Pergamon

Computers Math. Applic. Vol. 29, No. 5, pp. 21–42, 1995

0898-1221/95 $9.50 + 0.00

0898-1221(94)00246-0

The Estimation of Normalized Fuzzy Weights

P.-T. CHANG AND E. S. LEE*
Department of Industrial Engineering, Kansas State University
Manhattan, KS 66506, U.S.A.

(Received June 1994; accepted July 1994)

Abstract—The estimation of a normalized set of positive fuzzy weights constitutes the most important aspects in the fuzzy multiple attribute decision making process. A systematic treatment of this problem is carried out in this paper. The concept of fuzzy normalization is first defined and the meaning of consistency in a fuzzy environment is discussed. Based on these definitions and discussions, the various approaches in the literature are examined and several improvements or new approaches are proposed. Numerical examples are used to evaluate and to compare the various existing and the newly proposed approaches.

Keywords—Normalized set of fuzzy weights, Positive fuzzy reciprocal matrix, Fuzzy pairwise comparison, Fuzzy normalization, Approximate fuzzy eigenvalue.

1. INTRODUCTION

One of the most important problems in fuzzy multiple attribute decision making is the estimation of a normalized set of fuzzy weights from a positive fuzzy reciprocal matrix, which was obtained by the use of fuzzy pairwise comparisons of the various factors based on the opinion of an expert. However, there exists no systematic treatment of this important problem in the literature. The purpose of this paper is to examine the existing methods in the literature systematically and to propose some improvements based on this examination.

In order to treat the fuzzy version systematically, the crisp version of this problem is first summarized in the following. Suppose we have N factors $F_1, F_2, \dots, F_N$ and a crisp (nonfuzzy) matrix $M = [r_{ij}]_{N\times N}$, we wish to estimate a set of positive weights, $w = (w_1, w_2, \dots, w_N)$, $\sum_{i=1}^N w_i = 1$, from M. The elements r_{ij} in M are obtained in such a way that they represent the estimates of the relative significance between the factors F_i and F_j, or $r_{ij} = w_i/w_j$. Therefore, r_{ij} can be assumed as a ratio from $[1/9, 1] \cup [1, 9]$ [1] and $r_{ij} = 1/r_{ji}$ by the reciprocal property. Because of these assumptions, if M is consistent (i.e., $r_{ij}r_{jk} = r_{ik}$), we can determine the set of crisp weights w_i, $i = 1, \dots, N$, such that

$$\frac{w_i}{w_j} = r_{ij}, \qquad \forall i, j.$$

Unfortunately, in real-world situations, we usually only have the estimates of r_{ij}, and the actual value for r_{ij} may not be known. The question then arises: how to find w_i, $i = 1, \dots, N$, such that

$$r_{ij} \approx \frac{w_i}{w_j}, \qquad \forall i, j. \tag{1}$$

*Author to whom all correspondence should be addressed.

Typeset by AMS-TeX

Various approaches have been proposed to obtain this set of crisp weights. They can be roughly classified as:

(a) the eigenvector method [1–5],
(b) the least squares method (LSM) [6,7],
(c) the logarithmic least squares method (LLSM) [8,9],
(d) the geometric row means method (GRM) [8–12],
(e) the weighted least squares method (WLSM) [13], and
(f) a category of methods that involve only arithmetic operations: the row means of normalized columns approach [1, p. 239], the normalized row sums and the inverted column sums methods [9, p. 429].

The fuzzy version of this problem with fuzzy pairwise comparisons can be depicted as follows: How to find W_i, $i = 1, \dots, N$, such that

$$R_{ij} \approx \frac{W_i}{W_j}, \qquad \forall i, j, \tag{2}$$

where R_{ij}, W_i and W_j represent fuzzy numbers, and $M = (R_{ij})_{N \times N}$ is a fuzzy matrix obtained from fuzzy pairwise comparisons.

Several fuzzy extensions of the nonfuzzy methods of (1) have been proposed in the literature [14–18]. These extensions will be discussed in detail in this paper. One important advantage of the fuzzy approach is the fact that opinions can be represented fuzzily. Since experts frequently cannot give clear cut opinions, there is a need to develop approaches which have the ability to represent fuzzy or vague opinions. With the use of the fuzzy approach, data can now be estimated and given with fuzziness, tolerance, or vagueness by the experts.

In the investigation of the fuzzy approaches, the following issues are emphasized:

(i) the meaning of fuzzy positive reciprocal matrix, fuzzy normalization, and fuzzy consistency in a fuzzy environment,
(ii) the proper estimation or translation of the fuzziness of the original data into a normalized set of fuzzy weights,
(iii) the relationships between the estimated fuzziness of the original data and the method of normalization used, and
(iv) the limitations and the possible approaches to overcome these limitations for some of the existing methods.

As a result of these investigations, several improvements or new approaches are proposed. Some of them are:

(i) defined and investigated two different approaches for fuzzy normalization,
(ii) fuzzy extension of the crisp (nonfuzzy) row means of the normalized columns method (RMNC) [1, p. 239] is defined and investigated, and
(iii) a procedure for estimating an approximate fuzzy eigenvalue is proposed.

Finally, numerical examples are used to evaluate and to compare the various existing and the newly proposed methods.

2. POSITIVE FUZZY RECIPROCAL MATRICES

In order to construct the positive fuzzy reciprocal matrices, fuzzy arithmetic operations must be properly defined. Furthermore, since the representations are fuzzy and approximate, some approximations to simplify the arithmetic operations are also proposed in the following.

2.1. Fuzzy Arithmetics

The fuzzy numbers will be represented by the L-R type representation as was proposed by Dubois and Prade (see reference [19]). Two equivalent forms are usually used. The standard form,

$A = (m, \beta, \gamma)_{\text{L-R}}$, where β and γ represent the left and right spreads, respectively, m represents the mode, and the subscripts L and R represent the left and right reference (shape) functions of A, respectively. This standard form will be referred to as the *spread form.* The second equivalent representation of A is $A = (l, m, u)_{\text{L-R}}$, where $l \leq m \leq u$, and $l(= m - \beta)$ and $u(= m + \gamma)$ represent the lower and upper bounds of A, respectively. This second form will be referred to as the *bounded form.* Triangular fuzzy numbers with the following membership functions are typical examples:

$$\mu_A(x) = \begin{cases} \dfrac{x-l}{m-l} = \dfrac{x-l}{\beta}, & x \in [l, m], \\ \dfrac{u-x}{u-m} = \dfrac{u-x}{\gamma}, & x \in [m, u], \\ 0, & \text{otherwise.} \end{cases}$$

Flat fuzzy numbers are also used in the literature. In the bounded form, the flat fuzzy number can be represented by $A = (l, m, n, u)_{\text{L-R}}$ where $l \leq m \leq n \leq u$, and l and u represent the lower and upper bounds, respectively, and m and n represent the lower and upper endpoints of the modal interval of A, respectively, and $\mu_A(x) = 1$, $\forall x[m, n]$. Trapezoidal fuzzy numbers with trapezoidal shaped membership functions are typical flat fuzzy numbers.

Fuzzy arithmetics can be defined by the use of Zadeh's extension principle. A unary operation, in the general form of $B = f(A)$, has the membership function

$$\mu_B(y) = \sup_{y=f(x)} \min(\mu_A(x)). \tag{3}$$

Thus, the membership function of the resulting fuzzy number B from a unary operation is obtained by mapping the fuzzy number A through f. For the binary operation, $C = f(A, B)$, we have

$$\mu_C(z) = \sup_{z=f(x,y)} \min(\mu_A(x), \mu_B(y)). \tag{4}$$

Let A be in the bounded form, $A = (l, m, u)_{\text{L-R}}$ (or $A = (l, m, n, u)_{\text{L-R}}$ as a flat fuzzy number) and be positive (i.e., $l \geq 0$ or $l > 0$), then the following unary fuzzy arithmetics can be obtained based on equation (3):

(a) Scalar division, $B = A/\sigma$, $\forall \sigma \in (1, +\infty)$:

$$\mu_B(y) = \mu_A(x)|_{x=y_\sigma} \quad \text{and} \quad B = \left(\frac{l}{\sigma}, \frac{m}{\sigma}, \frac{u}{\sigma}\right)_{\text{L-R}} \quad \left(\text{or } B = \left(\frac{l}{\sigma}, \frac{m}{\sigma}, \frac{n}{\sigma}, \frac{u}{\sigma}\right)_{\text{L-R}}\right). \tag{5}$$

(b) Inverse, $B = A^{-1}$:

$$\mu_B(y) = \mu_A(x)|_{x=1/y} \quad \text{and} \quad B = \left(\frac{1}{u}, \frac{1}{m}, \frac{1}{l}\right)_{\text{R}'\text{-L}'} \quad \left(\text{or } B = \left(\frac{1}{u}, \frac{1}{n}, \frac{1}{m}, \frac{1}{l}\right)_{\text{R}'\text{-L}'}\right). \tag{6}$$

(c) Logarithm, $B = \ln(A)$:

$$\begin{aligned} &\mu_B(y) = \mu_A(x)|_{x=\exp(y)}, \quad \text{and} \\ &\qquad B = (\ln(l), \ln(m), \ln(u))_{\text{L}'\text{-R}'} \ (\text{or } B = (\ln(l), \ln(m), \ln(n), \ln(u))_{\text{L}'\text{-R}'}). \end{aligned} \tag{7}$$

(d) Exponential, $B = \exp(A)$:

$$\begin{aligned} &\mu_B(y) = \mu_A(x)|_{x=\ln(y)}, \quad \text{and} \\ &\qquad B = (\exp(l), \exp(m), \exp(u))_{\text{L}'\text{-R}'} \ (\text{or } B = (\exp(l), \exp(m), \exp(n), \exp(u))_{\text{L}'\text{-R}'}). \end{aligned} \tag{8}$$

(e) N-root, $B = A^{1/N}$:

$$\begin{aligned} &\mu_B(y) = \mu_A(x)|_{x=y^n}, \quad \text{and} \\ &\qquad B = \left(l^{1/N}, m^{1/N}, u^{1/N}\right)_{\text{L}'\text{-R}'} \ \left(\text{or } B = \left(l^{1/N}, m^{1/N}, n^{1/N}, u^{1/N}\right)_{\text{L}'\text{-R}'}\right). \end{aligned} \tag{9}$$

It is important to notice that in (6)–(9), the shapes of the resulting membership functions, $\mu_B(y)$, are different from that of $\mu_A(y)$ of the original fuzzy number A. Therefore, different subscript notations L′ and R′ are used for the resulting fuzzy number B.

The shapes of the resulting membership functions for B in (6)–(9) may be very complicated and difficult to manipulate. However, because of the fuzzy or approximate nature of the representation, approximations have been proposed in the literature for the resulting fuzzy numbers in (6)–(9). The general idea for the approximation is to use directly the original L and R reference (shape) functions of the fuzzy number A for the bounds (or spreads) and the mode of the resulting fuzzy number B. In other words,

(b′) Inverse: $$B \cong \left(\frac{1}{u}, \frac{1}{m}, \frac{1}{l}\right)_{\text{R-L}}. \tag{10}$$

(c′) Logarithm: $$B \cong (\ln l, \ln m, \ln u)_{\text{L-R}}. \tag{11}$$

(d′) Exponential: $$B \cong (\exp(l), \exp(m), \exp(u))_{\text{L-R}}. \tag{12}$$

(e′) N-root: $$B \cong \left(l^{1/N}, m^{1/N}, u^{1/N}\right)_{\text{L-R}}. \tag{13}$$

As an example, consider the fuzzy inverse of a triangular fuzzy number. Let $A = (l, m, u)_{\text{L-R}}$ be a triangular fuzzy number and $B = A^{-1}$, then, according to (6), B has the following exact membership function

$$\mu_B(y) = \mu_A\left(x = \frac{1}{y}\right) = \begin{cases} \dfrac{u - 1/y}{u - m} = \dfrac{u - 1/y}{\gamma}, & y \in \left[\dfrac{1}{u}, \dfrac{1}{m}\right], \\ \dfrac{1/y - l}{m - l} = \dfrac{1/y - l}{\beta}, & y \in \left[\dfrac{1}{m}, \dfrac{1}{l}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

According to (10), the membership function for B can be approximated as

$$\mu_B(y) \cong \begin{cases} \dfrac{y - 1/u}{1/m - 1/u}, & y \in \left[\dfrac{1}{u}, \dfrac{1}{m}\right], \\ \dfrac{1/l - y}{1/l - 1/m}, & y \in \left[\dfrac{1}{m}, \dfrac{1}{l}\right], \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Obviously, flat fuzzy number arithmetic operations can be approximated in a similar manner.

Another approximate formula for fuzzy inverse has also been proposed [19]. This approximate formula used essentially the same idea as that used in formulating (10) except in the spread form. Let $A = (m, \beta, \gamma)_{\text{L-R}}$, then $B = A^{-1}$ was approximated as:

$$(m, \beta, \gamma)^{-1}_{\text{L-R}} \left(\frac{1}{m}, \frac{\gamma}{m^2}, \frac{\beta}{m^2}\right)_{\text{R-L}}. \tag{16}$$

The differences between these two approximations are: $\gamma/m^2 > (1/m - 1/u)$ and $\beta/m^2 < (1/l - 1/m)$, for $\gamma, \beta, l > 0$. Formula (16) is best applied to situations where the fuzzy inverse (the resulting fuzzy number) occurs in the neighbor of its mode; i.e., its spreads must be relatively small compared to its mode.

The arithmetics of binary operations can also be obtained based on the extension principle. Assuming all the fuzzy numbers have a same type of reference (shape) functions, then the well known fuzzy addition and fuzzy subtraction formulas are:

$$(l_1, m_1, u_1)_{\text{L-R}} + (l_2, m_2, u_2)_{\text{L-R}} = (l_1 + l_2, m_1 + m_2, u_1 + u_2)_{\text{L-R}}, \tag{17a}$$

$$(l_1, m_1, n_1, u_1)_{\text{L-R}} + (l_2, m_2, n_2, u_2)_{\text{L-R}} = (l_1 + l_2, m_1 + m_2, n_1 + n_2, u_1 + u_2)_{\text{L-R}}, \tag{17b}$$

$$(l_1, m_1, u_1)_{\text{L-R}} - (l_2, m_2, u_2)_{\text{L-R}} = (l_1 - u_2, m_1 - m_2, u_1 - l_2)_{\text{L-R}}, \tag{17c}$$

$$(l_1, m_1, n_1, u_1)_{\text{L-R}} - (l_2, m_2, n_2, u_2)_{\text{L-R}} = (l_1 - u_2, m_1 - n_2, n_1 - m_2, u_1 - l_2)_{\text{L-R}}. \tag{17d}$$

Fuzzy multiplication changes the shape of the fuzzy number because of the nonlinear effect of multiplication. Suppose we have $A_i = (l_i, m_i, u_i)_{\text{L-R}}$, $l_i \geq 0$, $i = 1, \dots, N$ and $C = \prod_{i=1}^{N} A_i$. Let $A_i = [l_i(\alpha), u_i(\alpha)]$ represent the α-level set of A_i, $i = 1, \dots, N$, $\alpha \in (0,1]$. Then, according to the extension principle, C has the α-level set,

$$C^\alpha = \left[\prod_{i=1}^{N} l_i(\alpha), \prod_{i=1}^{N} u_i(\alpha)\right], \qquad \alpha \in (0,1]. \tag{18}$$

Thus, in general, the shape of the resulting membership function of C differs from that of A_i. The same results hold for the flat fuzzy numbers $A_i = (l_i, m_i, n_i, u_i)_{\text{L-R}}$, $i = 1, \dots, N$. The exact formula for multiplication is

$$\mu_C(z) = \alpha\big|_{z=\prod_{i=1}^{N} x_i(\alpha)}, \qquad \text{where } x_i(\alpha) \in \{l_i(\alpha), u_i(\alpha)\}, \quad i = 1, \dots, N, \quad \alpha \in (0,1], \quad \text{and}$$

$$C = \left(\prod_{i=1}^{N} l_i, \prod_{i=1}^{N} m_i, \prod_{i=1}^{N} u_i\right)_{\text{L}'\text{-R}'} \left(\text{or } C = \left(\prod_{i=1}^{N} l_i, \prod_{i=1}^{N} m_i, \prod_{i=1}^{N} n_i, \prod_{i=1}^{N} u_i\right)_{\text{L}'\text{-R}'}\right). \tag{19}$$

As an example, consider the trapezoidal fuzzy number $A_i = (l_i, m_i, n_i, u_i)_{\text{L-R}}$; from (18), we have

$$C^\alpha = \left[\prod_{i=1}^{N} (m_i - l_i)\alpha + l_i), \prod_{i=1}^{N} (u_i - (u_i - n_i)\alpha)\right]. \tag{20}$$

For triangular fuzzy numbers, simply let $n_i = m_i$, $i = 1, \dots, N$, in the above formula.

Analogous to the approximations for the unary operations, an approximate formula for fuzzy multiplication can be defined as

$$C \cong \left(\prod_{i=1}^{N} l_i, \prod_{i=1}^{N} m_i, \prod_{i=1}^{N} u_i\right)_{\text{L-R}} \left(\text{or } C \cong \left(\prod_{i=1}^{N} l_i, \prod_{i=1}^{N} m_i \prod_{i=1}^{N} n_i, \prod_{i=1}^{N} u_i\right)_{\text{L-R}}\right), \tag{21}$$

in which the reference functions L and R of A_i are directly applied to the lower and upper bounds and the mode of the resulting fuzzy number.

Dubois and Prade [19] also proposed an approximate formula for fuzzy multiplication. For fuzzy numbers $A_i = (m_i, \beta_i, \gamma_i)_{\text{L-R}}$, $i = 1, 2$, rewrite (21) in the following equivalent form

$$C \cong (m_1 m_2, m_1\beta_2 + m_2\beta_1 - \beta_1\beta_2, m_1\gamma_2 + m_2\gamma_1 + \gamma_1\gamma_2)_{\text{L-R}}. \tag{22}$$

When $\beta_1, \gamma_1, \beta_2$, and γ_2 are small compared to m_1 and m_2, $C = A_1 \times A_2$ may be approximated by

$$C(m_1 m_2, m_1\beta_2 + m_2\beta_1, m_1\gamma_2 + m_2\gamma_1)_{\text{L-R}}. \tag{23}$$

2.2. Positive Fuzzy Reciprocal Matrices

The positive reciprocal matrices are generally formed by pairwise comparisons among the various factors and is usually carried out by an expert based on certain criterion. A problem with N factors, $F_1, \dots, F_N$, forms an N by N comparison matrix. In nonfuzzy pairwise comparisons, two crisp factors are compared and a ratio (r_{ij}) is determined from $[1/9,\ 1] \cup [1,\ 9]$ (see [1]). Fuzzy pairwise comparisons can be conducted and defined similarly. The fuzzy ratio, R_{ij}, can be defined as $R_{ij} = (l, m, u)_{\text{L-R}}$, where $l \leq m \leq u$ and $l, m, u \in [1/9,\ 1] \cup [1,\ 9]$. As an example, $R_{ij} = (1/2,\ 1,\ 1.5)_{\text{L-R}}$. If flat fuzzy numbers are used, then $R_{ij} = (l, m, n, u)_{\text{L-R}}$, where $l \leq m \leq n \leq u$ and $l, m, n, u \in [1/9,\ 1] \cup [1,\ 9]$. An example can be $(1/2,\ 1,\ 1.5,\ 2)_{\text{L-R}}$.

The comparison matrix will be represented by $\mathbf{M} = (R_{ij})_{N \times N}$. After the fuzzy pairwise comparisons for all these factors have been carried out, the elements of the upper triangle of the fuzzy matrix $\mathbf{M}$ are obtained. The next step is to obtain the other half of the matrix $\mathbf{M}$ by using the fuzzy inverse, $R_{ji} = 1/R_{ij}$, $i, j = 1, \ldots, N$. As has been discussed earlier, three fuzzy inverse formulas may be used: the exact formula (6) and the approximate formulas (10) and (16). Thus, three variations of the fuzzy matrix $\mathbf{M}$ are obtained. Let

$$\mathbf{M} = \begin{matrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{matrix}$$

where $R_{ji} = 1/R_{ij}$, for $j = 2, \ldots, N$ and $i = 1, \ldots, j-1$. The three variations are:

(a) Using the exact formula (6), we have

$$\mathbf{M}_{(6)} = \begin{matrix} (1,1,1) & (l_{12}, m_{12}, u_{12})_{\text{L-R}} & \cdots & (l_{1N}, m_{1N}, u_{1N})_{\text{L-R}} \\ \left(\frac{1}{u_{12}}, \frac{1}{m_{12}}, \frac{1}{l_{12}}\right)_{\text{R}'\text{-L}'} & (1,1,1) & \cdots & (l_{2N}, m_{2N}, u_{2N})_{\text{L-R}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{1}{u_{1N}}, \frac{1}{m_{1N}}, \frac{1}{l_{1N}}\right)_{\text{R}'\text{-L}'} & \left(\frac{1}{u_{2N}}, \frac{1}{m_{2N}}, \frac{1}{l_{2N}}\right)_{\text{R}'\text{-L}'} & \cdots & (1,1,1) \end{matrix}.$$

(b) Using the approximate formula (10), we have

$$\mathbf{M}_{(10)} = \begin{matrix} (1,1,1) & (l_{12}, m_{12}, u_{12})_{\text{L-R}} & \cdots & (l_{1N}, m_{1N}, u_{1N})_{\text{L-R}} \\ \left(\frac{1}{u_{12}}, \frac{1}{m_{12}}, \frac{1}{l_{12}}\right)_{\text{R-L}} & (1,1,1) & \cdots & (l_{2N}, m_{2N}, u_{2N})_{\text{L-R}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{1}{u_{1N}}, \frac{1}{m_{1N}}, \frac{1}{l_{1N}}\right)_{\text{R-L}} & \left(\frac{1}{u_{2N}}, \frac{1}{m_{2N}}, \frac{1}{l_{2N}}\right)_{\text{R-L}} & \cdots & (1,1,1) \end{matrix}.$$

(c) Using the approximate formula (16), we have

$$\mathbf{M}_{(16)} = \begin{matrix} (1,0,0) & (m_{12}, \beta_{12}, \gamma_{12})_{\text{L-R}} & \cdots & (m_{1N}, \beta_{1N}, \gamma_{1N})_{\text{L-R}} \\ \left(\frac{1}{m_{12}}, \frac{\gamma_{12}}{m_{12}^2}, \frac{\beta_{12}}{m_{12}^2}\right)_{\text{R-L}} & (1,0,0) & \cdots & (m_{2N}, \beta_{2N}, \gamma_{2N})_{\text{L-R}} \\ \vdots & \vdots & \ddots & \vdots \\ \left(\frac{1}{m_{1N}}, \frac{\gamma_{1N}}{m_{1N}^2}, \frac{\beta_{1N}}{m_{1N}^2}\right)_{\text{R-L}} & \left(\frac{1}{m_{2N}}, \frac{\gamma_{2N}}{m_{2N}^2}, \frac{\beta_{2N}}{m_{2N}^2}\right)_{\text{R-L}} & \cdots & (1,0,0) \end{matrix}.$$

The matrices for fuzzy pairwise comparisons for flat (or trapezoidal) fuzzy numbers can be formed similarly.

Before we can proceed further, we need to introduce the following definitions about fuzzy reciprocal matrices.

DEFINITION 1. *A fuzzy matrix* $\mathbf{M} = (R_{ij})_{N \times N}$ *is a positive fuzzy reciprocal matrix if*

$$R_{ij} \times R_{ji} \approx 1 \quad \text{and} \quad m_{ij} \times m_{ji} = 1, \qquad \forall i, j = 1, \ldots, N,$$

where $\approx$ *means approximately equal. According to Definition 1,* $\mathbf{M}_{(6)}$, $\mathbf{M}_{(10)}$ *and* $\mathbf{M}_{(16)}$ *are positive fuzzy reciprocal matrices. For example, in* $\mathbf{M}_{(6)}$, *we have*

$$(l_{ij}, m_{ij}, u_{ij})_{\text{L-R}} \times \left(\frac{1}{u_{ij}}, \frac{1}{m_{ij}}, \frac{1}{l_{ij}}\right)_{\text{R}'\text{-L}'} = \left(\frac{l_{ij}}{u_{ij}}, 1, \frac{u_{ij}}{l_{ij}}\right)_{\text{L}''\text{-R}''},$$

for $i, j = 1, \ldots, N$.

DEFINITION 2. *A positive fuzzy reciprocal matrix* $\mathbf{M} = (R_{ij})$ *is fuzzily consistent if*

$$R_{ij} \times R_{jk} \approx R_{ik} \quad \text{and} \quad m_{ij} \times m_{jk} = m_{ik}, \qquad \forall i, j, k = 1, \dots, N.$$

In other words, Definition 2 requires an exact consistency in the modes and approximate consistency in the spread, $R_{ij} \times R_{jk} \approx R_{ik}$. However, this approximate consistency may be interpreted differently under different circumstances.

Definitions 1 and 2 become even more fuzzier for flat or trapezoidal fuzzy numbers. It is because of the fact that, with flat fuzzy numbers, it can only be required that there are some $c_{ij} \in [m_{ij},\ \ n_{ij}]$ and some $c_{ji} \in [m_{ji},\ \ n_{ji}]$, satisfy $c_{ij} \times c_{ji} = 1$ for Definition 1. We have a similar problem for Definition 2, that is, m_{ji} and n_{ji} in $c_{ij} \in [m_{ij},\ \ n_{ij}]$, $c_{jk} \in [m_{jk},\ \ n_{jk}]$, and $c_{ik} \in [m_{ik},\ \ n_{ik}]$ can only be approximately fixed and thus, we cannot require the entire modal intervals of a flat fuzzy number satisfy $m_{ij} \times m_{jk} = m_{ik}$.

Another problem is that the lower bound of an element of R_{ji} in the matrix $\mathbf{M}_{(16)}$ (that is the fuzzy inverse of R_{ij}) may be negative. For example, if $R_{ij} = (m_{ij}, \beta_{ij}, \gamma_{ij})_{\text{L-R}} = (2, 1, 2.5)_{\text{L-R}}$ in the spread form, then the fuzzy inverse of R_{ij}, R_{ji} is $(1/2, 2.5/2^2, 1/2^2)_{\text{R-L}}$. Using (16), a negative lower bound, $1/2 - 2.5/2^2 < 0$, is resulted. This confirms the fact discussed before, that is, the approximate formula (16), or the matrix $\mathbf{M}_{(16)}$, is appropriate to use only when the fuzzy parameters are not too fuzzy.

3. NORMALIZATION

Normalization of a set of fuzzy weights is important not only for the reason of unbiasedness and easy interpretation, but also is necessary for reaching a unique solution for some methods such as the logarithmic least square method. In the crisp case, a set of positive crisp numbers is said to be normalized if their sum is equal to one. In the fuzzy situation, different meanings of "fuzzy normalization" can be formulated. Thus, we need a definition for fuzzy normalization. By using the α-level set concept, the following definition can be formulated.

DEFINITION 3. *Let* $[l_i^*(\alpha), u_i^*(\alpha)]$ *represent the* α*-level set of the positive fuzzy number* W_i *in the set* $\{W_i;\ i = 1, \dots, N\}$*. The set* $\{W_i;\ i = 1, \dots, N\}$ *is said to be fuzzily normalized if*

$$\left[\sum_{i=1}^{N} l_i^*(\alpha)\right] \left[\sum_{i=1}^{N} u_i^*(\alpha)\right] = 1, \qquad \forall \alpha \in (0, 1].$$

DEFINITION 4. *If Definition 3 is satisfied only at* $\alpha = 1$ *and* $\alpha = 0$*, then it is called the relaxed fuzzy normalization. The equations for the relaxed normalization for the fuzzy number* $(l_i, m_i, u_i)_{\text{L-R}}$ *are:*

$$\sum_{i=1}^{N} m_i^* = \sum_{i=1}^{N} \frac{m_i}{\sum_{i=1}^{N} m_i} = 1, \qquad \text{for } \alpha = 1,$$

$$\sum_{i=1}^{N} l_i^* \sum_{i=1}^{N} u_i^* = \sum_{i=1}^{N} \frac{l_i}{\sum_{i=1}^{N} u_i} \sum_{i=1}^{N} \frac{u_i}{\sum_{i=1}^{N} l_i} = 1, \qquad \text{for } \alpha = 0.$$

and for the flat fuzzy number, the first equation in the above two equations is replaced by:

$$\sum_{i=1}^{N} m_i^* \sum_{i=1}^{N} n_i^* = \sum_{i=1}^{N} \frac{m_i}{\sum_{i=1}^{N} n_i} \sum_{i=1}^{N} \frac{n_i}{\sum_{i=1}^{N} m_i} = 1, \qquad \text{for } \alpha = 1.$$

The second equation for $\alpha = 0$ *remains unchanged.*

In the following, two different definitions, namely, fuzzy normalization with fuzzy division (FNFD) and geometric fuzzy normalization (GFN), are defined and discussed.

3.1. Fuzzy Normalization with Fuzzy Division (FNFD)

Fuzzy normalization can be considered as a fuzzy extension of the crisp normalization by the use of fuzzy addition and fuzzy division. But, in crisp normalization, only the modes appear in the equations and need to be considered. In fuzzy normalization, we must consider both the mode and the spreads and thus is much more complicated. In the following, the normalization of both the fuzzy number $(l,m,u)_{\text{L-R}}$ and the flat fuzzy number $(l,m,n,u)_{\text{L-R}}$ in the bounded form are discussed.

3.1.A. Fuzzy number $(l_i,m_i,u_i)_{\text{L-R}}$

To normalize the fuzzy number $X_i = (l_i,m_i,u_i)_{\text{L-R}}$, $l_i \geq 0$, $i = 1,\dots,N$, we first apply fuzzy addition (17a).

$$T = \sum_{i=1}^{N} X_i = \left(\sum_{i=1}^{N} l_i, \sum_{i=1}^{N} m_i, \sum_{i=1}^{N} u_i\right)_{\text{L-R}}. \tag{24}$$

Then, each X_i is divided by T—or multiplied by the inverse of T; that is, by applying the exact formula (6) for fuzzy inverse and (19) for fuzzy multiplication. The final results are

$$\frac{X_i}{T} = \left(\frac{l_i}{\sum_{i=1}^{N} u_i}, \frac{m_i}{\sum_{i=1}^{N} m_i}, \frac{u_i}{\sum_{i=1}^{N} l_i}\right)_{\text{L}'\text{-R}'}, \qquad i = 1,\dots,N. \tag{25}$$

It is easy to show that the results in equation (25) satisfy Definition 3. Let $[X_i/T]_\alpha$ denote the α-level set of X_i/T, $i = 1,\dots,N$, $0 \leq \alpha \leq 1$; then by the extension principle, $[X_i/T]_\alpha$ can be represented as

$$\left[\frac{X_i}{T}\right]_\alpha = \left[\frac{l_i(\alpha)}{\sum_{i=1}^{N} u_i(\alpha)}, \frac{u_i(\alpha)}{\sum_{i=1}^{N} l_i(\alpha)}\right]. \tag{26}$$

Summing these lower and upper endpoints in the right-hand-side of equation (26) over all $i = 1,\dots,N$, we obtain the desired result.

If we use the approximate formula (10) for fuzzy inverse and the approximate (21) for fuzzy multiplication, we have

$$\frac{X_i}{T} \cong \left(\frac{l_i}{\sum_{i=1}^{N} u_i}, \frac{m_i}{\sum_{i=1}^{N} m_i}, \frac{u_i}{\sum_{i=1}^{N} l_i}\right)_{\text{L-R}}, \qquad i = 1,\dots,N. \tag{27}$$

Equation (27) satisfies only the relaxed fuzzy normalization, Definition 4. For $0 < \alpha < 1$, Definition 3 does not hold.

Moreover, if we use the approximate formula (16) for fuzzy inverse and (23) for fuzzy multiplication, the results do not satisfy even the relaxed normalization definition.

3.1.B. Flat fuzzy number $(l_i,m_i,n_i,u_i)_{\text{L-R}}$

By using the same procedure as that used to obtain (25); we obtain, for flat fuzzy number $Xi = (l_i,m_i,n_i,u_i)_{\text{L-R}}$, $i = 1,\dots,N$, the following result:

$$\frac{X_i}{T} = \left(\frac{l_i}{\sum_{i=1}^{N} u_i}, \frac{m_i}{\sum_{i=1}^{N} n_i}, \frac{n_i}{\sum_{i=1}^{N} m_i}, \frac{u_i}{\sum_{i=1}^{N} l_i}\right)_{\text{L}'\text{-R}'}, \qquad i = 1,\dots,N. \tag{28}$$

It is easy to prove that Equation (28) satisfies the fuzzy normalization, Definition 3.

If we use the approximate formula (10) for fuzzy inverse and (21) for fuzzy multiplication, we have

$$\frac{X_i}{T} \cong \left(\frac{l_i}{\sum_{i=1}^{N} u_i}, \frac{m_i}{\sum_{i=1}^{N} n_i}, \frac{n_i}{\sum_{i=1}^{N} m_i}, \frac{u_i}{\sum_{i=1}^{N} l_i}\right)_{\text{L-R}}, \qquad i = 1,\dots,N. \tag{29}$$

Equation (29) satisfies only relaxed fuzzy normalization, Definition 4.

3.2. Geometric Fuzzy Normalization (GFN)

The second fuzzy normalization to be introduced is a variation of the FNFD through the use of "geometric fuzzy division" instead of the regular fuzzy division.

The "geometric fuzzy division" uses the concept of geometric mean. Let the bounded form of fuzzy numbers be represented by $A_1 = (l_1, m_1, u_1)_{\text{L-R}}$ and $A_2 = (l_2, m_2, u_2)_{\text{L-R}}$ (or $A_1 = (l_1, m_1, n_1, u_1)_{\text{L-R}}$, $A_2 = (l_2, m_2, n_2, u_2)_{\text{L-R}}$), then A_1/A_2 may be approximated by the use of the concept of geometric mean as:

$$\frac{A_1}{A_2} \cong \left(\frac{l_1}{(l_2u_2)^{1/2}}, \frac{m_1}{m_2}, \frac{u_1}{(l_2u_2)^{1/2}}\right)_{\text{L-R}} \tag{30}$$

$$\left(\text{or } \frac{A_1}{A_2} \cong \left(\frac{l_1}{(l_2u_2)^{1/2}}, \frac{m_1}{(m_2n_2)^{1/2}}, \frac{n_1}{(m_2n_2)^{1/2}}, \frac{u_1}{(l_2u_2)^{1/2}}\right)_{\text{L-R}}\right), \tag{31}$$

where the reference (shape) functions L and R for A_1 and A_2 are directly applied to the lower and upper bounds and to the mode of the resulting A_1/A_2.

Thus, for the fuzzy number $\{X_i;\ i = 1, \dots, N\}$, $X_i = (l_i, m_i, u_i)_{\text{L-R}}$, the GFN can be defined as:

$$\frac{X_i}{T} \cong \left(\frac{l_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}}, \frac{m_i}{\sum_{i=1}^N m_i}, \frac{u_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}}\right)_{\text{L-R}}, \qquad i = 1, \dots, N. \tag{32}$$

Notice that equation (32) does not alter the normalization of the modes of X_i/T. The geometric mean of the sums are only used in the normalization of the lower and upper bounds. Equation (32) satisfies the relaxed concept of fuzzy normalization, Definition 4.

$$\sum_{i=1}^N \frac{l_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}} \sum_{i=1}^N \frac{u_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}} = 1. \tag{33}$$

For flat fuzzy numbers, $X_i = (l_i, m_i, n_i, u_i)_{\text{L-R}}$, $i = 1, \dots, N$, the geometric fuzzy normalization is defined in a similar manner as:

$$\frac{X_i}{T} \cong \left(\frac{l_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}}, \frac{m_i}{\left(\sum_{i=1}^N m_i \sum_{i=1}^N n_i\right)^{1/2}}, \frac{n_i}{\left(\sum_{i=1}^N m_i \sum_{i=1}^N n_i\right)^{1/2}}, \frac{u_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}}\right)_{\text{L-R}}, \tag{34}$$

for $i = 1, \dots, N$. It can be shown that equation (34) satisfies the relaxed concept of fuzzy normalization.

3.3. Comparisons between FNFD and GFN

First, we shall show that GFN always provides a result of normalized set of positive fuzzy numbers that are less fuzzier than those provided by FNFD.

From equations (27) and (32) for $(l_i, m_i, u_i)_{\text{L-R}}$, we obtain the following inequalities:

$$\sum_{i=1}^N l_i \le \left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2} \le \sum_{i=1}^N u_i \tag{35}$$

and

$$\frac{l_i}{\sum_{i=1}^N u_i} \le \frac{l_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}} \le \frac{u_i}{\left(\sum_{i=1}^N l_i \sum_{i=1}^N u_i\right)^{1/2}} \le \frac{u_i}{\sum_{i=1}^N l_i}. \tag{36}$$

Thus, FNFD always gives a result that is fuzzier than that obtained by GFN.

For flat fuzzy numbers, in addition to inequality (35), we also obtain the following inequalities from equations (29) and (34):

$$\sum_{i=1}^N m_i \le \left(\sum_{i=1}^N m_i \sum_{i=1}^N n_i\right)^{1/2} \le \sum_{i=1}^N n_i, \tag{37}$$

and

$$\frac{m_i}{\sum_{i=1}^N n_i} \le \frac{m_i}{\left(\sum_{i=1}^N m_i \sum_{i=1}^N n_i\right)^{1/2}} \le \frac{n_i}{\left(\sum_{i=1}^N m_i \sum_{i=1}^N n_i\right)^{1/2}} \le \frac{n_i}{\sum_{i=1}^N m_i}. \tag{38}$$

Thus, for flat fuzzy numbers, GFN not only gives a result which is less fuzzier, but also has a narrower modal interval than that obtained by FNFD.

Next, we wish to discuss the limitations of the GFN approach. The main problem with this approach is that the results may contain expressions that are not fuzzy numbers, or, violating the assumptions of fuzzy numbers. For example, the resulting fuzzy number may not obey the requirements: *lower-bound* $\le$ *mode* $\le$ *upper-bound.*

This limitation becomes even more critical for flat fuzzy numbers, because for flat fuzzy numbers, the modal intervals are also influenced by the GFN approach.

4. NORMALIZED FUZZY WEIGHTS

The approaches proposed in the literature for estimating a normalized set of fuzzy weights from a positive fuzzy reciprocal matrix are summarized in the following. Emphasis is placed on the advantages and problems of these approaches. As a result of these summaries, two new approaches are proposed. These various approaches will be examined in more detail in the next section by the use of numerical examples.

4.1. The Fuzzy Logarithmic Least Square Methods (FLLSM)

The fuzzy logarithmic least square methods (FLLSM) were proposed by van Laarhoven and Pedrycz [16] and Boender *et al.* [14].

Laarhoven and Pedrycz used triangular fuzzy numbers, the approximate formulas (10)–(12) and (21), and the exact formula (17). The fuzzy pairwise comparison matrix was constructed as $\mathbf{M}_{(10)}$. Assuming the matrix was $\mathbf{M} = (R_{ij})_{N\times N}$ with $R_{ij} = (l_{ij}, m_{ij}, u_{ij})_{\text{L-R}}$ of the bounded form and also assuming δ_{ij} expert(s) were used to give their opinions for the pairwise comparison, $R_{ij}(i,j = 1,\dots,N)$. Denote each of these R_{ij} as R_{ijk} $(k = 1,\dots,\delta_{ij})$ and $R_{ijk} = (l_{ijk}, m_{ijk}, u_{ijk})_{\text{L-R}}$. Generally, δ_{ij} can be $1, 2, \dots,$ etc. The problem is to estimate the normalized set of positive fuzzy weights $\{W_i;\, i = 1,\dots,N\}$, $W_i = (w_{il}, w_{im}, w_{iu})_{\text{L-R}}$ of the bounded form.

These investigators used the following logarithmic least square minimization:

$$\sum_{i,j}\sum_{k=1}^{\delta_{ij}} \left(\ln(R_{ijk}) - \ln\left(\frac{W_i}{W_j}\right)\right)^2 \to \min.$$

By using the fuzzy division (equations (10) and (22)) for W_i/W_j and equation (11) for the fuzzy logarithmic operation, the above problem was transformed into

$$\sum_{i<j}\sum_{k=1}^{\delta_{ij}} \{\ln(l_{ijk}) - \ln(w_{il}) + \ln(w_{ju})\}^2 + \{\ln(m_{ijk}) - \ln(w_{im}) + \ln(w_{jm})\}^2 + \{\ln(u_{ijk}) - \ln(w_{iu}) + \ln(w_{jl})\}^2 \to \min. \tag{39}$$

From (39), the following system of normal equations can be obtained:

$$\ln(w_{il})\sum_{j\neq i}^{N}\delta_{ij}-\sum_{j\neq i}^{N}\delta_{ij}\ln(w_{ju})=\sum_{j\neq i}^{N}\sum_{k=1}^{\delta_{ij}}\ln(l_{ijk}),\qquad i=1,\dots,N, \tag{40a}$$

$$\ln(w_{im})\sum_{j\neq i}^{N}\delta_{ij}-\sum_{j\neq i}^{N}\delta_{ij}\ln(w_{jm})=\sum_{j\neq i}^{N}\sum_{k=1}^{\delta_{ij}}\ln(m_{ijk}),\qquad i=1,\dots,N, \tag{40b}$$

$$\ln(w_{iu})\sum_{j\neq i}^{N}\delta_{ij}-\sum_{j\neq i}^{N}\delta_{ij}\ln(w_{jl})=\sum_{j\neq i}^{N}\sum_{k=1}^{\delta_{ij}}\ln(u_{ijk}),\qquad i=1,\dots,N. \tag{40c}$$

Notice that (40b) is for the modes $\ln(w_{im})$, and (40a) and (40c) for the lower and upper bounds, respectively. Since $\ln(l_{ijk})+\ln(l_{jik})=\ln(u_{ijk})+\ln(u_{jik})=0$, equations (40a) and (40c) sum to zero and are thus linearly dependent. The subsystem (40b) for the modes coincides with the conventional nonfuzzy LLSM. The system has an indefinite number of solutions.

To solve the system (40); we first set any one of the unknowns in $\ln(w_{im})$, $i=1,\dots,N$, equal to an arbitrary given constant (say, $c\geq 0$); and also set any one of the other unknowns in $\ln(w_{il})$ and $\ln(w_{iu})$, $i=1,\dots,N$, equal to an arbitrary given constant (say, $d\geq 0$); then the other unknowns are obtained by solving equations (40). Let X_i, $i=1,\dots,N$ and $X_i=(x_{il},x_{im},x_{iu})$ be an arbitrary solution thus obtained, then, the general solution for system (40) can be represented by

$$(x_{il}+a,\ x_{im}+b,\ x_{iu}+a),\qquad i=1,\dots,N, \tag{41}$$

where $a,b\geq 0$ are arbitrary constants. If $a=b=0$, then c and d must be nonzero.

After first taking the fuzzy exponential by using (12), and then normalizing the general solution (41), a unique solution can be obtained which is a normalized set of fuzzy weights:

$$W_i=\left(\frac{\exp(x_{il}+a)}{\sum_{i=1}^{N}\exp(x_{iu}+a)},\frac{\exp(x_{im}+b)}{\sum_{i=1}^{N}\exp(x_{im}+b)},\frac{\exp(x_{iu}+a)}{\sum_{i=1}^{N}\exp(x_{il}+a)}\right)_{\text{L-R}},\qquad i=1,\dots,N, \tag{42}$$

where fuzzy normalization by the use of fuzzy division (FNFD) has been carried out.

It was noted by van Laarhoven and Pedrycz, solution (41) may not always satisfy the condition: $x_{il}+a\leq x_{im}+b\leq x_{iu}+a$, $i=1,\dots,N$. But, in general after normalization, the final normalized results (42) form a set of correct fuzzy numbers.

Boender *et al.* [14] observed that the normalized set of fuzzy weights (42) does not actually minimize the least square problem (39). This is because the fuzzy weights were obtained by dividing the lower and upper bounds by different constants. Therefore, starting from the fuzzy exponential (12) of the general solution (41),

$$(\exp(x_{il})a',\exp(x_{im})b',\exp(x_{iu})a'),\qquad i=1,\dots,N, \tag{43}$$

where $a'=\exp(a)$ and $b'=\exp(b)$, Boender *et al.* proceeded to seek the values for a' and b' such that the optimal solution for (43) was also normalized. Their idea happens to coincide with the relaxed concept of fuzzy normalization; that is, if (43) represents a normalized set of fuzzy numbers, then we should have

$$\sum_{i=1}^{N}\exp(x_{il})a'\sum_{i=1}^{N}\exp(x_{iu})a'=1,\qquad \sum_{i=1}^{N}\exp(x_{im})b'=1,$$

which implies that

$$a'=\frac{1}{\left(\sum_{i=1}^{N}\exp(x_{il})\sum_{i=1}^{N}\exp(x_{iu})\right)^{1/2}},\qquad b'=\frac{1}{\sum_{i=1}^{N}\exp(x_{im})}.$$

Thus, we should obtain

$$W_i = \left(\frac{\exp(x_{il})}{\left(\sum_{i=1}^N \exp(x_{il}) \sum_{i=1}^N \exp(x_{iu})\right)^{1/2}}, \frac{\exp(x_{im})}{\sum_{i=1}^N \exp(x_{im})}, \frac{\exp(x_{iu})}{\left(\sum_{i=1}^N \exp(x_{il}) \sum_{i=1}^N \exp(x_{iu})\right)^{1/2}} \right)_{\text{L-R}}, \tag{44}$$

with $i = 1, 2, \dots, N$. As can be seen, equation (44) uses the geometric fuzzy normalization and satisfies the relaxed concept of fuzzy normalization.

As has been pointed before in discussing the Laarhoven-Pedrycz's FLLSM, the general solution for (41) may have some fuzzy numbers where $x_{il} > x_{im}$ or $x_{iu} > x_{im}$. This same phenomenon may also occur in the above modified FLLSM due to Boender *et al.* However, after experimenting with a number of examples, we reached the same conclusion as that reached by Laarhoven and Pedrycz for their FLLSM: in general, this phenomenon does not cause any problem in the applications of FLLSM due to Boender *et al.*

However, as was discussed in Section 3.3, the GFN does have some limitations. To illustrate, consider the following bounded form example.

$$\begin{aligned} &R_{12} = (3, 4.5, 5)_{\text{L-R}}, \quad R_{13} = (6, 8, 8.5)_{\text{L-R}}, \quad R_{14} = (5, 5.5, 6)_{\text{L-R}}, \\ &R_{23} = (3, 4, 4.5)_{\text{L-R}}, \quad R_{24} = (4, 4.5, 5)_{\text{L-R}}, \quad R_{34} = (4, 5, 5.5)_{\text{L-R}}, \end{aligned} \tag{45}$$

Using the above numerical values, solving (40) we obtain:

$$\begin{aligned} X_1 &= (2.862, 3.027, 3.051), \quad X_2 = (1.943, 2.051, 2.182), \\ X_3 &= (1.169, 1.241, 1.373), \quad X_4 = (0.5, 0.5, 0.531). \end{aligned}$$

Using (44) the following final results are obtained:

$$\begin{aligned} W_1 &= (0.541, 0.616, 0.654)_{\text{L-R}}, \quad W_2 = (0.216, 0.232, 0.274)_{\text{L-R}}, \\ W_3 &= (0.100, 0.103, 0.122)_{\text{L-R}}, \quad W_4 = (0.051, 0.049, 0.053)_{\text{L-R}}, \end{aligned}$$

in which, unfortunately, we have $w_{4l} = 0.051 > w_{4u} = 0.049$ for the fuzzy weight W_4. On the other hand, if formula (42) of Laarhoven-Pedrycz's FLLSM were used, a normalized set of correct fuzzy numbers would have been obtained.

The above example indicates that a more critical problem for formula (44) is due to the geometric fuzzy normalization. For a possible remedy for these limitations, we may have to return to the original least square problem (39) and solve it with additional conditions. For example, for the matrix (45), the following condition

$$\left(\sum_{i=1}^N \exp(x_{il}) \sum_{i=1}^N \exp(x_{iu}) \right)^{1/2} \geq \sum_{i=1}^N \exp(x_{im})$$

may have to be added.

Besides the above discussed FLLSM, Boender *et al.* [14] also proposed another FLLSM, in which geometric ratio scales were employed for quantifying the gradations of the decision maker's judgment. Because of our discussions restricted to the L-R type fuzzy numbers, this FLLSM shall not be discussed here.

4.2. The Fuzzy Geometric Row Means Methods (FGRM)

The fuzzy geometric row means method (FGRM) was proposed by Buckley [15] and Lootsma [17].

In Buckley's FGRM, exact formulas (6), (9), and (17)–(19) were used with trapezoidal fuzzy numbers. The fuzzy pairwise comparison matrix was constructed as $\mathbf{M}_{(6)}$ with trapezoidal fuzzy numbers. Assuming the matrix was represented by $\mathbf{M} = (R_{ij})_{N\times N}$ with the bounded form $R_{ij} = (l_{ij}, m_{ij}, n_{ij}, u_{ij})_{\text{L-R}}$, the normalized set of fuzzy weights $\{W_i;\, i = 1,\dots,N\}$, $W_i = (w_{il}, w_{im}, w_{in}, w_{iu})_{\text{L-R}}$ is to be estimated. If $m_{ij} = n_{ij}$, then the above trapezoidal number becomes the triangular fuzzy number.

The fuzzy GRM with fuzzy pairwise comparisons R_{ij} leads to:

$$W_i = \frac{\left(\prod_{j=1}^{N} R_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} R_{ij}\right)^{1/N}}, \qquad i = 1,\dots,N. \tag{46}$$

To determine the normalized set of fuzzy weights W_i for (46), the numerator on the right-hand-side of equation (46) was first obtained. Let $X_i = (\prod_{j=1}^{N} R_{ij})^{1/N}$ represents the geometric mean for row i. Applying the exact formulas (19) and (9) and the extension principle with α-level cut of X_i, denoted as $[X_i]_\alpha = [f_i(\alpha), g_i(\alpha)]$, the following expressions are obtained:

$$f_i(\alpha) = \left[\prod_{j=1}^{N}((m_{ij} - l_{ij})\alpha + l_{ij})\right]^{1/N}, \quad g_i(\alpha) = \left[\prod_{j=1}^{N}(u_{ij} - (u_{ij} - n_{ij})\alpha)\right]^{1/N}$$

for $0 \le \alpha \le 1$. The denominator on the right-hand-side of equation (46) can be expressed as $\sum_{i=1}^{N} X_i$, which can be determined by the use of the α-level cut, $[\sum_{i=1}^{N} X_i]_\alpha = [f(\alpha), g(\alpha)]$, where

$$f(\alpha) = \sum_{i=1}^{N} f_i(\alpha), \quad g(\alpha) = \sum_{i=1}^{N} g_i(\alpha).$$

The α-level cut for W_i can be determined as $[f_i(\alpha)/g(\alpha), g_i(\alpha)/f(\alpha)]$ and W_i can be expressed as:

$$W_i = \left(\frac{\left(\prod_{j=1}^{N} l_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} u_{ij}\right)^{1/N}}, \frac{\left(\prod_{j=1}^{N} m_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} n_{ij}\right)^{1/N}}, \frac{\left(\prod_{j=1}^{N} n_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} m_{ij}\right)^{1/N}}, \frac{\left(\prod_{j=1}^{N} u_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} l_{ij}\right)^{1/N}}\right)_{\text{L}'\text{-R}'}, \tag{47}$$

$i = 1,\dots,N$. Equation (47) reduced to an equation for triangular fuzzy numbers if $m_{ij} = n_{ij}$, $i, j = 1,\dots,N$. Fuzzy normalization with fuzzy division has been used to obtain equation (47) and this equation satisfies the concept of fuzzy normalization, Definition 3.

Another FGRM approach was proposed by Lootsma [17, p. 103]. Approximate formulas (10), (13), and (22), and the exact addition formula (17) with triangular fuzzy numbers were used. This FLLSM appears to be an approximate version of Buckley's FLLSM approach. The results were shown to be:

$$W_i \cong \left(\frac{\left(\prod_{j=1}^{N} l_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} u_{ij}\right)^{1/N}}, \frac{\left(\prod_{j=1}^{N} m_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} m_{ij}\right)^{1/N}}, \frac{\left(\prod_{j=1}^{N} u_{ij}\right)^{1/N}}{\sum_{i=1}^{N}\left(\prod_{j=1}^{N} l_{ij}\right)^{1/N}}\right)_{\text{L-R}}, \tag{48}$$

with $i = 1, \ldots, N$. Due to approximation, equation (48) satisfies only the relaxed concept of fuzzy normalization, Definition 4.

Another FGRM approach was proposed by Wagenknecht and Hartmann [18], where the bell shaped membership function was used. Extension principle was applied with the form of max-*product* instead of the usual max-min approach. This approach shall not be discussed here.

4.3. The Fuzzy Least Square Method (FLSM)

A fuzzy extension of the least square method (FLSM) was proposed by Wagenknecht and Hartmann [18].

The approximate formulas (16) for fuzzy inverse and (23) for fuzzy multiplication with the fuzzy number $(m, \beta, \gamma)_{\text{L-R}}$ in the spread form were used. The fuzzy pairwise comparison matrix was constructed as $\mathbf{M}_{(16)}$. Assuming the matrix was represented by $\mathbf{M} = (R_{ij})_{N\times N}$ with $R_{ij} = (m_{ij}, \beta_{ij}, \gamma_{ij})_{\text{L-R}}$ in the spread form, a normalized set of fuzzy weights $\{W_i;\ i = 1, \ldots, N\}$, $W_i = (w_{im}, w_{i\beta}, w_{i\gamma})_{\text{L-R}}$ is to be estimated.

The fuzzy LSM with fuzzy pairwise comparisons R_{ij} can be depicted as:

$$\sum_{j=1}^{N}\sum_{i\neq j}^{N}\left(R_{ij} - \frac{W_i}{W_j}\right)^2 \to \min .$$

Let $C_{ij} = W_i/W_j$, and applying the approximate formulas (16) and (23), $C_{ij} = W_i/W_j$ may be obtained as follows.

C_{ij} may also be written as $(c_{ijm}, c_{ij\beta}, c_{ij\gamma})_{\text{L-R}}$, where

$$c_{ijm} = \frac{w_{im}}{w_{jm}}, \quad c_{ij\beta} = \frac{(w_{j\gamma}w_{im} + w_{i\beta}w_{jm})}{w_{jm}^2}, \quad c_{ij\gamma} = \frac{(w_{j\beta}w_{im} + w_{i\gamma}w_{jm})}{w_{jm}^2}.$$

The fuzzy least square problem is then transformed into the following:

$$\begin{gathered}\sum_{j=1}^{N}\sum_{i\neq j}^{N}\left[(c_{ijm} - m_{ij})^2 + (c_{ij\beta} - \beta_{kj})^2 + (c_{ij\gamma} - \gamma_{ij})^2\right] \to \min, \\ \sum_{i=1}^{N} w_{im} = 1, \qquad w_{im} > 0.\end{gathered} \tag{49}$$

There are several problems with the use of equation (49). First, (49) normalizes only the modes of the fuzzy weights, and thus the results do not satisfy either the concept of fuzzy normalization, Definition 3, nor the relaxed normalization, Definition 4.

Second, problem (49) automatically assumed that all $w_{i\beta}, w_{i\gamma} \geq 0$. This prevents any possibility of *lower-bound* > *mode* or *mode* > *upper-bound* in the resulting fuzzy weights.

Third, as already indicated in Section 2.2, if the values for R_{ij} from some expert are too fuzzy, the fuzzy matrix $\mathbf{M}_{(16)}$ may contain some $R_{ji}(= 1/R_{ij})$ whose lower-bounds become negative $(m_{ji} - \beta_{ji} < 0)$. As will be shown later, in some cases this may in turn leads to negative lower bounds (i.e., $w_{im} - w_{i\beta} < 0$). These various factors will be shown more clearly in Section 5 with numerical examples.

One possible approach to prevent the fuzzy weights having negative lower bounds is to add the constraints,

$$w_{im} - w_{i\beta} \geq 0, \qquad i = 1, \ldots, N. \tag{50}$$

Another approach is to add the following normalization constraint on the lower and upper bounds of W_i:

$$\sum_{i=1}^{N}(w_{im} - w_{i\beta})\sum_{i=1}^{N}(w_{im} + w_{i\gamma}) = 1. \tag{51}$$

With (51) and the original constraint $\sum_{i=1}^{N} w_{im} = 1$, a normalized set of fuzzy weights can be obtained and the results satisfy the relaxed concept of fuzzy normalization, Definition 4.

4.4. The Fuzzy Row Means of Normalized Columns Methods (FRMNC)

Saaty pointed out in [1, p. 239] that the approach of row means of normalized columns may provide an estimation of a normalized set of crisp (nonfuzzy) weights. In the following, this approach will be extended to fuzzy problems. Two variations of the fuzzy extension will be considered: one uses fuzzy normalization with fuzzy division (FNFD), and the other uses geometric fuzzy normalization (GFN). A procedure for checking the consistency of a fuzzy matrix is also proposed based on the computation of an approximate fuzzy eigenvalue $\lambda = (\lambda_l, \lambda_m, \lambda_u)_{\text{L-R}}$ in the bounded form.

4.4.A. FRMNC with fuzzy normalization with fuzzy division

In this approach, matrix $\mathbf{M}_{(10)}$ is used. Assuming the matrix is represented by $\mathbf{M} = (R_{ij})_{N\times N}$ with $R_{ij} = (l_{ij}, m_{ij}, u_{ij})_{\text{L-R}}$ in the bounded form, a procedure for estimating a normalized set of approximate fuzzy weights $\{W_i;\ i = 1, \ldots, N\}$, $W_i = (w_{il}, w_{im}, w_{iu})_{\text{L-R}}$ from $\mathbf{M}$ is introduced in the following steps.

STEP 1. Compute a column-normalized matrix $\mathbf{M}^* = (R^*_{ij})_{N\times N}$ from $\mathbf{M}$ by

$$R^*_{ij} = \left(l^*_{ij}, m^*_{ij}, u^*_{ij}\right)_{\text{L-R}} = \frac{R_{ij}}{\sum_{i=1}^N R_{ij}} \cong \left(\frac{l_{ij}}{\sum_{i=1}^N u_{ij}}, \frac{m_{ij}}{\sum_{i=1}^N m_{ij}}, \frac{u_{ij}}{\sum_{i=1}^N l_{ij}},\right)_{\text{L-R}}, \tag{52}$$

with $i, j = 1, \ldots, N$. Fuzzy addition (17a) and approximate fuzzy division were used in equation (52), where fuzzy division is carried out by first using fuzzy inverse (10) and then using fuzzy multiplication (22).

STEP 2. Compute the row means of $\mathbf{M}^*$ by

$$W_i = \frac{\sum_{j=1}^N R^*_{ij}}{N} = \left(\frac{\sum_{j=1}^N l^*_{ij}}{N}, \frac{\sum_{j=1}^N m^*_{ij}}{N}, \frac{\sum_{j=1}^N u^*_{ij}}{N}\right)_{\text{L-R}}, \tag{53}$$

with $i = 1, \ldots, N$. Scalar division (5) is used in equation (53). The normalized set of fuzzy weights is thus obtained.

STEP 3. Compute an approximate fuzzy eigenvalue by the following procedure: Consider the following fuzzy version of the eigenvector problem,

$$\mathbf{MW} = \lambda \mathbf{W}, \tag{54}$$

where $\mathbf{M}$ is the fuzzy pairwise comparison matrix and $W = (W_1, W_2, \ldots, W_N)^\top$ is obtained in Step 2. Our problem is to find the fuzzy eigenvalue λ.

First, let $\mathbf{P} = \mathbf{MW}$, where $\mathbf{P} = (P_1, P_2, \ldots, P_N)^\top$ can be determined by

$$P_i = \sum_{j=1}^N R_{ij} W_j \cong \left(\sum_{j=1}^N l_{ij} w_{jl}, \sum_{j=1}^N m_{ij} w_{jm}, \sum_{j=1}^N u_{ij} w_{ju}\right)_{\text{L-R}}, \tag{55}$$

with $i = 1, \ldots, N$. Approximate fuzzy multiplication (22) and fuzzy addition (17a) are used in equation (55). From equation (54), we also have the following relations

$$P_i \approx \lambda_i W_i, \qquad \text{for each } P_i,\ i = 1, \ldots, N,$$

where λ_i represents an approximate fuzzy eigenvalue from the i^{th} relation. Let $P_i = (p_{il}, p_{im}, p_{iu})_{\text{L-R}}$ for all i, it follows from the i^{th} relation that

$$(p_{il}, p_{im}, p_{iu})_{\text{L-R}} \approx (\lambda_{il}, \lambda_{im}, \lambda_{iu})_{\text{L-R}} \times (w_{il}, w_{im}, w_{iu})_{\text{L-R}} \cong (\lambda_{il} w_{il}, \lambda_{im} w_{im}, \lambda_{iu} w_{iu})_{\text{L-R}}.$$

Using this relation and equation (55), we determine λ_i as

$$\lambda_i = (\lambda_{il}, \lambda_{im}, \lambda_{iu})_{\text{L-R}} = \left(\frac{\sum_{j=1}^{N} l_{ij} w_{jl}}{w_{il}}, \frac{\sum_{j=1}^{N} m_{ij} w_{jm}}{w_{im}}, \frac{\sum_{j=1}^{N} u_{ij} w_{ju}}{w_{iu}} \right)_{\text{L-R}}, \tag{56}$$

with $i = 1, \dots, N$. It should be emphasized that equation (56) was not obtained by the use of fuzzy division. A final approximate fuzzy eigenvalue may be determined as

$$\lambda = (\lambda_l, \lambda_m, \lambda_u)_{\text{L-R}} \cong \left(\sum_{i=1}^{N} \frac{\lambda_{il}}{N}, \sum_{i=1}^{N} \frac{\lambda_{im}}{N}, \sum_{i=1}^{N} \frac{\lambda_{iu}}{N} \right)_{\text{L-R}}. \tag{57}$$

This approximate fuzzy eigenvalue obtained from the above procedure can be used to check the consistency of the fuzzy pairwise comparison matrix.

4.4.B. FRMNC with geometric fuzzy normalization

The other variation of the FRMNC is obtained with the use of the geometric fuzzy normalization instead of using fuzzy normalization with fuzzy division. This variation can be obtained easily by simply replacing equation (52) in Step 1 by the following equation:

$$R_{ij}^* \cong \left(\frac{l_{ij}}{\left(\sum_{i=1}^{N} l_{ij} \sum_{i=1}^{N} u_{ij} \right)^{1/2}}, \frac{m_{ij}}{\sum_{i=1}^{N} m_{ij}}, \frac{u_{ij}}{\left(\sum_{i=1}^{N} l_{ij} \sum_{i=1}^{N} u_{ij} \right)^{1/2}} \right)_{\text{L-R}}, \tag{58}$$

and leaving Steps 2 and 3 unchanged.

Because geometric fuzzy normalization is used, some of the R_{ij}^* (particularly the R_{ii}^*) in Step 1 may exhibit the phenomenon *lower-bound > mode* or *mode > upper-bound*. In general, however, the final results in Step 2 will be a normalized set of correct fuzzy numbers. This is the same phenomenon which also occurs in both the Laarhoven-Pedrycz and the Boender *et al.* approaches.

5. NUMERICAL EVALUATIONS AND COMPARISONS

In this section, the various versions of the approaches, which were summarized in the previous section, will be evaluated and compared by the use of numerical examples. A total of eight methods were considered.

Since Buckley and Lootsma give exactly the same lower and upper bounds, and also the same mode, these two approaches will be treated as one approach and will be abbreviated as "B-L." Three separate versions can formed based on the approach of Wagenknecht-Hartmann:

(a) Equation (49) and abbreviated as "WH1,"
(b) equation (49) with constraint (50) and abbreviated as "WH2," and
(c) equation (49) with constraint (51) and abbreviated as "WH3."

The two variations of FRMNC are:

(a) fuzzy normalization with fuzzy division (FNFD) and abbreviated as "CL-D," and
(b) geometric fuzzy normalization (GFN) abbreviated as "CL-G."

The Laarhoven-Pedrycz approach will be abbreviated as "LP," and the Boender *et al.* approach will be abbreviated as "B."

Twelve numerical examples were solved. Six are for the comparisons of four factors and the other six are for the comparisons of six factors. Among the twelve examples, six (three for the four factors and three for the six factors) were used as the base examples. Based on these base examples, the other six examples which are the fuzzier versions of the base examples were constructed. They were constructed with an increased upper bound and a decreased lower bound

Table 1. Fuzzy pairwise comparison matrices[1,2,3] in 4 factors.

M	$R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, R_{34}$	ϕ $\mathbf{M}_{(10)}$	ϕ $\mathbf{M}_{(16)}$
4-1	(3, 4, 5), (3, 4.5, 5.5), (6, 6.5, 7.5), (1/5, 1/4, 1/3), (5.5, 6.5, 7), (1, 2, 2.5).	0.387	0.350
4-2[4]	(2, 4, 5.5), (3, 4.5, 6.5), (5, 6.5, 8.5), (1/6, 1/4, 1/2.5), (4.5, 6.5, 8), (1, 2, 4).	0.674	0.645
4-3	(6, 7, 8), (2, 3.5, 4), (1, 1.5, 2), (1/7, 1/6, 1/5.5), (4, 5, 6), (5, 6.5, 7).	0.339	0.308
4-4	(5, 7, 8.5), (2, 3.5, 5), (1/2, 1.5, 3), (1/8, 1/6, 1/4.5), (3.5, 5, 7), (4.5, 6.5, 8.5).	0.689	0.615
4-5	(1/7, 1/6, 1/5), (2, 3, 4), (6, 6.5, 7.5), (4, 5, 6), (1/5, 1/4, 1/3), (1/4.5, 1/3.5, 1/2.5).	0.349	0.350
4-6	(1/8, 1/6, 1/4), (1.5, 3, 5), (5, 6.5, 8.5), (3, 5, 7), (1/6, 1/4, 1/2), (1/5.5, 1/3.5, 1/1.5).	0.749	0.786[5]

[1]The entries of the upper triangle are listed.
[2]$R_{ij} = (l_{ij}, m_{ij}, u_{ij})_{\text{L-R}}$ of the bound form.
[3]With $R_{ii} = (1, 1, 1)$, $i = 1, 2, 3, 4$; $R_{ji} = 1/R_{ij}$, $j = 2, 3, 4$ and $i = 1, \dots, j-1$.
[4]M-(4-2) is the fuzzier version of M-(4-1), M-(4-4) is the fuzzier version of M-(4-3), etc.
[5]When constructed as $\mathbf{M}_{(16)}$ contains negative lower bound(s) on some fuzzy entries R_{ji}.

Table 2. Fuzzy pairwise comparison matrices[1,2,3] in 6 factors.

M	$R_{12}, R_{13}, R_{14}, R_{15}, R_{16}, R_{23}, R_{24}, R_{25}, R_{26}, R_{34}, R_{35}, R_{36}, R_{45}, R_{46}, R_{56}$	ϕ $\mathbf{M}_{(10)}$	ϕ $\mathbf{M}_{(16)}$
6-1	(3, 4, 5), (1, 2, 3), (4, 5.5, 6), (7, 8, 9), (6, 6.5, 7), (1, 2, 3), (2, 3, 4), (1/2, 1, 2), (4, 5, 6), (5, 6, 7), (6, 7, 8), (2, 3, 3.5), (6, 7, 8), (3, 4.5, 5.5), (2, 2.5, 4).	0.506	0.478
6-2[4]	(2, 4, 5.5), (1, 2, 4), (3.5, 5.5, 7), (6, 8, 9), (5, 6.5, 8), (1/2, 2, 4), (1, 3, 5), (1/2, 1, 3), (3, 5, 6.5), (4, 6, 8), (5, 7, 8.5), (1.5, 3, 4.5), (5, 7, 9), (2.5, 4.5, 6), (1, 2.5, 4.5).	0.952	0.836[5]
6-3	(3, 3.5, 4), (3, 4.5, 5), (7, 8, 9), (5, 6.5, 7), (2, 3, 4), (1/4, 1/3, 1), (1/7, 1/6.5, 1/5.5), (3.5, 4.5, 5), (5, 5.5, 7), (1, 2, 3), (2.5, 3, 4), (1/2, 1, 1.5), (1/8, 1/7, 1/6), (1/2, 1/1.5, 1), (3, 4, 5).	0.500	0.511[5]
6-4	(2.5, 3.5, 5), (2, 4.5, 6), (6, 8, 9), (4, 6.5, 8), (1.5, 3, 5), (1/4.5, 1/3, 2), (1/7.5, 1/6.5, 1/4.5), (3, 4.5, 5.5), (4, 5.5, 7.5), (1/2, 2, 3.5), (1.5, 3, 5), (1/3, 1, 2), (1/8, 1/7, 1/5), (1/3, 1/1.5, 1.5), (2, 4, 5.5).	1.025	1.019[5]
6-5	(6, 7, 8), (1/8, 1/7, 1/6), (1/5, 1/4, 1/3), (2, 2.5, 3.5), (6, 7, 8), (6, 7, 8), (1, 2, 3), (7, 7.5, 8), (4, 4.5, 5.5), (7, 8, 8.5), (1/6, 1/5, 1/4), (4, 4.5, 5), (1/4, 1/3, 1/2.5), (5, 6, 7), (7, 8, 8.5).	0.316	0.308
6-6	(5, 7, 9), (1/9, 1/7, 1/5.5), (1/6, 1/4, 1/2.5), (1, 2.5, 4), (5.5, 7, 8.5), (5, 7, 8.5), (1, 2, 4), (6, 7.5, 8.5), (3, 4.5, 6.5), (6, 8, 9), (1/7, 1/5, 1/3), (3.5, 4.5, 6), (1/5, 1/3, 1/2), (4, 6, 7.5), (6.5, 8, 9).	0.606	0.579

[1]The entries of the upper triangle are listed.
[2]$R_{ij} = (l_{ij}, m_{ij}, u_{ij})_{\text{L-R}}$ of the bound form.
[3]With $R_{ii} = (1, 1, 1)$, $i = 1, 2, 3, 4, 5, 6$; $R_{ji} = 1/R_{ij}$, $j = 2, 3, 4, 5, 6$ and $i = 1, \dots, j-1$.
[4]M-(6-2) is the fuzzier version of M-(6-1), M-(6-4) is the fuzzier version of M-(6-3), etc.
[5]When constructed as $\mathbf{M}_{(16)}$ contains negative lower bound(s) on some fuzzy entries R_{ji}.

as compared to the base examples. The modes of these constructed examples remain unchanged. Tables 1 and 2 listed these 12 examples by tabulating only the elements of the upper triangle of the fuzzy matrix.

In order to give some idea about the fuzziness of the examples, the following parameter is also computed and listed in Tables 1 and 2:

$$\phi = \frac{1}{N^2} \sum_{i,j=1}^{N} \frac{u_{ij} - l_{ij}}{m_{ij}} \tag{59}$$

Table 3. The results of normalized set of fuzzy weights of the six matrices in 4 factors. (LP: Laarhoven-Pedrycz FLLSM; B: Boender *et al.* FLLSM; B-L: Buckley and Lootsma FGRM; WH1: Wagenknecht-Hartmann FLSM, problem (49); WH2: Problem (49) and constraints (50); WH3: Problem (49) and constraint (51); CL-D: FRMNC with FNFD; CL-G: FRMNC with GFN.)

M	Method	W_1, W_2, W_3, W_4	ψ	Fuzzy eigenvalue
4-1	LP	(.478,.590,.716)(.122,.143,.176)(.139,.207,.294)(.052,.059,.079)	0.497	(3.98,4.72,5.87)
	B	(.538,.590,.636)(.137,.143,.156)(.156,.207,.262)(.058,.059,.070)	0.253	(3.98,4.72,5.87)
	B-L	(.412,.590,.833)(.104,.143,.206)(.131,.207,.314)(.045,.059,.092)	0.775	(3.95,4.72,5.84)
	WH1	(.461,.512,.577)(.196,.257,.301)(.121,.165,.208)(.067,.067,.067)	0.290	(4.22,5.35,6.25)
	WH2	ditto		
	WH3	(.469,.513,.582)(.200,.256,.304)(.124,.164,.210)(.067,.067,.067)	0.288	(4.22,5.35,6.25)
	CL-D	(.383,.545,.778)(.132,.177,.234)(.138,.215,.331)(.048,.064,.107)	0.786	(4.00,4.78,5.87)
	CL-G	(.455,.545,.639)(.155,.177,.204)(.169,.215,.272)(.057,.064,.088)	0.397	(3.98,4.78,5.89)
4-2	LP	(.406,.590,.810)(.103,.143,.222)(.125,.207,.357)(.042,.059,.091)	0.866	(3.45,4.73,6.71)
	B	(.494,.590,.666)(.126,.143,.183)(.152,.207,.293)(.051,.059,.075)	0.444	(3.45,4.73,6.71)
	B-L	(.315,.590,1.05)(.082,.143,.283)(.106,.207,.422)(.033,.059,.115)	1.39	(3.43,4.73,6.68)
	WH1,2	(.437,.543,.570)(.084,.193,.282)(.086,.193,.269)(.056,.071,.071)	0.609	(3.36,4.89,6.61)
	WH3	(.495,.543,.628)(.105,.193,.302)(.107,.193,.290)(.064,.071,.079)	0.609	(3.28,4.89,6.57)
	CL-D	(.305,.545,.969)(.099,.177,.327)(.108,.215,.468)(.036,.064,.123)	1.39	(3.46,4.78,6.76)
	CL-G	(.404,.545,.702)(.134,.177,.245)(.152,.215,.329)(.047,.064,.091)	0.677	(3.45,4.78,6.76)
4-3	LP	(.318,.469,.617)(.097,.112,.130)(.291,.348,.462)(.058,.072,.100)	0.502	(4.48,5.47,6.66)
	B	(.364,.469,.540)(.111,.112,.114)(.333,.348,.404)(.067,.072,.088)	0.224	(4.48,5.47,6.66)
	B-L	(.301,.469,.654)(.084,.112,.151)(.262,.348,.514)(.054,.072,.109)	0.713	(4.47,5.47,6.61)
	WH1,2	(.354,.381,.434)(.068,.070,.075)(.433,.459,.510)(.090,.090,.105)	0.163	(4.84,6.08,7.17)
	WH3	(.342,.381,.423)(.066,.070,.073)(.419,.459,.496)(.087,.090,.102)	0.163	(4.84,6.08,7.17)
	CL-D	(.295,.451,.641)(.096,.133,.192)(.233,.308,.440)(.072,.109,.176)	0.781	(4.59,5.63,6.79)
	CL-G	(.354,.451,.527)(.115,.133,.160)(.278,.308,.369)(.086,.109,.147)	0.394	(4.59,5.63,6.79)
4-4	LP	(.235,.469,.807)(.090,.112,.150)(.256,.348,.504)(.044,.072,.141)	0.957	(3.96,5.47,8.93)
	B	(.297,.469,.637)(.113,.112,.118)[1](.324,.348,.399)(.055,.072,.112)	0.441	(3.96,5.47,8.93)
	B-L	(.210,.469,.917)(.067,.112,.204)(.199,.348,.659)(.038,.072,.163)	1.446	(3.90,5.47,8.75)
	WH1,2	(.365,.384,.488)(.071,.071,.092)(.425,.456,.573)(.077,.089,.114)	0.336	(3.71,6.04,8.10)
	WH3	(.331,.384,.454)(.065,.071,.086)(.384,.456,.532)(.069,.089,.106)	0.336	(3.70,6.04,8.10)
	CL-D	(.222,.451,.816)(.072,.133,.255)(.173,.308,.564)(.042,.109,.353)	1.707	(3.92,5.63,9.27)
	CL-G	(.311,.451,.584)(.101,.133,.179)(.239,.308,.407)(.061,.109,.238)	0.841	(3.94,5.63,9.12)
4-5	LP	(.245,.293,.352)(.303,.361,.435)(.063,.081,.112)(.212,.265,.316)	0.431	(5.65,6.70,8.01)
	B	(.270,.293,.319)(.334,.361,.395)(.069,.081,.102)(.234,.265,.286)	0.233	(5.65,6.70,8.01)
	B-L	(.213,.293,.405)(.264,.361,.500)(.058,.081,.122)(.186,.265,.360)	0.689	(5.65,6.70,8.00)
	WH1,2	(.124,.168,.218)(.471,.487,.487)(.070,.086,.100)(.203,.259,.270)	0.301	(6.63,7.63,8.41)
	WH3	(.130,.169,.224)(.487,.487,.504)(.073,.086,.103)(.212,.259,.279)	0.301	(6.61,7.63,8.42)
	CL-D	(.229,.303,.409)(.253,.355,.501)(.036,.049,.074)(.188,.293,.449)	0.739	(5.81,6.94,8.37)
	CL-G	(.263,.303,.352)(.304,.355,.416)(.044,.049,.061)(.228,.293,.366)	0.360	(5.80,6.94,8.34)
4-6	LP	(.210,.293,.414)(.247,.361,.542)(.051,.081,.160)(.155,.265,.392)	0.938	(4.56,6.70,9.52)
	B	(.258,.293,.338)(.304,.361,.442)(.063,.081,.130)(.190,.265,.319)	0.493	(4.56,6.70,9.52)
	B-L	(.156,.293,.562)(.188,.361,.716)(.042,.081,.193)(.122,.265,.499)	1.533	(4.55,6.70,9.49)
	WH1	(.146,.263,.406)(.180,.359,.503)(.074,.078,.078)(-.039,.300,.374)[2]	0.829	(2.35,6.80,9.15)
	WH2	(.168,.258,.419)(.223,.364,.530)(.078,.078,.087)(0,.300,.393)	0.810	—[3]
	WH3	(.198,.242,.423)(.336,.388,.567)(.080,.080,.107)(.054,.290,.397)	0.762	(2.07,6.87,8.97)
	CL-D	(.167,.303,.579)(.179,.355,.722)(.028,.049,.119)(.113,.293,.704)	1.691	(4.66,6.94,10.05)
	CL-G	(.226,.303,.415)(.259,.355,.495)(.042,.049,.082)(.168,.293,.459)	0.776	(4.63,6.94,9.94)

[1]Incorrect fuzzy number: *lower-bound*>*mode* or *upper-bound*<*mode*. [2]Negative lower-bound.
[3]Due to zero lower-bound, mode, or upper-bound, the fuzzy eigenvalue cannot be computed.

Table 4. The results of normalized set of fuzzy weights of the six matrices in 6 factors. (LP: Laarhoven-Pedrycz FLLSM; B: Boender *et al.* FLLSM; B-L: Buckley and Lootsma FGRM; WH1: Wagenknecht-Hartmann FLSM, problem (49); WH2: Problem (49) and constraints (50); WH3: Problem (49) and constraint (51); CL-D: FRMNC with FNFD; CL-G: FRMNC with GFN.)

M	Method	$W_1, W_2, W_3, W_4, W_5, W_6$	ϕ	Fuzzy eigenvalue
6-1	LP	(.333,.434,.526)(.106,.167,.255)(.155,.212,.313)(.082,.099,.126) (.040,.051,.072)(.031,.037,.049)	0.604	(5.76,7.31,9.72)
	B	(.385,.434,.455)(.123,.167,.220)(.180,.212,.270)(.094,.099,.109) (.046,.051,.062)(.036,.037,.042)	0.298	(5.76,7.31,9.72)
	B-L	(.269,.434,.655)(.092,.167,.296)(.131,.212,.374)(.066,.099,.158) (.033,.051,.087)(.025,.037,.060)	1.033	(5.74,7.31,9.67)
	WH1,2	(.389,.389,.402)(.122,.150,.195)(.249,.269,.297)(.072,.092,.125) (.042,.045,.050)(.052,.054,.060)	0.269	(5.65,7.59,9.56)
	WH3	(.380,.389,.394)(.119,.150,.192)(.243,.269,.291)(.070,.092,.123) (.041,.045,.049)(.051,.054,.059)	0.269	(5.66,7.59,9.56)
	CL-D	(.244,.390,.603)(.095,.177,.327)(.139,.212,.355)(.080,.118,.175) (.037,.064,.131)(.026,.039,.063)	1.080	(5.78,7.39,9.77)
	CL-G	(.309,.390,.462)(.125,.177,.245)(.174,.212,.284)(.097,.118,.143) (.047,.064,.101)(.034,.039,.048)	0.529	(5.77,7.39,9.75)
6-2	LP	(.291,.434,.585)(.075,.167,.342)(.123,.212,.390)(.070,.099,.158) (.032,.051,.080)(.027,.037,.065)	1.068	(4.74,7.31,11.66)
	B	(.370,.434,.460)(.095,.167,.269)(.156,.212,.306)(.089,.099,.124) (.041,.051,.063)(.034,.037,.051)	0.534	(4.74,7.31,11.66)
	B-L	(.196,.434,.879)(.058,.167,.448)(.089,.212,.542)(.048,.099,.233) (.023,.051,.116)(.019,.037,.095)	1.967	(4.69,7.31,11.50)
	WH1,2	(.371,.384,.437)(.115,.156,.248)(.206,.253,.324)(.056,.110,.182) (.044,.044,.052)(.050,.053,.064)	0.514	(4.16,7.53,11.01)
	WH3	(.353,.384,.419)(.107,.156,.240)(.195,.253,.312)(.051,.110,.177) (.042,.044,.050)(.047,.053,.062)	0.514	(4.21,7.53,11.03)
	CL-D	(.184,.390,.849)(.059,.177,.504)(.099,.212,.503)(.060,.118,.256) (.024,.064,.178)(.020,.039,.098)	2.031	(4.71,7.39,11.56)
	CL-G	(.272,.390,.534)(.092,.177,.312)(.147,.212,.337)(.086,.118,.175) (.036,.064,.112)(.031,.039,.062)	0.929	(4.69,7.39,11.53)
6-3	LP	(.390,.476,.541)(.096,.109,.151)(.095,.163,.246)(.063,.075,.099) (.095,.107,.128)(.052,.069,.098)	0.533	(7.44,9.20,11.65)
	B	(.439,.476,.481)(.108,.109,.135)(.107,.163,.219)(.071,.075,.088) (.107,.107,.114)(.058,.069,.088)	0.289	(7.44,9.20,11.65)
	B-L	(.311,.476,.683)(.078,.109,187)(.084,.163,.280)(.051,.075,.122) (.076,.107,.163)(.044,.069,.118)	0.969	(7.41,9.20,11.57)
	WH1,2	(.470,.470,.470)(.100,.112,.161)(.086,.166,.226)(.055,.061,.068) (.080,.093,.118)(.077,.098,.115)	0.399	(6.87,9.68,11.96)
	WH3	(.470,.470,.470)(.100,.112,.160)(.085,.166,.226)(.054,.061,.068) (.080,.093,.118)(.077,.098,.115)	0.399	(6.87,9.68,11.95)
	CL-D	(.260,.388,.541)(.108,.152,.241)(.075,.134,.226)(.081,.113,.177) (.094,.136,.199)(.046,.077,.145)	0.940	(7.68,9.55,11.85)
	CL-G	(.318,.388,.436)(.132,.152,.194)(.092,.134,.185)(.100,.113,.142) (.115,.136,.162)(.057,.077,.116)	0.482	(7.68,9.55,11.83)

for $\mathbf{M} = (R_{ij})_{N\times N}$ with $R_{ij} = (l_{ij}, m_{ij}, u_{ij})_{\text{L-R}}$ in the bounded form and $i, j = 1, \dots, N$. This parameter, ϕ, may be called the *mean relative fuzziness of the data.* It should be noted that the value of ϕ may be different for $\mathbf{M}_{(10)}$ and $\mathbf{M}_{(16)}$, but it is always same for $\mathbf{M}_{(6)}$ and $\mathbf{M}_{(10)}$.

Another problem concerns the negative lower in the lower triangle matrix. When the fuzzy matrix was constructed according to $\mathbf{M}_{(16)}$, negative lower bound(s) may result on some of the fuzzy elements at the lower triangle, $R_{ji} (= 1/R_{ij})$. This is noted in Tables 1 and 2.

Tables 3 and 4 listed the calculated fuzzy weights $\{W_i;\ i = 1, \dots, 4 \text{ (or 6)}\}$, $W_i = (w_{il}, w_{im}, w_{iu})$ in the bounded form for each of the fuzzy matrix examples by the use of the eight different

Table 4. (contd.)

M	Method	W_1, W_2, W_3, W_4, W_5, W_6	ψ	Fuzzy eigenvalue
6-4	LP	(.319,.476,.624)(.085,.109,.189)(.059,.163,.326)(.055,.075,.130) (.081,.107,.153)(.042,.069,.138)	1.048	(6.13,9.20,14.26)
	B	(.399,.476,.499)(.106,.109,.151)(.074,.163,.261)(.069,.075,.104) (.101,.107,.123)(.053,.069,.110)	0.547	(6.13,9.20,14.26)
	B-L	(.212,.476,.958)(.057,.109,.284)(.047,.163,.422)(.038,.075,.194) (.053,.107.236)(.031,.069,.194)	2.015	(6.06,9.20,13.99)
	WH1	(.363,.372,.386)(.048,.055,.083)(-.174,.245,.310)[2](.062,.101,.123) (.063,.114,.134)(.067,.114,.114)	0.716	(5.07,10.89,14.33)
	WH2	(.351,.351,.383)(.054,.062,.119)(0,.298,.350)(.063,.084,.100) (.060,.103,.127)(.066,.102,.107)	0.633	—[3]
	WH3	(.350,.350,.472)(.066,.066,.137)(.021,.291,.423)(.085,.085,.111) (.090,.104,.142)(.097,.105,.124)	0.645	(1.11,10.37,13.74)
	CL-D	(.186,.388,.764)(.078,.152,.368)(.043,.134,.336)(.060,.113,.280) (.068,.136,.273)(.032,.077,.250)	1.980	(6.27,9.55,14.19)
	CL-G	(.267,.388,.500)(.115,.152,.238)(.066,.134,.230)(.087,.113,.181) (.097,.136,.186)(.047,.077,.160)	0.933	(6.25,9.55,14.13)
6-5	LP	(.148,.178,.222)(.221,.281,.347)(.169,.194,.221)(.095,.124,.168) (.158,.190,.226)(.028,.032,.037)	0.390	(9.79,11.50,13.44)
	B	(.164,.178,.201)(.244,.281,.314)(.187,.194,.200)(.105,.124,.152) (.174,.190,.204)(.031,.032,.033)	0.188	(9.79,11.50,13.44)
	B-L	(.132,.178,.250)(.198,.281,.389)(.147,.194,.255)(.087,.124,.184) (.139,.190,.257)(.025,.032,.042)	0.643	(9.79,11.50,13.42)
	WH1,2	(.189,.221,.257)(.170,.196,.235)(.125,.150,.176)(.107,.141,.175) (.222,.258,.278)(.033,.033,.033)	0.281	(10.3,12.24,13.83)
	WH3	(.192,.221,.260)(.173,.196,.237)(.127,.150,.178)(.109,.141,.177) (.225,.258,.281)(.033,.033,.033)	0.279	(10.3,12.24,13.83)
	CL-D	(.163,.221,.299)(.198,.261,.343)(.170,.227,.299)(.077,.112,.166) (.118,.162,.223)(.014,.018,.023)	0.614	(10.1,11.98,14.00)
	CL-G	(.188,.221,.259)(.227,.261,.298)(.201,.227,.253)(.090,.112,.142) (.137,.162,.191)(.016,.018,.020)	0.303	(10.2,11.98,14.00)
6-6	LP	(.115,.178,.254)(.191,.281,.404)(.150,.194,.260)(.078,.124,.192) (.132,.190,.294)(.025,.032,.043)	0.735	(8.35,11.50,15.18)
	B	(.139,.178,.211)(.229,.281,.336)(.180,.194,.216)(.094,.124,.160) (.159,.190,.244)(.031,.032,.036)	0.351	(8.35,11.50,15.18)
	B-L	(.093,.178,.316)(.153,.281,.506)(.116,.194,.336)(.064,.124,.235) (.106,.190,.365)(.020,.032,.056)	1.251	(8.34,11.50,15.15)
	WH1,2	(.165,.214,.269)(.148,.209,.286)(.091,.136,.206)(.075,.141,.194) (.199,.266,.307)(.034,.034,.034)	0.540	(8.74,12.35,14.92)
	WH3	(.175,.215,.275)(.157,.209,.292)(.097,.135,.209)(.082,.140,.198) (.211,.268,.315)(.034,.034,.034)	0.528	(8.70,12.39,14.94)
	CL-D	(.119,.221,.392)(.154,.261,.458)(.132,.227,.376)(.057,.112,.213) (.090,.162,.311)(.012,.018,.031)	1.219	(8.58,11.98,15.85)
	CL-G	(.157,.221,.298)(.205,.261,.344)(.179,.227,.278)(.076,.112,.160) (.119,.162,.232)(.015,.018,.023)	0.582	(8.59,11.98,15.85)

[2]Negative lower-bound. [3]Due to zero lower-bound, mode, or upper-bound, the fuzzy eigenvalue cannot be computed.

approaches. In order to obtain some idea of the relative fuzziness, the following parameter

$$\psi = \frac{1}{N} \sum_{i=1}^{N} \frac{w_{iu} - w_{il}}{w_{im}} \tag{60}$$

was computed and listed in the tables. ψ may be called the mean relative fuzziness of the fuzzy weights. In addition, the approximate fuzzy eigenvalues were calculated according to the procedure introduced in Section 4.4.A and are also listed in the tables.

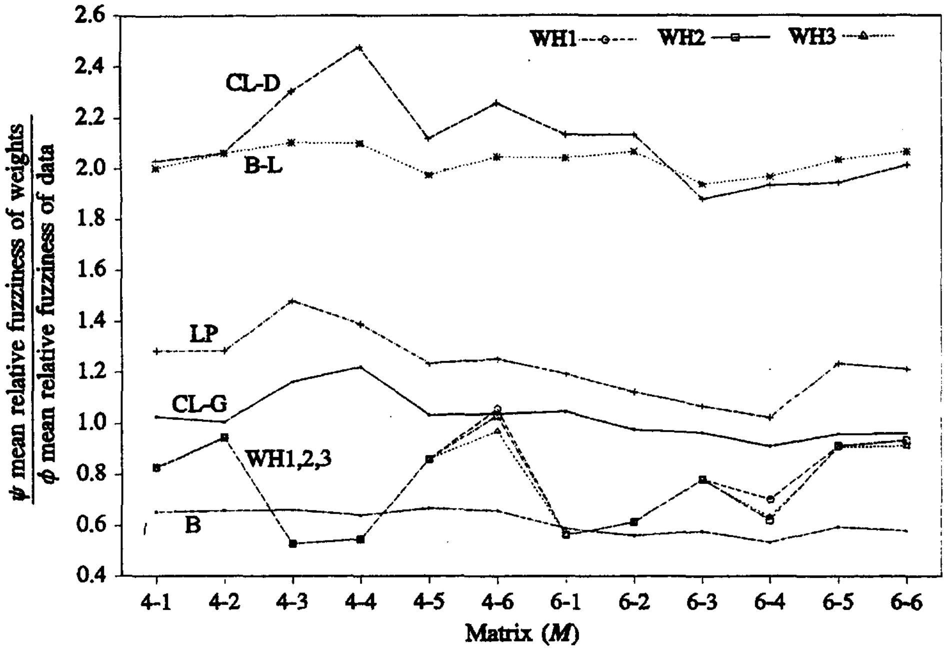

Figure 1. Mean relative fuzziness of fuzzy weights vs. data.

The following discussions are based on the observations and comparisons of these numerical results.

First, from the examinations of the fuzzy weights, we notice the following. The first two variations of Wagenknecht-Hartmann (WH1,2) frequently provide results with some of the spreads equal to zero. From Tables 3 and 4, it can be seen that with WH1,2, there are some fuzzy weights which have the lower bound and/or upper bound equal to the mode. The problem with the Buckley and Lootsma approach (B-L) is that some of the normalized fuzzy weights may have its upper bound larger than one; for example, $w_{1u} = 1.05$ of W_1 for M-4-2 in Table 3. This appears to be somewhat of a disturbing result. The approach of Boender *et al.* was shown again that an incorrect fuzzy weight, $w_{2l} = 0.113 > w_{2m} = 0.112$ for M-4-4, can result.

Next, the results of the fuzzy eigenvalues are examined. The same approximate fuzzy eigenvalues were obtained from the approach of Laarhoven-Pedrycz FLLSM (LP) with FNFD and from the approach of Boender *et al.* FLLSM (B) with GFN. In addition, both of these results satisfy at least the relaxed concept of fuzzy normalization.

A similar phenomenon discussed above also occurs in the two variations of FRMNC with FNFD (CL-D) and with GFN (CL-G). The fuzzy eigenvalues derived with these two methods are very close, even though they may not be exactly the same. This same phenomenon also appears with the three variations of Wagenknecht-Hartmann FLSM (WH1,2,3); fairly close approximate fuzzy eigenvalues were obtained by the use of these three methods.

In general, the approximate fuzzy eigenvalues derived with the results of these eight approaches appear to be approximately consistent. There are some exceptions. These exceptions are the relatively low values in the lower bounds of the fuzzy eigenvalue for M-4-6 and M-6-4 with the approach of Wagenknecht-Hartmann FLSM (WH1,3). These drastic decreases in the lower bounds in the WH1 case are caused by the negative lower bounds of the fuzzy weights.

Third, the fuzziness of the resulting weights by the use of the eight approaches is compared with the fuzziness of the original data. Figure 1 shows this comparison by plotting the ratio of mean relative fuzziness of fuzzy weight versus the mean relative fuzziness of the data, ψ/ϕ. The approaches of CL-D (FRMNC with FNFD) and B-L (Buckley and Lootsma) always give fuzzy weights that were fuzzier than the fuzziness of the original data. The method of Boender *et al.* FLLSM (B) gave fuzzy weights that are less fuzzier than the data's fuzziness. The ratios ψ/ϕ

for the three variations of Wagenknecht-Hartmann FLSM (WH1,2,3) fluctuated quite a lot; some indicate good representations of the data's fuzziness, while some others not. The method of Laarhoven-Pedrycz FLLSM (LP) appears somewhat good in grasping the data's fuzziness; but still with some over estimations. The method of FRMNC with GFN (CL-G) appeared to be fairly good in matching the fuzziness of fuzzy weights with the data's fuzziness.

REFERENCES

1. T.L. Saaty, A scaling method for priorities in hierarchical structures, *J. Math. Psychology* **15**, 234–281 (1977).
2. T.L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, (1980).
3. T.L. Saaty, *Decision Making for Leaders, Lifetime Learning*, Belmont, CA, (1982).
4. T.L. Saaty and L.G. Vargas, Inconsistency and rank preservation, *J. Math. Psychology* **28**, 205–214 (1984).
5. T.L. Saaty and L.G. Vargas, Comparison of eigenvalue, logarithmic least squares and least squares methods in estimating ratios, *Mathl. Modelling* **5** (5), 309–324 (1984).
6. K.O. Cogger and P.L. Yu, Eigen weight vectors and least distance approximation for revealed preference in pairwise weight ratios, School of Business, University of Kansas, Lawrence, KS, (1983).
7. G.C. McMeekin, The pairwise comparison approach to the estimation of a ratio scale: A stochastic interpretation of the best priority weights ratio scale estimator, Presented at the *Atlantic Economic Society Meeting*, Washington, DC, October 11, 1979.
8. J.G. de Grann, Extensions to the multiple criteria analysis method of T.L. Saaty, National Institute for Water Supply, Voorburg, Netherlands, (1980).
9. F.A. Lootsma, Performance evaluation of non-linear optimization methods via multi-criteria decision analysis and via linear model analysis, In *Nonlinear Optimization 1981*, (Edited by M.J.D. Powell), pp. 419–453, Academic Press, New York, (1982).
10. J. Aczel and T.L. Saaty, Procedures for synthesizing ratio judgements, *J. Math. Psychology* **27**, 93–102 (1983).
11. V.R.R. Uppuluri, Logarithmic least-squares approach to Saaty's decision problems, In Mathematics and Statistics Research Department Progress Report, (Edited by W.E. Lever, D.E. Shepherd, R.C. Ward and D.G. Wilson), Oak Ridge National Laboratory, Oak Ridge, TN, (1978).
12. V.R.R. Uppuluri, Expert opinion and ranking methods, NRC FIN No. B044, Oak Ridge National Laboratory, Oak Ridge, TN, (1983).
13. A.T.W. Chu, R.E. Kalaba and K. Spingarn, A comparison of two methods for determining the weights of belonging to fuzzy sets, *J. Optim. Theory Appl.* **27**, 531–538 (1979).
14. C.G.E. Boender, J.G. de Grann and F.A. Lootsma, Multi-criteria decision analysis with fuzzy pairwise comparisons, *Fuzzy Sets and Systems* **29**, 133–143 (1989).
15. J.J. Buckley, Fuzzy hierarchical analysis, *Fuzzy Sets and Systems* **17**, 233–247 (1985).
16. P.J.M. van Laarhoven and W. Pedrycz, A fuzzy extension of Saaty's priority theory, *Fuzzy Sets and Systems* **11**, 229–241 (1983).
17. F.A. Lootsma, Performance evaluation of nonlinear optimization methods via pairwise comparison and fuzzy numbers, *Mathematical Programming* **33**, 93–114 (1985).
18. M. Wagenknecht and K. Hartmann, On fuzzy rank-ordering in polyoptimization, *Fuzzy Sets and Systems* **11**, 253–264 (1983).
19. D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, (1980).