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Properties of Weak Solutions of Generalized Radial Transport Equations

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INTRODUCTION

This paper is concerned with singular integro-differential equations of a form including time-dependent transport equations with spherical symmetry. Weak solutions that have arbitrary initial values and satisfy suitable homogeneous boundary conditions are studied with the following principal results:

1. The weak solutions depend on their initial data continuously (and, therefore, uniquely).

2. The first derivatives of a weak solution, under certain conditions, can be estimated a priori from the first derivatives of the initial data. Such a weak solution actually is a solution of the problem almost everywhere.

3. When the quantities that enter the integro-differential equation satisfy certain conditions of positivity, the weak solutions of the equation are ordered like their initial data.

The existence of weak solutions will be discussed in a subsequent paper, now being prepared, dealing principally with finite difference schemes for calculation.

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1. STATEMENT OF PROBLEM. NOTATION AND DEFINITIONS

Let X_0 denote a fixed positive constant. For $0 < x \leqslant X_0$, $|y| \leqslant 1$, and $t \geq 0$, we shall be concerned with the solutions $u(x, y, t)$ of mixed initial- and boundary-value problems for integro-differential equations of the form

$$u_t + yu_x + \frac{1 - y^2}{x} u_y + c(x, y, t)\, u = g(x, y, t) + Su, \qquad (1.1)$$

where

$$Su(x, y, t) \equiv \int_{-1}^{1} K(x, y, t, y')\, u(x, y', t)\, dy'.$$

The initial conditions considered are of the form

$$u(x, y, 0) = \phi(x, y), \tag{1.2}$$

while the boundary condition is

$$u(X_0, y, t) = 0 \qquad \text{when} \qquad -1 \leqslant y < 0. \tag{1.3}$$

Such problems occur, for instance in the theory of neutron transport with spherical symmetry. Equation (1.1) usually appears in this context in the notation

$$\frac{1}{v}\phi_t + \mu\phi_r + \frac{1-\mu^2}{r}\phi_\mu + \sigma(r)\phi = g + S\phi,$$

where t denotes time, r radius, μ the cosine of the angle between the radius vector and the velocity vector, v the particle speed, $\phi(t, r, \mu)$ the particle density, σ a total cross section for particle loss, and $g + S\phi$ sources. The boundary condition (1.3) here requires that $\phi(t, R, \mu) = 0$ for $\mu < 0$ and means physically that no particle enters the sphere $0 \leqslant r \leqslant R$ from outside $(r > R)$.

The remainder of the paper is written in the notation introduced first. We shall restrict our considerations to a fixed parallelepiped

$$S_T: \qquad 0 < x \leqslant X_0, \qquad |y| \leqslant 1, \qquad 0 \leqslant t \leqslant T,$$

with arbitrary positive T, on which c and g are assumed to be defined and u will be studied. K is assumed to be given on a corresponding four-dimensional parallelepiped

$$\Sigma_T: \qquad 0 < x \leqslant X_0, \qquad |y| \leqslant 1, \qquad 0 \leqslant t \leqslant T, \qquad |y'| \leqslant 1,$$

and ϕ on the two dimensional base,

$$S_0: \qquad 0 < x \leqslant X_0, \qquad |y| \leqslant 1,$$

of S_T. The point set N_0 consisting of the planes $y = 1$ and $y = -1$ and of the line segment

$$x = X_0, \qquad y = 0, \qquad 0 \leqslant t \leqslant T,$$

will turn out to be singular. Hence, we frequently shall restrict u to the domain

$$S_{T,0} = S_T - N_0.$$

Our minimal assumptions, except in Section 5, are as follows:

(i) c is bounded and measurable in S_T.

(ii) g is bounded and measurable in S_T.

(iii) (a) $K(x, y, t, y')$ is integrable over Σ_T.

(b) For each x, y, t $K(x, y, t, y')$ is integrable with respect to y', and

$$\int_{-1}^{1} | K(x, y, t, y') | \, dy' \leqslant k_0 ,$$

where k_0 is a constant independent of x, y, t.

(iv) ϕ is bounded and measurable over S_0.

When a function $f(x, y, t)$, say continuous in a domain S, is absolutely continuous with respect to x when y and t are fixed, absolutely continuous with respect to y when x and t are fixed, and absolutely continuous with respect to t when x and y are held fixed, we shall say more briefly that $f(x, y, t)$ is absolutely continuous in S with respect to x, y, and t. For such a function, the first partial derivatives with respect to x, y, and t exist at almost all points of S and, moreover, are measurable (in the three-dimensional sense) on S.

DEFINITION 1. *A bounded function $u(x, y, t)$, absolutely continuous with respect to x, y, t* in $S_{T,0}$, *is a "solution almost everywhere" of (1.1)-(1.3) if (1.2) and (1.3) hold strictly and (1.1) holds at almost all points of S_T.*

Let W denote the class of continuous, piecewise differentiable functions $w(x, y, t)$ with support, for some positive δ, in the region

$$\begin{aligned} &0 \leqslant t \leqslant T - \delta, && \\ &\delta \leqslant x \leqslant X_0 - \delta && \text{when} \quad y \geqslant 0, \\ &\delta \leqslant x \leqslant X_0 && \text{when} \quad y < 0, \\ &| y | \leqslant 1. && \end{aligned}$$

DEFINITION 2. *A bounded, measurable function $u(x, y, t)$ is a "weak solution" of (1.1)-(1.3) if, for any function w belonging to W,*

$$\int_{t>0} \left\{ u \left(w_t + y w_x + \left(\frac{1 - y^2}{x} w \right)_y \right) + w(- cu + g + Su) \right\} dxdydt$$
$$+ \int w(x, y, 0) \, \phi(x, y) \, dxdy = 0. \tag{1.4}$$

The function u is called a weak solution of (1.1) if (1.4) holds at least for such w

in W as also vanish for $|y| \geqslant 1 - \delta$, $0 \leqslant t \leqslant \delta$, $X_0 - \delta \leqslant x \leqslant X_0$ *with positive* δ. *(In this case, the two-dimensional integral in (1.4) drops out.)*

Weak solutions will be alternatively characterized (Section 2) in terms of integrals over characteristic curves. A weak solution, if absolutely continuous in $S_{T,0}$, will be seen to be a solution almost everywhere, and conversely.

2. Characteristic Curves

A characteristic curve is a curve $\mathscr{C}$: $x = x(t)$, $y = y(t)$ such that

$$dx/dt = y, \qquad dy/dt = (1 - y^2)/x; \tag{2.1}$$

differentiation in the direction of the tangent to $\mathscr{C}$ thus is given by the operator

$$\frac{d}{dt} = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + \frac{1 - y^2}{x}\frac{\partial}{\partial y}. \tag{2.2}$$

One and but one characteristic, $\mathscr{C}_{x_1,y_1,t_1}$ or $\mathscr{C}_{P_1}$, passes through each point $P_1 = (x_1, y_1, t_1)$ for which $x_1 > 0$, $|y_1| \leqslant 1$. This characteristic can be represented as

$$x = X(t; P_1) \equiv [(t - t_1)^2 + 2x_1y_1(t - t_1) + x_1^2]^{1/2},$$
$$y = Y(t; P_1) \equiv (t - t_1 + x_1y_1)/X(t; P_1).$$

The directed segment of $\mathscr{C}_{P_1}$ from P_1 to another of its points, say $P_2 = (x_2, y_2, t_2)$, will be denoted by $\mathscr{C}(P_1, P_2)$. On $\mathscr{C}(P_1, P_2)$, clearly,

$$X(t; P_1) = X(t; P_2), \qquad Y(t; P_1) = Y(t; P_2).$$

For $x_1 > 0$, $|y_1| < 1$, the projection of $\mathscr{C}_{P_1}$ on the xy-plane is a U-shaped curve, the branch of

$$\mathscr{C}_{Q_1} : \frac{x_1^2(1 - y_1^2)}{x^2} + y^2 = 1$$

contained in the strip $x > 0$, $|y| < 1$ and opening towards large values of x; here, $Q_1 = (x_1, y_1)$. The latter strip, $x > 0$, $|y| < 1$, is simply covered by the $\mathscr{C}_{Q_1}$, which, as $x_1 \to 0$, as $y_1 \to 1$, or as $y_1 \to -1$, approach the strip's boundary, the "curve" made up of the three segments, $y = -1$, $x = 0$, and $y = 1$.

We note that

$$
\begin{aligned}
\frac{\partial X(t; P_1)}{\partial x_1} &= \frac{x_1 + y_1(t - t_1)}{X(t; P_1)}, \\
\frac{\partial X(t; P_1)}{\partial y_1} &= \frac{x_1(t - t_1)}{X(t; P_1)}, \\
\frac{\partial Y(t; P_1)}{\partial x_1} &= \frac{x_1(1 - y_1^2)(t_1 - t)}{X(t; P_1)^3}, \\
\frac{\partial Y(t; P_1)}{\partial y_1} &= \frac{x_1^2(x_1 + y_1(t - t_1))}{X(t; P_1)^3}.
\end{aligned}
\tag{2.3}
$$

Thus, we have, in particular,

$$\left| \frac{\partial X(t; P_1)}{\partial x_1} \right| \leqslant 1. \tag{2.4}$$

Furthermore, for $0 \leqslant t < t_1$ and $| y_1 | < 1$, we have

$$0 \leqslant \frac{\partial Y(t; P_1)}{\partial x_1} < X(t; P_1)^{-1}. \tag{2.5}$$

Indeed,

$$X \left(\frac{\partial Y}{\partial x_1} \right) = - \frac{(t - t_1)\, x_1(1 - y_1^2)}{(t - t_1)^2 + 2x_1 y_1(t - t_1) + x_1^2},$$

an expression that, after the substitution $y_1 = 1 - z$, $0 < z < 2$, becomes

$$\frac{(t_1 - t)\, x_1 z(2 - z)}{(t_1 - t - x_1)^2 + 2(t_1 - t)\, x_1 z},$$

which, under the stipulated conditions, certainly is less than 1. In addition, for any a, b,

$$\left| \int_a^b \frac{\partial Y(t; P_1)}{\partial x_1}\, dt \right| \leqslant 2; \tag{2.6}$$

this is because

$$\frac{\partial Y(t; P_1)}{\partial x_1} = \frac{\partial^2 X(t; P_1)}{\partial t\, \partial x_1}$$

by (2.1) and because of (2.4).

As above, let $P = (x, y, t)$ denote a variable point along the characteristic $\mathscr{C}_{P_1}$ passing through an arbitrary point $P_1 = (x_1, y_1, t_1)$ of S_T. The last such P belonging to S_T as t decreases from the value t_1 will be called the "foot" of the characteristic and will be denoted by the symbol $P^* = (x^*, y^*, t^*)$. Either $0 < x^* \leqslant X_0$ and

$$t^* = 0,$$

in which case

$$x^* = (t_1^2 - 2x_1y_1t_1 + x_1^2)^{1/2},$$
$$y^* = (x_1y_1 - t_1)/x^*,$$

or $0 < t^* \leqslant t_1$ and

$$x^* = X_0, \qquad y^* < 0,$$

so that, from the equation $X_0 = X(t^*; P_1)$,

$$t^* = t^{**}(P_1),$$
$$y^* = y^{**}(P_1),$$

where

$$t^{**}(P_1) \equiv t_1 - x_1y_1 - \sqrt{X_0^2 - (1 - y_1^2)\, x_1^2},$$

and

$$y^{**}(P_1) = -\sqrt{X_0^2 - (1 - y_1^2)\, x_1^2}/X_0 .$$

Regarded as functions of P_1, x^*, y^*, and t^* thus are continuous with piecewise continuous derivatives such that, in the domain in which $t^* = 0$,

$$\frac{\partial t^*}{\partial t_1} = 0, \qquad \frac{\partial x^*}{\partial t_1} = -y^*, \qquad \frac{\partial y^*}{\partial t_1} = -\frac{1 - y^{*2}}{x^*},$$
$$\frac{\partial t^*}{\partial x_1} = 0, \qquad \frac{\partial t^*}{\partial y_1} = 0$$

and, in the domain in which $t^* > 0$,

$$\frac{\partial t^*}{\partial t_1} = 1, \qquad \frac{\partial x^*}{\partial t_1} = 0, \qquad \frac{\partial y^*}{\partial t_1} = 0,$$
$$\frac{\partial t^*}{\partial x_1} = -y_1 - \frac{(1 - y_1^2)\, x_1}{X_0 y^{**}}, \qquad \frac{\partial t^*}{\partial y_1} = -x_1 + \frac{x_1^2 y_1}{X_0 y^{**}} .$$

Hence, x^* and t^* are Lipschitz continuous with respect to t_1 with Lipschitz constants equal to 1. Likewise, y^* is Lipschitz-continuous with respect to t_1, but nonuniformly. Since

$$Y_{t_1}(t; P_1) = -Y_t(t; P_1),$$

we have from (2.1) that

$$Y_{t_1}(0; P_1) = -\frac{1 - Y(0; P_1)^2}{X(0; P_1)};$$

hence, for $A < B$

$$| Y(0; x_1, y_1, B) - Y(0; x_1, y_1, A) | = \int_A^B \frac{1 - Y(0; x_1, y_1, s)^2}{X(0; x_1, y_1, s)} ds$$

$$< \frac{B - A}{\min_{A \leqslant s \leqslant B} X(0; x_1, y_1, s)}.$$

Consequently, on any segment consisting of points $P_1 = (x_1, y_1, t_1)$ such that $x_1 = \text{constant}$, $y_1 = \text{constant}$, and $A \leqslant t_1 \leqslant B$, we see that $y^*(P_1)$ is Lipschitz-continuous with respect to t_1 with Lipschitz constant

$$1/\min_{A \leqslant s \leqslant B} X(0; x_1, y_1, s).$$

We also note that, for $h > 0$,

$$0 \leqslant t^*(x_1, y_1, t_1 + h) - t^*(x_1, y_1, t_1) \leqslant h. \tag{2.7}$$

When

$$0 < x_1 < x_2 \leqslant X_0, \qquad X_0 - x_2 + | y_2 | > 0,$$

we have from the expressions for $\partial t^*/\partial x_1$ and y^{**} that

$$| t^*(x_2, y, t) - t^*(x_1, y, t) | < \frac{2(x_2 - x_1)}{| y^{**}(x_2, y, t) |}. \tag{2.8}$$

3. Reduction of Problem to an Integral Equation Alternative Definition of Weak Solution

Under hypotheses (i)-(iv) of Section 1, our problem is reduced to an integral equation by the traditional means of integrating 1.1 along $\mathscr{C}(P^*, P_1)$. To this end, for any function $f(x, y, t)$ define the line integral

$$\int_{\mathscr{C}(P^*, P_1)} f dt \equiv \int_{t^*}^{t_1} f(X(t; P^*), Y(t; P^*), t)\, dt.$$

Evidently, for any sufficiently smooth, say continuously differentiable, function v,

$$\int_{\mathscr{C}(P^*, P_1)} \left(v_t + y v_x + \frac{(1 - y^2)}{x} v_y \right) dt = v(P_1) - v(P^*).$$

Hence, if u in particular were sufficiently smooth, then for almost every P^* we would have by the integration of (1.1) over $\mathscr{C}(P^*, P_1)$ the integral relation

$$u(P_1) = u(P^*) + \int_{\mathscr{C}(P^*, P_1)} (-cu + g + Su)\, dt. \tag{3.1}$$

Below, every weak solution of (1.1)-(1.3) will be seen, for almost every P^*, to satisfy (3.1) on $\mathscr{C}_{P^*}$ with

$$\begin{aligned} u(P^*) \equiv u(x^*, y^*, t^*) &= \phi(x^*, y^*) \qquad \text{when} \qquad t^* = 0 \\ &= 0 \qquad \text{when} \qquad t^* > 0. \end{aligned} \tag{3.2}$$

Conversely, a function satisfying these conditions will be seen to be a weak solution of (1.1)-(1.3).

A function u satisfying (3.1) on a particular characteristic $\mathscr{C} = \mathscr{C}_{P^*}$ obviously is absolutely continuous along $\mathscr{C}$ and, at almost all points of $\mathscr{C}$, satisfies the differential condition

$$\left(\frac{du}{dt}\right)_{\mathscr{C}} = -cu + g + Su,$$

the left member of the last equation denoting the limit of the quotient $(u(P') - u(P))/(t' - t)$ as $P' = (x', y', t')$ tends to $P = (x, y, t)$ along $\mathscr{C}$. Multiplying this differential condition by any continuously differentiable function α and integrating the two members over an arbitrary segment of $\mathscr{C}$, we obtain a family of equivalent integral relations of the form

$$\alpha(P_1)\, u(P_1) = \alpha(P_2)\, u(P_2) + \int_{\mathscr{C}(P_2, P_1)} \left\{\left[\left(\frac{d\alpha}{dt}\right)_{\mathscr{C}} - \alpha c\right] u + \alpha g + \alpha S u\right\} dt. \tag{3.3}$$

We shall now prove that a weak solution of (1.1)-(1.3), if absolutely continuous with respect to x, y, t in $S_{T,0}$, is a solution almost everywhere in S_T, and conversely. Then we shall show that, as previously remarked, a function u is a weak solution of (1.1)-(1.3) if and only if u, for almost all P^*, satisfies (3.1) on $\mathscr{C}_{P^*}$ with $u(P^*)$ interpreted as in (3.2). Four theorems are formulated.

THEOREM 3.1. *If u is bounded and absolutely continuous with respect to x, y, t in $S_{T,0}$ and is a solution of (1.1)-(1.3) almost everywhere, then u is a weak solution of (1.1)-(1.3) in S_T.*

Proof. Multiply Eq. (1.1) by an arbitrary element w of W, integrate with respect to x, y, t over S_T, and integrate by parts to remove the differentiations of u. The result is relation (1.4).

THEOREM 3.2. *A weak solution of (1.1)-(1.3), if bounded and absolutely continuous with respect to* x, y, t *in* $S_{T,0}$, *is a solution of (1.1)-(1.3) almost everywhere in* S_T.

Proof. Integration by parts in (1.4) shows that

$$-\int_{t>0} w\left(u_t + yu_x + \frac{1-y^2}{x}u_y + cu - g - Su\right) dxdydt$$

$$+\int w(x, y, 0)\,(\phi(x, y) - u(x, y, 0))\,dxdy$$

$$+\int_{y<0} yw(X_0, y, t)\,u(X_0, y, t)\,dydt = 0.$$

Because of the arbitrariness of w, Eq. (1.1) holds almost everywhere, $u(x, y, 0) = \phi(x, y)$, and $u(X_0, y, t) = 0$ for $y < 0$. All the conditions thus are satisfied that u be a solution of (1.1)-(1.3) almost everywhere, as asserted.

THEOREM 3.3. *Suppose* u, *for almost all* P^*, *satisfies (3.1) on* $\mathscr{C}_{P^*}$, $u(P^*)$ *being defined as in (3.2). Then* u *is a weak solution of (1.1)-(1.3).*

Proof. The functions

$$\xi = x\sqrt{1 - y^2} \tag{3.4a}$$

and

$$\eta = xy - t \tag{3.4b}$$

are integrals of the differential equations (2.1): hence, the characteristic curves are described by simultaneous conditions of the form $\xi =$ constant, $\eta =$ constant. For this reason, we here consider new coordinates (ξ, η, τ), where

$$\tau = t. \tag{3.4c}$$

The inverse of the transformation (3.4a, b, c) being given by

$$x = \sqrt{\xi^2 + (\tau + \eta)^2},$$

$$y = (\eta + \tau)/\sqrt{\xi^2 + (\tau + \eta)^2},$$

$$t = \tau, \tag{3.5}$$

we easily verify that

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + \frac{1-y^2}{x}\frac{\partial}{\partial y}; \tag{3.6}$$

also

$$J = \frac{\partial(x, y, t)}{\partial(\xi, \eta, \tau)} = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\sqrt{1 - y^2}}{x}.$$

With $H = Jw$, we now multiply (3.1) by $H_{\tau_1}(P_1)$ and integrate with respect to ξ_1, η_1, τ_1 to obtain

$$\int H_{\tau_1}(P_1) \left[u(P_1) - u(P^*) - \int_{\mathscr{C}(P^*, P_1)} (-cu + g + Su)\, dt\right] d\xi_1 d\eta_1 d\tau_1 = 0. \tag{3.7}$$

Here, $H(P_1)$ is regarded as a function of ξ_1, η_1, τ_1, the ξ, η, τ that correspond to x_1, y_1, t_1. Since P^* is constant for constant ξ_1, η_1,

$$\begin{aligned} \int H_{\tau_1}(P_1)\, u(P^*)\, d\xi_1 d\eta_1 d\tau_1 &= -\int H(\xi_1, \eta_1, 0)\, u(P^*)\, d\xi_1 d\eta_1 \\ &= -\int w(x, y, 0)\, \phi(x, y)\, dxdy. \end{aligned}$$

The line integral in (3.7) is transformed by integration by parts with respect to τ_1. We thereby obtain

$$\int [H_{\tau_1} u + H(-cu + g + Su)]\, d\xi_1 d\eta_1 d\tau_1 + \int w(x, y, 0)\, \phi(x, y)\, dxdy = 0, \tag{3.8}$$

or, after a coordinate change,

$$\int \left\{ uJ^{-1} \left[(Jw)_t + y(Jw)_x + \frac{1 - y^2}{x} (Jw)_y \right] + w(-cu + g + Su) \right\} dx_1 dy_1 dt_1 + \int_{t=0} w\phi dxdy = 0.$$

This, as a little further computation shows, is equivalent to (1.4). Hence, u is indeed a weak solution of the problem, as contended.

THEOREM 3.4. *If u is a weak solution of (1.1)-(1.3), then, for almost all P^*, u satisfies (3.1) on $\mathscr{C}_{P^*}$ with $u(P^*)$ defined as in (3.2).*

Proof. In (1.4), make the substitution $w = H/J$ and change variables as in (3.4) thereby reducing (1.4) to the form (3.8). Integration by parts transforms (3.8) to

$$\int H_{\tau_1}(P_1) \left[u(P_1) - \int_{\mathscr{C}(P^*, P_1)} (-cu + g + Su)\, dt\right] d\xi_1 d\eta_1 d\tau_1 + \int w(x, y, 0)\, \phi(x, y)\, dxdy = 0,$$

the line integral being defined for almost all P^*.

This we rewrite as

$$\int H_{\tau_1}(P_1)\, I(P_1\,, P^*)\, d\xi_1 d\eta_1 d\tau_1 = 0, \tag{3.9}$$

where

$$I(P_1\,, P^*) = u(P_1) - \phi(P^*) - \int_{\mathscr{C}(P^*, P_1)} (-\,cu + g + Su)\, dt,$$

and

$$\begin{aligned} \phi(P^*) &= \phi(x^*, y^*) \qquad \text{when} \qquad t^* = 0, \\ &= 0 \qquad\qquad\quad \text{when} \qquad t^* > 0. \end{aligned}$$

From (3.9) it is easily seen, because of the arbitrariness of H, that, for almost all points P^*, $I(P_1\,, P^*)$ is independent of τ_1. For almost all P^*, therefore, u is continuous on $\mathscr{C}_{P^*}$, and the value of $I(P_1\,, P^*)$ in (3.9) is $u(P^*) - \phi(P^*)$, the symbol $u(P^*)$ here indicating the (as yet unknown) limiting value of $u(P_1)$ as P_1 tends to P^* along $\mathscr{C}_{P^*}$. If, in (3.9), we integrate with respect to τ_1, we have, because H is zero for $t \geqslant T - \delta$,

$$\int H(P^*)\,(u(P^*) - \phi(P^*))\, d\xi^* d\eta^* = 0,$$

a relation that proves

$$u(P^*) = \phi(P^*) \qquad \text{for almost all} \qquad P^*. \tag{3.10}$$

For any P^* such that $I(P_1\,, P^*)$ is independent of τ_1, u then being continuous on $\mathscr{C}_{P^*}$ and satisfying (3.9), we now see that $I(P_1\,, P^*) = I(P^*, P^*) = 0$. Hence, (3.1) holds, as contended.

4. Unique, Continuous Dependence of Weak Solutions upon Initial Data

Under hypotheses (i)-(iv) of Section 1, weak solutions will be seen to depend boundedly and, hence, continuously upon their initial data. This is implied by the theorem below, devoted to an estimate in which, for convenience, c has been assumed to be nonnegative. (This nonnegativity is merely a normalization arising, for instance, as a result of a substitution $u = e^{\lambda t} v$ with sufficiently large λ.)

Theorem 4.1 (Boundedness). *Let hypotheses (i), (ii), (iii), (iv) hold with*

$$0 \leqslant c \leqslant c_0\,, \qquad |\,g\,| \leqslant g_0\,, \qquad |\,\phi\,| \leqslant \phi_0$$

in S_T or S_0, the indexed symbols being constants. Then a weak solution of (1.1)-(1.3), which is assumed to be bounded, satisfies the particular condition

$$| u(x, y, t) | \leqslant \phi_0 e^{(c_0+k_0)t} + \frac{g_0}{c_0 + k_0} (e^{(c_0+k_0)t} - 1) \tag{4.1}$$

at almost every point of S_T.

COROLLARY (Uniqueness). *Under hypotheses (i) and (iii), a bounded weak solution of (1.1)-(1.3) in $S_{T,0}$ is uniquely determined by the choices of ϕ and g: i.e., the solution is zero almost everywhere if $\phi \equiv 0$ and $g \equiv 0$.*

We call attention to an additional result concerning uniqueness given in Theorem 5.2 below.

Proof of Theorem 4.1. For fixed t, $0 \leqslant t \leqslant T$, let Π_t denote the set of points $P = (x, y, t)$ such that u satisfies 3.1 on $\mathscr{C}_P$. Then define

$$U(t) = \sup_{P \in \Pi_t} | u(P) | .$$

This function is continuous. In fact, by (3.1) a uniform constant C exists such that, for any point $P' = (x', y', t')$ belonging to $\mathscr{C}_P$,

$$| u(P') - u(P) | \leqslant C | t' - t | .$$

Hence,

$$| u(P') | \leqslant | u(P) | + C | t' - t | \leqslant U(t) + C | t' - t | ,$$

and thus

$$U(t') \leqslant U(t) + C | t' - t | .$$

Since, in the last inequality, t and t' may be interchanged, it is clear that $U(t)$ is continuous, as asserted.

Because the $\mathscr{C}_P$ for $P \in \Pi_t$ simply cover S_T except for a subset of measure zero, we have

$$| u(x, y, t) | \leqslant U(t)$$

for almost all x, y in S_0.

Now consider any characteristic $\mathscr{C}_{P_1}$ along which (3.1) is valid. From the foregoing,

$$\left| \int_{\mathscr{C}(P^*, P_1)} cu dt \right| \leqslant c_0 \int_0^{t_1} U(t)\, dt$$

and

$$\left| \int_{\mathscr{C}(P^*, P_1)} g dt \right| \leqslant g_0 t_1 .$$

Furthermore,

$$\left| \int_{\mathscr{C}(P^*,P_1)} Sudt \right| = \left| \int_{\mathscr{C}(P^*,P_1)} dt \int_{-1}^{1} K(x, y, t, y')\, u(x, y', t)\, dy' \right|$$

$$\leqslant \int_{\mathscr{C}(P^*,P_1)} U(t)\, dt \int_{-1}^{1} | K(x, y, t, y') |\, dy'$$

$$\leqslant k_0 \int_0^{t_1} U(t)\, dt.$$

Using these estimates in (3.1) gives

$$| u(P_1) | \leqslant \phi_0 + g_0 t_1 + (c_0 + k_0) \int_0^{t_1} U(t)\, dt.$$

The left member, because P_1 is arbitrary, can be replaced by $U(t_1)$, and the resulting relation implies

$$U(t) \leqslant \phi_0 e^{(c_0+k_0)t} + \frac{g_0}{c_0 + k_0} (e^{(c_0+k_0)t} - 1);$$

this in turn implies (4.1)

5. Positivity. Monotonic Dependence of Solution upon Data, Coefficient, Kernel, and Inhomogeneous Part of Equation

When the data, the coefficient, the kernel, and the inhomogeneous part of the equation are non-negative, a weak solution too will be non-negative. This is true within a broader framework of assumptions than that heretofore considered, assumptions (i), (ii), and (iv), in particular, here being replaceable by the following three hypotheses:

(i)$_0$ c is integrable in S_T,

(ii)$_0$ g is integrable in S_T,

(iv)$_0$ ϕ is integrable in S_0.

Theorem 5.1. *Assume hypotheses* (i)$_0$, (ii)$_0$, (iii), (iv)$_0$, *and also assume*

$$\phi \geqslant 0, \qquad c \geqslant 0, \qquad K \geqslant 0, \qquad g \geqslant 0.$$

If u is a weak solution of (1.1)-(1.3), then

$$u(x, y, t) \geqslant 0$$

at almost all points of S_T.

The remark preceding Theorem 4.1 shows that this result is true if c is merely bounded below.

This result enables us to assert the uniqueness of the solutions of some equations with unbounded c or K:

THEOREM 5.2. *If Hypotheses* (i)$_0$ *and* (iii) *and the conditions*

$$c \geqslant 0, \qquad K \geqslant 0$$

are satisfied, a bounded weak solution of (1.1)-(1.3) in $S_{T,0}$ *is uniquely determined by the choices of* ϕ *and* g*: i.e., the solution is zero almost everywhere if* $\phi \equiv 0$ *and* $g \equiv 0$.

Another significant consequence of Theorem 5.1 on positivity is that, under suitable conditions, u depends monotonically in the same sense on $-c, g, K$, and ϕ:

THEOREM 5.3. *Consider two problems of the form specified in (1.1)-(1.3), each satisfying hypotheses* (i)$_0$, (ii)$_0$, (iii), (iv)$_0$. *Distinguishing corresponding quantities in the two problems by the subscript 1 or 2, assume*

$$c_2 \geqslant c_1 \geqslant 0, \qquad K_1 \geqslant K_2 \geqslant 0$$

and

$$g_1 \geqslant g_2, \qquad \phi_1 \geqslant \phi_2, \qquad g_1 \geqslant 0, \qquad \phi_1 \geqslant 0;$$

denote by u_1 *and* u_2 *weak solutions of the respective problems. At almost all points of* S_T, *it is then true that*

$$u_1 \geqslant u_2 .$$

Theorem 5.3 is proved by applying Theorem 5.1 to an equation for $u_1 - u_2$ (see[*1*], pp. 15-16).

Proof of Theorem 5.1. Let us apply (3.3) with

$$\alpha(P) \equiv \alpha(x, y, t) = \exp(-2k_0\delta t),$$

where δ is an arbitrary number > 1. With this choice of α and the substitution $u = \exp(2k_0\delta t)\, v$, relation (3.3) becomes

$$v(P_1) = v(P_2) + \int_{\mathscr{C}(P_2, P_1)} [-(2k_0\delta + c)\, v + e^{-2k_0\delta t} g + Sv]\, dt; \qquad (5.1)$$

it is valid (Theorem 3.4) for almost any characteristic $\mathscr{C}_{P_1}$ and for such segments $\mathscr{C}(P_2, P_1)$ as are contained in S_T.

Let Π denote the set of points $P_1 = (x_1, y_1, t_1)$ of S_T such that (5.1) holds. This set Π is a union of characteristic segments differing from S_T by a set of measure zero. To prove Theorem 5.1, it suffices to prove that $v \geqslant 0$ in Π. Suppose that, to the contrary,

$$m \equiv \inf_{P \in \Pi} v(P) < 0. \tag{5.2}$$

Then there is a point $P' = (x', y', t')$ of Π with

$$0 < x' < X_0, \qquad |y'| < 1, \qquad 0 < t' \leqslant T$$

such that

$$m' \equiv v(P') < m\delta^{-1}. \tag{5.3}$$

Let $\mathscr{C}'$ denote the characteristic curve passing through P'. From (5.1), v is continuous along $\mathscr{C}'$. For the moment, also let $P_0 = (x_0, y_0, t_0)$ denote the point with least ordinate t_0 on $\mathscr{C}'$ such that $v(P_0) = m'$. We see $t_0 > 0$ because $\phi \geqslant 0$. We see $0 < x_0 < X_0$ by considering that, if $x_0 = X_0$, as t decreased a variable point (x, y, t) of $\mathscr{C}'$ would cross the plane $x = X_0$ at the point P_0 in the direction of increasing x: hence, $(dx/dt)_{P_0} = y_0 < 0$, and by 1.3 $u(P_0) = 0$, a contradiction. Thus, $x_0 < X_0$. On the other hand, $x_0 > 0$, since, in fact, $x > 0$ at all points of $\mathscr{C}'$, and we conclude that $0 < x_0 < X_0$, as asserted.

Now we identify P' with P_0. Then $t' > 0$, $0 < x' < X_0$, and for any point $P = (x, y, t)$ on $\mathscr{C}'$,

$$v(P) > m' \qquad \text{if} \qquad t^* \leqslant t < t', \tag{5.4}$$

where t^* is the first value of t less than t' at which $\mathscr{C}'$ intersects either the initial plane $t = 0$ or the boundary $x = X_0$.

Since $K \geqslant 0$, by (5.3) we see

$$Sv \equiv \int K(x, y, t, y')\, v(x, y', t)\, dy' \geqslant k_0 m \geqslant k_0 \delta m'. \tag{5.5}$$

Let $P'' = (x'', y'', t'')$ be a point of $\mathscr{C}'$ such that $t^* < t'' < t'$ and

$$2v(P) - m' < 0 \tag{5.6}$$

for any point $P = (x, y, t)$ between P' and P'' on $\mathscr{C}'$. Thus, in particular, $v < 0$ on $\mathscr{C}(P'', P')$, and since $c \geqslant 0$ and $g \geqslant 0$ as well, from (5.1) and (5.5) we have

$$v(P') \geqslant v(P'') + \int_{\mathscr{C}(P'', P')} k_0 \delta (m' - 2v)\, dt.$$

This and (5.6) imply $v(P') > v(P'')$, a statement that eontradicts (5.3) and (5.4) and thus contradicts (5.2). We conclude that $m = 0$, this being the contention of the theorem.

6. A Priori Estimates for the First Derivatives of Weak Solutions

This section is devoted to the a priori estimation, under specified hypotheses, of Lipschitz constants with respect to t, x, and y for a weak solution u of (1.1)-(1.3). The bounds obtained are not all uniform in S_T, but suffice to show u to be absolutely continuous with respect to x, y, t in $S_{T,0}$ and thus to be a solution of (1.1)-(1.3) almost everywhere (Theorem 6.1).

We have already obtained (Theorem 4.1) an a priori bound for $|u|$ valid under Hypotheses (i) to (iv). Our estimates of Lipschitz constants for u require appropriate additional assumptions concerning difference quotients of c, g, K, and ϕ. We indicate these additional assumptions below, prefixing those pertaining to t-differences by the label (t), those pertaining to x-differences by the label (x), etc. The symbols c_1, g_1, k_1, ϕ_1 denote constants dependent only on T.

(i)$_1$(t) (or (x)) c is Lipschitz continuous with respect to t (or x) with Lipschitz constant c_1,

(y) $x^{-1}c$ is Lipschitz continuous with respect to y with Lipschitz constant c_1,

(ii)$_1$(t) (x) (y) same assumptions as (i)$_1$ concerning g, with Lipschitz constant g_1,

(iii)$_1$(t) $$\int_{-1}^{1} | K(x, y, t + h, y') - K(x, y, t, y') | \, dy' \leqslant k_1 | h |,$$

(x) $$\int_{-1}^{1} | K(x_1, y, t, y') - K(x_2, y, t, y') | \, dy' \leqslant k_1 | x_1 - x_2 |,$$

(y) $$\int_{-1}^{1} | K(x, y_1, t, y') - K(x, y_2, t, y') | \, dy' \leqslant k_1 x | y_1 - y_2 |$$

(′) Constants k' and δ, $0 < \delta < 1$, exist such that

$$| K(x, y, t, y') | \leqslant k'$$

for $| y' | \leqslant \delta$. (This requirement might be replaced by such an integral condition as

$$\left(\int_{-\delta}^{\delta} \frac{| K(x, y, t, y') |}{\sqrt{X_0^2 - (1 - y'^2)\, x^2}} \, dy' \leqslant \text{constant.} \right)$$

$(\text{iv})_1\,(x)\,(y)$ same assumptions as $(\text{i})_1(x)\,(y)$ concerning ϕ, with Lipschitz constant ϕ_1.

$(\text{iv})_{X_0}$ $\phi(x_0, y) = 0$ for $-1 \leqslant y < 0$.

The estimates obtainable under these hypotheses (Theorems 6,2, 6.3, 6.4), in the light of Theorem 3.2, justify the following result:

THEOREM 6.1. *Under all the hypotheses* (i) *to* (iv) *(Section 1) and* $(\text{i})_1$ *to* $(\text{iv})_1$ *above, a bounded weak solution of (1.1)-(1.3) in* S_T *is absolutely continuous with respect to* x, y, t *in* $S_{T,0}$ *and, moreover, is a solution of (1.1)-(1.3) almost everywhere in* S_T.

We now turn to the first estimate, which is concerned with a Lipschitz constant for u with respect to t.

THEOREM 6.2. *Under conditions* (i) *to* (iv) *(Section 1) and* $(\text{i})_1(t)$, $(\text{ii})_1(t)$, $(\text{iii})_1(t)$, $(\text{iv})_1$, *a bounded weak solution of (1.1)-(1.3) in* S_T *satisfies a uniform Lipschitz condition with respect to* t.

Proof. We must study the behavior of the component terms of the right side of (3.1), and we begin with the first, $u(P^*)$.

As seen in Section 2, x^* and t^* satisfy Lipschitz conditions with respect to t_1 with Lipschitz constant 1, while y^*, regarded as a function of t_1 alone on any interval $A \leqslant t_1 \leqslant B$, satisfies a condition of the form

$$| y^*(B) - y^*(A) | \leqslant (B - A)/\tilde{x},$$

where

$$\tilde{x} = \min_{A \leqslant s \leqslant B} X(0; x_1, y_1, s).$$

These and similar considerations, together with Hypothesis $(\text{iv})_1$, enable us to prove that $\phi(P^*)$ is Lipschitz-continuous with respect to t_1. Regarding x^*, as well as y^*, as a function of t_1 in the interval $A \leqslant t_1 \leqslant B$, x_1 and y_1 being held fixed, we have, in fact,

$$\begin{aligned} | \phi(x^*(B), y^*(B)) - \phi(x^*(A), y^*(A)) | & \\ \leqslant | \phi(\tilde{x}, y^*(B)) - \phi(\tilde{x}, y^*(A)) | & \\ + | \phi(x^*(B), y^*(B)) - \phi(\tilde{x}, y^*(B)) | & \\ + | \phi(\tilde{x}, y^*(A)) - \phi(x^*(A), y^*(A)) | & \\ \leqslant \phi_1(B - A + | \tilde{x} - x^*(A) | + | \tilde{x} - x^*(B) |). & \end{aligned}$$

From the definition of $\tilde{x}$ and the Lipschitz continuity of x^*, we have, however, that $| \tilde{x} - x^*(A) | \leqslant B - A$ and also $| \tilde{x} - x^*(B) | \leqslant B - A$. It follows that $\phi(P^*)$ is Lipschitz-continuous with respect to t_1, as contended, and with Lipschitz constant $3\phi_1$. Since $u(P^*)$ is continuous, coincides with $\phi(P^*)$ in one part of its domain, and vanishes in the other, $u(P^*)$ is Lipschitz continuous with respect to t_1.

Let us now consider the integral

$$G(P_1) = \int_{\mathscr{C}(P_1^*, P_1)} g dt,$$

here denoting the foot of $\mathscr{C}_{P_1}$ by $P_1^* = (x_1^*, y_1^*, t_1^*)$. In the notation

$$Q(t; P_1) = (X(t; P_1), Y(t; P_1)), \tag{6.1}$$

we write this integral more explicitly as

$$G(P_1) = \int_{t_1^*}^{t_1} g(Q(t; P_1), t)\, dt.$$

With $h > 0$, consider the point $P_2 = (x_1, y_1, t_1 + h)$, and denote the foot of $\mathscr{C}_{P_2}$ by $P_2^* = (x_2^*, y_2^*, t_2^*)$. We have

$$G(P_2) - G(P_1) = \int_{t_2^*}^{t_1+h} g(Q(t; P_2), t)\, dt - \int_{t_1^*}^{t_1} g(Q(t; P_1), t)\, dt.$$

Since

$$Q(t + h; P_2) = Q(t; P_1),$$

$$G(P_2) - G(P_1) = \int_{t_2^*}^{t_1^*+h} g(Q(t; P_2), t)\, dt$$

$$+ \int_{t_1^*+h}^{t_1+h} [g(Q(t; P_2), t) - g(Q(t; P_2), t - h)]\, dt.$$

Hence, and because $t_1^* \leqslant t_2^* \leqslant t_1^* + h$ by (2.7), we have

$$| G(P_2) - G(P_1) | \leqslant (g_0 + g_1 T)\, h, \tag{6.2}$$

g_0 here again denoting a bound for $| g |$. Thus, G is Lipschitz continuous with respect to t_1.

For the integral

$$C(P_1) = \int_{\mathscr{C}(P_1^*, P_1)} cu dt,$$

analogous reasoning shows that

$$| C(P_2) - C(P_1) | \leqslant u_0(c_0 + c_1 T) h$$
$$+ c_0 \int_{t_1^*}^{t_1} | u(Q(t; P_1), t + h) - u(Q(t; P_1), t) | \, dt,$$

where c_0 is an upper bound for $| c |$ and u_0 an upper bound for $| u |$ in S_T. Hence, if

$$v_h(t) = \sup_{\substack{0 < x \leqslant X_0 \\ |y| \leqslant 1}} \frac{| u(x, y, t + h) - u(x, y, t) |}{h},$$

where $0 \leqslant t < t + h \leqslant T$, we have

$$| C(P_2) - C(P_1) | \leqslant u_0(c_0 + c_1 T) h + c_0 h \int_0^{t_1} v_h(t) \, dt. \tag{6.3}$$

Lastly, consider

$$T(P_1) = \int_{\mathscr{C}(P_1^*, P_1)} Su \, dt$$
$$= \int_{\mathscr{C}(P_1^*, P_1)} dt \int_{-1}^{1} K(x, y, t, y') \, u(x, y', t) \, dy'.$$

By means similar to those above, we readily deduce

$$| T(P_2) - T(P_1) | \leqslant (k_0 + k_1 T) u_0 h + k_0 h \int_0^{t_1} v_h(t) \, dt. \tag{6.4}$$

In view of the foregoing considerations, from (3.1) we immediately have

$$\frac{| u(P_2) - u(P_1) |}{h} \leqslant C_1 + C_2 \int_0^{t_1} v_h(t) \, dt,$$

where C_1 and C_2 are constants depending on T, and it is easily seen that the left member can be replaced by $v_h(t)$. A bound for $v_k(t)$ follows as from Gronwall's inequality and this bound is a Lipschitz constant for u with respect to t, as demanded.

Our next, and principal, aim is to establish the following result concerning the Lipschitz continuity of u with respect to x:

THEOREM 6.3. *Under hypotheses* (i) *to* (iv) *and* $(\mathrm{i})_1(x)\,(y)$, $(\mathrm{ii})_1(x)\,(y)$, $(\mathrm{iii})_1(x)\,(y)\,(')$, $(\mathrm{iv})_1(x)\,(y)$, *a constant C exists with the following property: for any points $P_1 = (x_1, y_1, t_1)$ and $P_2 = (x_2, y_1, t_1)$ of S_T with $x_2 < x_1$,*

$$\sqrt{X_0^2 - x_1^2(1 - y_1^2)} \, | u(P_1) - u(P_2) | \leqslant C(x_1 - x_2).$$

Let f be a function on $S_T \cup \sigma$, where σ is the set of points $P = (x, y, t)$ such that $x > X_0$, $-1 \leqslant y < 0$, $0 \leqslant t \leqslant T$. Assume that on S_T, f satisfies Lipschitz conditions of the following type:

$$\text{(a)} \qquad | f(x_1, y, t) - f(x_2, y, t) | \leqslant L \, | x_1 - x_2 | ,$$

$$\text{(b)} \qquad | f(x, y_1, t) - f(x_2, y_2, t) | \leqslant Lx \, | y_1 - y_2 | .$$

Furthermore, let f be defined on σ by

$$f(x, y, t) = f(X_0, y, t).$$

Then f satisfies the conditions (a) and (b) on its entire domain of definition. In the course of the proof of Theorem 6.3, it will be necessary to compare the values of such f at corresponding points of neighboring characteristics. The characteristic curves, $\mathscr{C}_{P_1}$ and $\mathscr{C}_{P_2}$, on which the values of f will be compared, are those issuing from

$$P_1 = (x_1, y_1, t_1),$$

and

$$P_2 = (x_1 - h, y_1, t_1),$$

where

$$0 < h, \qquad 0 < t_1, \qquad | y_1 | < 1.$$

An arbitrary point of $\mathscr{C}_{P_i}$ will be denoted by

$$P(t; P_i) = (X(t; P_i), Y(t; P_i), t) \qquad (i = 1, 2).$$

The comparison referred to is stated in the following lemma.

Lemma 6.1. *Under assumptions* (a) *and* (b) *above,*

$$| f(P(t; P_1)) - f(P(t; P_2)) | \leqslant 2Lh.$$

Proof. For $0 \leqslant \theta \leqslant 1$, the characteristic curve with initial point $(x_1 - \theta h, y_1, t_1)$ does not meet the plane $x = 0$. Hence, if

$$\tilde{x} \equiv \tilde{x}(t) = \min_{0 \leqslant \theta \leqslant 1} X(t; x_1 - \theta h, y_1, t_1),$$

x_1, y_1, t_1, t, and h being regarded in the minimizing process as fixed, we see

$$\tilde{x} > 0.$$

Since

$$\begin{aligned} \Delta f &\equiv f(P(t; P_1)) - f(P(t; P_2)) \\ &= f(\tilde{x}, Y(t; P_1), t) - f(\tilde{x}, Y(t; P_2), t) \\ &\quad + f(X(t; P_1), Y(t; P_1), t) - f(\tilde{x}, Y(t; P_1), t) \\ &\quad + f(\tilde{x}, Y(t; P_2), t) - f(X(t; P_2), Y(t; P_2), t), \end{aligned}$$

we have by our assumptions

$$|\Delta f| \leqslant L\tilde{x}\,|\,Y(t; P_1) - Y(t; P_2)\,| + L(X(t; P_1) - \tilde{x}) + L(X(t; P_2) - \tilde{x}).$$

For some value $\tilde{\theta}$ between 0 and 1 we have, however,

$$\tilde{x} = X(t; x_1 - \tilde{\theta}h, y_1, t_1),$$

from which relation, and from (2.4),

$$X(t; P_1) - \tilde{x} \leqslant \tilde{\theta}\, h, \qquad X(t; P_2) - \tilde{x} \leqslant (1 - \tilde{\theta})\, h.$$

Furthermore, by (2.5)

$$|\,Y(t; P_1) - Y(t; P_2)\,| = h \left| \int_0^1 Y_{x_1}(t; x_1 - \theta h, y_1, t_1)\, d\theta \right| \leqslant \frac{h}{\tilde{x}}\,.$$

Hence, $|\Delta f| \leqslant 2Lh$, as asserted.

This lemma is easily applied to the following line integral of f defined for any point P_0 of S_{T_0}, with $T_0 > 0$, as

$$F(P_0) = \int_{\mathscr{C}(R_0, P_0)} f dt,$$

where $R_0 = P(0; P_0)$.

Corollary to Lemma 6.1. *With P_1 and P_2 as defined above, let $R_i = P(0; P_i)$, $i = 1, 2$. Under Hypotheses* (a) *and* (b),

$$|\,F(P(t; P_1)) - F(P(t; P_2))\,| \leqslant 2Lth.$$

Proof. If P' is any point of $\mathscr{C}_{P_1}$,

$$F(P_1) = \int_0^{t_1} f(P(t; P'))\, dt.$$

Hence,

$$F(P(t; P_1)) = \int_0^t f(P(s; P_1))\, ds,$$

and

$$F(P(t; P_1)) - F(P(t; P_2)) = \int_0^t (f(P(s; P_1)) - f(P(s; P_2)))\, ds.$$

The desired inequality now follows immediately from Lemma 6.1.

Our proof of Theorem 6.3 is based on the particular integral relation to which (3.3) reduces when we substitute $P_1{}^*$ for P_2 and select for $\alpha(P_1)$ the function

$$E(P_1) = \exp \left\{ \int_{\mathscr{C}(R_1, P_1)} c\,dt \right\},$$

where $R_1 = P(0; P_1)$ denotes as above the point of $\mathscr{C}_{P_1}$ for which $t = 0$. (Here we assume that the function c has been extended to $S_T \cup \sigma$ by the definition

$$c(x, y, t) = c(X_0, y, t)$$

for $P = (x, y, t)$ in σ. Then, on its new domain of definition, c still satisfies the conditions $(\mathrm{i})_1(x)$ and $(\mathrm{i})_1(y)$ and we may apply to it our earlier results concerning f with c_1 replacing L.) Since $(dE/dt)_{\mathscr{C}_{P_1}} = c(P_1)\,E(P_1)$, the resulting integral relation may be written as

$$u(P_1) = (E(P_1))^{-1}\,E(P_1{}^*)\,u(P_1{}^*) + (E(P_1))^{-1} \int_{\mathscr{C}(P_1^*, P_1)} E(g + Su)\,dt. \tag{6.5}$$

Its advantage over alternative forms is that difference quotients of its right member with respect to x can be estimated from a presumed bound for the difference quotients of u with respect to x, no similar bound for difference quotients of u with respect to y being involved.

The corollary to Lemma 6.1 shows that, for $0 \leqslant t \leqslant T$,

$$|\,E(P(t; P_1)) - E(P(t; P_2))\,| \leqslant Mh, \tag{6.6}$$

where M is a constant.

Proof of Theorem 6.3. We begin by proving that the first term in the right member of (6.5) is Lipschitz continuous. Let S^* denote the subset of S_T such that $P_1{}^* \in S_0$ for $P_1 \in S^*$. By $(\mathrm{iv})_{X_0}$, the term indicated is zero unless $P_1 \in S^*$; hence, Lipschitz-continuity need be proved just for S^*. We shall establish that each of the three factors of this term is Lipschitz-continuous with respect to x_1 in S^*. The first two factors are Lipschitz-continuous because, in the notation of Lemma 6.1,

$$E(P_1) - E(P_2) = E(P(t_1; P_1)) - E(P(t_1; P_2))$$

and

$$E(P_1{}^*) - E(P_2{}^*) = E(P(0; P_1)) - E((P(0; P_2))$$

when P_1, $P_2 \in S^*$, inequality (6.6) therefore applying in both instances. The third factor, $u(P_1{}^*)$, is Lipschitz continuous in S^* because of (2.4), (2.5), and Hypothesis $(\mathrm{iv})_1$. To see this, again assume $P_1{}^*$ and $P_1{}^*$ to lie on the initial plane, in which case $u(P_1{}^*) = \phi(P_1{}^*)$ and $u(P_2{}^*) = \phi(P_2{}^*)$. That

$u(P_1^*)$ is Lipschitz continuous then follows from the following estimations in which P_1' and P_2' are relabelings of P_1 and P_2 such that $X(0; P_2') \leqslant X(0; P_1')$:

$$\begin{aligned} | \phi(P_2^*) - \phi(P_1^*) | &= | \phi(X(0; P_1'), Y(0; P_1')) - \phi(X(0; P_2'), Y(0; P_2')) | \\ &\leqslant | \phi(X(0; P_1'), Y(0; P_1')) - \phi(X(0; P_2'), Y(0; P_1')) | \\ &\quad + | \phi(X(0; P_2'), Y(0; P_1')) - \phi(X(0; P_2'), Y(0; P_2')) | \\ &\leqslant \phi_1 | X(0; P_1') - X(0; P_2') | \\ &\quad + \phi_1 X(0; P_2') | Y(0; P_1') - Y(0; P_2') | \\ &\leqslant 2\phi_1 h. \end{aligned}$$

We conclude that the first term in the right member of (6.5) is Lipschitz continuous with respect to x_1, as asserted.

We shall next show for the integral

$$H(P_1) = \int_{\mathscr{C}(P_1^*, P_1)} E g dt$$

that

$$\sqrt{X_0^2 - (1 - y_1^2) x_1^2} \, | H(P_1) - H(P_2) | \leqslant Ch, \qquad (6.7)$$

where C is a suitable constant. To this end, for brevity set $E_i = E(P(t; P_i))$, $i = 1, 2$, and define g_i analogously. If t_i^* denotes the value of t at the foot of $\mathscr{C}_{P_i}$, $i = 1, 2$, we then have

$$H(P_i) = \int_{t_i^*}^{t_i} E_i g_i dt, \qquad i = 1, 2,$$

and, if

$$t_j^* = \max(t_1^*, t_2^*), \qquad t_k^* = \min(t_1^*, t_2^*),$$

$$\begin{aligned} H(P_1) - H(P_2) &= \int_{t_1^*}^{t_1} E_1 g_1 dt - \int_{t_2^*}^{t_1} E_2 g_2 dt \\ &= \int_{t_j^*}^{t_1} [E_1(g_1 - g_2) + g_2(E_1 - E_2)] \, dt + \int_{t_1^*}^{t_2^*} E_k g_k dt. \end{aligned}$$

For $0 \leqslant t \leqslant T$, E_1, E_2, g_1, g_2 are bounded, $g_1 - g_2$ is estimated by Lemma 6.1, and $E_1 - E_2$ is estimated by the corollary to this lemma. Hence, constants C_1 and C_2, which may depend upon T, exist such that

$$| H(P_1) - H(P_2) | \leqslant C_1 h + C_2 | t_1^* - t_2^* | .$$

Inequality (6.7) results from this when we now estimate $| t_1^* - t_2^* |$ from (2.8).

Our discussion up to this point shows the integral relation (6.5) to be of the form

$$u(P_1) = Z(P_1) + \int_{\mathscr{C}(P_1^*, P_1)} dt \int_{-1}^{1} H(P_1, P, y') \, u(x, y', t) \, dy', \qquad (6.5')$$

where

$$H(P_1, P, y') = (E(P)/E(P_1)) \, K(P, y'), \qquad P = P(t; P_1)$$

and where $Z(P_1)$ satisfies a Lipschitz condition of the type (6.7):

$$\sqrt{X_0^2 - (1 - y_1^2) \, x_1^2} \, | Z(P_1) - Z(P_2) | \leqslant \text{const.} \cdot h.$$

Our aim is to prove that $u(P_1)$ satisfies a Lipschitz condition of the type (6.7). To this end, again we consider the variable points

$$P_1 = (x_1, y_1, t_1), \qquad P_2 = (x_1 - h, y_1, t_1),$$

now with $0 < h < x_1 - \min(x_1, X_0/3)$. Also, we consider

$$P_2' = (x_1 - h', y_1, t_1)$$

for $0 < h' \leqslant h$, defining

$$M_h(t) = \sup_{\substack{P_1 \in S_t \\ 0 < h' \leqslant h}} \sqrt{X_0^2 - x_1^2(1 - y_1^2)} \, | u(P_1) - u(P_2') |$$

and

$$N_h(t) = \sup_{0 \leqslant t' \leqslant t} M_h(t').$$

Eventually, we shall obtain from (6.5′) an estimate for $N_h(t)$.

Again let

$$P = (x, y, t) = P(t; P_1),$$
$$Q = (\xi, \eta, t) = P(t; P_2)$$

denote variable points of $\mathscr{C}_{P_1}$ and $\mathscr{C}_{P_2}$, respectively. The values at P_1 and P_2 of the line integral in (6.5′) are

$$I(P_1) = \int_{\mathscr{C}(P_1^*, P_1)} dt \int_{-1}^{1} H(P_1, P, y') \, u(x, y', t) \, dy'$$

$$= \int_{t_1^*}^{t_1} dt \int_{-1}^{1} H(P_1, P, y') \, u(x, y', t) \, dy'$$

and

$$I(P_2) = \int_{\mathscr{C}(P_2^*, P_2)} dt \int_{-1}^{1} H(P_2, Q, y')\, u(\xi, y', t)\, dy'$$

$$= \int_{t_2^*}^{t_1} dt \int_{-1}^{1} H(P_2, Q, y')\, u(\xi, y', t)\, dy'.$$

Subtracting these values, and again defining j and k such that

$$t_j^* = \max(t_1^*, t_2^*), \qquad t_k^* = \min(t_1^*, t_2^*),$$

we readily obtain

$$I(P_1) - I(P_2) = \int_{t_j^*}^{t_1} dt \int_{-1}^{1} (H(P_1, P, y') - H(P_2, Q, y'))\, u(\xi, y', t)\, dy'$$

$$+ \int_{t_j^*}^{t_1} dt \int_{-1}^{1} H(P_1, P, y')\, (u(x, y', t) - u(\xi, y', t))\, dy'$$

$$+ \int_{t_1^*}^{t_2^*} dt \int_{-1}^{1} H(P_k, \bar{P}, y')\, u(\bar{x}, y', t)\, dy' = I_1 + I_2 + I_3,$$

where $\bar{P} = (\bar{x}, \bar{y}, t) = P(t; P_k)$.

Hypothesis $(\mathrm{iii})_1$, Theorem 4.1, and the estimate (2.8) prove that

$$\sqrt{X_0^2 - x_1^2(1 - y_1^2)}\, |\, I_3\, | \leqslant \text{const.} \cdot h.$$

Inequality (6.6) and Hypotheses $(\mathrm{iii})_1(x)\,(y)$ added to the previous reasons show that

$$|\, I_1\, | \leqslant \text{const.} \cdot h.$$

To estimate I_2, note that, because of (2.4), $|\, x - \xi\, | \leqslant h$. Hence, with

$$\hat{x} = \max(x, \xi),$$

for $x \neq \xi$ we have

$$\sqrt{X_0^2 - \hat{x}^2(1 - y'^2)}\, |\, u(x, y', t) - u(\xi, y', t)\, |/h \leqslant N_h(t). \qquad (6.8)$$

Next we shall require an additional result on the geometry of characteristic curves. Assuming $X_0 - x_1 + 1 - y_1^2 > 0$, denote by (X_0, y_0, t_0) and $(X_0, -y_0, t^0)$ $(y_0 \leqslant 0)$ the two points of intersection of $\mathscr{C}_{P_1}$ with the plane $x = X_0$. (Possibly $t_0 < 0$.) If ϵ is an arbitrary constant such that $0 < \epsilon < X_0/2$, the plane $x = X_0 - \epsilon$ intersects $\mathscr{C}_{P_1}$ in two points, in one point, or in none. If in two points, we denote these points by $(X_0 - \epsilon, y_\epsilon, t_\epsilon)$ and $(X_0 - \epsilon, -y_\epsilon, t^\epsilon)$ with $y_0 < y_\epsilon < 0$ and $t_\epsilon < t^\epsilon$. Let $P = (x, y, t)$ be a point of $\mathscr{C}_{P_1}$. The number t_ϵ then is the greatest value such that

$$X_0 - \epsilon \leqslant x \leqslant X_0 \qquad \text{for} \qquad t_0 \leqslant t \leqslant t_\epsilon ,$$

and t^ϵ the least value such that

$$X_0 - \epsilon \leqslant x \leqslant X_0 \qquad \text{for} \qquad t^\epsilon \leqslant t \leqslant t^0.$$

If $\mathscr{C}_{P_1}$ intersects the plane $x = X_0 - \epsilon$ in one point only, at this point $y = 0$ and, hence, $t = t_1 - x_1 y_1$. When the intersection consists of one point, we therefore define

$$t_\epsilon = t^\epsilon = t_1 - x_1 y_1 .$$

When the intersection is empty, we aslo define

$$t_\epsilon = t^\epsilon = t_1 - x_1 y_1 .$$

In all cases, $y_0 < Y(t_\epsilon; P_1) \leqslant 0$. The required result we now state in the following lemma:

LEMMA 6.2. *The quantities defined above satisfy the inequalities*

$$\left.\begin{matrix} t_\epsilon - t_0 \\ t^0 - t^\epsilon \end{matrix}\right\} \leqslant 2X_0\epsilon / \sqrt{X_0^2 - x_1^2(1 - y_1^2)}.$$

Proof. The functions $t - xy$ and $x^2(1 - y^2)$ have been observed (proof of Theorem 2.3) to be constant along any characteristic. Hence, in particular,

$$t_\epsilon - t_0 = x_\epsilon y_\epsilon - X_0 y_0 , \tag{6.9}$$

where

$$x_\epsilon = X(t_\epsilon; P_1) = X_0 - \epsilon \qquad \text{and} \qquad y_\epsilon = Y(t_\epsilon; P_1).$$

Similarly,

$$x_\epsilon^2(1 - y_\epsilon^2) = X_0^2(1 - y_0^2)$$

and, therefore,

$$\begin{aligned} -(x_\epsilon y_\epsilon + X_0 y_0)(x_\epsilon y_\epsilon - X_0 y_0) &= X_0^2 - x_\epsilon^2 \\ &= 2\epsilon X_0 - \epsilon^2. \end{aligned}$$

Since $y_\epsilon \leqslant 0$ and $y_0 < 0$, it follows that

$$x\, y_\epsilon - X_0 y_0 = \frac{\epsilon(2X_0 - \epsilon)}{-(x_\epsilon y_\epsilon + X_0 y_0)} \leqslant \frac{2\epsilon}{(-y_0)} .$$

Hence, by 6.9,

$$t_\epsilon - t_0 \leqslant 2\epsilon/(-y_0).$$

Furthermore, $X_0^2(1 - y_0^2) = x_1^2(1 - y_1^2)$, which implies

$$-y_0 = \sqrt{X_0^2 - x_1^2(1 - y_1^2)}/X_0 ,$$

this and the previous result proving the first inequality in the lemma. The second inequality is similarly obtained.

We now return to discussing I_2. Requiring $0 < \epsilon < X_0/3$, define t_ϵ and t^ϵ for $\mathscr{C}_{P_1}$ as was done for the lemma. Then set

$$\begin{aligned} T_\epsilon &= t_j^* && \text{if} \quad t_\epsilon \leqslant t_j^* \\ &= t_\epsilon && \text{if} \quad t_j^* < t_\epsilon < t_1 \\ &= t_1 && \text{if} \quad t_1 \leqslant t_\epsilon \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} T^\epsilon &= t_1 && \text{if} \quad t_1 \leqslant t^\epsilon \leqslant t^0 \\ &= t^\epsilon && \text{if} \quad t_j^* < t^\epsilon < t_1 \\ &= t_j^* && \text{if} \quad t^\epsilon \leqslant t_j^*. \end{aligned} \tag{6.11}$$

The main outcome of these definitions is that $[T_\epsilon , T^\epsilon]$ is the largest subinterval of $[t_j^*, t_1]$ on which $X(t; P_1) \leqslant X_0 - \epsilon$, unless this subinterval is empty or degenerate, in which case $[T_\epsilon , T^\epsilon]$ too is degenerate. Furthermore,

$$T_\epsilon - t_j^* \leqslant t_\epsilon - t_0 , \qquad t_1 - T^\epsilon \leqslant t^0 - t^\epsilon . \tag{6.12}$$

With the δ afforded by Hypothesis (iii)$_1$('), now decompose I_2 as follows:

$$\begin{aligned} I_2 &= \int_{t_j^*}^{T_\epsilon} \int_{-\delta}^{\delta} + \int_{T_\epsilon}^{T^\epsilon} \int_{-\delta}^{\delta} + \int_{T^\epsilon}^{t_1} \int_{-\delta}^{\delta} + \int_{t_j^*}^{t_1} \left\{ \int_{-1}^{-\delta} + \int_{\delta}^{1} \right\} \\ &= I' + I'' + I''' + I^{\text{iv}}. \end{aligned}$$

With $\hat{x} = \max(x, \xi)$, as before, we have, in the domain of integration of I^{iv},

$$\sqrt{X_0^2 - \hat{x}^2(1 - y'^2)} \geqslant X_0\delta$$

and, in the domain of integration of I'',

$$\sqrt{X_0^2 - \hat{x}^2(1 - y'^2)} \geqslant \sqrt{X_0^2 - (X_0 - \epsilon)^2} \geqslant \sqrt{\epsilon X_0}.$$

In view of (6.8), these integrals therefore can be estimated by inequalities of the form

$$|I''| \leqslant \frac{C}{\sqrt{\epsilon}} \int_0^{t_1} N_h(t)\,dt, \qquad |I^{\text{iv}}| \leqslant \frac{C}{\delta} \int_0^{t_1} N_h(t)\,dt,$$

where C is a constant independent of δ and ϵ.

We must yet estimate I' and I'''. The process is the same for both integrals, and we shall consider in detail only the first. Assumption (iii)$_1$(′) and inequality (6.8) show, respecting the inner integral in I', that a constant k'' exists for which

$$\left| \frac{1}{h} \int_{-\delta}^{\delta} H(P_1, P, y')\,(u(x, y', t) - u(\xi, y', t))\,dy' \right|$$

$$\leqslant k'' \int_{-\delta}^{\delta} | u(x, y', t) - u(\xi, y', t) |\,dy'$$

$$\leqslant k'' N_h(t) \int_{-\delta}^{\delta} dy' / \sqrt{X_0^2 - \hat{x}^2(1 - y'^2)}$$

$$= k'' N_h(t) \frac{1}{\hat{x}} \log \frac{(\sqrt{X_0^2 - (1 - \delta^2)\,\hat{x}^2} + \hat{x}\delta)^2}{X_0^2 - \hat{x}^2}$$

$$\leqslant 3k'' X_0^{-1} N_h(t) \log \frac{(1 + \delta)^2 X_0^2}{X_0^2 - \hat{x}^2},$$

the last inequality following because $\hat{x} \geqslant X_0/3$ ($x \geqslant X_0 - \epsilon > 2X_0/3$ and $\xi > x - h \geqslant x - X_0/3$) in the domain of integration for I'. Hence,

$$|I'| \leqslant 6k'' X_0^{-1} | \log (1 + \delta) X_0) | \int_{t_j^*}^{T_\epsilon} N_h(t)\,dt$$

$$+ 3k'' X_0^{-1} \int_{t_j^*}^{T_\epsilon} N_h(t) | \log (1/(X_0^2 - \hat{x}^2)) |\,dt.$$

Since $N_h(t)$ is a nondecreasing function,

$$|I'| \leqslant 6k'' X_0^{-1} N_h(t_1) \Big[(T_\epsilon - t_j^*) | \log ((1 + \delta) X_0) | + \int_{t_j^*}^{T_\epsilon} | \log (X_0^2 - \hat{x}^2) |\,dt \Big].$$

Since $\hat{x} = \max(x, \xi)$, the integral on the right above is not greater than

$$\int_{t_j^*}^{T_\epsilon} | \log (X_0^2 - x^2) |\,dt + \int_{t_j^*}^{T_\epsilon} | \log (X_0^2 - \xi^2) |\,dt.$$

We estimate these integrals, noting that

$$X_0^2 - x^2 = X_0^2 - X(t; P_1)^2$$
$$= X_0^2 - x_1^2(1 - y_1^2) - (t - t_1 + x_1 y_1)^2,$$

from the fact that the indefinite integral of $\log(a^2 - s^2)$ with respect to s is $(a + s)\log(a + s) - (a - s)\log(a - s) - 2s +$ constant. It follows that the integral of $|\log(X_0^2 - x^2)|$ with respect to t over an arbitrary interval of length η tends to zero, as $\eta \to 0$, uniformly with respect to t, x_1, y_1, t_1, the same obviously also being true of the integral of $|\log(X_0^2 - \xi^2)|$ with respect to t. Hence, to any positive ζ corresponds a positive η, independent of x_1, y_1, t_1, h, such that

$$\int_{t_j^*}^{T_\epsilon} |\log(X_0^2 - \hat{x}^2)|\, dt < \zeta \tag{6.13}$$

if $T_\epsilon - t_j^* < \eta$ and, hence, by (6.12), if $t_\epsilon - t_0 < \eta$. It follows that, if $t_\epsilon - t_0 < \eta$, then

$$|I'| \leqslant 6k'' X_0^{-1}(\eta\, |\log(1 + \delta) X_0| + \zeta)\, N_h(t_1).$$

The condition $t_\epsilon - t_0 \leqslant \eta$ (Lemma 6.2) is satisfied, however, when

$$2X_0\epsilon/\sqrt{X_0^2 - x_1^2(1 - y_1^2)} \leqslant \eta$$

and, hence, in particular, when

$$\epsilon = \frac{\sqrt{X_0^2 - x_1^2(1 - y_1^2)}}{2X_0}\,\eta.$$

Let us now impose this restriction upon ϵ. Then select ζ and, depending on ζ, η in order to have

$$|I'| \leqslant \left(\frac{1}{8X_0}\right) N_h(t_1);$$

neither ζ nor η depends on P_1. We similarly assure $|I'''| \leqslant (1/8X_0)\, N_h(t_1)$, these inequalities and the bounds obtained for I'' and I^{iv} implying that

$$|I_2| \leqslant \left(\frac{1}{4X_0}\right) N_h(t_1) + C\left[\frac{1}{\delta} + \frac{1}{\sqrt{\eta}\,[X_0^2 - x_1^2(1 - y_1^2)]^{1/4}}\right] \int_0^{t_1} N_h(t)\, dt.$$

in view of our estimates above of I_1 and I_3, we thus arrive at

$$| I(P_1) - I(P_2) | \leqslant (C_3(X_0^2 - x_1^2(1 - y_1^2))^{-1/2} + C_4)\, h$$

$$+ C \left[\frac{1}{\delta} + \frac{1}{\sqrt{\eta}\,[X_0^2 - x_1^2(1 - y_1^2)]^{1/4}}\right] \int_0^{t_1} N_h(t)\, dt + \left(\frac{1}{4X_0}\right) N_h(t_1), \tag{6.14}$$

the inequality needed to estimate $h^{-1}N_h(t)$ from (6.5′). Here and in the sequel, C_k always denotes a constant independent of P_1.

Differencing (6.5′), using (6.14), and multiplying both sides by $\sqrt{X_0^2 - x_1^2(1 - y_1^2)}$ gives

$$\sqrt{X_0^2 - x_1^2(1 - y_1^2)}\, | u(P_1) - u(P_2) | \leqslant C_6 h + C_7 \int_0^{t_1} N_h(t)\, dt + (\tfrac{1}{4}) N_h(t_1).$$

We apply this to two points $P_3 = (x_3, y_3, t_3)$ and $P_3' = (x_3 - h', y_3, t_3)$ so selected, in place of P_1 and P_2, that $0 \leqslant t_3 \leqslant t_1$, $x_3 > 0$, $0 < h' \leqslant h$, and

$$\sqrt{X_0^2 - x_3^2(1 - y_3^2)}\, | u(P_3) - u(P_3') | \geqslant (\tfrac{1}{2}) N_h(t_1).$$

Since $h' \leqslant h$ and $N_{h'} \leqslant N_h$, we thereby obtain

$$(\tfrac{1}{2}) N_h(t_1) \leqslant C_6 h + C_7 \int_0^{t_3} N_h(t)\, dt + (\tfrac{1}{4}) N_h(t_3)$$

$$\leqslant C_6 h + C_7 \int_0^{t_1} N_h(t)\, dt + (\tfrac{1}{4}) N_h(t_1)$$

and, therefore,

$$(\tfrac{1}{4}) N_h(t_1) \leqslant C_6 h + C_7 \int_0^{t_1} N_h(t)\, dt,$$

an integral inequality that immediately implies the existence of a uniform bound for $N_h(t)$ in the band $0 \leqslant t \leqslant T$. Theorem 6.3 with this is completey proved.

Our last task of a priori estimation has to do with the Lipschitz continuity of u with respect to y. The relevant estimate is formulated for convenience for the partial derivative u_y, which exists because of the Lipschitz continuity of u at almost every point.

THEOREM 6.4. *If hypotheses* (i) *to* (iv) *(Section 1) and* $(\mathrm{i})_1$ *to* $(\mathrm{iv})_1$ *are satisfied, then for each* x, t *in the region*

$$0 < x \leqslant X_0,$$
$$0 \leqslant t \leqslant T,$$

u_y exists at almost all values of y and satisfies

$$(1 - y^2)\sqrt{X_0^2 - x^2(1 - y^2)}\,|\,u_y(x, y, t)\,| \leqslant \text{const.} \cdot x. \tag{6.15}$$

Proof. With arbitrarily small, positive δ and δ', we first consider the subset of $S_{T,0}$

$$U_{T,\delta,\delta'}: \quad \begin{cases} 0 < x \leqslant X_0 - \delta', \\ 0 \leqslant t \leqslant T - \delta', \\ |\,y\,| \leqslant 1 - \delta, \qquad X_0 - x + |\,y\,| \geqslant \delta. \end{cases}$$

Let $P_1 = (x_1, y_1, t_1)$ be a point of $U_{T,\delta,\delta'}$ and $P_2 = (x_1, y_2, t_1)$ a point of $S_{T,0}$ with the same first and third coordinates. It suffices to prove that, for a sufficiently small, positive ϵ (depending on δ, δ'),

$$(1 - \max(y_1^2, y_2^2))\sqrt{X_0^2 - x_1^2(1 - y_1^2)}\,\frac{|\,u(P_2) - u(P_1)\,|}{y_2 - y_1} \leqslant \text{const.} \cdot x_1 \tag{6.16}$$

for $0 < y_2 - y_1 < \epsilon$.

We fix ϵ so small that, for $0 < y_2 - y_1 < \epsilon$, a positive value t_3 exists such that $P_3 = (x_1, y_2, t_3)$ is in $S_{T,0}$ with $t_1 < t_3 \leqslant T$ and

$$y_1 = Y(t_1; P_3). \tag{6.17}$$

By $P_4 = (x_4, y_1, t_1)$ we denote the intersection of $\mathscr{C}_{P_3}$ with the plane $t = t_1$; the abscissa of this point is

$$x_4 = X(t_1; P_3). \tag{6.18}$$

The points P_1, P_2, P_3, P_4 are indicated in Fig 1.

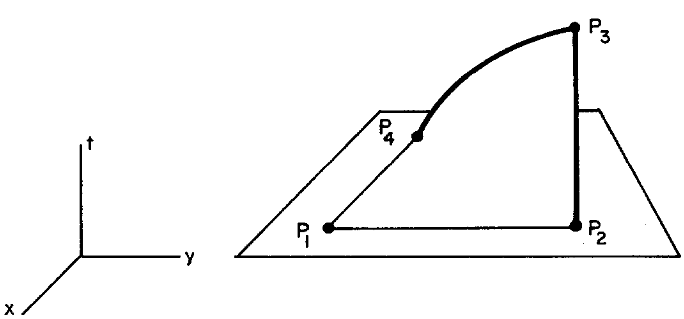

FIG. 1 Four points in the proof of Theorem 6.4

We now estimate $| u(P_2) - u(P_1) |$ as

$$| u(P_2) - u(P_1) | \leqslant | u(P_2) - u(P_3) | + | u(P_3) - u(P_4) | \\ + | u(P_4) - u(P_1) | \equiv u_{23} + u_{34} + u_{41} \qquad (6.19)$$

and shall find suitable bounds for the u_{ij} in terms of Lipschitz constants for u with respect to x and t. These Lipschitz constants, here denoted by L_x and L_t, respectively, are known from Theorems 6.2 and 6.3.

Condition (6.17) determining t_3 is

$$y_1 = \frac{t_1 - t_3 + x_1 y_2}{\sqrt{(t_1 - t_3)^2 + 2x_1 y_2(t_1 - t_3) + x_1^2}}.$$

Since we wish t_3 and t_1 to be equal when $y_2 = y_1$, we thus have

$$t_3 - t_1 = x_1 \left(y_2 - y_1 \sqrt{\frac{1 - y_2^2}{1 - y_1^2}} \right)$$

from which, by simple calculations, results the inequality

$$0 < \frac{t_3 - t_1}{y_2 - y_1} < \frac{x_1}{1 - \max(y_1^2, y_2^2)} \qquad (6.20)$$

and, in consequence, the estimate

$$u_{23} \leqslant L_t x_1 (y_2 - y_1)/(1 - \max(y_1^2, y_2^2)). \qquad (6.21)$$

Inequality (6.20) also implies

$$u_{34} \leqslant A x_1 (y_2 - y_1)/(1 - \max(y_1^2, y_2^2)), \qquad (6.22)$$

where A is a bound for the integrand in relation (3.1).

In view of our determination above of $t_3 - t_1$, from Eq. (6.18) defining x_4 we have

$$x_4 = x_1 \sqrt{\frac{1 - y_2^2}{1 - y_1^2}}.$$

Hence,

$$\frac{| x_4 - x_1 |}{y_2 - y_1} \leqslant \frac{x_1}{1 - \max(y_1^2, y_2^2)}$$

and, therefore,

$$u_{41} \leqslant \frac{L_x x_1 (y_2 - y_1)}{1 - \max(y_1^2, y_2^2)}.$$

This and the inequalities (6.21) and (6.22), taken in conjunction with Theorems 6.2 and 6.3 in which L_t and L_x are estimated, suffice to prove (6.16). Thus, the proof of Theorem 6.4 is complete.

Reference

1. Douglis, A. "The Existence and Calculation of Solutions of Certain Integro-Differential Equations in Several Dimensions." NOLTR 62-193, December 1962, pp. 1-59 (U.S. Naval Ordnance Laboratory, White Oak, Md.).

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