Let $q$ be even. Then $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)$ is nonempty if and only if

$$\min\{l, m - 2s - \gamma\} \geq \begin{cases} \max\{0, m - v - s - \min\{\delta, \gamma\}\} & \text{if } \delta \neq 1, \text{ or } \gamma \neq 1, \\ \text{or } \gamma = \delta = 1 \text{ and } \Gamma = 1, \\ \max\{0, m - v - s\} & \text{if } \gamma = \delta = 1 \text{ and } \Gamma = 0. \end{cases}$$

Moreover, $\mathcal{M}(m, 2s + \gamma, s, \Gamma; 2v + \delta + l)$ is the disjoint union of all the orbits $\mathcal{M}(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l)$ under $O_{2v+\delta+t}(\mathbb{F}_q)$, where $t$ satisfies (4).

Now, the numbers of subspaces or flats are given as follows:

Lemma 7. Suppose that $t \leq l$ and that (2) or (4) holds when $q$ is odd or even, respectively, then

$$N(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + l) = N(m - t, 2s + \gamma, s, \Gamma; 2v + \delta) \times q^{(m-t)(l-t-1)} \prod_{i=1}^{t} (q^i - 1) \prod_{i=1}^{2s} (q^i - 1),$$

where

$$N(m - t, 2s + \gamma, s, \Gamma; 2v + \delta) = q^{2s(\gamma + s - m + t) + s t + (\delta + \gamma) - (m - t - 2s - \gamma)} \times \frac{\prod_{i=s}^{v+\gamma-1} (q^i - 1)(q^{i+\delta-1} + 1)}{\prod_{i=1}^{s} (q^i - 1) \prod_{i=0}^{s+\gamma-1} (q^i + 1) \prod_{i=1}^{m-t-2s-\gamma} (q^i - 1)} \times n_0(m - t, 2s + \gamma, s, \Gamma; 2v + \delta),$$

where

$$n_0(m - t, 2s, s; 2v + \delta) = 1;$$

$$n_0(m - t, 2s + 1, s, \Gamma; 2v + \delta) = \begin{cases} q^{v-s-1}(q^{s+s-m+t+1} - 1) & \text{if } \delta = 0, \\ q^{v-s}(q^{s+s-m+t+1} - 1) & \text{if } \delta = 1 \text{ and } \Gamma = z, \\ q^{v-s}(q^{s+s-m+t+1} + 1) & \text{if } \delta = 1 \text{ and } \Gamma = 1, \\ q^{v-s}(q^{s+s-m+t+2} + 1) & \text{if } \delta = 2, \end{cases}$$

when $q$ is odd; and

$$n_0(m - t, 2s + 1, s, \Gamma; 2v + \delta) = \begin{cases} 2q^{v-s-1}(q^{s+s-m+t+1} - 1) & \text{if } \delta = 0, \\ 2q^{m-t-2s-1}(q^{2v+s-m+t+1} - 1) & \text{if } \delta = 1 \text{ and } \Gamma = 0, \\ 2q^{m-t-2s-1} & \text{if } \delta = 1 \text{ and } \Gamma = 1, \\ 2q^{v-s}(q^{s+s-m+t+2} + 1) & \text{if } \delta = 2, \end{cases}$$
when $q$ is even; and

$$n_0(m - t, 2s + 2, s; 2v + \delta)$$

$$= \begin{cases} q^{2(v-s)-2}(q^{v+s-m+t+1} - 1)(q^{v+s-m+t+2} - 1) & \text{if } \delta = 0, \\
q^{2(v-s)-1}(q^{v+s-m+t+2} - 1)(q^{v+s-m+t+2} + 1) & \text{if } \delta = 1, \\
q^{2(v-s)}(q^{v+s-m+t+2} + 1)(q^{v+s-m+t+3} + 1) & \text{if } \delta = 2. \end{cases}$$

**Lemma 8.** Let $q$ be odd. Suppose that (3) holds, then

$$N(m, 2s + \gamma, s; 2v + \delta + 1) = \sum_t N(m, 2s + \gamma, s, t; 2v + \delta + 1),$$

where the summation range of $t$ takes all values in

$$[\max\{0, m - v - s - \min\{\delta, \gamma\}\}, \min\{l, m - 2s - \gamma\}],$$

if $\delta \neq 1$, or $\gamma \neq 1$, or $\delta = \gamma = 1$ and $\Gamma = 1$, or

$$[\max\{0, m - v - s\}, \min\{l, m - 2s - \gamma\}],$$

if $\gamma = \delta = 1$ and $\Gamma = z$.

**Lemma 9.** Let $q$ be even. Suppose that (5) holds, then

$$N(m, 2s + \gamma, s, \Gamma; 2v + \delta + 1) = \sum_t N(m, 2s + \gamma, s, \Gamma, t; 2v + \delta + 1),$$

where the summation range of $t$ takes all values in

$$[\max\{0, m - v - s - \min\{\delta, \gamma\}\}, \min\{l, m - 2s - \gamma\}],$$

if $\delta \neq 1$, or $\gamma \neq 1$, or $\delta = \gamma = 1$ and $\Gamma = 1$, or

$$[\max\{0, m - v - s\}, \min\{l, m - 2s - \gamma\}],$$

if $\gamma = \delta = 1$ and $\Gamma = 0$.

5. Some lemmas

**Lemma 10.** Let $m \geq 1$ and $P$ be an $(m - 1)$-flat in $\text{PG}(2v + \delta + l - 1, \mathbb{F}_q)$. Then

$$|P \cap Q_{2v+\delta+l}| = |PQ_{2v+\delta+l}P|. $$

**Proof.** Let $x = \iota(x_1, x_2, \ldots, x_{2v+\delta+l})$ be a point of $\text{PG}(2v + \delta + l - 1, \mathbb{F}_q)$, and $x \in P \cap Q_{2v+\delta+l}$, then $x \in P$ and $\iota x Q_{2v+\delta+l} x = 0$. From $x \in P$ we deduce that $x = P y$, where $y = \iota(y_1, y_2, \ldots, y_m)$. Then $\iota^t y PQ_{2v+\delta+l} P y = 0$, that is $y \in PQ_{2v+\delta+l}$. 
Conversely, let \( y = (y_1, y_2, \ldots, y_m) \) be a point of \( \text{PG}(m - 1, \mathbb{F}_q) \) and \( y \in \mathcal{P}_2^{v+\delta+1} \), then \( \mathcal{P}_2^{v+\delta+1} \). Let \( x = (x_1, x_2, \ldots, x_{2v+\delta+1}) = Py \). Then \( x_{2v+\delta+1}y = 0 \). So, there exist a bijection from \( \mathcal{P}_2^{v+\delta+1} \) to \( \mathcal{P}_2^{v+\delta+1} \) which maps \( y \in \mathcal{P}_2^{v+\delta+1} \) to \( x = Py \in \mathcal{P}_2^{v+\delta+1} \). 

**Lemma 11.** Let \( m \geq 1 \), and \( P_1, P_2 \) both be \((m-1)\)-flats of \( \text{PG}(k - 1, \mathbb{F}_q) \), which belong to the same orbit under \( O_{2v+\delta+1}(\mathbb{F}_q) \). Then

\[
|\mathcal{P}_2^{v+\delta+1}P_1| = |\mathcal{P}_2^{v+\delta+1}P_2|.
\]

**Proof.** \( P_1 \) and \( P_2 \) belong to the same orbit under \( O_{2v+\delta+1}(\mathbb{F}_q) \) if and only if there is an \( m \times m \) nonsingular matrix \( A \) and element \( T \in O_{2v+\delta+1}(\mathbb{F}_q) \), such that \( P_1 = TP_2A \). Thus

\[
|\mathcal{P}_2^{v+\delta+1}P_1| = |\mathcal{P}_2^{v+\delta+1}P_2|.
\]

**Lemma 12.** Let \( m \geq 1 \) and \( P \) be an \((m-1)\)-flat of type \((m, 2s + \gamma, s, \Gamma, t)\). Then

\[
|\mathcal{P}_2^{v+\delta+1}P| = \frac{q^{m-1} + q^{m-\gamma} - q^{m-\gamma-1} - 1}{q - 1},
\]

which is independent of \( \Gamma \) and \( t \).

**Proof.** By Lemma 10, we have

\[
|\mathcal{P}_2^{v+\delta+1}P| = |\mathcal{P}_2^{v+\delta+1}P|.
\]

By Lemma 11, when \( q \) is odd, we can assume that

\[
\mathcal{P}_2^{v+\delta+1}P = \begin{pmatrix}
0 & f(s) \\
f(s) & \\
\Gamma & \end{pmatrix},
\]

where

\[
\Gamma = \begin{cases}
\emptyset & \text{if } \gamma = 0, \\
(1) \text{ or } (z) & \text{if } \gamma = 1,
\end{cases}
\]

And when \( q \) is even, we can assume that

\[
\mathcal{P}_2^{v+\delta+1}P = \begin{pmatrix}
0 & f(s) \\
f(s) & \\
\Gamma & \end{pmatrix},
\]

where

\[
\Gamma = \begin{cases}
\emptyset & \text{if } \gamma = 0, \\
(1) \text{ or } (z) & \text{if } \gamma = 1,
\end{cases}
\]

And when \( q \) is even, we can assume that
where
\[
\Gamma = \begin{cases} 
\emptyset & \text{if } \gamma = 0, \\
(1) & \text{if } \gamma = 1, \\
(x \ 1) & \text{if } \gamma = 2.
\end{cases}
\]

So, we obtain
\[
|P \cap Q_{2v+\delta+l}| = \frac{(q^t - 1)(q^{s+\gamma-1} + 1)q^{(m-2s-\gamma)} + q^{m-2s-\gamma} - 1}{q-1} = \frac{q^{m-1} + q^{m-s-\gamma} - q^{m-s-1} - 1}{q-1}.
\]

**Corollary 13.** Let \(m \geq 1\) and \(P\) be an \((m-1)\)-flat of type \((m,2s+\gamma,s,F)\). Then
\[
|P \cap Q_{2v+\delta+l}| = \frac{q^{m-1} + q^{m-s-\gamma} - q^{m-s-1} - 1}{q-1},
\]
which is independent of \(\Gamma\).

**Corollary 14.** Let \(m \geq 1\) and \(P\) be an \((m-1)\)-flat of type \((m,2s+1,s,F,t)\). Then
\[
|P \cap Q_{2v+\delta+l}| = \frac{q^{m-1} - 1}{q-1},
\]
which is independent of \(s\) and \(t\).

6. The weight hierarchy of \(C_{2v+\delta+l}\)

**Lemma 15.** Assume that \(t \leq l\), \(0 < m < 2v+\delta+l\), and that subspaces of type \((m,2s+1,s,F,t)\) exist in \(\mathbb{F}_q^{(2v+\delta+l)}\). Then \(\max \{|P_{m-1} \cap Q_{2v+\delta+l}|\}\), where \(P_{m-1}\) runs through all \((m-1)\)-flats of \(\text{PG}(k-1,\mathbb{F}_q)\), cannot be achieved by \((m-1)\)-flats of type \((m,2s+1,s,F,t)\), unless, \(\delta = 2, m = 2v+1+l\), and \(t = l\). In the later case it is actually achieved by \((2v+l)\)-flat of type \((2v+1+l,2v+1,v,F,l)\).

**Proof.** Let \(P\) be an \((m-1)\)-flat of type \((m,2s+1,s,F,t)\), where \(t \leq l\). When \(q\) is odd, (2) becomes
\[
2s + 1 \leq m - t \leq \begin{cases} 
v + s + \min\{\delta, 1\} & \text{if } \delta = 0, \text{ or } \delta = 2, \text{ or } \delta = 1 \text{ and } F = 1, \\
v + s & \text{if } \delta = 1 \text{ and } F = 0.
\end{cases}
\]
When \( q \) is even, (4) becomes

\[
2s + 1 \leq m - t \leq \begin{cases} 
  v + s + \min\{\delta, 1\} & \text{if } \delta = 0, \text{ or } \delta = 2, \text{ or } \delta = 1 \text{ and } \Gamma = 1, \\
  v + s & \text{if } \delta = 1 \text{ and } \Gamma = 0.
\end{cases}
\]

We distinguish the following cases:

(a) \( \delta = 0, \) or \( \delta = 1 \) and \( \Gamma = z \) when \( q \) is odd, or \( \delta = 1 \) and \( \Gamma = 0 \) when \( q \) is even. Then both (6) and (7) become

\[
2s + 1 \leq m - t \leq v + s.
\]

Thus subspaces of type \((m, 2s, s, t)\) exist in \( \mathbb{F}_q^{(2v+\delta+1)} \). Let \( R \) be one of them. By Lemma 12,

\[
|R \cap Q_{2v+\delta+1}| = \frac{q^{m-1} + q^{m-s} - q^{m-s-1} - 1}{q - 1}.
\]

But

\[
|P_t \cap Q_{2v+\delta+1}| = \frac{q^{m-1} - 1}{q - 1},
\]

so \( |P_t \cap Q_{2v+\delta+1}| < |R \cap Q_{2v+\delta+1}| \), hence, \( \max\{|P_{m-1} \cap Q_{2v+\delta+1}|\} \) cannot be achieved by \( P_t \).

(b) \( \delta = 2, \) or \( \delta = 1 \) and \( \Gamma = 1 \). Then both (6) and (7) become

\[
2s + 1 \leq m - t \leq v + s + 1.
\]

We distinguish further the following cases:

(b.1) \( m - t < v + s + 1 \). Then we have

\[
2s + 1 \leq m - t \leq v + s.
\]

As in case (a), \( \max\{|P_{m-1} \cap Q_{2v+\delta+1}|\} \) cannot be achieved by \( P_t \).

(b.2) \( m - t = v + s + 1 \). Then

\[
2s + 1 \leq m - t = v + s + 1.
\]

(i) If \( s < v \), then

\[
2(s + 1) \leq m - t = v + (s + 1).
\]

Hence subspaces of type of \((m, 2(s + 1), s + 1, t)\) exist in \( \mathbb{F}_q^{(2v+\delta+1)} \). As in case (a), \( \max\{|P_{m-1} \cap Q_{2v+\delta+1}|\} \) cannot be achieved by \( P_t \).

(ii) If \( s = v \), then \( m - t = 2v + 1 \), i.e., \( m = 2v + 1 + t, \) and \( P_t \) is of type \((2v + 1 + t, 2v + 1, v, \Gamma, t)\). In \( \text{PG}(2v + \delta + t - 1, \mathbb{F}_q) \), all \((m - 1)\)-flats which may exist are of type \((2v + 1 + t, 2v + 1, v, \Gamma, t)\) or of type \((2v + 1 + t, 2s' + y', s', \Gamma, t')\), where \( t' \leq l \). Let \( W_{t'} \)
be a subspace of \((2v + 1 + t, 2s', t')\). Then by Corollary 14 and Lemma 12,
\[
|P_t \cap Q_{2v+\delta+t+1}| = \frac{q^{2v+t} - 1}{q - 1},
\]
\[
|W_{t'} \cap Q_{2v+\delta+t+1}| = \frac{q^{2v+t} + q^{2v+1+t-t'-s'} - q^{2v+1+t-s'-1} - 1}{q - 1}.
\]

When \(y' \geq 1\),
\[
|W_{t'} \cap Q_{2v+\delta+t+1}| < |P_t \cap Q_{2v+\delta+t+1}|.
\]

Because we are studying whether \(\max\{|P_m \cap Q_{2v+\delta+t+1}\}\) can be achieved by \(P_t, W_{t'}\), can be neglected.

When \(y' = 0\), \(W_{t'}\) is of type \((2v + 1 + t, 2s', t')\) and
\[
|W_{t'} \cap Q_{2v+\delta+t+1}| = \frac{q^{2v+t} + q^{2v+1+t-t'} - q^{2v+t-s'} - 1}{q - 1}.
\]

By (2) and (3), \(W_{t'}\) exist in \(PG(2v + \delta + l - 1, F_q)\) if and only if
\[
2s' \leq 2v + 1 + t - t' \leq v + s',
\]
which is equivalent to
\[
v + 1 + t - t' < t' \leq v + \frac{1}{2}(1 + t - t'),
\]
from which it follows that
\[
t' \geq t + 1.
\]

But
\[
|W_{t'} \cap Q_{2v+\delta+t+1}| = \frac{q^{2v+t} + q^{2v+1+t-t'} - q^{2v+t-s'} - 1}{q - 1}.
\]
\[
\leq \frac{q^{2v+t} + q^{v-1+t'}(q - 1) - 1}{q - 1},
\]

therefore when \(t' = l\), \(|W_{t'} \cap Q_{2v+\delta+t+1}|\) attains its maximum
\[
|W_{t} \cap Q_{2v+\delta+t+1}| = \frac{q^{2v+l-1} + q^{v+l-1}(q - 1) - 1}{q - 1}.
\]

On the other hand, for all \(t \leq l\), \((m - 1)\)-flats of type \((2v + 1 + t, 2v + 1, v, \Gamma, t)\) exist in \(PG(2v + \delta + l - 1, F_q)\). By hypothesis, \(m < 2v + \delta + l\). When \(\delta = 1\) and \(\Gamma = 1\), we must have \(t < l\). But
\[
|P_t \cap Q_{2v+1+l}| = \frac{q^{2v+t} - 1}{q - 1} < \frac{q^{2v+l-1} + q^{v+l-1}(q - 1) - 1}{q - 1}.
\]
hence \( \max\{|P_{m-1} \cap Q_{2v+1+l}|\} \) cannot be achieved by \( P_t \). When \( \delta = 2 \), we may take \( t = 1 \). Then
\[
|P_t \cap Q_{2v+\delta+l}| = \frac{q^{2v+l} - 1}{q - 1}.
\]
It is easy to verify that
\[
|P_t \cap Q_{2v+\delta+l}| \geq |W_t \cap Q_{2v+\delta+l}|.
\]
Hence, we conclude that \( \max\{|P_{m-1} \cap Q_{2v+\delta+l}|\} \) is achieved by \((2v+l)\)-flat of type \((2v+1+l,2v+1,v,\Gamma,l)\), when \( \delta = 2, m = 2v+1+l, t = l \).

**Lemma 16.** Assume that \( t \leq l, 0 < m < 2v+\delta+l \), and that subspaces of type \((m,2s+2,s,t)\) exist in \( \mathbb{F}_q^{2v+\delta+l} \). Then \( \max\{|P_{m-1} \cap Q_{2v+\delta+l}|\} \), where \( P_{m-1} \) runs through all \((m-1)\)-flats of \( \text{PG}(2v+\delta+l-1,q) \), cannot be achieved by \((m-1)\)-flats of type \((m,2s+2,s,t)\), unless \( \delta = 2, t = l-1 \), and \( m = 2v+1+l \). In the later case it is actually achieved by \((2v+l)\)-flat of type \((2v+1+l,2v+2,v,l-1)\).

**Proof.** It is similar to the proof of Lemma 15, we omit the details.

**Lemma 17.** Assume that subspaces of type \((m,2s,s,t)\) and of type \((m,2s',s',t')\) exist in \( \mathbb{F}_q^{2v+\delta+l} \). Let \( P \) and \( P' \) be subspaces of type \((m,2s,s,t)\) and \((m,2s',s',t')\) respectively. Then
\[
|P \cap Q_{2v+\delta+l}| > |P' \cap Q_{2v+\delta+l}|
\]
if and only if \( s < s' \).

**Proof.** By Lemma 12,
\[
|P \cap Q_{2v+\delta+l}| = \frac{q^{m-1} + q^{m-s} - q^{m-s-1} - 1}{q - 1}.
\]
Clearly,
\[
q^{m-s} - q^{m-s-1} = q^{m-s-1}(q - 1),
\]
so the lemma follows.

**Theorem 18.** The weight hierarchy of \( C_{2v+\delta+l} \) is as follows:
(a) \( \delta = 0, \)
\[
d_r(C_{2v+l}) = \begin{cases} 
q^{2v-1} - \frac{q^{2v-r-1}}{q-1}q', & r = 1,2,\ldots,v, \\
q^{2v-1} - \frac{q^{2v-r} + q' - q'^{-1}}{q-1}q', & r = v + 1,\ldots,2v+l,
\end{cases}
\]
(b) $\delta = 1,$

$$d_r(C_{2v+1+l}) = \begin{cases} 
q^{2v} - q^{2v-r} - q^r + q^{r-1}q', & r = 1, 2, \ldots, v + 1, \\
q^{2v} - q^{2v+1-r}q', & r = v + 2, \ldots, 2v + 1 + l,
\end{cases}$$

(c) $\delta = 2,$

$$d_1(C_{2v+2+l}) = q^{v+1}(q^v - 1),$$

$$d_r(C_{2v+2+l}) = \begin{cases} 
q^{2v+1} - q^{2v+1-r} - q^{v+1} + q^{v-1}q', & r = 2, 3, \ldots, v + 2, \\
q^{2v+1} - q^{2v+2-r} - q^{v+1} + q^r q', & r = v + 3, \ldots, 2v + 2 + l.
\end{cases}$$

**Proof.** By Theorem 2,

$$d_r(C_{2v+\delta+l}) = n - \max\{|P_{2v+\delta+l-1-r} \cap Q_{2v+\delta+l}|\},$$

where $P_{2v+\delta+l-1-r}$ runs through all $(2v + \delta + l - 1 - r)$-flats of $\text{PG}(2v + \delta + l - 1, F_q)$ and

$$n = \frac{q^{2v+\delta+l-1} + q^{v+l} - q^{v+\delta+l-1} - 1}{q - 1}.$$

At first, we discuss the case $\delta = 2$ and $r = 1.$ By Lemmas 15 and 16, we have

$$d_1(C_{2v+2+l}) = n - |P \cap Q_{2v+\delta+l}|,$$

where $P$ is $(2v + l)$-flat of type $(2v + 1 + l, 2v + 1, v, \Gamma, l).$ By Corollary 14,

$$|P \cap Q_{2v+\delta+l}| = \frac{q^{2v+l} - 1}{q - 1},$$

hence,

$$d_1(C_{2v+2+l}) = \frac{q^{2v+1} + q^{v+l} - q^{v+1} - q^{2v+l}}{q - 1} = q^{v+l}(q^v - 1).$$

Now, we consider all the other cases. By Lemmas 15–17, $\max\{|P_{2v+\delta+l-1-r} \cap Q_{2v+\delta+l}|\},$ where $P_{2v+\delta+l-1-r}$ runs through all $(2v + \delta + l - 1 - r)$-flats of $\text{PG}(2v + \delta + l - 1, F_q),$ is achieved by $(2v + \delta + l - 1 - r)$-flats of type $(2v + \delta + l - r, 2s, s, t)$ with $s$ as small as possible. Let $P$ be a $(2v + \delta + l - 1 - r)$-flat of type $(2v + \delta + l - r, 2s, s, t),$ then

$$2s \leq 2v + \delta + l - r - t \leq v + s.$$  

The least possible value of $s$ satisfying this inequality is $v + \delta - r,$ while $t = l,$ when $r \leq v + \delta,$ and 0 when $r > v + \delta,$ i.e.,

$$s = \begin{cases} 
v + \delta - r, & \text{when } r \leq v + \delta, \\
0, & \text{when } r > v + \delta.
\end{cases}$$
We distinguish the following cases:

(a) $\delta = 0$. When $r \leq v$, $P$ is of type $(2v + l - r, 2(v - r), v - r, l)$. Then

$$d_r(C_{2v+l}) = \frac{q^{2v+l-1} + q^{v+l} - q^{v+l-1} - 1}{q - 1} - \frac{q^{2v+l-r-1} + q^{v+l} - q^{v+l-1} - 1}{q - 1}$$

$$= q^{v-1} - q^{2v-r-1} q' .$$

When $2v + l > r > v$, $P$ is of type $(2v + l - r, 0, 0, l)$. Then

$$d_r(C_{2v+l}) = \frac{q^{2v+l-1} + q^{v+l} - q^{v+l-1} - 1}{q - 1} - \frac{q^{2v+l-r} - 1}{q - 1}$$

$$= q^{v-1} - q^{2v-r} + q^v - q^{v-1} q' .$$

When $r = 2v + l$, we have

$$d_r(C_{2v+l}) = q^{2v+l-1} + q^v - q^{v-1} - 1 q' .$$

(b) $\delta = 1$. When $r \leq v + 1$, $P$ is of type $(2v + 1 + l - r, 2(v + 1 - r), v + 1 - r, l)$, so

$$d_r(C_{2v+1+l}) = \frac{q^{2v+l-1} + q^{v+l} - q^{v+l-1} - 1}{q - 1} - \frac{q^{2v+l-r} - 1}{q - 1}$$

$$= q^{v} - q^{2v-r} q' + q^{v-1} q' .$$

When $2v + 1 + l > r > v + 1$, $P$ is of type $(2v + 1 + l - r, 0, 0, l)$. Then

$$d_r(C_{2v+1+l}) = \frac{q^{2v+l-1} + q^{v+l} - q^{v+l-1} - 1}{q - 1} - \frac{q^{2v+1+l-r} - 1}{q - 1}$$

$$= q^{v} - q^{2v+1-r} q' .$$

When $r = 2v + 1 + l$, we have

$$d_r(C_{2v+1+l}) = |Q_{2v+1+l}| = \frac{q^{2v+1+l} - 1}{q - 1} .$$

(c) $\delta = 2$. When $2 \leq r \leq v + 2$, $P$ is of type $(2v + 2 + l - r, 2(v + 2 - r), v + 2 - r, l)$, so

$$d_r(C_{2v+2+l}) = \frac{q^{2v+2+l-1} + q^{v+l} - q^{v+2+l-1} - 1}{q - 1} - \frac{q^{2v+1+l-r} + q^{v+l} - q^{v+l-1} - 1}{q - 1}$$

$$= q^{v+1} - q^{2v+1-r} - q^{v+1} + q^{v-1} q' .$$
When $2v + 2 + l > r > v + 2$, $P$ is of type $(2v + 2 + l - r, 0, 0, l)$. Then
\[
d_r(C_{2v+2+l}) = \frac{q^{2v+2+l-1} + q^{r+l} - q^{r+2v+l-1} - 1}{q-1} - \frac{q^{2v+2+l-r} - 1}{q-1}.
\]
When $r = 2v + 2 + l$, we have
\[
d_{2v+2+l}(C_{2v+2+l}) = |Q_{2v+2+l}| = \frac{q^{2v+1+l+1} + q^{v+1} - q^{v+1+l} - 1}{q-1}.
\]

**Corollary 19.** Let $k = 2v + l$, $r < v$. Then $d_r(C_{2v+l})$ meets the Griesmer–Wei bound.

**Proof.** By Theorem 18, $d_l(C_{2v+l}) = q^{2v-2+l}$. Thus,
\[
\sum_{i=0}^{r-1} \left[ \frac{d_l(C_{2v+l})}{q^i} \right] = q^l(q^{2v-2} + q^{2v-3} + \ldots + q^{2v-r-1})
\]
\[
= \frac{q^{2v-1} - q^{2v-r-1}}{q-1} q^l.
\]
\[
= d_r(C_{2v+l}). \quad \square
\]

7. The code $C_{2v+t}$ satisfies the chain condition

**Theorem 20.** The code $C_{2v+\delta+l}$ satisfies the chain condition.

**Proof.** We give only the proof of the case $\delta = 2$. The proofs of the other two cases are similar. By (2) and (4), there are $(2v + l)$-flats of type $(2v + 1 + l, 2v + 1, v, l)$ in $PG(2v + 1 + l, \mathbb{F}_q)$. Let $P_{2v+l}$ be one of them. Then by Lemma 15,
\[
d_1(C_{2v+2+l}) = n - |P_{2v+l} \cap Q_{2v+2+l}|.
\]
Again by (2) and (4), there are $(2v - 1 + l)$-flats of type $(2v + l, 2v, v, l)$ in $PG(2v + 1 + l, \mathbb{F}_q)$. Let $P_{2v-1+l}$ be one of them, and we can assume that $P_{2v-1+l} \subset P_{2v+l}$. Then by Lemmas 15–17,
\[
d_2(C_{2v+2+l}) = n - |P_{2v-1+l} \cap Q_{2v+2+l}|.
\]
And, again by (2) and (4), there are $(2v - 2 + l)$-flats of type $(2v - 1 + l, 2(v - 1), v - 1, l)$ in $PG(2v + 1 + l, \mathbb{F}_q)$. Let $P_{2v-2+l}$ be one of them, and we can assume that $P_{2v-2+l} \subset P_{2v-1+l}$. Then by Lemmas 15–17,
\[
d_3(C_{2v+2+l}) = n - |P_{2v-2+l} \cap Q_{2v+2+l}|.
\]
Proceeding in this way, we can find a chain of flats

\[ P_{2v+1} \supset P_{2v-1+l} \supset P_{2v-2+l} \supset \cdots \supset P_0 \supset P_{-1}, \]

where \( P_{2v+l} \) is a \((2v + l)\)-flat of type \((2v + 1, 2v + 1, v, \Gamma, l)\), \( P_i \) is an \( i \)-flat of type \((i + 1, 2(i + v + 1 - l), i + v + 1 - l, l)\) or \((i + 1, 0, 0, l)\), when \( 2v - 1 + l \geq i \geq v - 1 + l \) or \( v - 1 + l > i \geq 0 \), respectively, and \( P_{-1} = 0 \), such that

\[ d_r(C_{2v+2+l}) = n - |P_{2v+1+i-r} \cap Q_{2v+2+l}| \quad \text{for } 1 \leq r \leq 2v + 2 + l. \]

For \(-1 \leq i \leq 2v + l\), let

\[ P_i^\perp = \{(x_1, \ldots, x_k) \in \mathbb{F}_q^{(k)} | (x_1, \ldots, x_k) \cdot (y_1, \ldots, y_k) = 0 \ \forall (y_1, \ldots, y_k) \in P_i \}, \]

where \( k = 2v + 2 + l \). Then \( P_i^\perp \) is a \((2v + l - i)\)-flat in \( \text{PG}(2v + 1 + l, \mathbb{F}_q) \) and

\[ P_i^\perp \supset P_{2v+1+i-l} \supset \cdots \supset P_0^\perp \supset P_{-1}^\perp = \mathbb{F}_q^{(2v+2+l)}. \]

Denote by \( P_i^\perp \) a matrix representation of the \((2v + l - i)\)-flat \( P_i^\perp \) and let

\[ C_{2v+1+i-l} = P_i^\perp G_{2v+2+l}, \]

where \( G_{2v+2+l} \) is a generator matrix of the projective code \( C_{2v+1+l} \). Then \( C_{2v+1+l-i} \) is a \((2v + 1 + l - i)\)-dimensional subcode of \( C_{2v+2+l} \), and we have

\[ C_1 \subset C_2 \subset \cdots \subset C_{2v+1+l} \subset C_{2v+2+l}. \]

Then for \(-1 \leq i \leq 2v + l\),

\[ |\chi(C_{2v+1+l-i})| = n - |P_i \cap Q_{2v+2+l}| = d_{2v+1+l-i}(C_{2v+2+l}). \]

So \( C_j, j = 1, 2, \ldots, 2v + 2 + l \), is a chain of subcodes of \( C_{2v+2+l} \) such that

\[ |\chi(C_j)| = d_j(C_{2v+2+l}), \quad 1 \leq j \leq 2v + 2 + l, \]

and \( C_j \subset C_{j+1} \). Therefore \( C_{2v+2+l} \) satisfies the chain condition. \( \square \)

8. The generalized weights spectra of \( C_{2v+\delta+l} \)

**Theorem 21.** For \( r = 1, 2, \ldots, k - 1 \), all \( r \)-dimensional subcodes of \( C_{2v+\delta+l} \), whose duals are of type \((2v + \delta + l - r, 2s + \gamma, s, \Gamma)\), have the same support weight. Denote such a subcode by \( D_{r,s,\gamma,\Gamma} \), then

\[
w_s(D_{r,s,\gamma,\Gamma}) \]

\[
= \frac{q^{2v+\delta+l-1} + q^{2v+\delta+l-r-\delta-1} - q^{2v+\delta+l-r-\gamma-1} - q^{2v+\delta+l-r-s-\gamma-1}}{q - 1}.
\]
Proof. Immediate from Theorem 2 and Corollary 13. □

For simplicity, write \( w_{r,s,0} \), \( w_{r,s,1,1} \), and \( w_{r,s,2} \) for \( w_s(D_{r,s,0}) \), \( w_s(D_{r,s,1,1}) \), and \( w_s(D_{r,s,2}) \), respectively. Clearly, \( w_{r,s,1,1} \) does not depend on \( s \) and \( I \), and we may write \( w_{r,1} \) for \( w_{r,s,1,1} \). Then

\[
\begin{align*}
\frac{q^{2v+\delta+l-1}+q^{v+l}+q^{2v+\delta+l-r-s-1}-q^{v+\delta+l-1+r}-q^{2v+\delta+l-r-s}}{q-1}, \\
\frac{q^{2v+\delta+l-1}+q^{v+l}-q^{v+\delta+l-1}-q^{2v+\delta+l-r-1}}{q-1}, \\
\frac{q^{2v+\delta+l-1}+q^{v+l}+q^{2v+\delta+l-r-s-1}-q^{v+\delta+l-1+r}-q^{2v+\delta+l-r-s-2}}{q-1}.
\end{align*}
\]

Denote by \( w_r \) the set of support weights of \( r \)-dimensional subcodes of \( C_{2v+\delta+l} \), then \( w_r \) is the union of the following three sets

\[
\{w_{r,s,0} | s_0 \text{ satisfies } (8)\}, \\
\{w_{r,1}\}, \\
\{w_{r,s,2} | s_2 \text{ satisfies } (9)\},
\]

where (8) and (9) are

\[
\min\{l, 2v + \delta + l - r - 2s_0\} \geq \max\{0, v + \delta + l - r - s_0\}, \tag{8}
\]

and

\[
\min\{l, 2v + \delta + l - r - 2s_2 - 2\} \geq \max\{0, v + \delta + l - r - s_2\}, \tag{9}
\]

respectively. Therefore we have

**Theorem 22.** For \( r = 1, 2, \ldots, k - 1 \), the \( r \)-generalized weight distribution function of \( C_{2v+\delta+l} \) is

\[
A^{(r)}(z) = \sum_{s_0} N(2v + \delta + l - r, 2s_0, s_0; 2v + \delta + l)z^{w_{r,0}} + \sum_{s_1} N(2v + \delta + l - r, 2s_1, 1, s_1, I; 2v + \delta + l)z^{w_{r,1}} + \sum_{s_2} N(2v + \delta + l - r, 2s_2 + 2, s_2; 2v + \delta + l)z^{w_{r,2}},
\]
where the summation range of $\Gamma$ is $\{1,z\}$ when $q$ is odd, and is $\{1,0\}$ when $q$ is even, and the summation ranges of $s_0, s_2, s_1$ are given by (8), (9), and

$$\min\{i, 2v + \delta + 1 - r - 2s_1 - 1\}$$

$$\geq \begin{cases} \max\{0, v + \delta + 1 - r - s_1 - \min\{\delta, 1\}\}, \\
\text{if } \delta \neq 1, \text{ or } \delta = 1 \text{ and } \Gamma = 1, \\
\max\{0, v + \delta + 1 - r - s_1\} \\
\text{if } \delta = 1 \text{ and } \Gamma = z \text{ when } q \text{ is odd,} \\
\text{or if } \delta = 1 \text{ and } \Gamma = 0 \text{ when } q \text{ is even,} \end{cases}$$

respectively. Moreover,

$$A(k)(z) = z^{w_k(C_{2v+\delta+l})} = z^n,$$

where $n$ is given by (*).

**Proof.** We need only to prove that $w_4(C_{2v+\delta+l}) = n$, but this is trivial. $\Box$

**Corollary 23.** The weight spectrum of $C_{2v+\delta+l}$ is as follows:

1. The set of all weights achieved by some nonzero code-word in $C_{2v+\delta+l}$ is $W_0 \cup W_1 \cup W_2$, where

$$W_0 = \left\{ q^{2v+\delta+l-2} - q^{2v+\delta+l-s-2} - q^{v+1}q^{\delta-1} - 1 \right\} \min\{l, 2v + \delta + 1 - 2s\}$$

$$\geq \max\{0, v + \delta + 1 - s\},$$

$$W_1 = \left\{ q^{2v+\delta+l-2} - q^{v+1}q^{\delta-1} - 1 \right\} \min\{l, 2v + \delta + 1 - 2s\}$$

$$\geq \max\{0, v + 1 - s\}, \delta = 0, 1,$$

$$\max\{0, v + l - s\}, \delta = 2,$$

and

$$W_2 = \left\{ q^{2v+\delta+l-2} - q^{2v+\delta+l-s-3} - q^{v+1}q^{\delta-1} - 1 \right\} \min\{l, 2v + \delta + 1 - 2s\}$$

$$\geq \max\{0, v + \delta + 1 - s\}.$$ 

2. For every weight $i$, denote by $A_i$ the number of codewords which are of weight $i$. Then

$$A_0 = 1,$$

$$A_i = 0, \text{ when } i \neq 0 \text{ and } i \notin W_0 \cup W_1 \cup W_2,$$
\[ A_i = (q - 1) \sum_{i,s} N(2v + \delta + l - 1, 2s + \gamma, s, i; 2v + \delta + l), \]

where the summation ranges of \( s \) are as in \( W_0, W_1, \) and \( W_2, \) \( i \in W_0 \cup W_1 \cup W_2, \) \( \Gamma = 1 \) or \( z \) when \( q \) is odd, \( \Gamma = 1 \) or \( 0 \) when \( q \) is even. \]

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References


