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Characterizing all models in infinite cardinalities

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ABSTRACT

Fix a cardinal κ. We can ask the question: what kind of a logic L is needed to characterize all models of cardinality κ (in a finite vocabulary) up to isomorphism by their L-theories? In other words: for which logics L it is true that if any models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ satisfy the same L-theory then they are isomorphic?
It is always possible to characterize models of cardinality κ by their L_{κ^+,κ^+}-theories, but we are interested in finding a "small" logic L, i.e., the sentences of L are hereditarily of smaller cardinality than κ. For any cardinal κ it is independent of *ZFC* whether any such small definable logic L exists. If it exists it can be second order logic for $\kappa = \omega$ and fourth order logic or certain infinitary second order logic $L^2_{\kappa,\omega}$ for uncountable κ. All models of cardinality κ can always be characterized by their theories in a small logic with generalized quantifiers, but the logic may be not definable in the language of set theory. Our work continues and extends the work of Ajtai [Miklos Ajtai, Isomorphism and higher order equivalence, Ann. Math. Logic 16 (1979) 181–203].

1. Introduction

We shall investigate whether second order equivalence of two models, or equivalence in some stronger logic than second order logic, implies isomorphism of the models in certain cardinalities. We always assume that our vocabulary is finite. The notation which is not yet explained can be found under the heading "Notation" below.

Remark 1.1. We are assuming throughout this paper that the vocabulary is finite. This is because if the vocabulary is finite, then the isomorphism type of the model is characterizable inside the model in second order logic. In infinitary second order logic $L^2_{\kappa,\omega}$ the isomorphism type of the model is characterizable if the vocabulary is smaller than κ, and our assumption is stronger than what is needed.

The following lemma of Shelah demonstrates that not all countable models with countable vocabularies can be characterized by their second order theories.

Lemma 1.2 *(Shelah). There are two countable non-isomorphic second order equivalent models in a countably infinite vocabulary. The models are also L^n-equivalent for any n.*

Proof. The vocabulary of the models contains infinitely many constants $\{c_n\colon n \in \omega\}$. Let $\mathfrak{A}$ be a model such that $\mathrm{dom}(\mathfrak{A}) = \{a_n\colon n \in \omega\}$ and $c_n^{\mathfrak{A}} = a_n$ for each n. Let $\mathrm{dom}(\mathfrak{B}) = \{a_n\colon n \in \omega\} \cup \{a_\omega\}$ and $c_n^{\mathfrak{B}} = a_n$ for each n.

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The models are not isomorphic as in the model $\mathfrak{A}$ every element is an interpretation of some constant but in $\mathfrak{B}$ the element a_ω is not an interpretation of any constant. We claim that the models $\mathfrak{A}$ and $\mathfrak{B}$ are L^n-equivalent for any n. So take an arbitrary L^n-sentence ϕ. Let τ be the finite set of constants in ϕ. Now $\mathfrak{A} \upharpoonright \tau$ is isomorphic to $\mathfrak{B} \upharpoonright \tau$ and it follows that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L^n-sentences in the vocabulary τ. Thus $\mathfrak{A} \models \phi \Leftrightarrow \mathfrak{B} \models \phi$. □

Suppose L is a logic [3] (Chapter 2, Definition 1.1.1). The *L-theory* of a model is the set of L-sentences true in the model. Two models are said to satisfy the same L-theory if they satisfy the same L-sentences.

Definition 1.3. We use the expression $A(L,\kappa)$ to refer to the following condition: For any models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same L-theory then they are isomorphic.

We use $A(ZF,\kappa)$ to denote the condition "for all models $\mathfrak{A}$ and $\mathfrak{B}$ of cardinality κ in a finite vocabulary, if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences (with the model as a parameter) in the language of set theory then $\mathfrak{A} \cong \mathfrak{B}$." Note that *ZF* is not a logic as two isomorphic models can satisfy different sentences in the language of set theory.

Definition 1.4. We call $A(L^2,\omega)$ when restricted to ordinals the *Fraïssé Hypothesis*. This is the Hypothesis: All countable ordinals have different second order theories.

Ajtai [2] has proved that $A(L^2,\omega)$ is independent of *ZFC*. We are looking for related results in the cardinality $\aleph_0$ and similar results in higher cardinalities. The name "Fraïssé Hypothesis" has been introduced by Wiktor Marek. The Fraïssé Hypothesis has been studied by Fraïssé [4] and Marek [12,13].

Our results are relative to the consistency of *ZFC*. If we assume more than the consistency of *ZFC* it is always explicitly mentioned.

In Section 3 we will recall the proof of Ajtai and make some observations related to $A(L^2,\omega)$.

In Section 4 we will develop a forcing technique for coding subsets of ordinals by collapsing certain cardinals. This forcing is used to prove for example the following: If κ is a cardinal in L, then there is a transitive model of *ZFC* in which $A(L^4,\lambda)$ holds for exactly cardinals λ smaller than or equal to κ.

In Section 5 we will show that if κ is a cardinal, then there is a language $L^{\kappa *}$ with κ many generalized quantifiers such that $A(L^{\kappa *},\kappa)$ holds. Given a cardinal κ the language $L^{\kappa *}$ may be different for different models of *ZFC* containing κ and it is also possible that no such $L^{\kappa *}$ is definable in the language of set theory. This result for $\kappa = \omega$ is due to Scott Weinstein (Personal communication with Jouko Väänänen) and the generalization for uncountable κ is based on an idea of Per Lindström (Personal letter to Jouko Väänänen, 1 August 1974).

In Section 6 we will use Ajtai's method to prove that it is independent of *ZFC* whether $A(L^2_{\kappa,\omega},\kappa)$ holds for a regular cardinal κ. We will also prove that for different regular cardinals κ and λ, $A(L^2_{\kappa,\omega},\kappa)$ and $A(L^2_{\lambda,\omega},\lambda)$ are independent of each other. We will also give an analogous result for singular cardinals.

In Section 7 we will investigate the relation between $A(L^2,\omega)$ and various large cardinal axioms. If there are infinitely many Woodin cardinals and a measurable cardinal above them, then $A(L^2,\omega)$ fails. Assuming the consistency of relevant large cardinal axioms, if n is a natural number, then there is a model of *ZFC* in which there are n Woodin cardinals and $A(L^2,\omega)$ holds. As n grows bigger, more complex second order sentences seem to be needed to characterize all countable models up to isomorphism. $A(L^3,\omega)$ is consistent with Martin's Maximum and practically all large cardinal axioms.

For a discussion of the role of second order characterizations in the foundations of mathematics see [21].

Notation

The expression *ZF-formulas* refers to formulas in the language of set theory, i.e., first order language in a vocabulary with one binary relation $\in$. *ZF-equivalence* of two structures, denoted by $\mathfrak{A} \equiv_{ZF} \mathfrak{B}$, refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same formulas of the language of set theory, i.e., for any formula $\phi(x)$ in the language of set theory $V \models \phi(\mathfrak{A}) \leftrightarrow \phi(\mathfrak{B})$. If L is a logic $\mathfrak{A} \equiv_L \mathfrak{B}$ refers to the condition that $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same sentences of L. The expression $H(\kappa)$ refers to the set of sets hereditarily smaller than κ, i.e., $\{X$: the transitive closure of X has cardinality less than $\kappa\}$. The symbol $\upharpoonright$ means "restricted to". Depending on context this can mean a reduct of a model to a smaller vocabulary or restriction of some operations to some set. The notation $\phi^{\mathfrak{M}}(\cdot)$ refers to the set of tuples which satisfy the formula ϕ in model $\mathfrak{M}$. A forcing name of a given set X is denoted by $\dot{X}$. Interpretation of a definable set in a given model of *ZFC* is denoted by the set with the model of *ZFC* as superscript: for example ω_1^{L} means the ω_1 of L. If t is a term, $\mathfrak{A}$ is a model and s is an assignment which maps the free variables of t to elements of $\mathfrak{A}$, $t_s^{\mathfrak{A}}$ refers to the interpretation of t in $\mathfrak{A}$ with assignment s. Analogously if R is a higher order variable $R_s^{\mathfrak{A}}$ refers to the interpretation of the higher order variable R in the model $\mathfrak{A}$ with assignment s. If the higher order variable has subscripts and superscripts, we use parentheses for clarity: for example $(R^i_j)^{\mathfrak{A}}_s$. The expression $\mathfrak{A} \models_s \phi$ refers to the condition that the formula ϕ with the assignment s is satisfied in the model $\mathfrak{A}$. By *the reals* we mean the power set of ω.

Notation which is not explained is standard as used for example in Jech's book [9].

2. Preliminaries

2.1. The logics L^n

In this section we will present some fundamental definitions and lemmas about higher order logics. This section does not contain any new results. In the rest of the paper we have indicated results from the literature when we use such. All the other results are, according to our knowledge, new.

Definition 2.1. An n-ary relation $R_i^n \subseteq (dom(\mathfrak{A}))^n$ is *definable* in a language L in a model $\mathfrak{A}$ if there is an L-formula $\phi(x_1,\dots,x_n)$ such that

$$R_i^n = \{(a_1,\dots,a_n)\colon \mathfrak{A} \models \phi(a_1,\dots,a_n)\}.$$

An isomorphism-closed class of structures C is *characterizable* in a logic L if there is an L-formula $\phi_C(X_1,\dots,X_m)$ such that for any model $\mathfrak{A}$, $\mathfrak{A} \models_s \phi_C(X_1,\dots,X_m)$ iff $(A,(X_1)_s^{\mathfrak{A}},\dots,(X_m)_s^{\mathfrak{A}}) \in C$. When C is a singleton class $\{\mathfrak{B}\}$ we say that the model $\mathfrak{B}$ is characterizable in L.

A model $\mathfrak{B} = (B, R_1,\dots,R_n)$ is *definable up to isomorphism in model $\mathfrak{A}$ in a logic L* if there is an L-formula $\phi_{\mathfrak{B}}$ such that the following hold:

- $\mathfrak{A} \models_s \phi_{\mathfrak{B}}(X_1,\dots,X_m)$ iff $((X_1)_s^{\mathfrak{A}},\dots,(X_m)_s^{\mathfrak{A}})$ is isomorphic to $\mathfrak{B}$,
- there is an assignment s such that $\mathfrak{A} \models_s \phi_{\mathfrak{B}}(X_1,\dots,X_m)$.

We use L^2 to refer to second order logic. In L^2 we can quantify over all finitary relations over the universe of the model, thus our second order logic means the second order logic with full semantics. There are also other second order logics which do not use full semantics such as *monadic second order logic* where we can quantify over unary relations only, and second order logic with Henkin semantics [7]. More generally L^n refers to nth order logic with full semantics.[1]

Definition 2.2. We say that a second order formula $\phi(X,Y)$ defines a well-order of the reals if in the model $(\mathbb{N},+,\cdot,0,1)$ the formula ϕ defines a well-order of the subsets of $\mathbb{N}$.

We say that a second order formula $\phi(X,Y)$ defines a well-order of the power set of κ if in the model $(\kappa,\in)$ the formula ϕ defines a well-order of the subsets of κ.

Let $\tau = \{R_1,\dots,R_n\}$ be a relational vocabulary and let the arity of R_i be k_i for each i. We will next introduce a way to code a model of infinite cardinality κ in vocabulary τ into a subset of κ^m, where $m = \sum_{1\leqslant i\leqslant n} k_i$.

Definition 2.3 *(Coding a model into a subset of κ^m).* Let $\mathfrak{B} = (B, R_1^{\mathfrak{B}},\dots,R_n^{\mathfrak{B}})$ be a model of cardinality κ in the vocabulary τ and let $<$ be a well-order of B in order-type κ. We don't expect $<$ to be definable in $\mathfrak{B}$. Later in the proof of Theorem 3.1 we will in a way build (or guess) $<$ by second order quantifiers, which is possible as the structure $(\kappa,<)$ is definable up to isomorphism by an L^2-formula in a model of cardinality κ. The relations of $\mathfrak{B}$ can be coded into an m-ary relation $X_n^m \subseteq B^m$ in the following way: any sequence of ordinals belongs to X_n^m iff for some i it is of the form

$$(\underbrace{0,0,\dots}_{\sum_{j<i} k_j \text{ times}}\ \alpha_1+1,\alpha_2+1,\dots,\alpha_{n_i}+1,\ \underbrace{0,0,\dots}_{\sum_{i<j\leqslant n} k_j \text{ times}})$$

for some ordinals $\alpha_1,\dots,\alpha_{n_i}$ such that $\mathfrak{B} \models R_i(\alpha_1,\dots,\alpha_{n_i})$. The ordinals α_i, α_i+1 etc. refer to elements of B which have order-type α_i, α_i+1, etc. with respect to $<$.

Lemma 2.4. *Let $\mathfrak{A} = (A, R_1^{\mathfrak{A}},\dots,R_n^{\mathfrak{A}})$ be a model of infinite cardinality κ in a finite vocabulary τ. Let $\#R_i = k_i$ for each i and $m = \sum_{1\leqslant i\leqslant n} k_i$. Then:*

- *$\mathfrak{A}$ is isomorphic to some models[2] which have κ as universe.*
- *The set $I_{\mathfrak{A}}$ of those subsets of κ^m which are codes of models isomorphic to $\mathfrak{A}$ is definable up to isomorphism in $\mathfrak{A}$ by an L^2-formula.*

Proof. Obviously any bijection from A to κ generates a model isomorphic to $\mathfrak{A}$ which has κ as universe.

[1] There are several ways to define L^n. By and large they are all equivalent (at least as long as they allow to prove Lemma 2.5).

[2] We think models here as set theoretic objects. Thus there are many models isomorphic to $\mathfrak{A}$ with κ as universe, though up to isomorphism there is only one.

Recall that we assume, as in Definition 2.3, that $<$ is a well-order of A in order-type κ. In $(\mathfrak{A}, <)$ the relation X_n^m (introduced in Definition 2.3) is L^2-definable and each relation $R_i^{\mathfrak{A}}$ is second order definable from X_n^m. Let $\psi(X_n^m, \mathfrak{A}, <)$ be the following second order formula which says that X_n^m is the code of $\mathfrak{A}$ with respect to $<$:

$$\forall x_1, \ldots, \forall x_m \left(X_n^m(x_1, \ldots, x_m) \leftrightarrow \bigvee_{1 \leqslant i \leqslant m} \phi_i \right)$$

where ϕ_i is the conjunction of the following formulas[3]:

- $\bigwedge_{j \leqslant \sum_{t<i} k_t} x_j = 0.$
- $\bigwedge_{j > \sum_{t \leqslant i} k_t} x_j = 0.$
- $\bigwedge_{j \in 1+\sum_{t \leqslant i} k_t, \ldots, k_i + \sum_{t \leqslant i} k_t} x_j \neq 0 \wedge R(x_{1+\sum_{t \leqslant i} k_t} - 1, \ldots, x_{k_i + \sum_{t \leqslant i} k_t} - 1).$

Let $C \subseteq dom(<)^m$. The relation C is a code of a model isomorphic to $\mathfrak{A}$ (with respect to $<$) iff

$$\exists P_1^{k_1} \exists P_2^{k_2}, \ldots, \exists P_n^{k_n} \exists T$$
$$\left(\left(\phi_{\mathrm{bij}}\left(T, A, dom(<)\right) \wedge \bigwedge_{1 \leqslant i \leqslant n} \phi_i\right) \wedge \psi\left(C, \left(dom(<), P_1^{k_1}, \ldots, P_n^{k_n}\right), <\right)\right),$$

where $\phi_{\mathrm{bij}}(T, A, dom(<))$ says that T is a bijection from A to $dom(<)$, ψ is defined above and ϕ_i is the following formula:

$$\forall x_1, \ldots, \forall x_{k_i} \left(R_i(x_1, \ldots, x_{k_i}) \leftrightarrow P_i^{k_i}\left(T(x_1), \ldots, T(x_{k_i})\right)\right). \qquad \square$$

Lemma 2.5. (a) *Let ϕ be a second order formula in a finite vocabulary τ, $\mathfrak{A}$ a model of cardinality κ in $H(\kappa^+)$ with vocabulary τ, and let s be an assignment of the free variables of ϕ in $\mathfrak{A}$. Then there is a formula θ in the language of set theory such that $\mathfrak{A} \models_s \phi \Leftrightarrow H(\kappa^+) \models \theta(\mathfrak{A}, s)$. More generally for any nth order formula ϕ and an assignment s containing the free variables of ϕ in its domain, there is a formula θ in the language of set theory such that $(H((\beth_{n-2}(\kappa))^+), \in) \models \theta(\mathfrak{A}, s) \Leftrightarrow \mathfrak{A} \models_s \phi$.*

(b) *Let τ be a finite vocabulary and assume we have fixed some second order definable way to code models in the vocabulary τ by subsets of κ^m with respect to a given well-order in order-type κ of the universe of the model. There is a mapping $\phi \mapsto \phi^*$ from the set of $ZF(I)$-sentences[4] to the set of L^2-sentences in vocabulary τ such that $(H(\kappa^+), I_{\mathfrak{A}}, \in) \models \phi \Leftrightarrow \mathfrak{A} \models \phi^*$ for any model $\mathfrak{A}$ of cardinality κ in vocabulary τ. More generally, there is a mapping $\phi \mapsto \phi^{*n}$ from the set of $ZF(I)$-sentences to the set of L^n-sentences in vocabulary τ such that for any model $\mathfrak{A}$ of cardinality κ with vocabulary τ*

$$\left(H\left(\beth_{n-2}(\kappa)^+\right), I_{\mathfrak{A}}, \in\right) \models \phi \quad \Leftrightarrow \quad \mathfrak{A} \models \phi^{*n}.$$

Proof. (a) Assignments of finitely many first order and second order variables in the model $\mathfrak{A}$ belong to $H(\kappa^+)$. To formalize truth definition of ϕ in $\mathfrak{A}$ with an assignment s we need only quantify over those assignments which are in $H(\kappa^+)$. Generally third order variables are sets of second order variables and have cardinality at most 2^κ, fourth order variables have cardinality at most $2^{(2^\kappa)}$, and so on. It follows that interpretations of finitely many nth order variables belong to $H((\beth_{n-2}(\kappa))^+)$ and the truth definition of an nth order formula ϕ with an assignment s in a model $\mathfrak{A}$ can be formalized in $H((\beth_{n-2}(\kappa))^+)$ with $\mathfrak{A}$ and s as parameters.

(b) In second order logic we can quantify over transitive closures of the sets in $H(\kappa^+)$ in the following way. If R is a well-founded binary relation which satisfies the extensionality axiom $\forall x \forall y(\forall z(Rzx \leftrightarrow Rzy) \to x = y)$, then $(dom(R), R)$ is by Mostowski's Collapsing Theorem isomorphic to a transitive set. If R is also either empty or has a greatest element, then $(dom(R), R)$ is isomorphic to $(TC(a), \epsilon)$ for some $a \in H(\kappa^+)$. On the other hand, if $a \in H(\kappa^+)$ then $|TC(a)| \leqslant \kappa$ and there is a well-founded and extensional relation $R_a \subset A \times A$ such that $(dom(R_a), R_a)$ is isomorphic to $(TC(a), \epsilon)$. Thus in second order logic we can in a sense quantify over transitive closures of sets in $H(\kappa^+)$.

Let $\psi(R)$ be a second order formula which says that R is a well-founded binary relation which satisfies the extensionality axiom and is either empty or has a greatest element.

We can define two sets R and R' to be equal if and only if there is an isomorphism from $(dom(R), R)$ to $(dom(R'), R')$. Now we can define $x =^* y$ to be $\psi(R_x) \wedge \psi(R_y) \wedge (dom(R_x), R_x) \cong (dom(R_y), R_y)$.

We can define $\in$ as follows: $R_x \in^* R_y$ iff $\exists v \exists w \exists Q \exists T \theta$, where θ expresses the conjunction of the following:

1. v is the greatest element of R_y.
2. $R_y(w, v)$.

[3] To be precise the formulas below, such as $x_j = 0$, are not formulas in our language. However 0, immediate predecessor of an element and immediate successor of an element (all with respect to $<$) are definable so it is possible to write the expressions below formally.

[4] This means we have added an extra unary predicate I to $H(\kappa^+)$ and interpreted it as the set of those subsets of κ^m which are codes of models isomorphic to $\mathfrak{A}$. A $ZF(I)$-sentence is a first order sentence in the vocabulary $\{\in, I\}$, where $\in$ is a binary relation symbol and I is a unary relation symbol.

3. $Q = (P, R_y \restriction P)$ is the transitive closure of w with respect to R_y.
4. T is an isomorphism from $(dom(R_x), R_x)$ to Q.

Let then $x \in^* y = \psi(R_x) \wedge \psi(R_y) \wedge \exists x \exists y \exists Q \exists T \theta$.

Let then $(\neg\phi)^* = \neg(\phi^*)$, $(\phi \wedge \theta)^* = \phi^* \wedge \theta^*$ and $(\exists x\phi)^* = \exists R_x \phi^*$.

We define $I(x)^* = \psi(R_x) \wedge \exists K \exists < \exists R_n^m \exists R_H \exists R_{<'} \exists T(\psi_\kappa(K, <) \wedge \psi_{code}(R_n^m, \mathfrak{A}, <) \wedge \theta^*(R_H, R_{<'}, R_x) \wedge \phi_{\cong}(T, K, <, R_n^m, R_H, R_{<'}, R_x))$. Here are explanations for the parts of the formula.

- The formula ψ_κ defines the model $(\kappa, \in)$ up to isomorphism.
- The formula $\psi_{code}(R_n^m, \mathfrak{A}, <)$ says that R_n^m is a code, with respect to $<$, of a model isomorphic to $\mathfrak{A}$.
- The *ZF*-formula θ says that H is a cardinal, $<'$ is the natural order of H and $x \subseteq H^m$.
- The formula $\phi_{\cong}(T, K, <, R_n^m, R_H, R_{<'}, R_x)$ says that T is an isomorphism from $(K, <, R_n^m)$ to $(R_H, R_{<'}, R_x)$.

Next we will generalize the above result for $n > 2$. First we will define the concept of a *hereditarily monadic variable*. For a start we say that a monadic second order variable is hereditarily monadic. If we have defined what it means for an nth order variable to be hereditarily monadic, we define an $n+1$st order variable to be hereditarily monadic iff it has arity 1 and its only argument is a relation of one type: hereditarily monadic nth order variable. It is easy to prove by induction that in a model of cardinality κ there are $\beth_{n-1}(\kappa)$ hereditarily monadic nth order relations, i.e., interpretations of hereditarily monadic nth order variables.

Now the above proof works for L^{n+1} when we replace first order variables by hereditarily monadic nth order variables and second order variables of arity m by $n+1$st order variables which have as arguments only hereditarily monadic nth order variables.

We denote $A_n = \{A\colon A$ is a hereditarily monadic nth order relation over $\mathfrak{A}\}$. Thus $|A_n| = \beth_{2^{n-1}}(\kappa)$ and in L^{n+1} we can quantify over subsets of $(A_n \times A_n)$. Certain subsets B of $A_n \times A_n$ correspond to transitive closures of sets in $H((\beth_{2^{n-1}}(\kappa))^+)$, namely those sets B such that $(dom(B), B)$ satisfies the axiom of extensionality, B has a largest element and B is well-founded. As before, we can define two sets of the above form to be the same if they are isomorphic and a set a belongs to another set b if and only if there is an element b_0 in the domain of b which belongs to the greatest element in b, and the transitive closure of b_0 with respect to b is isomorphic to a. The relation $I_{\mathfrak{A}}$ is definable up to isomorphism in the same way as in the case of second order logic. □

2.2. Infinitary second order languages

We will next define the second order infinitary language $L^2_{\kappa,\omega}$. The nth order infinitary languages $L^n_{\kappa,\omega}$ can be defined in an analogous way.

Definition 2.6. Let κ be a regular cardinal. The logic $L^2_{\kappa,\omega}$ is the smallest logic which

1. Contains all second order atomic formulas.
2. Is closed under negation, conjunctions of size less than κ, disjunctions of size less than κ, first order existential and universal quantifiers and second order existential and universal quantifiers.[5]

Lemma 2.7. *Let κ be a regular cardinal. In the logic $L_{\kappa,\omega}$ all ordinals $(\alpha, <)$ smaller than κ are characterizable.*

Proof. We define by induction formulas $\theta_\alpha(x)$ for $\alpha < \kappa$.

$$\bigwedge_{\beta<\alpha} \exists y\big(y < x \wedge \theta_\beta(y)\big) \wedge \forall y\left(y < x \to \bigvee_{\beta<\alpha} \theta_\beta(y)\right).$$

The formula $\exists x(\psi_{LO} \wedge \theta_\alpha(x) \wedge \neg\exists y\, x < y)$, where ψ_{LO} is the conjunction of the axioms of linear order, characterizes the ordinal α. □

We will now present a lemma which is needed to show the independence of $A(L^2_{\kappa,\omega}, \kappa)$ from *ZFC* at a regular cardinal κ. In [10] in Definition 1.2.13 an exact coding of $L^2_{\kappa,\omega}$-formulas as set theoretic objects is given to prove the lemma.

Lemma 2.8. *Let $n \in \omega$. Every formula of $L^2_{\kappa^+,\omega}$ can be defined in $(V, \in)$ (or in $(H((\kappa^+)^+), \in)$) by a ZF-formula using a subset of κ as a parameter.*

If κ is an inaccessible cardinal, every formula of $L^2_{\kappa,\omega}$ can be defined in $(V, \in)$ (or in $(H(\kappa^+), \in)$) by a ZF-formula using a subset of some $\lambda < \kappa$ as a parameter.

[5] We allow here both second order relation variables and second order function variables.

3. Ajtai's result, the countable case

3.1. $A(L^2, \omega)$ and L^2-definable well-order of the reals

In this section we will present two theorems of Ajtai which show that $A(L^2, \omega)$ is independent of *ZFC*. After that we will discuss some related topics concerning countable models. We will now present for completeness the proof of the first part of the independence result:

Theorem 3.1. *(See Ajtai [2].) If there is a second order definable well-order of the power set of ω, then $A(L^2, \omega)$ holds. If the well-order is Σ^1_n for $n \geqslant 2$, then $A(\Sigma^1_{n+1}, \omega)$ holds.*

Proof. We will show that if there is a second order definable well-order of the reals, $A(\Sigma^1_k, \omega)$ holds for certain k. Let us assume our second order definable well-order of the reals is Δ^1_n for some $n \geqslant 2$.

As we have shown in Lemma 2.4, a model of cardinality $\aleph_0$ in a finite vocabulary is isomorphic to some models which have ω as universe. These models can be coded into n-ary relations on ω in an up to isomorphism second order definable way, and the set I of codes of models which have ω as their universe and are isomorphic to the model in question is up to isomorphism second order definable in the model in question. As there is a second order definable well-order of the reals and a second order characterizable bijection from ω^n to ω, we can talk in second order logic about the least subset A_0 of ω which is mapped to a set in I by the bijection. For each natural number n we can say in second order logic that n belongs to A_0, and also that n does not belong to A_0. If two countable models in a finite vocabulary have the same second order theory then they have the same set A_0. Consequently they have the same isomorphism type and they are isomorphic.

We will next write formally the sentences described in the above paragraph and calculate the complexity of them. Let Φ be the second order sentence:

$$\begin{aligned}&\exists N \exists 0' \exists 1' \exists +' \exists \cdot' \exists < \exists \pi_n \exists \bar{A}_0 \exists A'_0 \exists A^*_0\\ &\quad \big(\mathrm{def}\big(N, 0', 1', +', \cdot'\big) \wedge \mathrm{def}(\pi_n) \wedge \theta_{\bar{A}_0,\tau}\\ &\quad\ \wedge \psi_{\cong}(\bar{A}_0) \wedge \phi_{code}\big(\bar{A}_0, A'_0\big) \wedge \eta_n\big(A'_0, A^*_0\big)\\ &\quad\ \wedge \forall \bar{A}_1 \forall A'_1 \forall A^*_1 \big(\big(\theta_{\bar{A}_1,\tau} \wedge \psi_{\cong}(\bar{A}_1) \wedge \phi_{code}\big(\bar{A}_1, A'_1\big) \wedge \eta_n\big(A'_1, A^*_1\big)\big)\\ &\qquad \rightarrow \big(\phi'\big(A^*_0, A^*_1\big) \vee \forall x\big(A^*_0(x) \leftrightarrow A^*_1(x)\big)\big)\big) \wedge A^*_0(47)\big).\end{aligned} \tag{3.2}$$

Here are explanations of the different components of the sentence:

- $\mathrm{def}(N, 0', 1', +', \cdot')$ is the Π^1_1-formula which defines the structure $(\mathbb{N}, 0, 1, +, \cdot)$.
- $\mathrm{def}(\pi_n)$ is the first order formula which defines a bijection from N^n to N.
- $\theta_{\bar{A}_0,\tau}$ is a first order formula which says that $\bar{A}_0$ is a sequence of relations on N such that the arities correspond to arities of relations in τ.
- $\psi_{\cong}(\bar{A}_0)$ is a Σ^1_1 formula which says that $\mathfrak{A}$ (i.e., the model itself) is isomorphic to $\bar{A}_0$.
- $\phi_{code}(\bar{A}_0, A'_0)$ is the first order formula which says that A'_0 is the subset of N^n which codes $\bar{A}_0$, see Lemma 2.4.
- $\eta_n(A'_0, A^*_0)$ is the first order formula which says that A^*_0 is the image of A'_0 under π_n.
- $\phi'(A^*_0, A^*_1)$ is the Δ^1_n-formula which says that A^*_0 is strictly smaller than A^*_1 in the well-order of the power set of N defined by ϕ'. The formula ϕ' is formed from ϕ by replacing 0 by $0'$, 1 by $1'$, $+$ by $+'$, $\cdot$ by $\cdot'$ and by relativising all the first order and second order quantifiers to N.
- $A^*_0(47)$ is the first order formula which says that the natural number 47 (in the sense of N) belongs to A^*_0. Similarly we could say by a first order formula that n belongs to (or does not belong to) A^*_0 for any chosen n.

The formula

$$\begin{aligned}&\big(\big(\theta_{\bar{A}_1,\tau} \wedge \psi_{\cong}(\bar{A}_1) \wedge \phi_{code}\big(\bar{A}_1, A'_1\big) \wedge \eta_n\big(A'_1, A^*_1\big)\big)\\ &\quad \rightarrow \big(\phi'\big(A^*_0, A^*_1\big) \vee \forall x\big(A^*_0(x) \leftrightarrow A^*_1(x)\big)\big)\big)\end{aligned}$$

has the same complexity as $\neg\phi'(A^*_0, A^*_1)$, which is Δ^1_n, as ϕ' is Δ^1_n. Then the formula

$$\begin{aligned}&\forall \bar{A}_1 \forall A'_1 \forall A^*_1 \big(\big(\theta_{\bar{A}_1,\tau} \wedge \psi_{\cong}(\bar{A}_1) \wedge \phi_{code}\big(\bar{A}_1, A'_1\big) \wedge \eta_n\big(A'_1, A^*_1\big)\big)\\ &\quad \rightarrow \big(\phi'\big(A^*_0, A^*_1\big) \vee \forall x\big(A^*_0(x) \leftrightarrow A^*_1(x)\big)\big)\big)\end{aligned}$$

has complexity Π^1_n. Now the formula

$$\begin{aligned}&\big(\mathrm{def}\big(N,0',1',+',\cdot'\big)\wedge\mathrm{def}(\pi_n)\wedge\theta_{\bar{A}_0,\tau}\\&\quad\wedge\psi_{\cong}(\bar{A}_0)\wedge\phi_{code}\big(\bar{A}_0,A'_0\big)\wedge\eta_n\big(A'_0,A^*_0\big)\\&\quad\wedge\forall\bar{A}_1\forall A'_1\forall A^*_1\big(\big(\theta_{\bar{A}_1,\tau}\wedge\psi_{\cong}(\bar{A}_1)\wedge\phi_{code}\big(\bar{A}_1,A'_1\big)\wedge\eta_n\big(A'_1,A^*_1\big)\big)\\&\qquad\to\big(\phi'\big(A^*_0,A^*_1\big)\vee\forall x\big(A^*_0(x)\leftrightarrow A^*_1(x)\big)\big)\big)\wedge A^*_0(47)\big)\end{aligned}$$

has complexity Π^1_n and the formula (3.2) has complexity Σ^1_{n+1}. The sentence Φ is true in $\mathfrak{A}$, hence true in $\mathfrak{B}$. So $\mathfrak{A}\cong\mathfrak{B}$. Thus $A(\Sigma^1_{n+1},\omega)$ has been proved. □

Corollary 3.3. *(See Ajtai [2].) If $V=\mathrm{L}$ then $A(L^2,\omega)$ holds.*

Proof. In L there is a Δ^1_2 well-ordering of the power set of ω. □

As there is a Δ^1_2 well-order of the reals in L, Σ^1_3-equivalence implies isomorphism for countable models in L. More generally, if there is a Σ^1_n well-order of the reals, any two countable Σ_{n+1}-equivalent models are isomorphic. Hence they are second order equivalent and the full second order theory of a countable model is determined by its Σ_{n+1}-theory.

However, it does not follow that every second order sentence is equivalent to a Σ^1_{n+1} sentence for countable models [18] (Corollary 14.5 VIII(b)).

Corollary 3.4. *(See Ajtai [2], Harrington [5].) $A(L^2,\omega)$ is consistent with $V\neq\mathrm{L}$, even with the failure of the Continuum Hypothesis.*

Proof. By a result of Harrington [5] it is consistent with *ZFC* that the continuum is as big as desired but has a Δ^1_3-definable well-order. □

If we have a second order definable well-order of the reals with a parameter[6] r then any two countable models which satisfy the same second order theory with parameter r are isomorphic. This can be seen by just adding a parameter to the proof of Theorem 3.1. However, in this article we do not give much attention to the case where we allow parameters: We are generally interested in the possibility to determine isomorphism types of models by their theories in languages having sentences smaller than the cardinality of the model. Thus using a real parameter in a language to determine isomorphism type of a countable model (a real) is a bit disappointing.

However, we note the following result of Harrington [5]: It is consistent with *ZFC* that Martin's Axiom holds, the continuum is as big as wanted and there is a second order definable well-order of the reals using a real parameter. It follows that there is a model of *ZFC* in which the following hold:

1. Martin's Axiom.
2. For some real parameter r, second order equivalence with the real parameter r implies isomorphism for countable models.

Open Question 3.5. *Is Martin's Axiom consistent with $A(L^2,\omega)$?*

A second order definable well-order of the reals is also consistent with measurable and Woodin cardinals, which cannot exist in L. We will return to these large cardinals in Section 7.

By Theorem 3.1 $A(L^2,\omega)$ is consistent. In all our examples where $A(L^2,\omega)$ holds this is based on a second order definable well-order of the reals.

Open Question 3.6. *Is it consistent with ZFC that $A(L^2,\omega)$ holds, but there is no second order definable well-order of the reals?*

3.2. Optimality

We proved before that $A(\Sigma^1_3,\omega)$ is consistent with *ZFC*. Let us observe that $A(\Sigma^1_1,\omega)$ is simply false in *ZFC*.

Theorem 3.7. *For any infinite cardinal κ there are two non-isomorphic Σ^1_1-equivalent models of Peano Axioms of cardinality κ. In particular there are two non-isomorphic countable Σ^1_1-equivalent models of Peano Axioms.*

[6] The logic for second order logic with a real parameter is $L^2(Q_r)$, the second order logic with a generalized quantifier Q_r. The quantifier Q_r is defined as $\mathfrak{A}\models Q_r(x)\phi(x)\Leftrightarrow|\{x\colon\ \mathfrak{A}\models\phi(x)\}|\in r$. Note that if we have $(\omega,<)$ in the model (either in the vocabulary of the model or as interpretation of second order variables) then the formula $\psi(X)=\forall x(X(x)\leftrightarrow Q_r y(y<x))$ defines the real r as a subset of ω.

Sketch. We construct an elementary chain of length ω_1 of countable models of Peano Axioms. We put $\mathfrak{A}_0$ to be the standard model of arithmetic. If $\mathfrak{A}_\alpha$ has been defined we choose $\mathfrak{A}_{\alpha+1}$ to be an elementary extension which realizes some new type. Thus all the models in the chain become pairwise non-isomorphic. To make some of the models in chain Σ^1_1-equivalent, we make sure that the Σ^1_1 sentences true in the models of the chain are increasing. Thus for each Σ^1_1 formula $\exists R\phi$ which is true in the standard model of arithmetic we put a new relation to the vocabulary of $\mathfrak{A}_0$ and interpret it in such a way that the formula ϕ is satisfied. If $\mathfrak{A}_{\alpha+1}$ satisfies some Σ^1_1 sentences (in the original vocabulary) which are not true in $\mathfrak{A}_\alpha$ then we add new relations to the model so that every Σ^1_1 sentence is satisfied by a relation in the model. Since there are only countably many Σ^1_1 sentences, there is such an $\alpha < \omega_1$ that from α forward all models in the chain are Σ^1_1-equivalent in the original vocabulary. Thus from some α forward, all models in the chain are Σ^1_1-equivalent but not isomorphic.

The above proof works for all cardinalities $\aleph_\alpha < 2^{\aleph_0}$. In any cardinality greater than or equal to $2^{\aleph_0}$ there are more than $2^{\aleph_0}$ many non-isomorphic models of arithmetic, so the claim follows.

The theorem above is formulated for Peano Axioms, but the proof works equally well for any theory which has $2^{\aleph_0}$ many types and more than continuum many non-isomorphic models in all cardinalities greater than or equal to the continuum.

Open Question 3.8. *Is it consistent with ZFC that $A(\Sigma^1_2, \omega)$ holds?*

Open Question 3.9. *If $V = \mathrm{L}$, are there two countable non-isomorphic models which have the same monadic second order theory?*

3.3. Failure of $A(L^2, \omega)$

We will now recall the second part of the independence proof of Ajtai [2]:

Theorem 3.10. *(See Ajtai [2].) It is consistent with ZFC, that there are two countable non-isomorphic models which satisfy the same formulas of the language of set theory. In particular the models are second order equivalent and L^n-equivalent for all n.*

In the proof of Theorem 3.10 one Cohen-generic real is added to the set theoretic universe, and as a result there will be two second order equivalent non-isomorphic countable models in the generic extension. But actually by a little modification of the proof, we can add many generic reals to the universe and get many countable second order equivalent non-isomorphic models.

Theorem 3.11. *Let κ^+ be an infinite cardinal. There is a notion of forcing P that preserves cardinals and forces that there are κ^+ countable ZF-equivalent non-isomorphic models.*

Proof. We add κ^+ many Cohen-generic reals to the ground model. Forcing conditions are finite functions from $\kappa^+ \times \omega$ to $\{0, 1\}$. A forcing condition p is stronger than another forcing condition q iff p extends q. If G is a generic set for this notion of forcing, for all $\alpha < \kappa^+$, $f_\alpha = \{n\colon\ G(\alpha, n) = 1\}$ is a generic real. Note that for all $\alpha < \beta < \kappa^+$, f_α and f_β differ in infinitely many points. If A is a subset of ω, we denote by F^A the set of all subsets of ω which differ from A only in finitely many points. We are discussing the models $(F^{f_\alpha} \cup \omega, <_\omega, P_{f_\alpha})$, where $<_\omega$ is the natural order of ω and P_{f_α} is the relation which tells which natural numbers n belong to which sets in F^{f_α}.[7]

Let us denote the models constructed around f_α and f_β as described above by M^{f_α} and M^{f_β}. We will show that the models are *ZF*-equivalent. Assume not: then there is a forcing condition p and a *ZF*-sentence ϕ with possibly parameters from the ground model such that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$. So, suppose G is a generic filter over V containing p such that $V^G \models \phi(M^{f_\alpha}) \wedge \neg\phi(M^{f_\beta})$. Consider the following mapping $H_{p,\alpha,\beta} : dom(P) \to dom(P)$. Let $H_{p,\alpha,\beta}(f) = g$ such that the following hold:

- $g(\gamma, n) = f(\gamma, n)$, if $(\gamma, n) \in dom(p)$ and $(\gamma, n) \in dom(f)$.
- $g(\gamma, n) = f(\gamma, n)$, if $\gamma \in \kappa^+ \setminus \{\alpha, \beta\}$ and $(\gamma, n) \in dom(f)$.
- $g(\alpha, n) = f(\alpha, n)$, if $(\beta, n) \in dom(p)$ and $(\alpha, n) \in dom(f)$.
- $g(\beta, n) = f(\beta, n)$, if $(\alpha, n) \in dom(p)$ and $(\beta, n) \in dom(f)$.
- $g(\alpha, n) = f(\beta, n)$, if $(\beta, n) \in dom(f) \setminus dom(p)$ and $(\alpha, n) \notin dom(p)$.
- $g(\beta, n) = f(\alpha, n)$, if $(\alpha, n) \in dom(f) \setminus dom(p)$ and $(\beta, n) \notin dom(p)$.
- $g(\gamma, n)$ not defined, if none of the above conditions applies.

Let us denote the image of G under $F_{p,\alpha,\beta}$ by G'. As $F_{p,\alpha,\beta}$ is an automorphism of P, G' is a generic filter. The generic filter G' agrees with G in all ordinals different from α and β, and if at least one of $\{(\alpha, n), (\beta, n)\}$ belongs to $dom(p)$ then

[7] In fact the union of the relations $<_\omega$ and P_{f_α} is $\in$, so we could also form the model in vocabulary $\{\in\}$ instead of $\{<_\omega, P_{f_\alpha}\}$. We follow here Ajtai, whose vocabulary is maybe more intuitive than the alternative vocabulary.

G' agrees with G in (α,n) and (β,n). For those digits n for which neither (α,n) nor (β,n) belongs to $dom(p)$ the generic filter G' changes digits of α to digits of β and vice versa. Now $V^G = V^{G'}$, $p \in G'$ and the interpretations of $\dot{M}^{f_\alpha}$ and $\dot{M}^{f_\beta}$ swap places in the two generic extensions. Thus it is impossible that $p \Vdash \phi(\dot{M}^{f_\alpha}) \wedge \neg\phi(\dot{M}^{f_\beta})$.

But $(F^{f_\alpha} \cup \omega, <_\omega, P_{f_\alpha})$ and $(F^{f_\beta} \cup \omega, <_\omega, P_{f_\beta})$ are non-isomorphic: Since ω is a rigid structure, in an isomorphism every set in F^{f_α} should be mapped to exactly the same set in F^{f_β}. But this is impossible because $f_\alpha \notin F^{f_\beta}$. $\square$

The proof of Theorem 3.10 is the same: the non-isomorphic second order equivalent models M^G and M^{-G} are constructed from the Cohen-generic real G and its complement $-G$.

4. Fourth order logic

4.1. Coding subsets by collapsing cardinals

In Section 3 we saw that $A(L^2,\omega)$ is independent of *ZFC*. A natural question is whether analogous results can be proved for other higher order logics L^n or various uncountable cardinals κ. Our results in this section were inspired by the following theorem of Ajtai [2]:

Theorem 4.1. *(See Ajtai [2].) There is a model of ZFC in which $A(L^n,\omega)$ fails for every $n \in \omega$ but $A(ZF,\omega)$ holds.*

Next we will give some motivation to our definition of a forcing $P_{X',\kappa}$, which is used a lot in this section. The forcing uses some ideas of Kenneth McAloon [15].

Assume $M = \mathrm{L}[X]$, λ is a cardinal and $X \subseteq \lambda$. Assume also that M and L have the same cardinals, $\kappa = \aleph_\alpha^M$ is a cardinal in M and *GCH* holds above κ in M. We will next introduce the forcing $P_{X',\kappa}$, which makes X definable from κ, but does not add any new subsets to κ. Let X' be a subset of $\lambda \setminus \{\beta\colon \beta \text{ is a limit ordinal}\}$ such that X' and X contain the same information.[8] The forcing is an iterated forcing of length λ with full support at all limit stages. The idea is that $P_{X',\kappa}$ collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$ for $\beta \in X'$, and does not collapse any other cardinals. After the forcing, X' (and hence X) is definable from α as $X' = \{\beta < \lambda\colon \aleph^{\mathrm{L}}_{\alpha+\omega\cdot\beta+2} \text{ is not a cardinal}\}$. Next we will give an exact definition of the forcing conditions.

Definition 4.2 *($P_{X',\kappa}$).* The forcing conditions are sequences $(p_\beta)_{\beta<\lambda}$ such that the following hold:

1. If $0 \in X'$, then P_0 is the set of partial functions from $\aleph_{\alpha+1}$ to $\aleph_{\alpha+2}$ of cardinality smaller than $\aleph_{\alpha+1}$. A forcing condition p is stronger than a forcing condition q if and only if p extends q. If $0 \notin X'$, then P_0 is the trivial forcing.
2. Assume $\beta = \gamma + 1$ and $P_{\gamma'}$ has been defined for all $\gamma' \leqslant \gamma$.
   If $\beta \in X'$, we define P_β to be the set of sequences $p_\gamma, \gamma \leqslant \beta$ where the γth coordinate belongs to P_γ for each $\gamma < \beta$ and the βth coordinate is a forcing name $\dot{Y}$ such that $p \restriction \beta \Vdash$ "$\dot{Y}$ is a partial function from $\aleph_{\alpha+\omega\cdot\beta+1}$ to $\aleph_{\alpha+\omega\cdot\beta+2}$ of cardinality smaller than $\aleph_{\alpha+\omega\cdot\beta+1}$". If p and q are two conditions of length β then p is stronger than q if and only if $p \restriction \gamma$ is stronger than $q \restriction \gamma$ and $p \restriction \beta \Vdash$ "$p(\beta)$ and $q(\beta)$ are partial functions from $\aleph_{\alpha+\omega\cdot\beta+1}$ to $\aleph_{\alpha+\omega\cdot\beta+2}$ of cardinality smaller than $\aleph_{\alpha+\omega\cdot\beta+1}$ and $p(\alpha) \supseteq q(\alpha)$".
   If $\beta \notin X'$ then P_β is the trivial forcing.
3. If β is a limit ordinal, the forcing conditions in P_β are the sequences p of length β such that for each $\gamma < \beta$ $p \restriction \gamma \Vdash p(\gamma) \in P_\gamma$. This forcing has full support in all limit stages, which means that in limit stages all coordinates of a forcing condition may be non-zero. A forcing condition p is stronger than q if and only if $p \restriction \gamma$ is stronger than $q \restriction \gamma$ for each $\gamma < \beta$.

Lemma 4.3. *Assume $M = \mathrm{L}[X]$, λ is a cardinal, $X \subseteq \lambda$ and $X' = \{\alpha + 1\colon \alpha \in X\}$. Assume also that M and* L *have the same cardinals and GCH holds above κ in M. Let G be a $P_{X',\kappa}$-generic set over M. $M[G] \models X' = \{\beta < \lambda\colon \aleph^{\mathrm{L}}_{\alpha+\omega\cdot\beta+2}$ is not a cardinal$\}$.*

Proof. We prove by induction on β that after P_β the claim holds for all $\gamma \leqslant \beta$, i.e., for all $\gamma \leqslant \beta$, $\aleph^{\mathrm{L}}_{\alpha+\omega\cdot\gamma+2}$ is a cardinal iff $\gamma \in X'$. The rest of the iterated forcing is $< \aleph_{\alpha+\omega\cdot(\beta+1)}$-closed and does not add subsets of $\aleph_{\alpha+\omega\cdot\beta+1}$ so the claim follows.

1. Let $\beta = 0$. If $0 \notin X'$ then P_0 is the trivial forcing and the claim holds. If $\beta \in X'$ then P_β collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$. The forcing P_β is $< \aleph_{\alpha+\omega\cdot\beta+1}$-closed and has cardinality $\aleph_{\alpha+\omega\cdot\beta+2}$ (because *GCH* holds above $\kappa = \aleph_\alpha$), so other cardinals and *GCH* above κ are preserved.
2. Let $\beta = \gamma + 1$ and assume Induction Hypothesis holds for γ. If $\beta \notin X'$ then P_β is the trivial forcing and the claim holds. If $\beta \in X'$ then P_β collapses $\aleph_{\alpha+\omega\cdot\beta+2}$ to $\aleph_{\alpha+\omega\cdot\beta+1}$. Note that P_β is $< \aleph_{\alpha+\omega\cdot\beta+1}$-closed and has cardinality $\aleph_{\alpha+\omega\cdot\beta+2}$, because *GCH* above κ holds. It follows that P_β preserves other cardinals. Also *GCH* above κ is preserved so the claim holds.

[8] For example for all $\alpha < \lambda : \alpha \in X \leftrightarrow \alpha + 1 \in X'$.

3. Assume β is a limit ordinal and the Induction Hypothesis holds for all smaller ordinals. The forcing P_β has cardinality at most $\aleph_{\alpha+\omega\cdot\beta}$ so it does not collapse any cardinals greater than $\aleph_{\alpha+\omega\cdot\beta}$. Also $\aleph_{\alpha+\omega\cdot\beta}$ is not collapsed because there are cofinally many cardinals below which are not collapsed. *GCH* above κ is also preserved.
4. The whole forcing $P_{X',\kappa}$ has cardinality at most $\aleph_{\alpha+\omega\cdot\lambda}$ so cardinals greater than $\aleph_{\alpha+\omega\cdot\lambda}$ are preserved. The cardinal $\aleph_{\alpha+\omega\cdot\lambda}$ itself is preserved, as cofinally many cardinals below it are preserved. $\square$

Theorem 4.4. *Let κ be a cardinal in* L. *There is a model of ZFC in which $2^\kappa = 2^{\aleph_0}$, $A(L^4,\kappa)$ holds, $A(L^2,\kappa)$ fails and all cardinals $\leqslant \kappa$ of* L *are preserved.*

Proof. Let L be the ground model. We make an iterated forcing which has three parts and length $\kappa^+ + 1$. After the forcing fourth order equivalence implies isomorphism in cardinality κ but second order equivalence does not:

1. First we add 2^κ Cohen-subsets of ω. This forcing does not collapse any cardinals and after the forcing $2^{\aleph_0} = 2^\kappa$.
2. Now let G be the generic set we added in step 1. and let Π be a bijection from $2^{\aleph_0}$ to 2^κ in $V[G]$. We want to make G and Π definable from κ in the language of set theory, but not to make them up to isomorphism second order definable in cardinality κ.
   As G and Π are of cardinality $2^\kappa = \kappa^+$, there is a subset X of 2^κ which codes them both. Let X' be a subset of $2^\kappa \setminus \{\gamma\colon \gamma$ is 0 or a limit ordinal$\}$ such that X and X' are definable from each other. We will now make one such X' definable from κ in the language of set theory. Let $\dot{X}'$ be a forcing name for X'. After step 1. of this iterated forcing the *GCH* holds above κ, $\dot{X}'$ has cardinality 2^κ, and the cardinals are the same as in L, so by Lemma 4.3 $P_0 * P_{\dot{X}',\kappa}$ does not add any new subsets of κ and makes $\dot{X}'$ definable from κ in the language set theory.
3. In the last step we add $\aleph_{\alpha+\omega\cdot\kappa^+ +1}$ Cohen subsets of κ^+. This does not collapse cardinals or add new subsets of κ. Now $\dot{X}'$ is definable in $(H((\beth_2(\kappa))^+), \in)$ with α as parameter, as $\dot{X}' = \{\beta\colon \aleph^{\mathrm{L}}_{\alpha+\omega\cdot\beta+2}$ is not a cardinal$\}$. In $(H((\beth_2(\kappa))^+), \in, I_{\mathfrak{A}})$ $\dot{X}'$ is definable without parameters. As $\dot{X}'$ codes G, the canonical well-order of $L[G]$ up to sets of cardinality κ is definable as well. As $\dot{X}'$ codes also a bijection from $2^{\aleph_0}$ to 2^κ, by Lemma 2.5, there is a fourth order definable well-order of the power set of κ and an up to isomorphism fourth order definable bijection from the power set of κ to the reals. Now, as in Theorem 3.1, we can have fourth order sentences which say "There are $R_0 \subseteq \kappa$ and $R_0' \subseteq \omega$ such that R_0 is the least subset in the well-order isomorphic to the model in question and Π maps R_0 to R_0' and $R_0'(8743)$". Sentences of this form determine the isomorphism type of the model so $A(L^4,\kappa)$ holds after the forcing. $A(L^2,\kappa)$ fails after the forcing as it fails after the first Cohen forcing[9] and we did not add any subsets of κ after that. $\square$

Theorem 4.5. *Let κ be a cardinal in* L. *There is a model of ZFC in which $2^\kappa = 2^{\aleph_0}$, and $A(L^4,\lambda)$ holds and $A(L^2,\lambda)$ fails in any cardinality $\lambda \leqslant \kappa$.*

Proof. Let L be the ground model. We use an iterated forcing which has the following steps:

1. We add $2^\kappa = \kappa^+$ Cohen subsets of ω. Cardinals are preserved in this forcing and after this forcing $2^\lambda = 2^\kappa = \kappa^+$ for any $\lambda \leqslant \kappa$. Also $A(L^2,\lambda)$ fails for all $\lambda \leqslant \kappa$, see Theorem 6.7 below.
2. Now let G be the generic set we added in step 1 and let $\{\Pi_\lambda\colon \lambda \leqslant \kappa\}$ be a set such that each Π_λ is a bijection from $2^{\aleph_0}$ to 2^λ in $V[G]$. Let $\dot{X}'$ be a subset of $2^\kappa \setminus \{\gamma\colon \gamma$ is 0 or a limit ordinal$\}$ which codes G and all the bijections Π_λ.
   As in the previous theorem $P_0 * P_{\dot{X}',\kappa}$ makes X' definable from κ and adds the same subsets of κ as P_0 alone.
3. In the last step we add $\aleph_{\alpha+\omega\cdot\kappa^+ +1}$ Cohen subsets of 2^κ.
   After the forcing $A(L^2,\lambda)$ fails for every $\lambda \leqslant \kappa$ as we did not add any new subsets of λ after step 1. After the forcing $A(L^4,\lambda)$ holds for all $\lambda \leqslant \kappa$ as in $H(\beth_2(\lambda)^+)$ there is a definable well-order of the power set of λ and a definable bijection from 2^λ to 2^ω. $\square$

Theorem 4.6. *Let κ be a cardinal in* L *and let n be a natural number greater than or equal to* 2. *There is a model of ZFC in which $A(L^n,\kappa)$ fails but $A(L^{n+2},\kappa)$ holds.*

Proof. Let L be the ground model. Our iterated forcing has the following steps:

1. We add 2^κ Cohen subsets of ω. After this step $A(L^n,\kappa)$ fails for every n, $2^\kappa = 2^\omega$, *GCH* holds above κ and all cardinals of L remain cardinals.
2. Let $\dot{X}'$ be a forcing name for a subset of $\kappa^+ \setminus \{\gamma\colon \gamma$ is a limit ordinal or 0$\}$ which codes the generic set added in step 1 and a bijection Π from 2^ω to 2^κ. The second step is $P_{\dot{X}',\beth_{n-2}(\kappa)}$ This step does not add any subsets of $\beth_{n-2}(\kappa)$.
3. Cohen forcing which adds $\aleph_{\alpha+\omega\cdot 2^\kappa+1}$ subsets of $\beth_{n-1}(\kappa)$.

[9] Theorem 3.10 proves this for $\kappa = \omega$ and Theorem 6.7 below proves the uncountable case.

After the forcing $A(L^n,\kappa)$ fails as it fails after the first Cohen forcing and no subsets of $\beth_{n-2}(\kappa)$ are added after that. After the forcing $\dot{X}'$ is definable in $H((\beth_n(\kappa))^+)$ and thus there is an up to isomorphism L^{n+2}-definable well-order of the power set of κ and an up to isomorphism L^{n+2}-definable bijection from 2^κ to 2^ω. It follows that $A(L^{n+2},\kappa)$ holds. $\square$

Note that there are several open questions left, for example the following:

Open Question 4.7. *Does $A(L^{n+1},\kappa)$ hold after the above forcing? Or does it depend on κ and n whether $A(L^{n+1},\kappa)$ holds after the above forcing?*

Theorem 4.8. *Let κ be a cardinal definable in* L*. There is a model of ZFC in which $A(L^n,\kappa)$ fails for every n but $A(ZF,\kappa)$ holds and all cardinals $\leqslant\kappa$ of* L *are preserved.*

Proof. This is just an obvious generalization of Ajtai's Theorem 4.1. Note that the theorem could be also proved by using the forcing $P_{X,\kappa}$. Let L be the ground model. We do an iterated forcing with two steps:

1. Let P_0 be a forcing which adds $2^\kappa=\kappa^+$ Cohen subsets of ω. After this forcing there are two *ZF*-equivalent non-isomorphic models of cardinality κ in a finite vocabulary. The models are also L^n-equivalent for any natural number n. This forcing does not collapse any cardinals and also *GCH* above κ is preserved. After this forcing $2^\kappa=2^\omega$.
2. In the second step we make the Cohen subset G which we added in step 1 and a bijection Π from 2^κ to 2^ω definable in the language of set theory. We make this in such a way, that the truth of all L^n sentences in models of cardinality κ is preserved, and after the forcing the power set of κ has a *ZF*-definable well-order and there is a *ZF*-definable bijection from 2^κ to 2^ω. Consequently $A(ZF,\kappa)$ holds after the forcing.
   As κ is a definable cardinal in L, also $\aleph_{\kappa+\omega}$ is a definable cardinal in L. As *GCH* holds above κ in $L[G]$, the truth of L^n sentences in models of cardinality κ in $L[G]$ is determined by sets which are hereditarily smaller than $\aleph_{\kappa+\omega}$. We will introduce a forcing which makes G and Π definable in the language of set theory but does not add any sets which are hereditarily smaller than $\aleph_{\kappa+\omega}$.
   Let $X\subseteq\kappa^+$ be a set which codes G and Π. Let P_1 be a forcing which adds $\aleph_{\kappa+\omega+\alpha+2}$ Cohen subsets of $\aleph_{\kappa+\omega+\alpha}$ for those α for which $\alpha\in X$. After the forcing we can read X as the function from κ^+ to $\{0,1\}$ which maps α to 0 if *GCH* holds at $\aleph_{\kappa+\omega+\alpha}$ and to 1 otherwise. Now as X is definable by a *ZF*-formula we have a *ZF*-definable well-order of the power set of κ and a *ZF*-definable bijection from 2^κ to 2^ω. It follows that $A(ZF,\kappa)$ holds. $\square$

5. Generalized quantifiers

5.1. The countable case

In this section we investigate whether higher order logics can be replaced in the above results by a logic with generalized quantifiers. A clear limitation is provided by the following result [6]:

Theorem 5.1. *(See Hella [6].) Let n be a natural number. Let $\{Q_i\colon i\in I\}$ be a set of generalized quantifiers of arity $\leqslant n$ and let κ be any infinite cardinal. Then there are two models of cardinality κ which are $L(\{Q_i\colon i\in I\})$-equivalent but not isomorphic.*

In view of the above theorem, in order to characterize all models of an infinite cardinality by their theories in a logic $L(\{Q_i\colon i\in I\})$, the arity of the generalized quantifiers of the logic has to increase beyond any finite bound. On the other hand, if we let the arity grow we can find a generalized quantifier logic L such that $A(L,\kappa)$ holds provably in ZFC. We will next give the definition of the above mentioned language in case $\kappa=\omega$.

Definition 5.2. Let $(\mathfrak{A}_r)_{r\in\mathbb{R}}$ be an indexing of all countable models in finite vocabularies by real numbers, i.e., for any countable model $\mathfrak{A}$ in a finite vocabulary there is exactly one $r\in\mathbb{R}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_r$.

The language $L^*=L(Q_{r,s}\colon r,s\in\mathbb{Q})$ contains atomic formulas, is closed under negation, conjunction and first order existential and universal quantifiers. L^* is also closed under the quantifiers

$$Q_{r,s}\bar{x}^1,\dots,\bar{x}^n\big(\phi_1\big(\bar{x}^1\big),\dots,\phi_n\big(\bar{x}^n\big)\big) \tag{$*$}$$

for all $r,s\in\mathbb{Q}$. The formulas ϕ_i above can be any formulas of L^* with the given free variables. The notation $\bar{x}^k$ is a shorthand for $x^k_1,\dots,x^k_{N_k}$, where $x^n_m\neq x^p_o$ whenever $m\neq o$ or $n\neq p$.

Recall that $\phi^{\mathfrak{M}}(\cdot)=\{\bar{x}\colon \mathfrak{M}\models\phi(\bar{x})\}$. The formula $(*)$ is defined to be true in a model $\mathfrak{M}$ if and only if $|\mathfrak{M}|=\aleph_0$ and $(M,\phi_1^{\mathfrak{M}}(\cdot),\dots,\phi_n^{\mathfrak{M}}(\cdot))$ is isomorphic to a structure $\mathfrak{A}_t$ such that $r<t<s$.

We cannot prove in *ZFC* that there is any such indexing $(\mathfrak{A}_r)_{r\in\mathbb{R}}$ of the countable models, which is definable in the language of set theory. But we fix one such indexing no matter whether it is definable or not.

Theorem 5.3. *(See Weinstein (unpublished).)*[10] *In any model of ZFC there is a countable language L^* such that $A(L^*, \omega)$ holds.*

Proof. Let L^* be as above. Let $\mathfrak{A}$ be a countable model in a finite relational vocabulary $(R_1, \dots, R_n)$. Note that constants can be coded into unary relations and n-ary functions can be coded into $n+1$-ary relations so restriction to relational vocabularies does not make the result less general. The sentence $Q_{r_0,s_0}\bar{x}^1, \dots, \bar{x}^n(R_1(\bar{x}^1), \dots, R_n(\bar{x}^n))$ is true in $\mathfrak{A}$ if and only if the r such that $\mathfrak{A}_r$ is isomorphic to $(A, R_1^{\mathfrak{A}}, \dots, R_n^{\mathfrak{A}})$ is between r_0 and s_0. Suppose now $\mathfrak{A}$ and $\mathfrak{B}$ are two countable non-isomorphic models in vocabulary τ. Then $\mathfrak{A}$ is isomorphic to some $\mathfrak{A}_p$ and $\mathfrak{B}$ is isomorphic to some $\mathfrak{A}_q$ for different p and q. Let r_0 and s_0 be such that $r_0 < p < s_0$ and either $q < r_0$ or $s_0 < q$. Then

$$\mathfrak{A} \models Q_{r_0,s_0}\bar{x}^1, \dots, \bar{x}^n\big(R_1\big(\bar{x}^1\big), \dots, R_n\big(\bar{x}^n\big)\big)$$

but

$$\mathfrak{B} \models \neg Q_{r_0,s_0}\bar{x}^1, \dots, \bar{x}^n\big(R_1\big(\bar{x}^1\big), \dots, R_n\big(\bar{x}^n\big)\big). \qquad \square$$

5.2. The uncountable case

Theorem 5.3 can be generalized to any infinite cardinality as we will do next.

The proof is based on an idea of Per Lindström (Personal letter to Jouko Väänänen, 1 August 1974). First we will give the definition of the relevant logic.

Definition 5.4. Let $(\mathfrak{A}_f)_{f:\kappa\to\{0,1\}}$ be an indexing of all models of cardinality κ in finite vocabularies.

Define the sets X_α and X'_α as follows: $X_\alpha = \{f : \kappa \to \{0,1\}\colon\ f(\alpha) = 0\}$, $X'_\alpha = \{f : \kappa \to \{0,1\}\colon\ f(\alpha) = 1\}$.

Let $L^{\kappa *} = L(Q^S_\alpha, R^S_\alpha\colon\ \alpha < \kappa$, S a finite set of variables) contain atomic formulas, be closed under negation, conjunction and first order existential and universal quantifiers. Let S be any finite sequence of finite sequences of distinct variables $(S = (\bar{x}^1, \dots, \bar{x}^k))$. Let $L^{\kappa *}$ be also closed under the following quantifiers Q^S_α and R^S_α:

$$Q^S_\alpha\bar{x}^1, \dots, \bar{x}^k\big(\phi_1\big(\bar{x}^1\big), \dots, \phi_k\big(\bar{x}^k\big)\big)$$
$$R^S_\alpha\bar{x}^1, \dots, \bar{x}^k\big(\phi_1\big(\bar{x}^1\big), \dots, \phi_k\big(\bar{x}^k\big)\big).$$

The formula $Q^S_\alpha\bar{x}^1, \dots, \bar{x}^k(\phi_1(\bar{x}^1), \dots, \phi_k(\bar{x}^k))$ is true in a model $\mathfrak{M}$ iff $|\mathfrak{M}| = \kappa$ and $(M, \phi_1^{\mathfrak{M}}(\cdot), \dots, \phi_k^{\mathfrak{M}}(\cdot))$ is isomorphic to an $\mathfrak{A}_f$ such that $f \in X_\alpha$.

The formula $R^S_\alpha\bar{x}^1, \dots, \bar{x}^k(\phi_1(\bar{x}^1), \dots, \phi_k(\bar{x}^k))$ is true in a model $\mathfrak{M}$ iff $|\mathfrak{M}| = \kappa$ and $(M, \phi_1^{\mathfrak{M}}(\cdot), \dots, \phi_k^{\mathfrak{M}}(\cdot))$ is isomorphic to an $\mathfrak{A}_f$ such that $f \in X'_\alpha$.

Note that there are countably many finite vocabularies, and for any finite vocabulary there are at most 2^κ pairwise non-isomorphic models of cardinality κ with the vocabulary. Thus an indexing $(\mathfrak{A}_f)_{f:\kappa\to\{0,1\}}$ of all models of cardinality κ in finite vocabularies always exists though may be impossible to define in the language of set theory.

Theorem 5.5. *Let κ be an infinite cardinal. There is a language $L^{\kappa *}$ of cardinality κ such that $A(L^{\kappa *}, \kappa)$ holds.*

Proof. Let $L^{\kappa *}$ be as above. Any $f : \kappa \to \{0,1\}$ can be expressed as an intersection of κ many sets of the form X_α and X'_α, namely $\{f\} = \bigcap\{X_\alpha\colon\ f(\alpha) = 0\} \cap \bigcap\{X'_\alpha\colon\ f(\alpha) = 1\}$. On the other hand, if f and g are two different functions from κ to $\{0,1\}$, there is an X_α such that one of f and g belongs to X_α and the other does not.

As in the previous theorem, assume without loss of generality that a model $\mathfrak{A}$ of cardinality κ has a relational vocabulary $R_1, \dots, R_n$.

The sentence $Q_\alpha\bar{x}^1, \dots, \bar{x}^n(R_1(\bar{x}^1), \dots, R_n(\bar{x}^n))$ is true in a model $\mathfrak{A}$ if and only if the f such that $\mathfrak{A}_f$ is isomorphic to $(A, R_1^{\mathfrak{A}}, \dots, R_n^{\mathfrak{A}})$ belongs to X_α. The sentence $R_\alpha\bar{x}^1, \dots, \bar{x}^n(R_1(\bar{x}^1), \dots, R_n(\bar{x}^n))$ is true in a model $\mathfrak{A}$ if and only if the f such that $\mathfrak{A}_f$ which is isomorphic to $(A, R_1^{\mathfrak{A}}, \dots, R_n^{\mathfrak{A}})$ belongs to X'_α.

Suppose τ is a finite relational vocabulary $(R_1^{m_1}, \dots, R_n^{m_n})$ where the superscripts denote the arities of the relation symbols. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two non-isomorphic models of cardinality κ with vocabulary τ. Let P be a sequence of variables which corresponds to arities of the relation symbols, i.e., $P = \bar{x}^1, \dots, \bar{x}^n$ such that each $\bar{x}^p$ contains m_p variables. Now $(A, R_1(\cdot)^{\mathfrak{A}}, \dots, R_n(\cdot)^{\mathfrak{A}})$ is isomorphic to some $\mathfrak{A}_f$ and $(B, R_1(\cdot)^{\mathfrak{B}}, \dots, R_n(\cdot)^{\mathfrak{B}})$ is isomorphic to some $\mathfrak{A}_g$ and $f \neq g$. So there is a β such that one of f and g (say f) gets value 0 at β and the other gets value 1. Now $\mathfrak{A} \models Q^P_\beta\bar{x}^1, \dots, \bar{x}^n(R_1(\bar{x}^1), \dots, R_n(\bar{x}^k))$ but $\mathfrak{B} \models \neg Q^P_\beta\bar{x}^1, \dots, \bar{x}^n(R_1(\bar{x}^1), \dots, R_n(\bar{x}^k))$. $\square$

[10] This result and its proof is presented here with the permission of Professor Scott Weinstein.

Hella's result 5.1 showed that in order to characterize all models of cardinality κ the arity of the quantifiers in the language must be unbounded. However, if we look at the proof above we see that if we want to characterize all models of cardinality κ in a fixed finite vocabulary, the arity of quantifiers in the language can be bounded.

6. Infinitary second order languages

6.1. Discussion

We noted in Theorem 3.1 that $A(L^2, \omega)$ is consistent with *ZFC*. But is $A(L^2, \aleph_1)$ consistent with *ZFC*? It is easy to show by a simple cardinality argument that $A(L^2, \aleph_1)$ does not necessarily hold.

In any finite vocabulary with a binary predicate there are $2^{\aleph_0}$ many L^2-theories. In a finite vocabulary with a binary predicate there are $2^{\aleph_1}$ models of cardinality $\aleph_1$ which are pairwise non-isomorphic. Thus if $2^{\aleph_0} < 2^{\aleph_1}$, then there are two second order equivalent non-isomorphic models of cardinality $\aleph_1$. However, if $2^{\aleph_0} = 2^{\aleph_1}$ we don't know what happens.

Open Question 6.1. *Is it consistent with ZFC that $2^{\aleph_0} = 2^{\aleph_1}$ and $A(L^2, \aleph_1)$ holds?*

By appropriate coding, sentences of second order logic are natural numbers and second order theories are real numbers. Via coding, countable models are also real numbers, so the question whether any two different reals of the latter type correspond to two different reals of the former type is meaningful.

It is well known that all models of cardinality κ can be characterized up to isomorphism by an L_{κ^+,κ^+} sentence. However, these sentences have the same cardinality as the model in question. In this paper we are interested in the possibility of characterizing models up to isomorphism by theories, where the sentences have cardinality smaller than the model.

We make the following observations about the possibility to characterize models up to isomorphism by infinitary languages. In the countable cardinality of the models $L_{\omega_1,\omega}$-equivalence implies isomorphism. Generally $L_{\infty,\omega}$-[11]equivalence of two models is equivalent to the existence of a winning strategy for player II in Ehrenfeucht–Fraïssé game of length ω between the models. Ehrenfeucht–Fraïssé game of length ω can distinguish any two non-isomorphic models only in the countable cardinality, so $L_{\infty,\omega}$ is not good in characterizing uncountable models up to isomorphism. Nadel and Stavi [17] have investigated logics $L_{\infty,\lambda}$ and showed that these are not successful in characterizing all models up to isomorphism in cardinality λ, where λ is an uncountable successor cardinal.

Thus infinitary languages are not sufficient for characterizing all models up to isomorphism in an uncountable cardinality λ, if we don't allow the infinitary language to have sentences of cardinality λ. The logics L^n are also not very successful. As they have only continuum many theories they cannot characterize all models in a cardinality which has more than continuum-many non-isomorphic models.

6.2. Regular cardinals

We have introduced the infinitary second order language $L^2_{\kappa,\omega}$ for a regular cardinal κ in the preliminaries. We will now prove that it is independent of *ZFC* whether all models of cardinality κ in any finite vocabulary can be characterized up to isomorphism by their $L^2_{\kappa,\omega}$-theories. Sentences of $L^2_{\kappa,\omega}$ correspond to subsets of cardinals $\lambda < \kappa$ so this logic is within the scope of this paper.

If $\mathfrak{A}$ is a model of cardinality κ, all sets in $H(\kappa)$ are up to isomorphism $L^2_{\kappa,\omega}$-definable in $\mathfrak{A}$. This is because the structure of any set in $H(\kappa)$ can be coded into a subset of an ordinal $\alpha < \kappa$ and, as we will see below in the proof of Theorem 6.3, such a subset of α can be defined up to isomorphism in $\mathfrak{A}$ by an $L^2_{\kappa,\omega}$-formula.

Theorem 6.2. *If κ is a regular cardinal and there is a second order definable well-order of the power set of κ, then $A(L^2_{\kappa,\omega}, \kappa)$ holds. In particular $A(L^2_{\kappa,\omega}, \kappa)$ holds if $V = \mathrm{L}$.*

Proof. We omit the details as the proof is entirely similar to the proof of Theorem 3.1. See also the proof of Theorem 6.3 below.

As in Theorem 3.1, a model can be coded into an n-ary relation $R \subseteq \kappa^n$. By Lemma 2.7 all ordinals smaller than κ are characterizable. For all n-tuples of ordinals smaller than κ we can say whether the tuple belongs to or does not belong to the least subset of κ^n in the well-order which is isomorphic with the model. The canonical well-order of L up to sets of cardinality κ is up to isomorphism second order definable in any cardinality κ. $\square$

In Theorem 6.2 we saw that $A(L^2_{\kappa,\omega}, \kappa)$ holds in L at any regular cardinal κ as there is a second order definable well-order of the power set of κ. In fact we will get a better result.

[11] $L_{\infty,\omega}$ is the union of the languages $L_{\kappa,\omega}$, κ a regular cardinal.

Theorem 6.3. *Let κ be a regular cardinal and let $H(\kappa^+) \subset \mathrm{L}[X]$ for some set X with $X \subseteq \lambda < \kappa$. Then $A(L^2_{\kappa,\omega}, \kappa)$ holds.*

Proof. The idea is the following. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of cardinality κ. By assumption $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic to some sets in $\mathrm{L}[X]$. In the infinitary second order language $L^2_{\kappa,\omega}$ we can talk about the least subset of κ^n in the canonical well-order of $\mathrm{L}[X]$ which is isomorphic to $\mathfrak{A}$. We will now describe how this is done.

In the next formula, which defines the set X up to isomorphism in a model of cardinality κ, we will use the formulas θ_α from Lemma 2.7 to define the ordinals.

$$\exists <^* \left(\phi_{(\kappa,\epsilon)}\big(A, <^*\big) \wedge \forall x\left(P(x) \leftrightarrow \bigvee_{\alpha\in X} \theta_\alpha(x)\right)\right).$$

In the above formula $\phi_{(\kappa,\epsilon)}$ denotes the formula which defines $(\kappa, <)$ up to isomorphism and A denotes the domain of the model in question. We denote this formula which defines X up to isomorphism by ϕ_X.

If the set X and an ordinal $\alpha < \kappa^+$ are given, the αth level of the sets constructible from X is up to isomorphism second order definable from these parameters. Also the canonical well-order of $\mathrm{L}_\alpha[X]$ is second order definable on $L_\alpha[X]$ from X and α. Let $\phi_{\mathrm{L}_\alpha[X]}(Y, X, \alpha)$ be a second order formula which says that Y is the αth level of the sets constructible from X (up to isomorphism) and let $\phi_{<_{\mathrm{L}_\alpha[X]}}(Z, X, \alpha)$ be a second order formula which says that Z is the canonical well-order of the αth level of the sets constructible from X (up to isomorphism).

As usual, we assume that the model in question has been coded into an n-ary relation R. We are interested in sentences of the following form:

There are X, a, M, $<$ and R_0 such that the following hold:

1. $\phi_X(X)$.
2. a is an ordinal.
3. $\phi_{\mathrm{L}_\alpha[X]}(M, X, a)$.
4. $\phi_{<_{\mathrm{L}_\alpha[X]}}(<, X, a)$.
5. $R_0 \in M \wedge R_0 \cong R \wedge \forall R_1((R_1 \in M \wedge R_1 \cong R) \rightarrow (R_0 < R_1 \vee R_0 = R_1))$.
6. $(\alpha_1, \ldots, \alpha_n) \in R_0$.

The first four formulas say that a is an ordinal, X is what it is supposed to be (up to isomorphism), M is $\mathrm{L}_a[X]$ (up to isomorphism) and $<$ is $<_{\mathrm{L}_a[X]}$ (up to isomorphism). The fifth formula says that R_0 belongs to $\mathrm{L}_a[X]$ and it is the least model in the canonical well-order of $\mathrm{L}_a[X]$ which is isomorphic to the model in question. The sixth formula says that a tuple $(\alpha_1, \ldots, \alpha_n)$ belongs to R_0. Similarly we can say that a tuple does not belong to R_0.

If two models of cardinality κ are now $L^2_{\kappa,\omega}$-equivalent, then they satisfy all the same sentences of the form above. Thus they have the same set R_0 and consequently they are isomorphic. □

Corollary 6.4. *It is consistent relative to the consistency of a measurable cardinal that there is a measurable cardinal κ and $A(L^2_{\lambda,\omega}, \lambda)$ holds for any $\lambda > 2^\kappa$.*

Proof. There is a model of *ZFC* [19] such that there is a measurable cardinal κ and every set is constructible from a certain subset of the power set of κ. □

Open Question 6.5. *Are the following conditions equivalent?*

1. *There is an $L^2_{\kappa,\omega}$-definable well-order of the power set of κ.*
2. $A(L^2_{\kappa,\omega}, \kappa)$.

Ajtai proved the following theorem in case $\kappa = \omega$, see Theorem 3.10.

Theorem 6.6. *Let κ be a regular cardinal. It is consistent with ZFC that there are two ZF-equivalent non-isomorphic models of cardinality κ. The models are also $L^n_{\kappa,\omega}$-equivalent for all n.*

Proof. We add a Cohen-generic subset G of κ. The forcing conditions are mappings of cardinality smaller than κ from κ to $\{0, 1\}$. We define the model $(F^G \cup \kappa, <_\kappa, R_G)$. Here F^G is the set of all subsets of κ which agree with G everywhere except in a set of cardinality smaller than κ, $<_\kappa$ is the natural order of κ and R_G is a relation which tells which elements of κ belong to which sets in F^G. The model $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ is defined in an analogous way for the complement $-G$ of G.

We note that this forcing is $<\kappa$-closed so it does not add any new subsets of cardinals smaller than κ. If κ is inaccessible, all cardinals below κ are preserved and κ remains inaccessible.

No forcing condition can determine the model $(F^G \cup \kappa, <_\kappa, R_G)$ in any way, as a forcing condition defines the value of G only in a subset of κ which has cardinality less than κ. For any forcing condition p there are two generic filters G and G' containing p such that

$$V^G = V^{G'}, \quad \left(F^G \cup \kappa, \dot{<}_\kappa, R_G\right)^{V^G} = \left(F^{-G} \cup \kappa, \dot{<}_\kappa, R_{-G}\right)^{V^{G'}}$$

and

$$\left(F^G \cup \kappa, \dot{<}_\kappa, R_G\right)^{V^{G'}} = \left(F^{-G} \cup \kappa, \dot{<}_\kappa, R_{-G}\right)^{V^G}.$$

Thus the models $(F^G \cup \kappa, <_\kappa, R_G)$ and $(F^{-G} \cup \kappa, <_\kappa, R_{-G})$ are *ZF*-equivalent with parameters from the ground model. As the forcing does not add any new subsets of any cardinals smaller than κ, by Lemma 2.8 the models are $L^2_{\kappa,\omega}$-equivalent. But they are not isomorphic: the well-ordered structure $(\kappa, <_\kappa)$ is rigid, so every subset of κ would be mapped in an isomorphism to itself. However $G \in (F^G \cup \kappa, <_\kappa, R_G)$ and $G \notin (F^{-G} \cup \kappa, <_\kappa, R_{-G})$, so there is no isomorphism. □

Theorem 6.7. *Let M be a transitive model of ZFC and let κ be a regular cardinal in M. If we force a Cohen subset of κ in M, in the generic extension there are two ZF-equivalent non-isomorphic models of cardinality λ in all cardinalities $\lambda \geqslant \kappa$.*

Proof. We have proved that adding a Cohen subset of a regular cardinal κ produces two models of cardinality κ which are non-isomorphic but satisfy the same formulas of the language of set theory with parameters from the ground model. In fact Cohen subsets produce such models in all cardinalities $\lambda \geqslant \kappa$. This is because we can extend the universes of the models defined in Theorem 6.6 by adding λ new elements, which do not belong to any of the relations of the model. These new models can be constructed from the models introduced in Theorem 6.6 and the term λ, and thus they are *ZF*-equivalent. □

6.3. Independence

We have proved that it is independent of *ZFC* whether $A(L^2_{\kappa,\omega}, \kappa)$ holds at a regular cardinal κ. It happens that these are also relatively independent of each other, as the following theorem demonstrates.

Theorem 6.8. *Let J be a finite set of regular cardinals. It is consistent that $A(L^2_{\kappa,\omega}, \kappa)$ fails for all cardinals κ in J and holds at every regular cardinal κ not in J.*

Proof. We start from L and use iterated forcing to add Cohen subsets of all cardinals in J, adding a Cohen subset first to the smallest cardinal in J and proceeding this way upwards. We note that *GCH* holds in L and adding a single Cohen subset preserves *GCH* so *GCH* is preserved all the way through our forcing. Also cardinals are preserved. Let κ be a cardinal in J. It follows from the Factor Lemma that the iterated forcing can be decomposed into $P_{<\kappa} * P_\kappa * P_{>\kappa}$ as follows. The forcing $P_{<\kappa}$ preserves *GCH* and cardinals and P_κ adds a Cohen subset of κ. Thus after $P_{<\kappa} * P_\kappa$ we have *GCH*, cardinals are preserved and $A(L^2_{\kappa,\omega}, \kappa)$ fails because of the proof of Theorem 6.6 applied after $P_{<\kappa}$. The forcing $P_{>\kappa}$ is κ^+ closed and thus does not add any subsets of cardinals smaller than or equal to κ. Consequently, $P_{>\kappa}$ does not change the truth value of $A(L^2_{\kappa,\omega}, \kappa)$, which is false after the forcing $P_{<\kappa} * P_\kappa$.

Let now $\kappa \notin J$. Our forcing can be decomposed into $P_{<\kappa} * P_{>\kappa}$. The forcing $P_{<\kappa}$ adds some Cohen subsets below κ and $P_{>\kappa}$ adds only subsets of cardinals greater than κ. Thus after the forcing $H(\kappa^+) \subseteq L[X]$ for some $X \subseteq \lambda < \kappa$ and from Theorem 6.3 it follows that $A(L^2_{\kappa,\omega}, \kappa)$ holds. □

Theorem 6.9. *Let J be a set which contains some successor cardinals and possibly ω. It is consistent that $A(L^2_{\kappa,\omega}, \kappa)$ fails for all $\kappa \in J$, and holds for all successor cardinals outside J and for all inaccessible cardinals which do not have a cofinal subset in J.*

Proof. Let L be the ground model. We use a *backward Easton forcing* [16] with full support in all limit stages, which proceeds upwards and adds Cohen subsets of all cardinals in J.

The forcing conditions are as follows:

1. If $\omega \in J$, then P_0 is the set of finite partial functions from ω to $\{0, 1\}$. A forcing condition p is stronger than forcing condition q if and only if p extends q. If $\omega \notin J$, then P_0 is the trivial forcing.
2. Assume $\alpha = \beta^+$ and P_γ has been defined for all $\gamma \leqslant \beta$.
   If $\aleph_\alpha \in J$, we define P_α to be the set of sequences $p_\gamma, \gamma \leqslant \alpha$ where the γth coordinate belongs to P_γ for each $\gamma < \alpha$, and the αth coordinate is a forcing name $\dot{X}$ such that $p \restriction \alpha \Vdash$ "$\dot{X}$ is a partial function from $\aleph_\alpha$ to $\{0, 1\}$ and $|\dot{X}| < \aleph_\alpha$". If p and q are two conditions of length α, then p is stronger than q if and only if $p \restriction \alpha$ is stronger than $q \restriction \alpha$ and $p \restriction \alpha \Vdash$ "$p(\alpha)$ and $q(\alpha)$ are partial functions from $\aleph_\alpha$ to $\{0, 1\}$ which have cardinality smaller than $\aleph_\alpha$ and $p(\alpha) \supseteq q(\alpha)$".
   If $\aleph_\alpha \notin J$ then P_α is the trivial forcing.
3. If α is a limit ordinal, forcing conditions in P_α are tuples p of length α such that for each $\beta < \alpha$, $p \restriction \beta \Vdash$ "$p(\beta) \in P_\beta$".
   This forcing has full support in all limit stages, which means that in limit stages all coordinates of a forcing condition may be non-zero. A forcing condition p is stronger than a forcing condition q if and only if $p \restriction \beta$ is stronger than $q \restriction \beta$ for each $\beta < \alpha$.

We will inductively show that for all cardinals κ the following conditions will hold after the forcing:

1. κ remains a cardinal.
2. If κ is ω or a successor cardinal, $A(L^2_{\kappa,\omega},\kappa)$ fails iff $\kappa \in J$. If κ is inaccessible cardinal and there is no cofinal subset of κ in J then $A(L^2_{\kappa,\omega},\kappa)$ holds.
3. The Generalized Continuum Hypothesis holds up to cardinal κ.

Let us assume the claim holds for all cardinals below κ. The forcing can be decomposed into:

$$P_{<\kappa} * P_\kappa * P_{>\kappa}$$

in such a way that after the forcing $P_{<\kappa}$ the Induction Hypothesis holds below κ and if $\kappa \in J$, then P_κ adds a Cohen subset of κ, and if $\kappa \notin J$, then P_κ is the trivial forcing. The forcing $P_{>\kappa}$ is κ^+-closed, so it does not make any chance to Induction Hypothesis in cardinals less or equal to κ.

If $\kappa \in J$, then the Cohen forcing makes $A(L^2_{\kappa,\omega},\kappa)$ false, and adding a single Cohen subset does not make *GCH* false at κ.

If $\kappa \notin J$, the trivial forcing does not make *GCH* false at κ. Also $H(\kappa^+) \subseteq \mathrm{L}(X)$ for $X \subseteq \lambda < \kappa$ which codes all the previously added generic subsets, so from Theorem 6.3 it follows that $A(L^2_{\kappa,\omega},\kappa)$ holds.

We still need to show that *GCH* is preserved at limit cardinals:

1. Assume λ is a singular limit cardinal. From the Induction Hypothesis we know that *GCH* holds below λ. Because our ground model was L and the failure of the $SCH(\lambda)$ implies $0^\sharp$ exists, after our forcing it cannot be that $\neg SCH(\lambda)$. Thus $SCH(\lambda)$. Now λ is a strong limit cardinal so $2^\lambda = \lambda^{cf(\lambda)} = \lambda^+$ by $SCH(\lambda)$.
2. Let κ be an inaccessible cardinal. All subsets of κ in V^G are constructible from a single set of cardinality κ which codes all the generic sets added below κ. Thus the power set of κ has cardinality κ^+. $\square$

Remark 6.10. If we allow J to be a proper class in the assumption of Theorem 6.9, the theorem seems still to be valid. Then we just need to use a proper class of forcing conditions and the length of the iteration is a proper class.

Ajtai's original proof (see Theorem 3.10) did not only show the independence of $A(L^2,\omega)$, but it showed the independence of whether nth order equivalence implies isomorphism for countable models for arbitrary $n \geqslant 2$. This is also true for the generalization of Ajtai's result to arbitrary regular cardinals, Theorem 6.6, which we presented earlier in this section. When we use iterated forcing and add Cohen subsets first to smaller cardinals and then to bigger cardinals, adding Cohen subsets to bigger cardinals does not change (infinitary) second order equivalence of models at smaller cardinals. However, it might change (infinitary) higher order equivalence of models for some stronger higher order logics. The following question is an example of the problem:

Open Question 6.11. *Let P be the two-step iterated forcing which adds first a Cohen subset of $\aleph_0$ and then a Cohen subset of $\aleph_1$. Let M_0^G and M_0^{-G} be the usual models constructed from the Cohen real and its complement produced in the first step of P. Are the models M_0^G and M_0^{-G} third order equivalent after the forcing P?*

6.4. Singular cardinals

We have already given a generalization of Ajtai's result to regular cardinals. Next we will turn our attention to the case of singular cardinals. For the case of regular cardinals the languages $L^2_{\kappa,\omega}$ had an important role. For the singular cardinals κ we introduce a language which has the same role as the languages $L^2_{\kappa,\omega}$ had for regular cardinals κ.

Definition 6.12. Let κ be a singular cardinal. We define $L^2_\kappa = \bigcup_{\lambda<\kappa} L^2_{\lambda^+,\omega}$.

Note that the set of L^2_κ-formulas is closed under finitary first order connectives and quantifiers, but not under conjunctions or disjunctions of length $cf(\kappa)$.

Two important facts about the languages L^2_κ are the following:

1. Every ordinal $\alpha < \kappa$ is characterizable in L^2_κ.
2. Every formula of L^2_κ can be expressed as a formula of the language of set theory using a subset of some $\lambda < \kappa$ as a parameter.

As the formulas of L^2_κ are the formulas of $L^2_{\lambda,\omega}$ for regular cardinals $\lambda < \kappa$, the above facts follow from Lemma 2.8 and Lemma 2.7.

Theorem 6.13. *If $V = \mathrm{L}$ then $A(L^2_\kappa,\kappa)$ holds for any singular cardinal κ.*

Proof. We showed before in Theorem 6.2 that if $V = \mathrm{L}$ then all $L^2_{\kappa,\omega}$-equivalent models of cardinality κ are isomorphic for any regular cardinal κ. Because all ordinals less than κ are characterizable in L^2_κ, the proof we used there works without any changes for L^2_κ. $\square$

Theorem 6.14. *Let $\kappa = \aleph_\alpha$ be a singular cardinal. There is a forcing extension of* L *in which $A(L^2_\kappa, \kappa)$ fails and all cardinals are preserved.*

Proof. Let L be the ground model. As in Theorem 6.9, we use the full support iterated Cohen forcing. This time we add generic subsets of all regular cardinals smaller than κ.

Recall that for each regular $\aleph_\beta < \kappa$ our forcing creates two models M^G_β and M^{-G}_β of cardinality $\aleph_\beta$ which are $L^2_{\aleph_\beta,\omega}$-equivalent and non-isomorphic. We define the models M^G_κ and M^{-G}_κ as follows:

M^G_κ contains the α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the βth coordinate is either M^G_β or M^{-G}_β.
2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the βth coordinate is $\emptyset$.
3. The set of indexes β where the βth coordinate is M^{-G}_β is not cofinal in α.

Similarly we define M^{-G}_κ to contain those α-sequences which satisfy the following conditions:

1. If $\beta < \alpha$ and $\aleph_\beta$ is regular, the βth coordinate is either M^G_β or M^{-G}_β.
2. If $\beta < \alpha$ and $\aleph_\beta$ is singular, the βth coordinate is $\emptyset$.
3. The set of indexes β where the βth coordinate is M^G_β is not cofinal in α.

Clearly the models are non-isomorphic as there is no sequence in M^{-G}_κ which could be mapped to the sequence in M^G_κ which contains only the models M^G_β.

We will now prove that the models are L^2_κ-equivalent. Assume not. Then there is a forcing condition p such that $p \Vdash$ "$\dot\phi \in L^2_\kappa \wedge \dot\phi(\dot M^{G_\kappa}) \wedge \neg\dot\phi(\dot M^{-G_\kappa})$" for some forcing name $\dot\phi$. Thus, for any generic filter G such that $p \in G$ we have $V^G \models \phi(M^G) \wedge \neg\phi(M^{-G})$. The sentence ϕ is a sentence in the language of set theory with a subset of some $\aleph_{\gamma^+} < \kappa$ as a parameter.

We will now construct another generic filter G' which contains p such that $\dot\phi^{V^G} = \dot\phi^{V^{G'}}$. The elements of G' are made from elements of G by the following modification:

1. Up to stage γ^+ (where the formula ϕ appears) no modification is done.
2. In the domain of p no modification is done.
3. Above stage γ^+ outside the domain of p the forcing condition is chanced to its mirror image, i.e., the domain remains the same but zeros and ones chance places.

Clearly $p \in G'$. Also up to stage γ^+ the generic sets G' and G agree about everything, so $\dot\phi^{V^G} = \dot\phi^{V^{G'}}$. After stage γ^+ the generic set G' adds essentially complements of those sets which G adds to all regular cardinals. There is a difference only in the domain of p which is always of a smaller cardinality. In particular $M^G_\beta = M^{-G'}_\beta$ and $M^{-G}_\beta = M^{G'}_\beta$ for all $\gamma^+ < \beta < \alpha$. Also $V^G = V^{G'}$. Now $\dot M^{G^{V^{G'}}} = M^{-G}$ and $\dot M^{-G^{V^{G'}}} = M^G$, i.e., the models chance places in the generic extensions. However, the formula ϕ is the same and $V^G = V^{G'}$ so ϕ cannot be true in one model and false in the other. $\square$

We will next present a model of *ZFC* in which the infinitary second order languages cannot characterize all models in any cardinality.

Corollary 6.15. *Assuming the consistency of an inaccessible cardinal, there is a model of ZFC in which $A(L^2_\kappa, \kappa)$ fails for all singular cardinals κ and $A(L^2_{\kappa,\omega}, \kappa)$ fails for all regular cardinals κ.*

Proof. We start from a model of *ZFC* which satisfies $V = \mathrm{L}$ and there is an inaccessible cardinal. Let λ be the least inaccessible cardinal in that model. We proceed upwards and add by iterated Cohen forcing generic subsets of all regular cardinals smaller than λ. At limit stages we take full support. After the forcing $A(L^2_\kappa, \kappa)$ fails for all singular cardinals $\kappa < \lambda$ and $A(L^2_{\kappa,\omega}, \kappa)$ fails for all regular cardinals $\kappa < \lambda$ and λ remains inaccessible. Thus $V_\kappa^{(V^G)}$ satisfies *ZFC* and $A(L^2_\kappa, \kappa)$ fails for all singular cardinals κ and $A(L^2_{\kappa,\omega}, \kappa)$ fails for all regular cardinals κ. $\square$

Open Question 6.16. *Is it consistent with ZFC that there is a singular cardinal κ such that $A(L^2_\kappa, \kappa)$ fails but $A(L^2_{\lambda,\omega}, \lambda)$ does not fail in cofinally many regular cardinals λ below κ?*

7. $A(L^2, \omega)$ and large cardinal axioms

7.1. Large cardinals

In this section we will discuss how some large cardinal axioms are related to $A(L^2, \omega)$. First we will discuss consistency of some large cardinal axioms with second order definable well-orders of the reals. Then we will show that if there are enough large cardinals then $A(L^2, \omega)$ is false. In the end we will discuss third order definable well-orders of the reals and forcing axioms.

From the proof of Theorem 3.1 and some well-known facts about the consistency of well-orders of the reals with large cardinals we get the following results:

Theorem 7.1. *(See Ajtai [2], Silver [19], Martin and Steel [14].) It is consistent that there is a measurable cardinal and $A(\Sigma^1_4, \omega)$ holds. It is consistent that there are n Woodin cardinals and $A(\Sigma^1_{n+3}, \omega)$ holds. The above results are relative to the consistency of the relevant large cardinal axioms.*

Proof. The existence of a measurable cardinal with a Δ^1_3 well-order of the reals is consistent [19], so by Theorem 3.1 it is consistent that there is a measurable cardinal and $A(\Sigma^1_4, \omega)$ holds. Also for each natural number n it is consistent to have n Woodin cardinals and a Σ^1_{n+2} well-order of the reals [14]. From Theorem 3.1 it follows that it is consistent that there are n Woodin cardinals and $A(\Sigma^1_{n+3}, \omega)$ holds. □

We will next present a lemma which is needed to prove Theorem 7.4: "If there are enough large cardinals then $A(L^2, \omega)$ fails."

Lemma 7.2. *$A(L^2, \omega)$ is true in V if and only if it is true in* $\mathrm{L}(\mathbb{R})$.

Sketch. Countable models and isomorphisms between them are reals, and V and $\mathrm{L}(\mathbb{R})$ have the same reals. A truth predicate of L^2 for a countable model can be defined inductively and is determined by the reals, so V and $\mathrm{L}(\mathbb{R})$ have the same second order truth predicates of countable models. Consequently a counterexample for $A(L^2, \omega)$ in V or $\mathrm{L}(\mathbb{R})$ works also in the other.

Theorem 7.3. *(See Woodin [22].) If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ, then no forcing construction in V_δ can change the theory of* $\mathrm{L}(\mathbb{R})$.

Corollary 7.4. *If there is a measurable cardinal above a limit of Woodin cardinals then $A(L^2, \omega)$ fails.*

Proof. Assume there is a measurable cardinal above a limit of Woodin cardinals. We add a Cohen-generic real G to V as in Theorem 3.10. Now $A(L^2, \omega)$ is false in $V[G]$. By Lemma 7.2 $A(L^2, \omega)$ is false in $\mathrm{L}(\mathbb{R})^{V[G]}$. By assumption and Theorem 7.3 $A(L^2, \omega)$ is false in $\mathrm{L}(\mathbb{R})^V$ and by Lemma 7.2 $A(L^2, \omega)$ is false in V. □

Some large cardinal axioms imply that there is no second order definable well-order of the reals. In particular this holds for large cardinal axioms that imply Projective Determinacy. These axioms possibly imply that $A(L^2, \omega)$ fails. If that is the case, we can ask the question: does $A(L^3, \omega)$ hold? By the following theorem most large cardinal axioms are consistent with $A(L^3, \omega)$ (relative to the consistency of the large cardinal axiom in question).

Theorem 7.5. *$A(L^3, \omega)$ is consistent with practically all known consistent large cardinal axioms.*

Proof. Let the ground model be a model of *ZFC* which satisfies your favorite large cardinal axiom. By a result of Abraham and Shelah [1] it is possible to force a third order definable well-order of the reals with a small forcing[12]. In the generic extension $A(L^3, \omega)$ holds because of the same reasoning as in Theorem 3.1. If the large cardinal axiom was preserved in the forcing, then the generic extension satisfies the large cardinal axiom and $A(L^3, \omega)$. □

[12] If κ is a large cardinal we say that a notion of forcing P is small (relative to κ) if $|P| < \kappa$. Practically all large cardinals are preserved in small forcings [9] (Theorem 21.2).

7.2. Forcing axioms

As we already noted in section 3, it is an open question whether Martin's Axiom is consistent with $A(L^2,\omega)$. Unlike the consistency of the Proper Forcing Axiom and Martin's Maximum, the consistency of Martin's Axiom $+2^{\aleph_0}=2^{\aleph_1}=\aleph_2$ can be proved from the consistency of *ZFC*.

In all our examples of models of *ZFC* where $A(L^2,\omega)$ holds, there is a second order definable well-order of the reals. It is well known that Projective Determinacy implies that there is no second order definable well-order of the reals.

Theorem 7.6. *(See Steel [20].) The Proper Forcing Axiom implies that there is no second order definable well-order of the reals.*

Since the Proper Forcing Axiom implies Projective Determinacy [20], if $A(L^2,\omega)$ is consistent with the Proper Forcing Axiom, then $A(L^2,\omega)$ can hold without a second order definable well-order of the reals.

Open Question 7.7. *Is Proper Forcing Axiom consistent with $A(L^2,\omega)$?*

Theorem 7.8. *Assuming the consistency of the relevant large cardinal axioms it is consistent that Martin's Maximum holds with $A(L^3,\omega)$.*

Proof. By Paul Larson's result [11] Martin's Maximum is consistent with the existence of a well-order of the reals definable in $H(\aleph_2)$ without parameters.

By Lemma 2.5 we can quantify over elements of $H(\aleph_2)$ in third order logic thus Martin's Maximum is consistent with a third order definable well-order of the reals. Consequently it is consistent that Martin's Maximum holds and $A(L^3,\omega)$ holds. $\square$

8. Summary and future work

8.1. Summary

If κ is an infinite cardinal we can ask the question what is the least logic L such that every L-theory is κ categorical. If κ is a regular cardinal, adding a Cohen subset of κ makes sure that no such small definable logic L exists. If κ is a singular cardinal, adding Cohen subsets of cofinally many $\lambda<\kappa$ by an iterated forcing, taking full support at all limits, does essentially the same. However, there is always a small logic L with generalized quantifiers such that all L-theories are κ-categorical but L may be not definable in the language of set theory.

In the countable cardinality the "small" logic can be second order logic. If $V=\mathrm{L}$ even Σ^1_3 is enough. With n Woodin cardinals Σ^1_{n+3} can be enough. But if there are infinitely many Woodin cardinals and a measurable cardinal above them then $A(L^2,\omega)$ fails. However $A(L^3,\omega)$ is consistent with practically all large cardinal axioms. $A(L^3,\omega)$ is also consistent with Martin's Maximum.

In an uncountable regular cardinality the small logic can be $L^2_{\kappa,\omega}$ or L^n where $n\geqslant 4$. Whether $A(L^2_{\kappa,\omega},\kappa)$ holds for different cardinals κ is very much independent of each other.

The following table contains information about whether $A(L,\kappa)$ holds for certain language L and cardinal κ. In the intersection of an L-row and a κ-column we have described in the left-hand-side a model of *ZFC* where $A(L,\kappa)$ holds and on the right-hand-side a model of *ZFC* where $A(L,\kappa)$ fails (if they exist). The question mark means an open question. Cohen, iter., and $P_{X,\kappa}$ refer to suitable Cohen forcing, iterated Cohen forcing with full support in all limit stages and the forcing $P_{X,\kappa}$ defined in Section 4, respectively. Regular column refers to arbitrary uncountable regular cardinals and singular column refers to arbitrary uncountable singular cardinals. The ground model is L in all the forcings:

$A(L,\kappa)$	$\aleph_0$	*regular*	*singular*
FO	$-/always$	$-/always$	$-/always$
$L_{\kappa^+,\omega}$	$always/-$	$\kappa=\aleph_0/\kappa\neq\aleph_0$	$-/always$
L^2	$V=L/Cohen$	$?/V=L$	$?/V=L$
$L^2_{\kappa,\omega}/L^2_\kappa$	$V=L/Cohen$	$V=L/Cohen$	$V=L/iter.$
L^3	$V=L/Cohen$	$?/V=L$	$?/V=L$
L^4	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$
L^n	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$
ZF	$V=L/Cohen$	$P_{X,\kappa}/V=L$	$P_{X,\kappa}/V=L$

8.2. Future work

In this subsection we list the most important open questions and possible directions of future research.

Recall Question 3.6:

Open Question 8.1. *Is it consistent with ZFC that $A(L^2, \omega)$ holds, but there is no second order definable well-order of the reals?*

Shelah and Väänänen are preparing a paper with a positive answer to this question.

Adding a Cohen subset of a regular cardinal produces two *ZF*-equivalent non-isomorphic models of cardinality κ. When we do iterated Cohen forcings we have not been able to prove that the models remain *ZF*-equivalent. The following question is an example of that: Let L be the ground model and $P = P_0 * P_1$ be an iterated forcing which adds first a Cohen subset of ω and then a Cohen subset of $\aleph_1$. Let G be a P-generic set over L and G_0 the P_0-generic set over L determined by G and M^{G_0} and M^{-G_0} the models constructed from G_0 and $-G_0$ (see Theorem 3.10). Are M^{G_0} and M^{-G_0} third order equivalent in L$[G]$?

Open Question 8.2. *Is it consistent with ZFC that $A(L^2, \kappa)$ holds for an uncountable cardinal κ? If not, is it consistent that $A(L^3, \kappa)$ holds for an uncountable cardinal κ?*

Open Question 8.3. *Is it consistent with ZFC that Martin's axiom $+ 2^{\aleph_0} = \aleph_2$ holds with $A(L^2, \omega)$.*

Possible directions for future research:

1. Our results are often related to models which resemble L a lot (Theorem 6.3 is used in many results). An interesting question is whether our results could be generalized to inner models of some large cardinals.
2. The question about whether every L-theory is κ-categorical in a model class C. We have here only discussed briefly the Fraïssé Hypothesis, i.e., the above question in case $L = L^2$, $\kappa = \omega$ and C is the class of ordinals.
3. Ehrenfeucht–Mostowski models. Adding a Cohen real introduces two countable non-isomorphic *ZF*-equivalent linear orders [10]. Suitable cardinal collapse makes the Fraïssé Hypothesis fail [13]. For which theories T can we construct non-isomorphic *ZF*-equivalent Ehrenfeucht–Mostowski models over these linear orders (or ordinals)? Is this possible for all unstable theories? Tapani Hyttinen, Kaisa Kangas and Jouko Väänänen have worked on this question [8].
4. Bigger vocabularies. We have considered only finite vocabularies here. One could ask questions of the form "What is the least logic L such that every model of cardinality κ with vocabulary of cardinality λ can be characterized by its L-theory?".

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