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On the stability of rod with variable cross-section

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Abstract

There is a solution of the problem of the stability of a compressed rod with a variable cross-section. A rectangular cross-section with a variable width is selected as the section. The result is that the problem leads to a differential equation of the fourth order with variable coefficients. From the solution of this equation, a critical force for several particular cases. These cases reflect some different conditions fixing rod and function changes of the width of the cross-section.

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Keywords: The stability of rod; Variable cross-section; Critical force

Nomenclature

E	elastic modulus
J	axial moment of inertia
b	section width
h	depth of section
l	rod length
q	transverse load P axial force
α	dimensionless parameter of transverse load
β	dimensionless parameter of axial force
ρ	dimensionless coordinate
φ	dimensionless function of the deflection of the rod

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1. Introduction.

The rods with a variable cross section are often used as framings of unique buildings.

Let us consider a rectangular cross section with the constant modulus of elasticity. In this case, the second moment J of the cross section takes the form

$$J = \frac{b \cdot h^3}{12}, \tag{1}$$

where b – section width, h – section depth.

We assume that the width of the cross section varies as [1]

$$b = b(x) = b_0 \cdot \left(1 + (k-1) \cdot \left(\frac{x}{l}\right)^m\right), \tag{2}$$

where b_0 – a section width at $x = 0$, $k = b(l)/b(0)$.

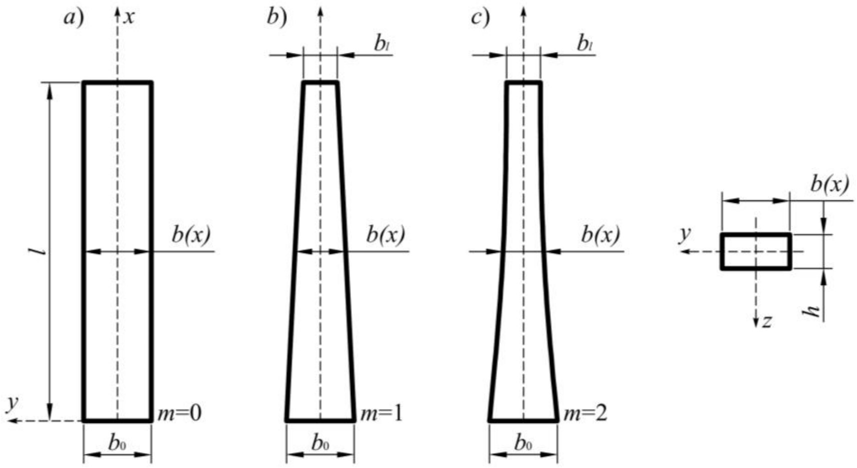

Fig. 1. The model.

2. Statement of the problem

The differential equilibrium equation of the compressed and bent rod in the case of a constant axial force P and the transverse load q takes the form [2]

$$\frac{d}{dx^2}\left(E \cdot J \cdot \frac{dy^2}{dx^2}\right) + P \cdot \frac{dy^2}{dx^2} - q = 0, \tag{3}$$

where E – modulus of elasticity, J – axial moment of inertia.

Let us express the transverse force q with the dimensionless parameter α and characteristics of the rod

$$q = \alpha \cdot \frac{E \cdot b_0 \cdot h^3}{12 \cdot l^3}, \tag{4}$$

where l – the length of the rod.
Similarly, we express the axial force P

$$P = \beta \cdot \frac{E \cdot b_0 \cdot h^3}{12 \cdot l^2}. \tag{5}$$

Substituting (2), (4) and (5) into (3) and dividing by $E \cdot b_0 \cdot h^3/12$, after some simplifications we obtain

$$\begin{gathered}\left[1+(k-1)\cdot\left(\frac{x}{l}\right)^m\right]\cdot\frac{d^4y}{dx^4}+\frac{2}{x}\cdot m\cdot(k-1)\cdot\left(\frac{x}{l}\right)^m\cdot\frac{d^3y}{dx^3}+\\+\left[\frac{m}{x^2}\cdot(k-1)\cdot(m-1)\cdot\left(\frac{x}{l}\right)^m+\beta\cdot\frac{1}{l^2}\right]\cdot\frac{d^2y}{dx^2}-\frac{\alpha}{l^3}=0.\end{gathered} \tag{6}$$

Next, we introduce a new dimensionless variable $\rho = x/l$ and a function $\varphi = y/l$. As result, the equation (6) takes the next form

$$\begin{gathered}\left[1+(k-1)\cdot\rho^m\right]\cdot\frac{d^4\varphi}{d\rho^4}+2\cdot m\cdot(k-1)\cdot\rho^{m-1}\cdot\frac{d^3\varphi}{d\rho^3}+\\+\left(m\cdot(k-1)\cdot(m-1)\cdot\rho^{m-2}+\beta\cdot\rho^2\right)\cdot\frac{d^2\varphi}{d\rho^2}-\alpha=0.\end{gathered} \tag{7}$$

3. Solution of the problem

The equation (7) is an equation with variable coefficients. It has a well-known analytical solutions only for some particular cases of parameters m and k.

Next, we write down the equations for cases $m = 0,\ 1,\ 2$ without any transverse load ($\alpha = 0$).

At $m = 0$

$$k \cdot \frac{d^4\varphi}{d\rho^4} + \beta \cdot \frac{d^2\varphi}{d\rho^2} = 0. \tag{8}$$

The critical force in this case is given by

$$P_{cr,0}(k) = \frac{\pi^2 \cdot E \cdot J}{(\mu \cdot l)^2} \cdot k, \tag{9}$$

where μ – coefficient of a free length, which depends on the fixing conditions (Fig. 2). It respectively equals $\mu = [0.5,\ 0.7,\ 1,\ 2]$.

Substituting(1), (2) and (5) into this expression we obtain

$$\beta_{cr,0}(k) = \frac{\pi^2}{\mu^2} \cdot k, \tag{10}$$

where β_{cr} – the critical value of β when the buckling takes place. We calculate these values at $k = 1$ $\beta_{cr,0}(1) = [39.48,\ 20.14,\ 9.87,\ 2.47]$.

At $m = 1$

$$[1 + (k-1) \cdot \rho] \cdot \frac{d^4\varphi}{d\rho^4} + 2 \cdot (k-1) \cdot \frac{d^3\varphi}{d\rho^3} + \beta \cdot \frac{d^2\varphi}{d\rho^2} = 0. \tag{11}$$

At $m = 2$

$$\left[1 + (k-1) \cdot \rho^2\right] \cdot \frac{d^4\varphi}{d\rho^4} + 4 \cdot (k-1) \cdot \rho \cdot \frac{d^3\varphi}{d\rho^3} + [2 \cdot (k-1) + \beta] \cdot \frac{d^2\varphi}{d\rho^2} = 0. \tag{12}$$

For this expressions we also need to write the boundary conditions.

Let us consider fixing variants (Fig. 2)

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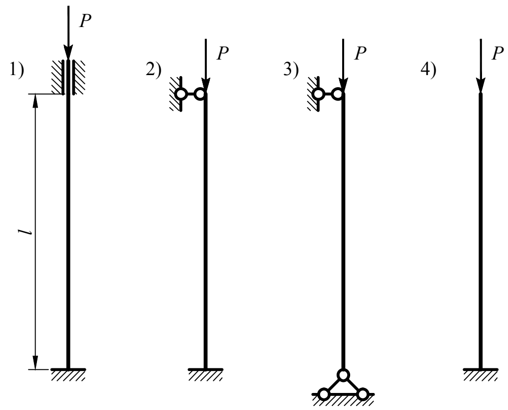

Fig. 2. Fixing variants.

There are boundary conditions for each of fixing variants below.

Table 1. Boundary conditions.

	$\rho = 0$	$\rho = 1$
1	$\varphi = 0;\ \frac{d\varphi}{d\rho} = 0;$	$\varphi = 0;\ \frac{d\varphi}{d\rho} = 0;$
2	$\varphi = 0;\ \frac{d\varphi}{d\rho} = 0;$	$\varphi = 0;\ \frac{d^2\varphi}{d\rho^2} = 0;$
3	$\varphi = 0;\ \frac{d^2\varphi}{d\rho^2} = 0;$	$\varphi = 0;\ \frac{d^2\varphi}{d\rho^2} = 0;$
4	$\varphi = 0;\ \frac{d\varphi}{d\rho} = 0;$	$\frac{d^2\varphi}{d\rho^2} = 0;\ [1 + (k-1)] \cdot \frac{d^3\varphi}{d\rho^3} + \beta \cdot \frac{d\varphi}{d\rho} = 0.$

Solutions of the equations (11) and (12) are cumbersome expressions. Let us write them in the general form.

$$\varphi = C_1 + C_2 \cdot \rho + C_3 \cdot f_1(\rho, k, \beta) + C_4 \cdot f_2(\rho, k, \beta). \tag{13}$$

Searching for a critical parameter β we used a non-triviality condition of the general solution. In this regard the constants C_1 and C_2 were defined from the boundary conditions. Next, using the rest of the boundary conditions we constituted the coefficient matrix at C_3 and C_4. Equating to zero the determinant of this matrix, we obtain an expression about β_{cr} and k.

$$\left.\begin{aligned} C_3 \cdot v_{11}(\beta_{cr}, k) + C_4 \cdot v_{12}(\beta_{cr}, k) = 0; \\ C_3 \cdot v_{21}(\beta_{cr}, k) + C_4 \cdot v_{22}(\beta_{cr}, k) = 0; \end{aligned}\right\} \tag{14}$$

$$A = \begin{bmatrix} v_{11}(\beta_{cr}, k) & v_{12}(\beta_{cr}, k) \\ v_{21}(\beta_{cr}, k) & v_{22}(\beta_{cr}, k) \end{bmatrix}; \tag{15}$$

$$\det(A) = \det(\beta_{cr}, k) = 0.$$

Getting an explicit dependence $\beta_{cr}(k)$ is not possible. For plotting this function were used the mathematical package "Maple". Below are the plots (Fig. 3, Fig. 4).
For the boundary conditions 1 and 2.

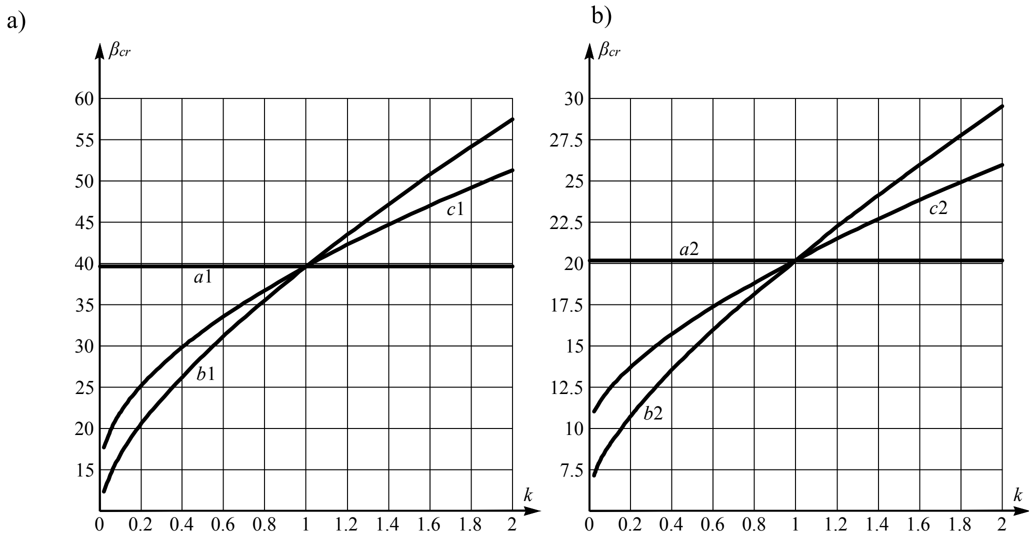

Fig. 3. Dependence $\beta_{cr}(k)$. a) boundary conditions 1. b) boundary conditions 2.

For the boundary conditions 3 and 4.

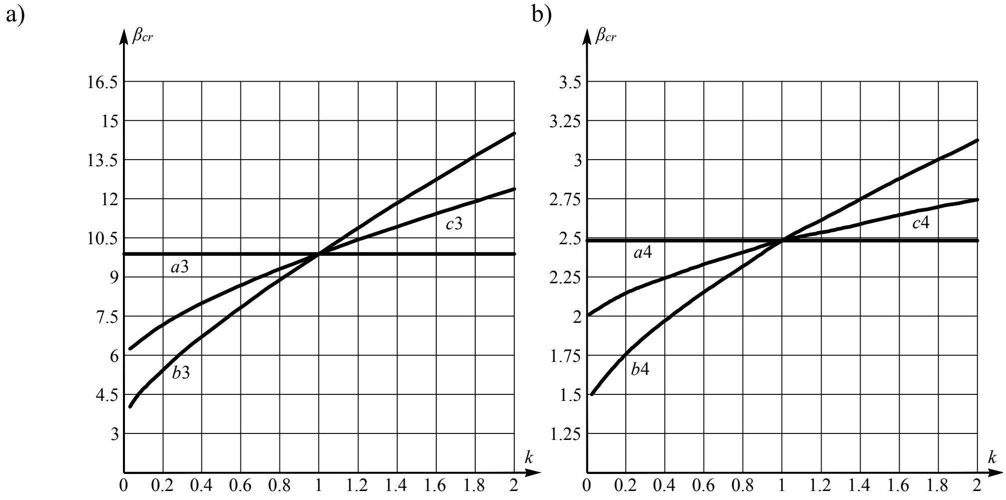

Fig. 4. Dependence $\beta_{cr}(k)$. a) boundary conditions 3. b) boundary conditions 4.

Cases in which $k < 0.2$ are rare. The graphs are given for reference.

Solving such problems is often regarded with a rod of constant cross-section width $b = b_{\min}$. In this case, a significant amount of a load capacity is not considered, This load capacity is determined by the expression

$$\delta_\beta = \begin{cases} \dfrac{\beta_{cr} - \beta_{cr,0}(k)}{\beta_{cr}} \cdot 100, & k < 1; \\ \dfrac{\beta_{cr} - \beta_{cr,0}(1)}{\beta_{cr}} \cdot 100, & k \ge 1. \end{cases} \tag{16}$$

Below there is a graph of the dependence for case $b1$. The plots for other case are similar.

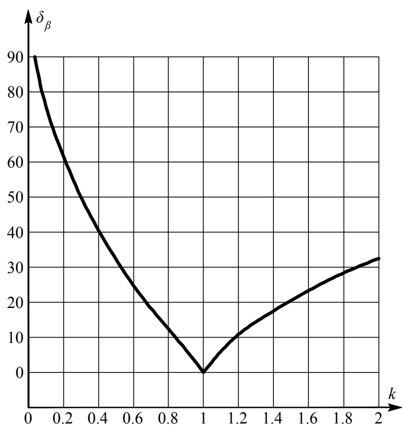

Fig. 5. Dependence $\delta_\beta(k)$ for case $b1$.

4. Conclusions

The Fig. 3 and Fig. 4 show that in the case $k<1$ is more reasonable to use the relationship (2) with the parameter $m=2$. In the case $k>1$ is more advantageous to use the value $m=1$.

Analyzing the Fig. 5 we come to the conclusion that the inclusion inhomogeneity section is necessary because the result is up to 60% extra load-carrying capacity.

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