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Iterative algorithms of domain decomposition for the solution of a quasilinear elliptic problem

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Abstract

This paper deals with iterative algorithms for domain decomposition applied to the solution of a quasilinear elliptic problem. Two iterative algorithms are examined: the first one is the Schwarz alternating procedure and the second algorithm is suitable for parallel computing. Convergence results are established in the two-domain and multidomain decomposition cases. Some issues of parallel implementation of these algorithms are discussed.

Keywords: Elliptic problem; Domain decomposition method; Parallel computing

AMS classification: 35J60, 65N55

1. Introduction

We are interested in iterative algorithms for domain decomposition that reduce a given problem to sequences of problems on subdomains.

The Schwarz alternating method [8] has attracted much attention as a convenient computational method for the solution of a large class of elliptic or parabolic problems. In recent years, various convergence results for the Schwarz method have been obtained by many authors. We note here the Lions's series of the papers [6, 7], where it can be found a systematic investigation of convergence properties of the Schwarz method and the bibliography for this theme.

This paper is devoted to the study of convergence properties of the Schwarz alternating method and the related computational method from [1]. The latter method is highly suitable for parallel computing. In [2–4] this method has been analyzed and illustrated by solving singularly perturbed boundary value problems. Here, we continue to study this method for the solution of a quasilinear elliptic problem and compare convergence properties of this method with the classical Schwarz alternating method. The iterative methods will be presented in the continuous form.

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The main object of this paper is concerned with convergence results of the geometric aspects for domain decomposition of the original domain, such as dependence of the convergence rate on the amount of overlapping, a number of subdomains and their sizes.

The structure of the paper is as follows. In Section 2, we present two iterative algorithms for domain decomposition in the case of decomposition of the original domain into two subdomains. The first one is the Schwarz method and the second one is the method from [1]. In Section 3, we generalize the two algorithms to more than two subdomains. We end the paper by discussing some issues of parallel implementation of these algorithms.

2. Two-domain decomposition case

In this section we introduce and analyze iterative algorithms for the case of two overlapping subdomains.

Let Ω_0 be the rectangular domain

$$\Omega_0 = \{(x,y) : 0 < x < x_*, 0 < y < y_*\}.$$

We consider the quasilinear elliptic equation

$$\Delta u(P) = f[P, u(P)], \;\; P \equiv (x,y) \in \Omega_0, \tag{2.1a}$$

with the homogeneous Dirichlet boundary conditions

$$u(P) = 0, \;\; P \in \partial\Omega_0 \text{ is the boundary of } \Omega_0, \tag{2.1b}$$

where $\Delta \equiv (\partial^2/\partial x^2 + \partial^2/\partial y^2)$. Assume that $f(P,u)$ is sufficiently smooth and satisfies

$$f_u \geqslant \beta_0^2, \qquad (P,u) \in \Omega_0 \times (-\infty, \infty), \qquad \beta_0 = \text{const.} > 0, \quad (f_u = \partial f/\partial u). \tag{2.1c}$$

Under suitable continuity and compatibility conditions on the data at the corners of $\bar{\Omega}_0$, a unique solution of (2.1) exists and $u(P) \in C^m(\bar{\Omega}_0) \cap C^{m+2}(\Omega_0), m \geqslant 0$ (see [5] for details).

We introduce the decomposition of the domain Ω_0 into the two overlapping subdomains Ω_1 and Ω_2 (see Fig.1):

$$\Omega_0 = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 \neq \emptyset,$$

$$\Gamma_i^0 = \partial\Omega_0 \cap \partial\Omega_i, \qquad \Gamma_i = \partial\Omega_i \backslash \Gamma_i^0, \quad i = 1,2,$$

$$\Gamma_1 = \{P : x = x_{\mathrm{r}}, \; 0 \leqslant y \leqslant y_*\}, \qquad \Gamma_2 = \{P : x = x_{\mathrm{l}}, \; 0 \leqslant y \leqslant y_*\},$$

$$0 < x_{\mathrm{l}} < x_{\mathrm{r}} < x_*.$$

2.1. Iterative algorithms

Consider the two sequences of functions $\{v^n\}, \{w^n\}$, $n \geqslant 1$, satisfying the problems:

$$\Delta v^n = f(P, v^n), \quad P \in \Omega_1, \tag{2.2a}$$

$$v^n(P) = 0, \quad P \in \Gamma_1^0, \qquad v^n(P) = \bar{v}^n(P), \quad P \in \Gamma_1;$$

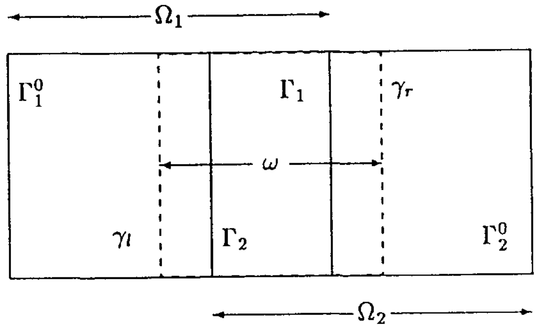

Fig. 1.

$$\Delta w^n = f(P, w^n), \quad P \in \Omega_2, \tag{2.2b}$$

$$w^n(P) = 0, \quad P \in \Gamma_2^0, \qquad w^n(P) = \bar{w}^n(P), \quad P \in \Gamma_2.$$

We now consider two iterative algorithms.

The first one, A1, is the Schwarz alternating procedure. Here the boundary conditions $\bar{v}^n$ and $\bar{w}^n$ from (2.2a) and (2.2b) are defined by

$$\bar{v}^{n+1}(P) = w^n(P), \quad P \in \Gamma_1, \qquad \bar{w}^n(P) = v^n(P), \quad P \in \Gamma_2, \ n \geqslant 1, \tag{2.3}$$

where an initial guess $\bar{v}^1$ must be prescribed.

The second algorithm, A2, is constructed using the following interfacial problem

$$\Delta z^n = f(P, z^n), \quad P \in \omega, \qquad z^n(P) = 0, \quad P \in \gamma_0, \tag{2.4a}$$

$$z^n(P) = v^n(P), \quad P \in \gamma_l, \qquad z^n(P) = w^n(P), \quad P \in \gamma_r,$$

where the subdomain ω is defined by (see Fig.1)

$$\omega \subset \Omega_0, \quad \Omega_l \cap \Omega_2 \subset \omega, \quad \gamma_0 = \partial\Omega_0 \cap \partial\omega, \quad \partial\omega = \gamma_0 \cup \gamma_l \cup \gamma_r,$$

$$\gamma_l = \{P: x = X_l, \ 0 \leqslant y \leqslant y_*\}, \qquad \gamma_r = \{P: x = X_r, 0 \leqslant y \leqslant y_*\},$$

$$0 < X_l < x_l < x_r < X_r < x_*.$$

The boundary conditions from (2.2a), (2.2b) are determined by

$$\bar{v}^{n+1}(P) = z^n(P), \quad P \in \Gamma_1, \qquad \bar{w}^{n+1}(P) = z^n(P), \quad P \in \Gamma_2, \ n \geqslant 1, \tag{2.4b}$$

where the initial guesses $\bar{w}^1$ and $\bar{v}^1$ are given.

Algorithm A1 is a serial procedure, since the solution v^n of (2.2a) must be obtained in order to determine the boundary condition $\bar{w}^n(P) = v^n(P), P \in \Gamma_2$ used in (2.2b). Thus, (2.2a) and (2.2b) are executed in lockstep fashion. Algorithm A2 can however be carried out by parallel processing, since on each iteration step problems (2.2a) and (2.2b) can be solved concurrently to give both v^n and w^n.

2.2. Convergence results

We now establish convergence properties of algorithms (2.2), (2.3) – A1 and (2.2), (2.4) – A2.

2.2.1. Preliminaries

In the following lemmas we obtain the required results necessary for the present discussion. Introduce the following one-dimensional linear two-point boundary value problems:

$$u''(x) - b(x)u(x) = 0, \quad x \in \Omega^x = (x_1, x_2), \qquad u(x_1) = u_1, \qquad u(x_2) = u_2, \tag{2.5}$$

where the coefficient $b(x)$ is smooth with $b(x) \geqslant \beta_0^2$; and

$$(\varphi_{\Omega^x}^{\mathrm{I,II}})'' - \beta_0^2(\varphi_{\Omega^x}^{\mathrm{I,II}}) = 0, \; x \in \Omega^x, \qquad \varphi_{\Omega^x}^{\mathrm{I}}(x_1) = \varphi_{\Omega^x}^{\mathrm{II}}(x_2) = 1, \qquad \varphi_{\Omega^x}^{\mathrm{I}}(x_2) = \varphi_{\Omega^x}^{\mathrm{II}}(x_1) = 0, \tag{2.6}$$

(the prime denotes differentiation).

Lemma 1. *If $u(x)$ and $\varphi_{\Omega^x}^{\mathrm{I,II}}(x)$ are the solutions to* (2.5), (2.6), *then the following estimates hold:*

$$| u(x) | \leqslant \varphi_{\Omega^x}^{\mathrm{I}}(x) \, | u_1 | + \varphi_{\Omega^x}^{\mathrm{II}}(x) \, | u_2 |, \quad x \in \bar{\Omega}^x; \tag{2.7a}$$

$$0 < \varphi_{\Omega^x}^{\mathrm{I,II}}(x) < 1, \quad x \in \Omega^x; \tag{2.7b}$$

$$c^{\mathrm{I}}\varphi_{\Omega^x}^{\mathrm{I}}(x) + c^{\mathrm{II}}\varphi_{\Omega^x}^{\mathrm{II}}(x) \leqslant \max(c^{\mathrm{I}}, c^{\mathrm{II}}), \quad x \in \bar{\Omega}^x, \tag{2.7c}$$

where constants $c^{\mathrm{I}}, c^{\mathrm{II}} \geqslant 0$;

$$\varphi_{\Omega^x}^{\mathrm{I}}(x) \leqslant \exp[-\beta_0(x - x_1)], \qquad \varphi_{\Omega^x}^{\mathrm{II}}(x) \leqslant \exp[-\beta_0(x_2 - x)], \quad x \in \bar{\Omega}^x. \tag{2.7d}$$

Proof. Can be found in [2]. Consider the two-dimensional linear problems:

$$\Delta u(P) - b(P)u = 0, \; P \in \Omega = (x_1, x_2) \times (y_1, y_2), \qquad u(P) = V(P), \quad P \in \partial\Omega \text{ is the boundary of } \Omega, \tag{2.8}$$

where $V(P), b(P)$ are smooth, $b(P) \geqslant \beta_0^2$; and

$$L_0 U \equiv \Delta U(P) - \beta_0^2 U(P) = 0, \quad P \in \Omega, \; U(P) = U_0(P), \quad P \in \partial\Omega, \tag{2.9a}$$

the boundary condition $U_0(P)$ is determined by

$$U_0(P) \,|_{\Gamma_i} = \{\max | V(P) | : P \in \Gamma_i\}, \quad 1 \leqslant i \leqslant 4, \quad \partial\Omega = \bigcup_i \Gamma_i, \tag{2.9b}$$

where Γ_i is the ith side of the rectangular domain Ω.

Note that the function $U_0(P)$ is piecewise smooth and may have points of discontinuity at the corners of Ω.

Lemma 2. *If $u(P), U(P)$ are the solutions to* (2.8) *and* (2.9), *respectively, then we have the following inequalities:*

$$| u(P) | \leqslant U(P), \quad P \in \bar{\Omega} \backslash \{\textit{the corners of } \bar{\Omega}\}; \tag{2.10a}$$

$$\max_{P\in\Omega} U(P) < \sup_{P\in\bar{\Omega}} U(P). \tag{2.10b}$$

Proof. Estimate (2.10a) follows immediately from the maximum principle. Estimate (2.10b) is well known as the strong maximum principle.

2.2.2. Convergence of algorithms A1 and A2 via the maximum principle

Theorem 1. *If $\Omega_1 \cap \Omega_2 \neq \emptyset$ $(x_l < x_r)$, then the iterative algorithm* (2.2), (2.3) (*i.e. the Schwarz alternating procedure*) *converges to the solution of problem* (2.1) *with linear* (*geometrical*) *rate $q = q_1 q_2 \in (0,1)$. The coefficients $q_1, q_2 \in (0,1)$ depend only on (Ω_1, Γ_2) and (Ω_2, Γ_1), respectively, and they are determined by*

$$q_i = \| U_i(P) \|_{\Gamma_{3-i}}, \qquad L_0 U_i = 0, \quad U_i \,|_{\Gamma_i^0} = 0, \quad U_i \,|_{\Gamma_i} = 1, \ i = 1,2, \tag{2.11}$$

$$L_0 \equiv \Delta - \beta_0^2, \qquad \| U_i(P) \|_{\Gamma_{3-i}} \equiv \max\{| U_i(P) | : P \in \Gamma_{3-i}\}.$$

Proof. Introduce the functions $\zeta^n(P) = v^n(P) - v^{n-1}(P)$, $\xi^n(P) = w^n(P) - w^{n-1}(P)$, $n \geqslant 2$. From (2.2), (2.3) and the mean-value theorem, it follows that $\zeta^n(P)$ and $\xi^n(P)$ satisfy the following problems:

$$\Delta\zeta^n(P) - f^n_{u\zeta}(P)\zeta^n(P) = 0, \quad P \in \Omega_1, \quad \zeta^n \,|_{\Gamma_1^0} = 0, \quad \zeta^n \,|_{\Gamma_1} = \xi^{n-1} \,|_{\Gamma_1};$$

$$\Delta\xi^n(P) - f^n_{u\xi}(P)\xi^n(P) = 0, \quad P \in \Omega_2, \quad \xi^n \,|_{\Gamma_2^0} = 0, \quad \xi^n \,|_{\Gamma_2} = \zeta^n \,|_{\Gamma_2},$$

where $f^n_{u\zeta}(P) \equiv \partial f(P, \Theta^n_\zeta(P))/\partial u$, $\Theta^n_\zeta(P)$ lies between $v^n(P)$ and $v^{n-1}(P)$. Analogously, $f^n_{u\xi}(P)$ is determined by an intermediate value between $w^n(P)$ and $w^{n-1}(P)$. Denote

$$\delta^n = \max\,[\| \zeta^n(P) \|_{\Gamma_1}, \| \xi^n(P) \|_{\Gamma_2}],$$

$$\| \zeta^n(P) \|_{\Gamma_1} \equiv \max\{| \zeta^n(P) | : P \in \Gamma_1\},$$

$$\| \xi^n(P) \|_{\Gamma_2} \equiv \max\{| \xi^n(P) | : P \in \Gamma_2\}.$$

Using the boundary conditions for $\zeta^n(P)$ and $\xi^n(P)$ and Lemma 2, it follows that

$$\begin{aligned}\| \zeta^n(P) \|_{\Gamma_1} &= \| \xi^{n-1}(P) \|_{\Gamma_1} \leqslant q_2 \| \xi^{n-1}(P) \|_{\Gamma_2} = q_2 \| \zeta^{n-1}(P) \|_{\Gamma_2} \\ &\leqslant q_2 q_1 \| \zeta^{n-1}(P) \|_{\Gamma_1},\end{aligned}$$

$$\begin{aligned}\| \xi^n(P) \|_{\Gamma_2} &= \| \zeta^n(P) \|_{\Gamma_2} \leqslant q_1 \| \zeta^n(P) \|_{\Gamma_1} = q_1 \| \xi^{n-1}(P) \|_{\Gamma_1} \\ &\leqslant q_1 q_2 \| \xi^{n-1}(P) \|_{\Gamma_1} .\end{aligned}$$

From this we obtain

$$\delta^n \leqslant q\delta^{n-1}, \quad n \geqslant 2, \ q = q_1 q_2,$$

where q_i, $i=1,2$ from (2.11). By Lemma 2 and (2.11), we conclude that $q_1, q_2 \in (0,1)$ and hence, $q \in (0,1)$. Now, from the maximum principle, we conclude

$$\sup_{\bar{\Omega}_1} |\zeta^n(P)| \leqslant \|\zeta^n(P)\|_{\Gamma_1} \leqslant \delta^n; \qquad \sup_{\bar{\Omega}_2} |\xi^n(P)| \leqslant \|\xi^n(P)\|_{\Gamma_2} \leqslant \delta^n.$$

This proves the convergence of algorithm (2.2), (2.3) with linear rate q.

Theorem 2. *If $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 \in \omega$ $(X_l < x_l < x_r < X_r)$, then the iterative algorithm* (2.2), (2.4) *converges to the solution of problem* (2.1) *with linear rate $q \in (0,1)$*

$$q = [\max(q_1^{\omega}, q_2^{\omega})][\max(q_1^{l}, q_2^{r})]. \tag{2.12a}$$

The coefficients $q_1^l, q_2^r \in (0,1)$ depend only on (Ω_1, γ_l) and (Ω_2, γ_r), respectively; $q_1^{\omega}, q_2^{\omega} \in (0,1)$ depend on (ω, Γ_1) and (ω, Γ_2), respectively. They are determined by

$$\begin{aligned}
&q_1^l = \|U_1(P)\|_{\gamma_l}, && L_0 U_1 = 0, && U_1|_{\Gamma_1^0} = 0, && U_1|_{\Gamma_1} = 1,\\
&q_2^r = \|U_2(P)\|_{\gamma_r}, && L_0 U_2 = 0, && U_2|_{\Gamma_2^0} = 0, && U_2|_{\Gamma_2} = 1,\\
&q_i^{\omega} = \|U(P)\|_{\Gamma_i}, && L_0 U = 0, && U|_{\gamma_0} = 0, && U|_{\gamma_l} = U|_{\gamma_r} = 1, \quad i=1,2.
\end{aligned} \tag{2.12b}$$

Proof. Analogously to the proof of Theorem 1, we introduce the functions $\zeta^n(P) = v^n(P) - v^{n-1}(P)$, $\xi^n(P) = w^n(P) - w^{n-1}(P)$ and $\chi^n(P) = z^n(P) - z^{n-1}(P)$, $n \geqslant 2$. From (2.2), (2.4) and the mean-value theorem, we conclude that $\zeta^n(P)$, $\xi^n(P)$ and $\chi^n(P)$ satisfy the problems:

$$\begin{aligned}
&\Delta\zeta^n(P) - f_{u\zeta}^n(P)\zeta^n(P) = 0, \quad P \in \Omega_1,\\
&\zeta^n|_{\Gamma_1^0} = 0, \qquad \zeta^n|_{\Gamma_1} = \chi^{n-1}|_{\Gamma_1};\\
&\Delta\xi^n(P) - f_{u\xi}^n(P)\xi^n(P) = 0, \quad P \in \Omega_2,\\
&\xi^n|_{\Gamma_2^0} = 0, \qquad \xi^n|_{\Gamma_2} = \chi^{n-1}|_{\Gamma_2};\\
&\Delta\chi^n(P) - f_{u\chi}^n(P)\chi^n(P) = 0, \quad P \in \omega,\\
&\chi^n|_{\gamma_0} = 0, \qquad \chi^n|_{\gamma_l} = \zeta^n|_{\gamma_l}, \qquad \chi^n|_{\gamma_r} = \xi^n|_{\gamma_r}.
\end{aligned}$$

Denote

$$\delta^n = \max[\|\zeta^n(P)\|_{\Gamma_1}, \|\xi^n(P)\|_{\Gamma_2}].$$

By using Lemma 2 and the boundary conditions for $\zeta^n(P)$, $\xi^n(P)$, we conclude

$$\begin{aligned}
\|\zeta^n(P)\|_{\Gamma_1} &= \|\chi^{n-1}(P)\|_{\Gamma_1} \leqslant q_1^{\omega} \max[\|\chi^{n-1}(P)\|_{\gamma_l}, \|\chi^{n-1}(P)\|_{\gamma_r}]\\
&= q_1^{\omega} \max[\|\zeta^{n-1}(P)\|_{\gamma_l}, \|\xi^{n-1}(P)\|_{\gamma_r}]\\
&\leqslant q_1^{\omega} \max[q_1^l\, \|\zeta^{n-1}(P)\|_{\Gamma_1};\ q_2^r\, \|\xi^{n-1}(P)\|_{\Gamma_2}].
\end{aligned}$$

Thus, we have

$$\|\zeta^n(P)\|_{\Gamma_1} \leqslant q_1^{\omega}[\max(q_1^l, q_2^r)]\delta^{n-1}.$$

In the same way, we can obtain

$$\| \xi^n(P) \|_{\Gamma_2} \leqslant q_2^{\omega}[\max(q_1^{\mathrm{l}}, q_2^{\mathrm{r}})]\delta^{n-1}.$$

Hence, $\delta^n \leqslant q\delta^{n-1}$, where q from (2.12a). Evaluating q_i^{ω}, $i = 1,2$, and q_1^{l}, q_2^{r} from (2.12b) with Lemma 2, it follows that $q \in (0,1)$. This proves the theorem.

2.2.3. Estimates on the rates of convergence

To illustrate convergence properties of algorithms (2.2), (2.3) for A1 and (2.2), (2.4) for A2, we estimate the rates of convergence for the following case of problem (2.1):

$$L_0 u \equiv \Delta u(P) - \beta_0^2 u(P) = f(P), \quad P \in \Omega_0 = (0, x_*) \times (0, y_*), \tag{2.13}$$

$$u(P) = 0, \quad P \in \partial\Omega_0.$$

Theorem 3. *If $\Omega_1 \cap \Omega_2 \neq \emptyset$ $(x_\mathrm{l} < x_\mathrm{r})$, then the iterative algorithm* (2.2), (2.3) *converges to the solution of problem* (2.13) *with linear rate $q \in (0,1)$. The following estimate on q holds*

$$q \leqslant \varphi^{\mathrm{II}}_{\Omega_1^x}(x_\mathrm{l})\varphi^{\mathrm{I}}_{\Omega_2^x}(x_\mathrm{r}), \tag{2.14}$$

where $\varphi^{\mathrm{I}}_{\Omega_2^x}(x)$, $\varphi^{\mathrm{II}}_{\Omega_1^x}(x)$ are determined by (2.6) *with the coefficient $\tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2$ instead of β_0^2 and $\Omega_1^x = \{x : 0 < x < x_\mathrm{r}\}$, $\Omega_2^x = \{x : x_\mathrm{l} < x < x_*\}$.*

Proof. Introduce the functions $\zeta^n(P) = v^n(P) - v^{n-1}(P)$, $\xi^n(P) = w^n(P) - w^{n-1}(P)$, $n \geqslant 2$. From (2.2), (2.3) and (2.13), we have

$$L_0\zeta^n(P) = 0, \quad P \in \Omega_1, \quad \zeta^n |_{\Gamma_1^0} = 0, \quad \zeta^n |_{\Gamma_1} = \xi^{n-1} |_{\Gamma_1};$$

$$L_0\xi^n(P) = 0, \quad P \in \Omega_2, \quad \xi^n |_{\Gamma_2^0} = 0, \quad \xi^n |_{\Gamma_2} = \zeta^n |_{\Gamma_2}.$$

Let $\{\psi_k(y), k \geqslant 1\}$ be the eigenfunctions of the eigenvalue problems

$$\psi_k''(y) = -\lambda_k^2\psi_k^2, \quad 0 < y < y_*, \quad \psi_k(0) = \psi_k(y_*) = 0.$$

Then, we obtain

$$\lambda_k = (\pi/y_*)k, \quad \psi_k(y) = (2/y_*)^{1/2}\sin(\lambda_k y), \quad k = 1,2,\ldots.$$

Let $\{\varphi^{\mathrm{I,II}}_{\Omega_2^x,k}(x),\ k \geqslant 1\}$ be the solutions of problem (2.6) with coefficients $\tilde{\beta}_k^2 = \beta_0^2 + \lambda_k^2$, $k = 1,2,\ldots$, instead of β_0^2.

By the method of separation of variables the solutions $\zeta^n(P)$, $\xi^n(P)$ can be expanded as follows:

$$\zeta^n(x,y) = \sum_k \delta_k^{\zeta,n}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(x)\psi_k(y),$$

$$\xi^n(x,y) = \sum_k \delta_k^{\xi,n}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(x)\psi_k(y),$$

where $\delta_k^{\zeta,n}$ and $\delta_k^{\xi,n}$ are the coefficients of the expansions of ζ^n on Γ_1 and ξ^n on Γ_2, respectively, i.e.

$$\delta_k^{\zeta,n} = \int_0^{y_*} \zeta^n(x_r, y)\psi_k(y)\,dy, \qquad \delta_k^{\xi,n} = \int_0^{y_*} \xi^n(x_l, y)\psi_k(y)\,dy.$$

Using the boundary conditions for ζ^n on Γ_1 and ξ^n on Γ_2 and the orthonormality of $\{\psi_k(y),\ k \geqslant 1\}$, we conclude that

$$\delta_k^{\xi,n-1} = \delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(x_l), \quad k = 1,2,\ldots$$

and

$$\delta_k^{\zeta,n} = q_k\delta_k^{\zeta,n-1}, \qquad q_k = \varphi^{\mathrm{II}}_{\Omega_1^x,k}(x_l)\varphi^{\mathrm{I}}_{\Omega_2^x,k}(x_r), \quad k = 1,2,\ldots$$

Thus, q_k is the reduction factor of the kth frequency of the error at the boundaries Γ_1 and Γ_2. From this and using the estimates (2.7b) from Lemma 1, it follows the convergence of algorithm (2.2), (2.3) with linear rate q, where

$$q \leqslant \max_{k\geqslant 1} q_k = q_1.$$

This concludes the proof.

Corollary 1. *For algorithm* (2.2), (2.3) *the following bound on q holds:*

$$q \leqslant \exp[-2\tilde{\beta}_1(x_r - x_l)], \quad \tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2.$$

Proof. Follows from (2.7d) and (2.14).

Theorem 4. *If $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cap \Omega_2 \subset \omega$ $(X_l < x_l < x_r < X_r)$, then the iterative algorithm* (2.2), (2.4) *converges to the solution of problem* (2.13) *with linear rate $q \in (0,1)$. The following estimate on q holds*

$$q \leqslant [\max(q_1^\omega, q_2^\omega)][\max(q_1^l, q_2^r)], \tag{2.15}$$

$$q_1^\omega = \varphi^{\mathrm{I}}_{\omega^x}(x_r) + \varphi^{\mathrm{II}}_{\omega^x}(x_r), \qquad q_2^\omega = \varphi^{\mathrm{I}}_{\omega^x}(x_l) + \varphi^{\mathrm{II}}_{\omega^x}(x_l),$$

$$q_1^l = \varphi^{\mathrm{II}}_{\Omega_1^x}(X_l), \qquad q_2^r = \varphi^{\mathrm{I}}_{\Omega_2^x}(X_r),$$

where $\varphi^{\mathrm{I}}_{\Omega_2^x}(x)$, $\varphi^{\mathrm{II}}_{\Omega_1^x}(x)$ and $\varphi^{\mathrm{I}}_{\omega^x}(x), \varphi^{\mathrm{II}}_{\omega^x}(x)$ are determined by (2.6) *with the coefficient $\tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2$ instead of β_0^2 and*

$$\Omega_1^x = \{x: 0 < x < x_r\}, \qquad \Omega_2^x = \{x : x_l < x < x_*\}, \qquad \omega^x = \{x: X_l < x < X_r\}.$$

Proof. Analogously to the proof of Theorem 2, we introduce the functions $\zeta^n(P) = v^n(P) - v^{n-1}(P)$, $\xi^n(P) = w^n(P) - w^{n-1}(P)$, $\chi^n(P) = z^n(P) - z^{n-1}(P)$, $n \geqslant 2$. From (2.2), (2.4) and (2.13), we have

$$L_0\zeta^n(P) = 0, \quad P \in \Omega_1,$$

$$\zeta^n\,|_{\Gamma_1^0} = 0, \qquad \zeta^n\,|_{\Gamma_1} = \chi^{n-1}\,|_{\Gamma_1};$$

$$L_0\xi^n(P) = 0, \quad P \in \Omega_2,$$

$$\xi^n \mid_{\Gamma_2^0} = 0, \qquad \xi^n \mid_{\Gamma_2} = \chi^{n-1} \mid_{\Gamma_2};$$

$$L_0\chi^n(P) = 0, \quad P \in \omega,$$

$$\chi^n \mid_{\gamma_0} = 0, \qquad \chi^n \mid_{\gamma_1} = \zeta^n \mid_{\gamma_1}, \qquad \chi^n \mid_{\gamma_r} = \xi^n \mid_{\gamma_r}.$$

Using the same notations as in Theorem 3, by the method of separation of variables, we can write the solutions $\zeta^n(P)$, $\xi^n(P)$, $\chi^n(P)$ in the following forms:

$$\zeta^n(x,y) = \sum_k \delta_k^{\zeta,n} \varphi^{\mathrm{II}}_{\Omega_1^x,k}(x)\psi_k(y),$$

$$\xi^n(x,y) = \sum_k \delta_k^{\xi,n} \varphi^{\mathrm{I}}_{\Omega_2^x,k}(x)\psi_k(y),$$

$$\chi^n(x,y) = \sum_k [\delta_k^{\chi,n} \varphi^{\mathrm{I}}_{\omega^x,k}(x) + \Delta_k^{\chi,n} \varphi^{\mathrm{II}}_{\omega^x,k}(x)]\psi_k(y),$$

where $\delta_k^{\zeta,n}$, $\delta_k^{\xi,n}$ and $\delta_k^{\chi,n}$, $\Delta_k^{\chi,n}$ are the coefficients of the expansions of ζ^n on Γ_1, ξ^n on Γ_2, χ^n on γ_1 and on γ_r, respectively, i.e.

$$\delta_k^{\zeta,n} = \int_0^{y_*} \zeta^n(x_r,y)\psi_k(y)\,\mathrm{d}y, \qquad \delta_k^{\xi,n} = \int_0^{y_*} \xi^n(x_1,y)\psi_k(y)\,\mathrm{d}y,$$

$$\delta_k^{\chi,n} = \int_0^{y_*} \chi^n(X_1,y)\psi_k(y)\,\mathrm{d}y, \qquad \Delta_k^{\chi,n} = \int_0^{y_*} \chi^n(X_r,y)\psi_k(y)\,\mathrm{d}y.$$

Using the boundary conditions for $\chi^{n-1}(P)$ and the orthonormality of $\{\psi_k(y),\ k \geqslant 1\}$, it follows that

$$\delta_k^{\chi,n-1} = \int_0^{y_*} \left[\sum_i \delta_i^{\zeta,n-1} \varphi^{\mathrm{II}}_{\Omega_1^x,i}(X_1)\psi_i(y)\right] \psi_k(y)\,\mathrm{d}y = \delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_1),$$

and analogously,

$$\Delta_k^{\chi,n-1} = \delta_k^{\xi,n-1}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r), \quad k = 1,2,\ldots.$$

From this and the boundary conditions for $\zeta^n(P)$ and $\xi^n(P)$, we conclude that

$$\begin{aligned}\zeta^n(x_r,y) &= \sum_k \delta_k^{\zeta,n}\psi_k(y) \\ &= \sum_k \Big[\delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_1)\varphi^{\mathrm{I}}_{\omega^x,k}(x_r) \\ &\quad + \delta_k^{\xi,n-1}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)\varphi^{\mathrm{II}}_{\omega^x,k}(x_r)\Big]\psi_k(y),\end{aligned}$$

$$\begin{aligned}\xi^n(x_l, y) &= \sum_k \delta_k^{\xi,n}\psi_k(y)\\ &= \sum_k \Big[\delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l)\varphi^{\mathrm{I}}_{\omega^x,k}(x_l)\\ &\quad +\delta_k^{\xi,n-1}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)\varphi^{\mathrm{II}}_{\omega^x,k}(x_l)\Big]\psi_k(y).\end{aligned}$$

Thus, we obtain

$$\delta_k^{\zeta,n} = \delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l)\varphi^{\mathrm{I}}_{\omega^x,k}(x_r) + \delta_k^{\xi,n-1}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)\varphi^{\mathrm{II}}_{\omega^x,k}(x_r),$$

$$\delta_k^{\xi,n} = \delta_k^{\zeta,n-1}\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l)\varphi^{\mathrm{I}}_{\omega^x,k}(x_l) + \delta_k^{\xi,n-1}\varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)\varphi^{\mathrm{II}}_{\omega^x,k}(x_l).$$

Denoting $\delta_k^n = \max(|\delta_k^{\zeta,n}|, |\delta_k^{\xi,n}|)$ and applying estimate (2.7c), we get

$$\delta_k^n \leqslant \delta_k^{n-1}[\varphi^{\mathrm{I}}_{\omega^x,k}(x_r) + \varphi^{\mathrm{II}}_{\omega^x,k}(x_r)]\max[\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l), \varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)],$$

$$\delta_k^n \leqslant \delta_k^{n-1}[\varphi^{\mathrm{I}}_{\omega^x,k}(x_l) + \varphi^{\mathrm{II}}_{\omega^x,k}(x_l)]\max[\varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l), \varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r)].$$

Hence, we obtain (compare with (2.12))

$$\delta_k^n \leqslant q_k\delta_k^{n-1}, \qquad q_k = [\max(q^{\omega}_{1,k}, q^{\omega}_{2,k})][\max(q^{\mathrm{l}}_{1,k}, q^{\mathrm{r}}_{2,k})],$$

$$q^{\omega}_{1,k} = \varphi^{\mathrm{I}}_{\omega^x,k}(x_r) + \varphi^{\mathrm{II}}_{\omega^x,k}(x_r), \qquad q^{\omega}_{2,k} = \varphi^{\mathrm{I}}_{\omega^x,k}(x_l) + \varphi^{\mathrm{II}}_{\omega^x,k}(x_l),$$

$$q^{\mathrm{l}}_{1,k} = \varphi^{\mathrm{II}}_{\Omega_1^x,k}(X_l), \qquad q^{\mathrm{r}}_{2,k} = \varphi^{\mathrm{I}}_{\Omega_2^x,k}(X_r).$$

where q_k is the reduction factor of the kth frequency of the error at the boundaries Γ_1 and Γ_2. Since

$$\max_{k\geqslant 1} q_k = q_1,$$

then estimate (2.15) follows. This proves the theorem.

Corollary 2. *For algorithm* (2.2), (2.4) *the following bound on* q *holds*

$$q \leqslant \max\{\exp[-\tilde{\beta}_1(X_r - x_l)],\ \exp[-\tilde{\beta}_1(x_r - X_l)]\}, \qquad \tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2.$$

Proof. Follows from (2.15) and Lemma 1.

3. Multidomain decomposition

In this section we generalize algorithms A1 and A2 from Section 2.1 to more than two subdomains.

Introduce the multidomain overlap decomposition of the domain $\bar{\Omega}_0 = [0,x_*]\times[0,y_*]$ into the subdomains Ω_j, $j = 1,2,\ldots,M$:

$$\Omega_j = (x^{\mathrm{l}}_{j-1}, x^{\mathrm{r}}_j)\times(0,y_*),\ \ \Omega_j\cap\Omega_{j+1}\neq\emptyset,$$

$$\Omega_j^x = (x^{\mathrm{l}}_{j-1}, x^{\mathrm{r}}_j), \quad 0 < x^{\mathrm{l}}_j < x^{\mathrm{r}}_j < 1, \quad x^{\mathrm{l}}_0 = 0, \quad x^{\mathrm{r}}_M = 1, \quad j = 1,2,\ldots,M-1.$$

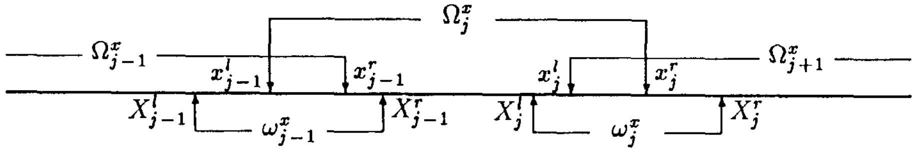

Fig. 2.

$$\Gamma_j^0 = \partial\Omega_0 \cap \partial\Omega_j, \qquad \Gamma_j = \partial\Omega_j \backslash \Gamma_j^0 = \Gamma_j^{\mathrm{l}} \cup \Gamma_j^{\mathrm{r}}, \qquad \Gamma_j^{\mathrm{l}} = \{P : x = x_{j-1}^{\mathrm{l}},\ 0 < y < y_*\},$$
$$\Gamma_j^{\mathrm{r}} = \{P : x = x_j^{\mathrm{r}},\ 0 < y < y_*\}.$$

Fig. 2 illustrates the x-section of the multidomain decomposition.

3.1. *Iterative algorithms for multidomain decomposition*

On each subdomain Ω_j, $j = 1,2,\ldots,M$ define the sequence $\{v_j^n(P)\}$, $n \geqslant 1$ satisfying the following problems:

$$\Delta v_j^n = f(P, v_j^n), \quad P \in \Omega_j, \tag{3.1}$$
$$v_j^n(P) = 0, \quad P \in \Gamma_j^0, \qquad v_j^n(P) = \bar{v}_j^n(P), \quad P \in \Gamma_j,\ j = 1,2,\ldots,M.$$

We next introduce a (parallel) extension of the Schwarz alternating procedure. The boundary conditions $\bar{v}_j^n$ are defined by

$$\bar{v}_j^n = \begin{cases} v_{j-1}^n(P), & P \in \Gamma_j^{\mathrm{l}} \\ v_{j+1}^n(P), & P \in \Gamma_j^{\mathrm{r}} \end{cases}, \quad j = 2i+1,\ i = 0,\ 1,\ldots,I-1; \tag{3.2}$$
$$\bar{v}_j^{n+1} = \begin{cases} v_{j-1}^n(P), & P \in \Gamma_j^{\mathrm{l}} \\ v_{j+1}^n(P), & P \in \Gamma_j^{\mathrm{r}} \end{cases}, \quad j = 2i,\ i = 0,\ 1,\ldots,I;$$

where initial guesses v_j^1, $j = 2i+1, i = 0,1,\ldots,I-1$ must be prescribed. We assume here that $M = 2 \times I$.

To generalize algorithm A2 to a multidomain decomposition, we introduce the $(M-1)$ interfacial problems:

$$\Delta z_j^n = f(P, z_j^n), \quad P \in \omega_j, \qquad z_j^n(P) = 0, \quad P \in \gamma_j^0, \tag{3.3a}$$
$$z_j^n(P) = v_j^n(P), \quad P \in \gamma_j^{\mathrm{l}}, \qquad z_j^n(P) = v_{j+1}^n(P), \quad P \in \gamma_j^{\mathrm{r}},$$

where the subdomain ω_j is defined by (see Fig. 2)

$$\omega_j = (X_j^{\mathrm{l}}, X_j^{\mathrm{r}}) \times (0, y_*), \qquad \Omega_j \cap \Omega_{j+1} \subset \omega_j, \qquad \omega_j \cap \omega_{j+1} = \emptyset, \qquad \partial\omega_j = \gamma_j^0 \cup \gamma_j^{\mathrm{l}} \cup \gamma_j^{\mathrm{r}},$$
$$\gamma_j^0 = \partial\Omega_0 \cap \partial\omega_j, \qquad \gamma_j^{\mathrm{l}} = \{P : x = X_j^{\mathrm{l}}, 0 \leqslant y \leqslant y_*\},$$
$$\gamma_j^{\mathrm{r}} = \{P : x = X_j^{\mathrm{r}}, 0 \leqslant y \leqslant y_*\},$$
$$0 < X_j^{\mathrm{l}} < x_j^{\mathrm{l}} < x_j^{\mathrm{r}} < X_j^{\mathrm{r}} < x_*, \quad j = 1,2,\ldots,M-1.$$

The boundary conditions in (3.1) are given by

$$\bar{v}_j^{n+1}(P) = z_{j-1}^n(P), \quad P \in \Gamma_j^{\mathrm{l}}, \quad j = 2,\dots,M; \tag{3.3b}$$

$$\bar{v}_j^{n+1}(P) = z_j^n(P), \; P \in \Gamma_j^{\mathrm{r}}, \; j = 1,2,\dots,M-1.$$

The initial guesses $v_j^1(P), P \in \Gamma_j$, $j = 1,2,\dots,M$, should be prescribed.

Algorithm (3.1), (3.2) is a parallel version of the Schwarz alternating procedure for a multidomain decomposition. On each iteration step of this algorithm, problems (3.1) with $j = 2i$, $i = 1,2,\dots,I$, are solved concurrently; the same is true for the problems with $j = 2i+1, i = 0,1,\dots,I-1$.

Algorithm (3.1), (3.3) can be carried out by parallel processing because the M problems for (3.1) can be solved concurrently; the same is true for the $(M-1)$ interfacial problems from (3.3).

3.2. Convergence results

Here we give convergence results for algorithms (3.1), (3.2) and (3.1), (3.3).

We denote

$$q_j^{\mathrm{l}} = \| U_j \|_{\Gamma_{j+1}^{l}}, \qquad L_0 U_j = 0, \quad U_j |_{\Gamma_j^0} = 0, \quad U_j |_{\Gamma_j} = 1,$$

$$q_j^{\mathrm{r}} = \| U_j \|_{\Gamma_{j-1}^{r}}, \qquad L_0 U_j = 0, \; U_j |_{\Gamma_j^0} = 0, \quad U_j |_{\Gamma_j} = 1,$$

and suppose that $q_0^{\mathrm{l}} = 0$, $q_{M+1}^{\mathrm{r}} = 0$.

Theorem 5. *If $\Omega_j \cap \Omega_{j+1} \neq \emptyset$, $j = 1,2,\dots,M-1$ and $\Omega_j \cap \Omega_{j+2} = \emptyset$, $j = 1,2,\dots,M-2$, then the multidomain version* (3.1), (3.2) *of the Schwarz alternating procedure converges to the solution of problem* (2.1) *with linear rate $q \in (0,1)$. The following estimate on q holds:*

$$q \leqslant \max(q_1, q_2), \tag{3.4}$$

$$q_1 \leqslant \max_{2 \leqslant j \leqslant M} [q_{j-1}^{\mathrm{l}} \max(q_j^{\mathrm{r}}, q_{j-2}^{\mathrm{l}})], \qquad q_2 \leqslant \max_{1 \leqslant j \leqslant M-1} [q_{j+1}^{\mathrm{r}} \max(q_j^{\mathrm{l}}, q_{j+2}^{\mathrm{r}})].$$

Proof. This result is established using a procedure similar the one used for Theorem 1. Introduce the functions $\zeta_j^n(P) = v_j^n(P) - v_j^{n-1}(P)$, $j = 1,2,\dots,M$, $n \geqslant 2$. From (3.1), (3.2), by the mean-value theorem, it is easy to verify that $\zeta_j^n(P)$, $j = 1,2,\dots,M$ satisfy

$$\Delta \zeta_j^n(P) - f_{u\zeta}^n \zeta_j^n(P) = 0, \quad P \in \Omega_j, \qquad \zeta_j^n(P) = 0, \quad P \in \Gamma_j^0;$$

$$\zeta_j^n(P) = \begin{cases} \zeta_{j-1}^n(P), & P \in \Gamma_j^{\mathrm{l}}, \\ \zeta_{j+1}^n(P), & P \in \Gamma_j^{\mathrm{r}}, \end{cases} \quad j = 2i+1, \; i = 0,1,\dots, \; I-1;$$

$$\zeta_j^{n+1}(P) = \begin{cases} \zeta_{j-1}^n(P), & P \in \Gamma_j^{\mathrm{l}}, \\ \zeta_{j+1}^n(P), & P \in \Gamma_j^{\mathrm{r}}, \end{cases} \quad j = 2i, \; i = 0,1,\dots,I.$$

Denote

$$\delta^n = \max_j [\| \zeta_j^n(P) \|_{\Gamma_j}], \qquad \| \zeta_j^n(P) \|_{\Gamma_j} \equiv \max\{ | \zeta_j^n(P) | : P \in \Gamma_j \}.$$

Using Lemma 2 and the boundary conditions for $\zeta_j^n(P)$, it follows that for $j = 2i$

$$\begin{aligned}
\| \zeta_{2i}^n(P) \|_{\Gamma_{2i}^{\mathrm{r}}} &= \| \zeta_{2i+1}^{n-1}(P) \|_{\Gamma_{2i}^{\mathrm{r}}} \\
&\leqslant q_{2i+1}^{\mathrm{r}} \max[\| \zeta_{2i+1}^{n-1}(P) \|_{\Gamma_{2i+1}^{\mathrm{l}}}, \| \zeta_{2i+1}^{n-1}(P) \|_{\Gamma_{2i+1}^{\mathrm{r}}}] \\
&= q_{2i+1}^{\mathrm{r}} \max[\| \zeta_{2i}^{n-1}(P) \|_{\Gamma_{2i+1}^{\mathrm{l}}}, \| \zeta_{2i+2}^{n-1}(P) \|_{\Gamma_{2i+1}^{\mathrm{r}}}] \\
&\leqslant q_{2i+1}^{\mathrm{r}} \max \Big\{ q_{2i}^{\mathrm{l}} \max[\| \zeta_{2i}^{n-1}(P) \|_{\Gamma_{2i}^{\mathrm{l}}}, \| \zeta_{2i}^{n-1}(P) \|_{\Gamma_{2i}^{\mathrm{r}}}] ; \\
&\qquad q_{2i+2}^{\mathrm{r}} \max[\| \zeta_{2i+2}^{n-1}(P) \|_{\Gamma_{2i+2}^{\mathrm{l}}}, \| \zeta_{2i+2}^{n-1}(P) \|_{\Gamma_{2i+2}^{\mathrm{r}}})] \Big\} \\
&\leqslant q_{2i+1}^{\mathrm{r}} \max(q_{2i}^{\mathrm{l}}, q_{2i+2}^{\mathrm{r}}) \delta^{n-1}.
\end{aligned}$$

Analogously, it can be proved

$$\| \zeta_{2i}^n(P) \|_{\Gamma_{2i}^{\mathrm{l}}} \leqslant q_{2i-1}^{\mathrm{l}} \max(q_{2i-2}^{\mathrm{l}}, q_{2i}^{\mathrm{r}}) \delta^{n-1}.$$

Hence, we have

$$\| \zeta_{2i}^n(P) \|_{\Gamma_{2i}} \leqslant \max[q_{2i-1}^{\mathrm{l}} \max(q_{2i-2}^{\mathrm{l}}, q_{2i}^{\mathrm{r}}); q_{2i+1}^{\mathrm{r}} \max(q_{2i}^{\mathrm{l}}, q_{2i+2}^{\mathrm{r}})] \delta^{n-1}.$$

It is easy to see that this estimate holds for $j = 2i + 1$. Thus, we obtain $\delta^n \leqslant q\delta^{n-1}$, $n \geqslant 2$, where q is determined by (3.4). We suppose that

$$q_j^{\mathrm{l}} = \| U_j \|_{\Gamma_{j+1}^{l}} = 0, \quad j = 1; \qquad q_j^{\mathrm{r}} = \| U_j \|_{\Gamma_{j-1}^{\mathrm{r}}} = 0, \quad j = M.$$

From Lemma 2, we conclude that $q_j^{\mathrm{l}}, q_j^{\mathrm{r}} \in (0, 1)$ and hence, $q \in (0, 1)$. This completes the proof of the theorem.

We now introduce the following notations:

$$q_j^{\omega,\mathrm{l}} = \| U_j(P) \|_{\Gamma_{j+1}^{\mathrm{l}}}, \qquad L_0 U_j = 0, \quad U_j |_{\gamma_j^0} = 0, \quad U_j |_{\gamma_j} = 1;$$

$$q_j^{\omega,\mathrm{r}} = \| U_j(P) \|_{\Gamma_j^{\mathrm{r}}}, \qquad L_0 U_j = 0, \quad U_j |_{\gamma_j^0} = 0, \quad U_j |_{\gamma_j} = 1;$$

$$q_j^{\mathrm{l}} = \| U_j(P) \|_{\gamma_j^{\mathrm{l}}}, \qquad L_0 U_j = 0, \quad U_j |_{\Gamma_j^0} = 0, \quad U_j |_{\Gamma_j} = 1;$$

$$q_j^{\mathrm{r}} = \| U_j(P) \|_{\gamma_{j-1}^{\mathrm{r}}}, \qquad L_0 U_j = 0, \quad U_j |_{\Gamma_j^0} = 0, \quad U_j |_{\Gamma_j} = 1.$$

Theorem 6. *If* $\Omega_j \cap \Omega_{j+1} \neq \emptyset$, $\Omega_j \cap \Omega_{j+1} \in \omega_j, j = 1, 2, \ldots, M-1$, *and* $\omega_j \cap \omega_{j+1} = \emptyset$, $j = 1, 2, \ldots, M-2$, *then algorithm* (3.1), (3.3) *converges to the solution of problem* (2.1) *with linear rate* $q \in (0, 1)$. *The following estimate on* q *holds:*

$$q \leqslant \max(q_1, q_2), \tag{3.5}$$

$$q_1 \leqslant \max_{2 \leqslant j \leqslant M} [q_{j-1}^{\omega,\mathrm{l}} \max(q_{j-1}^{\mathrm{l}}, q_j^{\mathrm{r}})]; \qquad q_2 \leqslant \max_{1 \leqslant j \leqslant M-1} [q_j^{\omega,\mathrm{r}} \max(q_j^{\mathrm{l}}, q_{j+1}^{\mathrm{r}})].$$

Proof. Introducing the functions $\zeta_j^n(P) = v_j^n(P) - v_j^{n-1}(P)$, $j = 1,2,\ldots,M$ and $\chi_j^n(P) = z_j^n(P) - z_j^{n-1}(P)$, $j = 1,2,\ldots,M-1$, $n \geqslant 2$, and using the mean-value theorem, from (3.1), (3.3), we conclude that $\zeta_j^n(P)$, $j = 1,2,\ldots,M$ satisfy

$$\Delta\zeta_j^n(P) - f_{u\zeta}^n\zeta_j^n(P) = 0, \quad P \in \Omega_j, \qquad \zeta_j^n(P) = 0, \quad P \in \Gamma_j^0,$$

$$\zeta_j^n(P) = \chi_{j-1}^{n-1}(P), \quad P \in \Gamma_j^{\mathrm{l}}, \; j = 2,3,\ldots,M, \qquad \zeta_1^n(P) = 0, \quad P \in \Gamma_1^{\mathrm{l}},$$

$$\zeta_j^n(P) = \chi_j^{n-1}(P), \quad P \in \Gamma_j^{\mathrm{r}}, \; j = 1,2,\ldots,M-1, \qquad \zeta_M^n(P) = 0, \quad P \in \Gamma_M^{\mathrm{r}};$$

and $\chi_j^n(P)$, $j = 1,2,\ldots,M-1$, satisfy

$$\Delta\chi_j^n(P) - f_{u\chi}^n\chi_j^n(P) = 0, \quad P \in \omega_j, \qquad \chi_j^n(P) = 0, \quad P \in \gamma_j^0,$$

$$\chi_j^n(P) = \zeta_j^n(P), \quad P \in \gamma_j^{\mathrm{l}}, \qquad \chi_j^n(P) = \zeta_{j+1}^n, \quad P \in \gamma_j^{\mathrm{r}}.$$

Denote

$$\delta^n = \max_j \| \zeta_j^n(P) \|_{\Gamma_j}, \qquad \| \zeta_j^n(P) \|_{\Gamma_j} \equiv \max\{| \zeta_j^n(P) | : P \in \Gamma_j\}.$$

From Lemma 1 and the boundary conditions for $\zeta_j^n(P)$ and $\chi_j^n(P)$, we conclude

$$\begin{aligned}
\| \zeta_j^n(P) \|_{\Gamma_j^{\mathrm{r}}} &= \| \chi_j^{n-1}(P) \|_{\Gamma_j^{\mathrm{r}}} \leqslant q_j^{\omega,\mathrm{r}} \max[\| \chi_j^{n-1}(P) \|_{\gamma_j^{\mathrm{l}}}; \| \chi_j^{n-1}(P) \|_{\gamma_j^{\mathrm{r}}}] \\
&= q_j^{\omega,\mathrm{r}} \max[\| \zeta_j^{n-1}(P) \|_{\gamma_j^{\mathrm{l}}}; \| \zeta_{j+1}^{n-1}(P) \|_{\gamma_j^{\mathrm{r}}}] \\
&\leqslant q_j^{\omega,\mathrm{r}} \max\Big\{q_j^{\mathrm{l}} \max[\| \zeta_j^{n-1}(P) \|_{\Gamma_j^{\mathrm{l}}}; \| \zeta_j^{n-1}(P) \|_{\Gamma_j^{\mathrm{r}}}]; \\
&\qquad q_{j+1}^{\mathrm{r}} \max[\| \zeta_{j+1}^{n-1}(P) \|_{\Gamma_{j+1}^{\mathrm{l}}}; \| \zeta_{j+1}^{n-1}(P) \|_{\Gamma_{j+1}^{\mathrm{r}}}]\Big\} \\
&\leqslant q_j^{\omega,\mathrm{r}} \max(q_j^{\mathrm{l}}, q_{j+1}^{\mathrm{r}})\delta^{n-1}.
\end{aligned}$$

Analogously, the following can be obtained:

$$\| \zeta_j^n(P) \|_{\Gamma_j^{\mathrm{l}}} \leqslant q_{j-1}^{\omega,\mathrm{l}} \max(q_{j-1}^{\mathrm{l}}, q_j^{\mathrm{r}})\delta^{n-1}.$$

Hence, we have

$$\| \zeta_j^n(P) \|_{\Gamma_j} \leqslant \max[q_{j-1}^{\omega,\mathrm{l}} \max(q_{j-1}^{\mathrm{l}}, q_j^{\mathrm{r}}); q_j^{\omega,\mathrm{r}} \max(q_j^{\mathrm{l}}, q_{j+1}^{\mathrm{r}})]\delta^{n-1}.$$

Thus, we obtain $\delta^n \leqslant q\delta^{n-1}$, $n \geqslant 2$, where q is determined by (3.5). This proves the theorem.

3.2.1. Estimates on the rates of convergence in the multidomain decomposition case

As in Section 2.2.3, we estimate the rates of convergence for algorithms (3.1), (3.2) and (3.1), (3.3), considering the linear problem (2.13).

Theorem 7. *In the case of problem* (2.13), *under the assumptions of Theorem 5, the rate of convergence q for the Schwarz alternating procedure* (3.1), (3.2) *has estimate* (3.4) *with*

$$q_j^{\mathrm{l}} = \varphi_{\Omega_j^{\mathrm{r}}}^{\mathrm{I}}(x_j^{\mathrm{l}}) + \varphi_{\Omega_j^{\mathrm{r}}}^{\mathrm{II}}(x_j^{\mathrm{l}}), \qquad q_j^{\mathrm{r}} = \varphi_{\Omega_j^{\mathrm{r}}}^{\mathrm{I}}(x_{j-1}^{\mathrm{r}}) + \varphi_{\Omega_j^{\mathrm{r}}}^{\mathrm{II}}(x_{j-1}^{\mathrm{r}}),$$

where $\varphi^{\mathrm{I}}_{\Omega^y_j}(x)$, $\varphi^{\mathrm{II}}_{\Omega^y_j}(x)$ *are determined by* (2.6) *with the coefficient* $\tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2$ *instead of* β_0^2 *and* $\Omega^x_j = \{x : x^{\mathrm{l}}_{j-1} < x < x^{\mathrm{r}}_j\}$.

Proof. This theorem can be proved by the same approach as in Theorem 3, using the method of separation of variables.

Theorem 8. *In the case of problem* (2.13), *under the assumptions of Theorem 6, the rate of convergence q for algorithm* (3.1), (3.3) *has estimate* (3.5) *with*

$$q^{\mathrm{l}}_j = \varphi^{\mathrm{I}}_{\Omega^y_j}(X^{\mathrm{l}}_j) + \varphi^{\mathrm{II}}_{\Omega^y_j}(X^{\mathrm{l}}_j), \qquad q^{\mathrm{r}}_j = \varphi^{\mathrm{I}}_{\Omega^y_j}(X^{\mathrm{r}}_{j-1}) + \varphi^{\mathrm{II}}_{\Omega^y_j}(X^{\mathrm{r}}_{j-1}),$$

$$q^{\omega,\mathrm{l}}_j = \varphi^{\mathrm{I}}_{\omega^y_j}(x^{\mathrm{l}}_j) + \varphi^{\mathrm{II}}_{\omega^y_j}(x^{\mathrm{l}}_j), \qquad q^{\omega,\mathrm{r}}_j = \varphi^{\mathrm{I}}_{\omega^y_j}(x^{\mathrm{r}}_j) + \varphi^{\mathrm{II}}_{\omega^y_j}(x^{\mathrm{r}}_j),$$

where $\varphi^{\mathrm{I}}_{\Omega^y_j}(x)$, $\varphi^{\mathrm{II}}_{\Omega^y_j}(x)$ *and* $\varphi^{\mathrm{I}}_{\omega^y_j}(x)$, $\varphi^{\mathrm{II}}_{\omega^y_j}(x)$ *are determined by* (2.6) *with the coefficient* $\tilde{\beta}_1^2 = \beta_0^2 + (\pi/y_*)^2$ *instead of* β_0^2 *and*

$$\Omega^x_j = \{x : x^{\mathrm{l}}_{j-1} < x < x^{\mathrm{r}}_j\}, \qquad \omega^x_j = \{x : X^{\mathrm{l}}_j < x < X^{\mathrm{r}}_j\}.$$

Proof. Analogous to the proof of Theorem 4.

3.3. *A simplified analysis of parallel performance for algorithms* (3.1), (3.2) *and* (3.1), (3.3)

Here we present a simplified parallel analysis for algorithms (3.1), (3.2) – A1 and (3.1), (3.3) – A2, focusing only on computer architecture-independent factors. For simplicity, we consider the linear problem (2.13), where $\Omega_0 = (0,1) \times (0,1)$.

We assume that the communication, synchronization and load-balancing costs can be ignored and that each subdomain is undivided when mapped onto processors.

Introduce the uniform decomposition of the original domain Ω_0 into the subdomains Ω_j, $j = 1,2,\dots,M$ and the interfacial subdomains ω_j, $j = 1,2,\dots,M-1$:

$$\Omega_j = ((j-1)D, jD) \times (0,1), \quad D = (1-d)/M + d, \quad j = 1,2,\dots,M,$$

$$d = x^{\mathrm{r}}_j - x^{\mathrm{l}}_j, \quad j = 1,2,\dots,M-1;$$

$$\omega_j = (jD - d - H, jD + H) \times (0,1),$$

$$d_{\mathrm{inf}} = X^{\mathrm{r}}_j - X^{\mathrm{l}}_j = d + 2H, \quad j = 1,2,\dots,M-1,$$

where d, d_{inf} are the sizes of the uniform overlapping subdomains and of the interfacial subdomains in x-direction, respectively.

We also suppose that the elliptic operator from (2.13) is discretized by the usual five-point difference scheme and that the mesh on Ω_0 is a square of side h. Let N_Ω, N_ω denote a number of gridpoints in $\Omega_j, j = 1,2,\dots,M$ and in $\omega_j, j = 1,2,\dots,M-1$, respectively.

$K_{\mathrm{A1}}, K_{\mathrm{A2}}$ are the minimum number of iterations for A1, A2 to achieve an error of ε, i.e.

$$e^K \equiv \max \| e^K_j \|_{\Gamma_j} \leqslant \varepsilon, \qquad e^n_j = V^n_j(P) - V^{n-1}_j(P), \quad P \in \{\text{gridpoints}\},$$

where $V_j^n(P)$ is the solution of the difference scheme on the subdomain $\Omega_j, j = 1,2,\dots,M$. We assume here that the subdomain interfaces $\Gamma_j, j = 1,2,\dots,M-1$ belong to the mesh-lines in y-direction.

The execution times T_{A1}, T_{A2} for algorithms A1, A2 can be defined by

$$T_{AI} = K_{AI} t_{AI}, \quad I = 1,2,$$

where t_{A1}, t_{A2} are unit time (or a number of arithmetic operations) on each iteration step for A1 and A2, respectively. t_{A1}, t_{A2} depend not only on how many unknowns N_Ω (and on N_ω in the case of A2) each subdomain has but also on the method used to solve the discretization problem.

Let us suppose that the number of processors equals to the numbers of subdomains.

In the case of algorithm A1, we assume that each problem from (3.1) is solved by one processor (in other words, each subdomain is mapped onto one processor). The unit time t_{A1} is determined by

$$t_{A1} = t_{A1}^{\text{even}} + t_{A1}^{\text{odd}},$$

$$t_{A1}^{\text{even}} = \max_{1 \leqslant j \leqslant [M/2]} \{t_{2j}\}, \qquad t_{A1}^{\text{odd}} = \max_{0 \leqslant j \leqslant [(M+1)/2]+1} \{t_{2j+1}\},$$

where t_j is the unit time on Ω_j. Since the decomposition of the original domain Ω_0 is uniform, then it follows that

$$t_{A1} = 2t_I, \quad t_j = t_I, \quad j = 1,2,\dots, M.$$

In the case of algorithm A2, we assume that each processor is loaded by one problem from (3.1) and by one interfacial problem from (3.3) (one of them is loaded only by one problem from (3.1)). This assumption will be realistic, if the cost of the interfacial problem from (3.3) come to a small part of the cost of the problem from (3.1). Analogously, t_{A2} is determined by

$$t_{A2} = t_{A2}^{\Omega} + t_{A2}^{\omega}, \qquad t_{A2}^{\Omega} = \max_{1 \leqslant j \leqslant M} \{t_j\}, \qquad t_{A2}^{\omega} = \max_{1 \leqslant j \leqslant M-1} \{t_j^{\omega}\},$$

where t_j^{ω} is the unit time on ω_j and hence,

$$t_{A2} = t_I + \tau_I, \qquad t_j^{\omega} = \tau_I, \quad j = 1,2,\dots,M-1.$$

The times t_I and τ_I depend on N_Ω and N_ω, respectively, and on the kind of solver being used, i.e. $t_I = F(N_\Omega), \tau_I = F(N_\omega)$. Consequently, we have

$$T_{A1} = 2K_{A1} F(N_\Omega), \qquad T_{A2} = K_{A2}[F(N_\Omega) + F(N_\omega)],$$

$$N_\Omega = D/h^2 = [(1-d)/M + d]/h^2, \qquad N_\omega = d_{\inf}/h^2 = [(d+2H)/M]/h^2.$$

If we use the solvers ADI, SOR, SLOR, SSOR for problems like (2.13), then the computational work is approximately equal $F(N) = O(N^{3/2})$, where N is a number of interior nodes.

If the stepsize h is sufficiently small, then the sequences of the solutions $\{V_j^n(P)\}, j = 1,2,\dots,M$ for algorithms A1 and A2 have the same rate of convergence q as in the continuous case. Consequently, we can evaluate K_{A1}, K_{A2} using the following relationship:

$$e^K \leqslant (Q)^K e^1 \leqslant \varepsilon, \qquad e^n \equiv \max \| e_j^n \|_{\Gamma_j},$$

Table 1
Execution times $T_{A1}, T_{A2}^{1/2}, T_{A2}^{1}$ for $\beta_0 = 1$

$d \setminus M$	3	4	5	6	7	8	9	10
	6.295	5.399	4.877	4.529	4.282	4.096	3.953	3.839
2×10^{-2}	4.329	3.789	3.494	3.315	3.201	3.129	3.084	3.059
	3.326	2.959	2.772	2.671	2.618	2.596	2.595*	2.608
	3.391	2.990	2.769	2.634	2.547	2.489	2.451	2.427
4×10^{-2}	2.466	2.276	2.204	2.188*	2.203	2.238	2.285	2.342
	1.974	1.883	1.878*	1.913	1.971	2.044	—	—
	2.434	2.201	2.085	2.024	1.994	—	—	—
6×10^{-2}	1.884	1.825*	1.842	1.894	1.964	—	—	—
	1.577	1.596	1.669	—	—	—	—	—
	1.963	1.817	1.757	—	—	—	—	—
8×10^{-2}	1.624	1.639	1.710	—	—	—	—	—
	1.419	—	—	—	—	—	—	—

Table 2
Execution times $(\times 10^{-1}) T_{A1}, T_{A2}^{1/2}, T_{A2}^{1}$ for $\beta_0 = 100$

$d \setminus M$	3	4	5	6	7	8	9	10
	1.020	0.6819	0.5017	0.3923	0.3198	0.2688	0.2312	0.2026
2×10^{-2}	0.5448	0.3709	0.2784	0.2217	0.1849	0.1587	0.1394	0.1248
	0.3722	0.2571	0.1958	0.1585	0.1339	0.1165	0.1039	0.0943
	0.5397	0.3702	0.2792	0.2235	0.1863	0.1599	0.1404	0.1254
4×10^{-2}	0.2988	0.2139	0.1683	0.1403	0.1217	0.1087	0.0995	0.0932
	0.2149	0.1583	0.1279	0.1095	0.0982	0.0935	—	—
	0.3800	0.2669	0.2057	0.1680	0.1427	—	—	—
6×10^{-2}	0.2246	0.1681	0.1375	0.1189	0.1079	—	—	—
	0.1691	0.1314	0.1133	—	—	—	—	—
	0.3004	0.2156	0.1695	—	—	—	—	—
8×10^{-2}	0.1902	0.1478	0.1254	—	—	—	—	—
	0.1491	—	—	—	—	—	—	—

where the coefficient Q is determined by Theorem 7 for A1 and by Theorem 8 for A2. Hence, we have

$$K = \left[\left| \frac{\ln(\varepsilon/e^1)}{\ln(Q)} \right| \right] + 1.$$

Tables 1, 2 present the results for $\beta_0 = 1$ and 10^2, respectively, where β_0 is from (2.13). In the tables, we give the execution times for the case $\varepsilon/e^1 = 10^{-5}$ and $F(N)$=const.$\times N^{3/2}$, const.=1, for various values of d and M. The tables contain the values of T_{A1} and T_{A2} $(T_{A2}^{1/2}, T_{A2}^{1})$ for the cases $H = 0.5d$ and $H = d$, $(d_{\text{inf}} = d + 2H)$. Absolute values of the execution times are not important themselves. In the tables, the sign — denotes that one of the relationships $\omega_j \cap \omega_{j+1} = \emptyset$, $j = 1, 2, \ldots, M-2$ is not fulfilled.

It should be noted that these results indicate that for $\beta_0 = 1$ (Table 1) and for small values d, the execution time T_{A2} has the minimum value (* indicates this value). However, in the case of large values d, $T_{A2} = T_{A2}(M)$ is a monotonically increasing function. $T_{A1}(M)$ decreases monotonically for all values d and H.

If the coefficient β_0 from (2.13) is sufficiently large (see Table 2), then $T_{A1}(M)$ and $T_{A2}(M)$ decrease monotonically for all values d and H. Indeed, for d and H fixed, the coefficients Q_{A1}, Q_{A2} (consequently, the numbers of iterations K_{A1}, K_{A2}) are independent of the number of processors M and approximately equal

$$Q_{A1} \cong \exp(-\tilde{\beta}_1 d), \qquad Q_{A2} \cong \exp[-\tilde{\beta}_1(d+2H)].$$

These relationships follow from Theorems 7, 8 and from the uniform decomposition of the original domain Ω_0. Thus, for d and H fixed, K_{A1}, K_{A2} are independent of M, but $F(N_\Omega)$ decreases with M. Hence, $T_{A1}(M)$ and $T_{A2}(M)$ decrease monotonically with M.

It is worthy to note here that convergence properties of algorithms A1 and A2 applied to the singularly perturbed problem

$$\mu^2 \Delta u(P) = f[P, u(P)], \quad P \in \Omega_0; \quad u(P) = 0, \quad P \in \partial\Omega_0,$$

where μ is a small parameter, are analogous to the results mentioned in the case of a large value of the coefficient β_0 from problem (2.13).

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