Oscillation of a quasilinear impulsive delay parabolic equation with two different boundary conditions

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Abstract

In this paper, we discuss the oscillation for a class of quasilinear impulsive delay parabolic equations with two different boundary conditions and obtain several oscillation criteria. © 2004 Elsevier Inc. All rights reserved.

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1. Introduction

In recent years, there has been increasing interest in studying the oscillatory behavior of solutions of impulsive partial functional differential equation (see [1–14]). Erbe et al. [11] obtained comparison principles for impulsive parabolic equations, and this was the earliest work in this field of research. In [1], Bainov has considered the impulsive delay partial differential system with monotone iterative methods. In [7], Deng and Ge have studied
the oscillation of the impulsive delay parabolic equation, and obtained some sufficient conditions of oscillation. So far this was the unique work about impulsive delay distributed parameter system.

In this paper, we consider the following quasilinear impulsive delay parabolic equation of neutral type, and obtain several oscillation criteria:

\[
\begin{aligned}
\frac{\partial}{\partial t} \left[ u(x,t) - \sum_{i=1}^{m} p_i(t) u(x, t - \tau_i) \right] &= a(t) \Delta u(x,t) + b(t) \Delta u(x,t - \delta) - h(x,t) u(x, t - \sigma) - q(x,t) f[u(x,t - \rho)], \\
u(x, t_k^+) - u(x, t_k^-) &= C_k u(x, t_k), \quad k = 1, 2, 3, \ldots,
\end{aligned}
\]

with Robin boundary condition

\[
\frac{\partial u}{\partial n} + \beta u = 0, \quad x \in \partial \Omega, \ t \neq t_k,
\]

or with Dirichlet boundary condition

\[
u = 0, \quad x \in \partial \Omega, \ t \neq t_k,
\]

where \( \Delta u \) is the Laplacian in \( \mathbb{R}^N \), \( R_+ = [0, +\infty) \), \( \Omega \) is a boundary domain in \( \mathbb{R}^N \) with a piecewise continuous smooth boundary \( \partial \Omega \). \( n \) denotes the unit exterior vector normal to \( \partial \Omega \).

We assume that the following conditions (H) hold throughout the paper:

(i) \( 0 < t_1 < t_2 < \cdots < t_k < \cdots \) and \( \lim_{k \to +\infty} t_k = +\infty \);
(ii) \( p_i(t), a(t), b(t) \in PC(R_+, R_+), h(x,t), q(x,t) \in PC[\bar{\Omega} \times R_+, R_+] \), \( i = 1, 2, \ldots, m \);
(iii) \( \tau_i, \delta, \sigma, \rho \) are positive constants, \( i = 1, 2, \ldots, m \); \( C_k > -1, k = 1, 2, \ldots \);
(iv) \( u(x, t) \) is a piecewise continuous function,

\[
u(x, t_k) = u(x, t_k^-), \quad k = 1, 2, \ldots;
\]

(v) \( \beta(x) \in C(\partial \Omega \cup \{0, +\infty\}) \).

The paper is organized as follows. In the next section, we give three propositions. In Section 3, we give several theorems about the sufficient conditions of the oscillation and their proofs.

2. Preliminaries

In this section, we give the definition of oscillation and state three lemmas that will be used in Section 3.

Definition. A nonzero solution \( u(x, t) \) of the problem (1.1) with some boundary condition is called oscillatory in the domain \( G = \Omega \times R_+ \) if for each positive number \( T \) there exists
a point \((x_0, T_0) \in \Omega \times [T, +\infty)\) such that \(u(x_0, T_0) = 0\). Otherwise, we say the solution is nonoscillatory.

**Lemma 2.1.** Assume that condition (v) holds and \(\lambda_1\) is the first eigenvalue of the Robin eigenvalue problem

\[
\Delta \varphi + \lambda \varphi = 0, \quad x \in \Omega, \quad \frac{\partial \varphi}{\partial n} + \beta(x) \varphi = 0, \quad x \in \partial \Omega, \tag{2.1}
\]

and \(\varphi_1(x)\) is the eigenvalue corresponding to \(\lambda_1\); then \(\lambda_1 > 0\) and \(\varphi_1(x) > 0, x \in \Omega\). (See [15, Theorem 3.3.22].)

**Lemma 2.2.** Assume that condition (v) holds and \(\lambda_2\) is the first eigenvalue of the Dirichlet eigenvalue problem

\[
\Delta \varphi + \lambda \varphi = 0, \quad x \in \Omega, \quad \varphi = 0, \quad x \in \partial \Omega, \tag{2.2}
\]

and \(\varphi_2(x)\) is the eigenvalue corresponding to \(\lambda_2\); then \(\lambda_2 > 0\) and \(\varphi_2(x) > 0, x \in \Omega\). (See [15, Theorem 3.3.22].)

**Lemma 2.3.** Let \(\mu\) be a positive constant, \(p(t) \in C([0, +\infty), (0, +\infty))\), and \(y(tk) = y(t - k)\), \(k = 1, 2, \ldots\). If hypotheses

(a) \(\limsup_{t \to +\infty} \prod_{t - \mu < x < t} (1 + \tilde{C}_k) < +\infty, \tilde{C}_k = \max\{0, C_k\}\);
(b) \(\liminf_{t \to +\infty} \int_{t - \mu}^t p(s) ds > \frac{1}{\mu} \limsup_{t \to +\infty} \prod_{t - \mu < x < t} (1 + C_k)\),

hold, then differential inequality

\[
\begin{cases}
    y'(t) + p(t)y(t - \mu) \leq 0, & t \geq 0, t \neq t_k,
    \\
    y(t_k^+) - y(t_k^-) \leq C_k y(t_k), & k = 1, 2, \ldots,
\end{cases}
\]

has no nonoscillatory solution. (See [16, Lemma 1].)

3. Main results

In this section, we state three oscillation criteria and present their proofs.

**Theorem 1.** Let \(H(t) = \min_{x \in \Omega} h(x, t), Q(t) = \min_{x \in \Omega} q(x, t)\). Assume that (H) and the following hypotheses hold:

\((H_1)\) \(f(u)\) is a positive convex function in \(R_+\), \(f(-u) = -f(u) < 0\), and there exists a positive continuous function \(\alpha(t)\) such that \(f(u)/u > \alpha(t), u \in R_+, t \in R_+\);
\((H_2)\) \(\sum_{i=1}^m p_i(t) = 1\) and there exists \(1 \leq r \leq m\) such that \(\lim_{t \to +\infty} p_r(t) = p_0 > 0\.\)
If
\[
\limsup_{t \to +\infty} \prod_{t-\delta < t_k < t} (1 + \tilde{C}_k) < +\infty, \quad \tilde{C}_k = \max \{0, C_k\},
\] (3.1)
\[
\liminf_{t \to +\infty} \int_{t-\delta}^t H(s) \, ds > \frac{1}{e} \limsup_{t \to +\infty} \prod_{t-\delta < t_k < t} (1 + C_k),
\] (3.2)
then all nonzero solutions of the boundary value problem (1.1), (B_j), j = 1, 2, are oscillatory in G.

**Proof.** Assume that there exists a nonoscillatory solution of the problem (1.1), (B_j).

Let \( u(x, t) \) be a nonoscillatory positive solution of the boundary value problem (1.1), (B_j), \( \Omega \times (T_0, +\infty) \), for \( T_0 > 0 \).

By the condition (ii), there exists a \( T_1 \geq T_0 \) such that
\[
t - \tau_i \geq T_0, \quad t - \delta \geq T_0, \quad t - \sigma \geq T_0, \quad t - \rho \geq T_0,
\]
for \( t \geq T_1, i = 1, 2, \ldots, m; \)
then
\[
u(x, t - \tau_i) > 0, \quad u(x, t - \delta) > 0, \quad u(x, t - \sigma) > 0, \quad u(x, t - \rho) > 0,
\]
for \( (x, t) \in \Omega \times (T_1, +\infty) \).

When \( t \neq t_k \), multiplying both sides of Eq. (1.1), (B_j), by the eigenfunction \( \varphi_j(x) \) of eigenvalue problem (2.1) \( (j = 1, 2) \), then integrating with respect to \( x \) over the domain \( \Omega \), we have
\[
\frac{\partial}{\partial t} \left( \int_{\Omega} u \varphi_j(x) \, dx - \sum_{i=1}^{m} p_i(t) \int_{\Omega} u(x, t - \tau_i) \varphi_j(x) \, dx \right)
\]
\[
= a(t) \int_{\Omega} \varphi_j(x) \Delta u(x, t) \, dx + b(t) \int_{\Omega} \varphi_j(x) \Delta u(x, t - \delta) \, dx
\]
\[- \int_{\Omega} \varphi_j(x) h(x, t) u(x, t - \sigma) \, dx - \int_{\Omega} \varphi_j(x) q(x, t) f[u(x, t - \rho)] \, dx,
\]
\[t > T_1, \quad t \neq t_k, \quad j = 1, 2,\] (3.3)
Using Green’s formula, Lemmas 2.1 and 2.2, and Jensen inequality, we have
\[
\int_{\Omega} \Delta u \varphi_j \, dx = \int_{\partial \Omega} \left( \varphi_j(s) \frac{\partial u}{\partial n} - u \frac{\partial \varphi_j(s)}{\partial n} \right) \, ds + \int_{\Omega} u \Delta \varphi_j \, dx
\]
\[- \lambda_j \int_{\Omega} u \varphi_j \, dx, \quad t > T_1, \quad t \neq t_k, \quad j = 1, 2,
\]
\[
\int_{\Omega} \Delta u(x, t - \delta) \varphi_j \, dx = \int_{\Omega} u(x, t - \delta) \Delta \varphi_j \, dx = -\lambda_j \int_{\Omega} u(x, t - \delta) \varphi_j \, dx,
\]
\[t > T_1, \quad t \neq t_k, \quad j = 1, 2,
\]
\begin{equation}
\Omega \int q(x,t)f[u(x,t-\rho)]\varphi_j(x)dx \\
\geq Q(t)\left(\int_{\Omega} \varphi_j(x)dx \right)f\left[\left(\int_{\Omega} \varphi_j(x)dx \right)^{-1} \int_{\Omega} u(x,t-\rho)\varphi_j(x)dx \right],
\end{equation}

\text{for } t > T_1, t \neq t_k.

So we get

\begin{equation}
\frac{\partial}{\partial t} \left( \int_{\Omega} u(x,t)\varphi_j(x)dx - \sum_{i=1}^{m} p_i(t) \int_{\Omega} u(x,t-\tau_i)\varphi_j(x)dx \right)
\leq -\lambda_j a(t) \int_{\Omega} u(x,t)\varphi_j(x)dx - \lambda_j b(t) \int_{\Omega} u(x,t-\delta)\varphi_j(x)dx \\
- H(t) \int_{\Omega} u(x,t-\sigma)\varphi_j(x)dx \\
- Q(t) \left( \int_{\Omega} \varphi_j(x)dx \right)f\left[\left(\int_{\Omega} \varphi_j(x)dx \right)^{-1} \int_{\Omega} u(x,t-\rho)\varphi_j(x)dx \right],
\end{equation}

\text{for } t > T_1, t \neq t_k, j = 1, 2.

(3.4)

Let

\begin{equation}
V(t) = \left( \int_{\Omega} \varphi_j(x)dx \right)^{-1} \int_{\Omega} u\varphi_j(x)dx.
\end{equation}

We have

\begin{equation}
\frac{d}{dt} \left( V(t) - \sum_{i=1}^{m} p_i(t) V(t-\tau_i) \right) + \lambda_j a(t) V(t) + \lambda_j b(t) V(t-\delta) + H(t) V(t-\sigma) \\
+ Q(t) a(t-\rho) V(t-\rho) \leq 0, \quad t > T_1, t \neq t_k, j = 1, 2.
\end{equation}

(3.5)

Let

\begin{equation}
Z(t) = V(t) - \sum_{i=1}^{m} p_i(t) V(t-\tau_i).
\end{equation}

We have

\begin{equation}
Z'(t) + \lambda_j a(t) Z(t) + \lambda_j b(t) Z(t-\sigma) + H(t) Z(t-\delta) + Q(t) a(t-\rho) Z(t-\rho) \\
\leq -\lambda_j a(t) \sum_{i=1}^{m} p_i(t) V(t-\tau_i) - \lambda_j b(t) \sum_{i=1}^{m} p_i(t-\sigma) V(t-\sigma-\tau_i) \\
- H(t) \sum_{i=1}^{m} p_i(t-\delta) V(t-\delta-\tau_i) \\
- Q(t) a(t-\rho) \sum_{i=1}^{m} p_i(t-\rho) V(t-\rho-\tau_i),
\end{equation}

\text{for } t > T_1, t \neq t_k, j = 1, 2.

(3.6)
We can find $T_2 > T_1$ such that $V(t - \sigma - \tau_i) > 0$, $V(t - \delta - \tau_i) > 0$, $V(t - \rho - \tau_i) > 0$, $t > T_2$. Then (3.6) implies that

$$
Z'(t) + \lambda_j a(t)Z(t) + \lambda_j b(t)Z(t - \sigma) + H(t)Z(t - \delta) + Q(t)a(t - \rho)Z(t - \rho) \leq 0, \quad t > T_2, \ t \neq t_k, \ j = 1, 2.
$$

(3.7)

We will prove $Z(t) > 0$ in the following.

If $Z(t) < 0$, we can prove that $V(t)$ is bounded. Otherwise, there exists a sequence $\{t_n\}$ such that

$$
t_n \in [T_2, +\infty), \quad \lim_{n \to +\infty} t_n = +\infty,
$$

and

$$
\lim_{n \to +\infty} V(t_n) = +\infty, \quad V(t_n) = \max_{T_2 \leq t \leq t_n} V(t),
$$

$$
0 > Z(t_n) = V(t_n) - \sum_{i=1}^{m} p_i(t_n)V(t_n - \tau_i) \geq V(t_n) - \sum_{i=1}^{m} p_i(t_n)V(t_n) = \left(1 - \sum_{i=1}^{m} p_i(t_n)\right)V(t_n) = 0.
$$

It leads to contradiction, so $V(t)$ is bounded. It is easy to get that $Z(t)$ is bounded from $(H_2)$.

Since $Z'(t) \leq 0$, $\lim_{t \to +\infty} Z(t)$ exists.

In fact, there exist two sequences $\{t_n\}$ and $\{t^*_n\} \subset R^+$ such that

$$
\lim_{n \to +\infty} t_n = \lim_{n \to +\infty} t^*_n = +\infty,
$$

and

$$
\lim_{n \to +\infty} V(t_n) = \lim_{n \to +\infty} V(t^*_n) = L, \quad \lim_{n \to +\infty} V(t^*_n) = \liminf_{n \to +\infty} V(t^*_n) = l.
$$

From $(H_2)$, we have

$$
p_r(t_n) > 0, \quad p_r(t^*_n) > 0,
$$

$$
V(t_n - \tau_i) \leq L + \varepsilon, \quad V(t^*_n - \tau_i) \geq l - \varepsilon.
$$

So

$$
V(t_n - \tau_r) = \frac{1}{p_r(t_n)} \left[V(t_n) - Z(t_n) - \sum_{i=1, i \neq r}^{m} p_i(t_n)V(t_n - \tau_i)\right]
$$

$$
\geq \frac{1}{p_r(t_n)} \left[V(t_n) - Z(t_n) - (L + \varepsilon)(1 - p_i(t_n))\right],
$$

$$
L \geq \lim_{n \to +\infty} V(t_n - \tau_r) \geq \frac{1}{p_0} \left[L - \lim_{n \to +\infty} Z(t_n) - (L + \varepsilon)(1 - p_0)\right].
$$

We get

$$
\lim_{n \to +\infty} Z(t_n) \geq -\varepsilon(1 - p_0),
$$
\[
V(t_n^* - \tau_r) = \frac{1}{p_r(t_n^*)} \left[ V(t_n^*) - Z(t_n^*) - \sum_{i=1, i \neq r}^{m} p_i(t_n^*)V(t_n^* - \tau_i) \right]
\leq \frac{1}{p_r(t_n^*)} \left[ V(t_n^*) - Z(t_n^*) - (l - \epsilon)(1 - p_r(t_n^*)) \right],
\]

\[
l \leq \lim_{n \to +\infty} V(t_n^* - \tau_r) \leq \frac{1}{p_0} \left[ l - \lim_{n \to +\infty} Z(t_n^*) - (l - \epsilon)(1 - p_0) \right].
\]

So we obtain
\[
\lim_{n \to +\infty} Z(t_n^*) \leq \epsilon(1 - p_0).
\]

Then
\[
-\epsilon(1 - p_0) \leq \lim_{n \to +\infty} Z(t_n) \leq \epsilon(1 - p_0).
\]

Since \( \epsilon \) is arbitrary, \( \lim_{n \to +\infty} Z(t_n) = 0 \), which contradicts with \( Z'(t) < 0 \) and \( Z(t) < 0, \ t > T_2 \).

Now we prove \( Z(t) > 0, \ t > T_2 \).

According to (3.7), we get
\[
Z'(t) + H(t)Z(t - \delta) \leq 0, \quad t > T_2, \ t \neq t_k.
\] (3.8)

For \( t = t_k, \ k = 1, 2, \ldots \), it is easy to get
\[
Z(t_k^+) - Z(t_k^-) = C_k Z(t_k).
\] (3.9)

Obviously, \( Z(t) \) is the nonoscillatory positive solution of the impulsive differential inequality (3.8) and condition (3.9). On the other side, from Lemma 2.3 and the hypotheses of Theorem 1, there is no eventually positive solution. It leads to contradiction. This completes the proof of Theorem 1. \( \square \)

From (3.7), it implies that
\[
Z(t) + \lambda_j b(t) Z(t - \sigma) \leq 0, \quad t > T_2, \ t \neq t_k, \ j = 1, 2.
\] (3.10)

We can obtain the following oscillation criterion.

**Theorem 2.** Assume that \( (H) \), \( (H_1) \), and \( (H_2) \) hold. If
\[
\limsup_{t \to +\infty} \prod_{t - \sigma < t_k < t} (1 + \tilde{C}_k) < +\infty, \quad \tilde{C}_k = \max\{0, C_k\},
\]
\[
\liminf_{t \to +\infty} \int_{t - \sigma}^{t} \lambda_j b(s) ds > \frac{1}{e} \limsup_{t \to +\infty} \prod_{t - \sigma < t_k < t} (1 + C_k),
\]
then all nonzero solutions of the boundary value problem (1.1), \( (B_j), \ j = 1, 2 \), are oscillatory in \( G \).
From (3.7), we can get
\[ Z'(t) + Q(t)\alpha(t - \rho)Z(t - \rho) \leq 0, \quad t > T_2, \ t \neq t_k. \] (3.11)

Similarly, we obtain the following theorem.

**Theorem 3.** Assume that \((H), (H_1), \) and \((H_2)\) hold. If
\[ \limsup_{t \to +\infty} \prod_{t - \rho < t_k < t} (1 + C_k) < +\infty, \quad C_k = \max\{0, C_k\}, \]
\[ \liminf_{t \to +\infty} \int_{t - \rho}^{t} Q(s)\alpha(s - \rho) ds > \frac{1}{e} \limsup_{t \to +\infty} \prod_{t - \rho < t_k < t} (1 + C_k), \]
then all nonzero solutions of the boundary value problem (1.1), \((B_j)\), \(j = 1, 2,\) are oscillatory in \(G.\)

The proofs of Theorems 2, 3 are similar to that of Theorem 1 and are omitted.

**Remark.** The method of integrating (1.1) directly with respect to \(x\) over the domain \(\Omega\) is used commonly. Using that method, we have
\[ b(t) \int_{\Omega} \Delta u(x, t - \tau) dx = b(t) \int_{\partial\Omega} \frac{\partial u}{\partial N} ds = -b(t) \int_{\partial\Omega} \beta(s)u(s, t - \tau) ds \leq 0; \]
where it always was neglected in dealing with the inequality (3.4). Hence, the oscillation criterion based on \(b(t)\) cannot be gained. In this paper, we obtain it in Theorem 2 by the eigenvalue method.

**Example.** Consider the following boundary value problem:
\[
\begin{cases}
\frac{\partial}{\partial t}[u - u(x, t - \pi)] \\
= b(t)u_{xx}(x, t - \pi) + 3t^2(1 + \sin \frac{\pi}{2})u(x, t - \pi)e^{\frac{1}{2}u(x, t - \pi)}, \\
\frac{\partial}{\partial n} u(0, t) + u(0, t) = 0, \quad \frac{\partial}{\partial n} u(\pi, t) + u(\pi, t) = 0,
\end{cases}
\]
where \(n = 1, \Omega = (0, \pi), \ q(x, t) = t^2(1 + \sin \frac{\pi}{2}), \ f[u(x, t - \rho)] = 3u(x, t - \pi)e^{\frac{1}{2}u(x, t - \pi)}\).

\[ b(t) = \begin{cases}
b_0, & \text{if } t = 0, \\
b_0 + \frac{1}{1 - \cos 4t}, & \text{if } t \neq \frac{2\pi}{9}, k = 1, 2, \ldots.
\end{cases}\]

It is easy to see that the conditions of Theorem 2 hold. Hence all the nonzero solutions of the boundary value problem (1.1), \((B_1)\), are oscillatory in \((0, \pi) \times (0, +\infty).\)

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