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Finite Gorenstein representation type implies simple singularity[☆]

Lars Winther Christensen^a, Greg Piepmeyer^{b,1}, Janet Striuli^{b,*},
Ryo Takahashi^c

^a Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409, USA

^b Department of Mathematics, University of Nebraska, Lincoln, NE 68588, USA

^c Department of Mathematical Sciences, Faculty of Science, Shinshu University, 3-1-1 Asahi, Matsumoto, Nagano 390-8621, Japan

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To Lucho Avramov on his sixtieth birthday

Abstract

Let R be a commutative noetherian local ring and consider the set of isomorphism classes of indecomposable totally reflexive R -modules. We prove that if this set is finite, then either it has exactly one element, represented by the rank 1 free module, or R is Gorenstein and an isolated singularity (if R is complete, then it is even a simple hypersurface singularity). The crux of our proof is to argue that if the residue field has a totally reflexive cover, then R is Gorenstein or every totally reflexive R -module is free.

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* Corresponding author.

E-mail addresses: lars.w.christensen@ttu.edu (L.W. Christensen), greg@math.missouri.edu (G. Piepmeyer), jstriuli2@math.unl.edu (J. Striuli), takahasi@math.shinshu-u.ac.jp (R. Takahashi).

¹ Current address: Department of Mathematics, University of Missouri, Columbia, MO 65211, USA.

0. Introduction

Remarkable connections between the module theory of a local ring and the character of its singularity emerged in the 1980s. They show how finiteness conditions on the category of maximal Cohen–Macaulay modules² characterize particular isolated singularities. We develop these connections in several directions.

A local ring with only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules is said to be of finite Cohen–Macaulay (CM) representation type. By work of Auslander [5], every complete Cohen–Macaulay local ring of finite CM representation type is an isolated singularity.

Specialization to Gorenstein rings opens to a finer description of the singularities; it centers on the simple hypersurface singularities identified in Arnol'd's work on germs of holomorphic functions [1]. By work of Buchweitz, Greuel, and Schreyer [12], Herzog [18], and Yoshino [32], a complete Gorenstein ring of finite CM representation type is a simple singularity in the generalized sense of [32]. Under extra assumptions on the ring, the converse holds by work of Knörrer [21] and Solberg [25].

In this introduction, R is a commutative noetherian local ring with maximal ideal \mathfrak{m} and residue field k . To avoid the *a priori* condition in [12,18,32] that R is Gorenstein, we replace finite CM representation type with a finiteness condition on the category $\mathcal{G}(R)$ of modules of Gorenstein dimension 0. Over a Gorenstein ring, these modules are precisely the maximal Cohen–Macaulay modules, but they are known to exist over any ring, unlike maximal Cohen–Macaulay modules.

Theorem A. *Let R be complete. If the set of isomorphism classes of non-free indecomposable modules in $\mathcal{G}(R)$ is finite and not empty, then R is a simple singularity.*

The category $\mathcal{G}(R)$ was introduced by Auslander and Bridger [4,6]. An R -module G is in $\mathcal{G}(R)$ if there is an exact complex of finitely generated free R -modules

$$F = \cdots \rightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \rightarrow \cdots,$$

such that G is isomorphic to $\text{Coker } \partial_0$ and the complex $\text{Hom}_R(F, R)$ is exact. Every finitely generated free R -module is in $\mathcal{G}(R)$, and the modules in this category have Gorenstein dimension 0 as in [4,6]; following [11] we call them *totally reflexive*.

The aforementioned works [12,18,32] show that Theorem A follows from the next result, which is proved as (4.3).

Theorem B. *If the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R)$ is finite, then R is Gorenstein or every module in $\mathcal{G}(R)$ is free.*

As this theorem does not require R to be complete, we considerably strengthen Theorem A using work of Huneke, Leuschke, and R. Wiegand [19,22,30]; this occurs in (4.5). Theorem B was conjectured by R. Takahashi [29], who proved it for henselian rings of depth at most two [27–29]. The class of rings over which all totally reflexive modules are free is poorly understood, but it is

² The finitely generated modules whose depth equals the Krull dimension of the ring.

known to include all Golod rings [11], in particular, all Cohen–Macaulay rings of minimal multiplicity.

To prove Theorem B we use a notion of $\mathcal{G}(R)$ -approximations, which is close kin to the CM-approximations of Auslander and Buchweitz [7]. When R is Gorenstein, a $\mathcal{G}(R)$ -approximation is exactly a CM-approximation. By [7], every module over a Gorenstein ring has a CM-approximation. Our proof of Theorem B goes via the following strong converse, proved as (3.4).

Theorem C. *Let R be a local ring and assume there is a non-free module in $\mathcal{G}(R)$. If the residue field k has a $\mathcal{G}(R)$ -approximation, then R is Gorenstein.*

This theorem complements recent developments in relative homological algebra. The notion of totally reflexive modules has two extensions to non-finitely generated modules; see [13] for details. One is Gorenstein projective modules, which allows arbitrary free modules in the definition above. By recent work of Jørgensen [20], every module over a complete local ring has a Gorenstein projective precover. The other extension is Gorenstein flat modules. By a result of Enochs and López-Ramos [15], every module has a Gorenstein flat precover.

Theorem C counterposes these developments; it shows that for finitely generated modules, the precovers found in [20] and [15] cannot, in general, be finitely generated. Assume that R is complete. Then a finitely generated R -module has a $\mathcal{G}(R)$ -approximation if and only if it has a $\mathcal{G}(R)$ -precover. Assume further that R is not Gorenstein. Theorem C shows that if $X \rightarrow k$ is a Gorenstein projective/flat precover and X is not free, then X is not finitely generated.

1. Categories and covers

In this paper, rings are commutative and noetherian; modules are finitely generated (unless otherwise specified). We write $\text{mod}(R)$ for the category of finitely generated modules over a ring R .

For an R -module M , we denote by M_i the i th syzygy in a free resolution. When R is local, we denote by $\Omega_i^R(M)$ the i th syzygy in the minimal free resolution of M . For an R -module M , set $M^* = \text{Hom}_R(M, R)$; we refer to this module as the *algebraic dual* of M .

We only consider full subcategories of $\text{mod}(R)$; this allows us to define a subcategory by specifying its objects. In the following, \mathcal{B} is a subcategory of $\text{mod}(R)$.

(1.1) Closures. Recall that the category \mathcal{B} is said to be closed under extensions if for every short exact sequence $0 \rightarrow B \rightarrow X \rightarrow B' \rightarrow 0$ with B and B' in \mathcal{B} also X is in \mathcal{B} . The closure of \mathcal{B} under extensions is by definition the smallest subcategory containing \mathcal{B} and closed under extensions. Recall also that \mathcal{B} is closed under direct sums and direct summands when a direct sum $M \oplus N$ is in \mathcal{B} if and only if both summands are in \mathcal{B} . The closure of \mathcal{B} under addition is by definition the smallest subcategory containing \mathcal{B} and closed under direct sums and direct summands; we denote it by $\text{add}(\mathcal{B})$.

We define the closure $\langle \mathcal{B} \rangle$ to be the smallest subcategory containing \mathcal{B} and closed under direct summands and extensions. It is straightforward to verify that the closure $\langle \mathcal{B} \rangle$ is reached by countable alternating iteration, starting with \mathcal{B} , between closure under addition and closure under extensions.

We say that \mathcal{B} is *closed under algebraic duality* if for every module B in \mathcal{B} the module B^* is also in \mathcal{B} . Similarly, we say that \mathcal{B} is *closed under syzygies* if for every module B in \mathcal{B} every first syzygy B_1 is in \mathcal{B} ; then every syzygy B_i is in \mathcal{B} .

(1.2) Precovers and covers. Let M be an R -module. A \mathcal{B} -precover of M is a homomorphism $\varphi : B \rightarrow M$, with $B \in \mathcal{B}$, such that every homomorphism $X \rightarrow M$ with $X \in \mathcal{B}$, factors through φ ; i.e., the homomorphism

$$\text{Hom}_R(X, \varphi) : \text{Hom}_R(X, B) \rightarrow \text{Hom}_R(X, M)$$

is surjective for each module X in \mathcal{B} . A \mathcal{B} -precover $\varphi : B \rightarrow M$ is a \mathcal{B} -cover if every $\gamma \in \text{Hom}_R(B, B)$ with $\varphi\gamma = \varphi$ is an automorphism.

Note that if the category \mathcal{B} contains R , then every \mathcal{B} -precover is surjective.

(1.3) If there are only finitely many isomorphism classes of indecomposable modules in \mathcal{B} , then every finitely generated R -module has a \mathcal{B} -precover; see [2, Prop. 4.2].

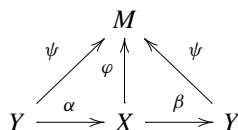
(1.4) Consider a diagram $B \xrightarrow{\varphi} M \oplus N \begin{matrix} \xleftarrow{\pi} \\ \xrightarrow{\iota} \end{matrix} M$, where $\pi\iota$ is the identity on M . If φ is a \mathcal{B} -precover, then so is $\pi\varphi : B \rightarrow M$.

The next two lemmas appear in Xu’s book [31, 2.1.1 and 1.2.8]. We include a proof of the second one since Xu left it to the reader.

(1.5) Wakamatsu’s lemma. Let \mathcal{B} be a subcategory of $\text{mod}(R)$, and let φ be a \mathcal{B} -cover of an R -module M . If \mathcal{B} is closed under extensions, then $\text{Ext}_R^1(X, \text{Ker } \varphi) = 0$ for all $X \in \mathcal{B}$.

(1.6) Lemma. Let \mathcal{B} be a subcategory of $\text{mod}(R)$, and let M be an R -module. If M has a \mathcal{B} -cover, then a \mathcal{B} -precover $\varphi : X \rightarrow M$ is a cover if and only if $\text{Ker } \varphi$ contains no non-zero direct summand of X .

Proof. Let $\psi : Y \rightarrow M$ be a \mathcal{B} -cover. For the “if” part, consider the commutative diagram below, where α and β are given by the precovering properties of φ and ψ .



Since $\psi\beta\alpha = \psi$ and ψ is a cover, the composite $\beta\alpha$ is an automorphism, so β is surjective. It also follows that X is isomorphic to $\text{Ker } \beta \oplus \text{Im } \alpha$. As $\text{Ker } \varphi$ contains no non-zero summand of X , the inclusion $\text{Ker } \beta \subseteq \text{Ker } \varphi$ implies that β is also injective. Consequently, φ is a \mathcal{B} -cover.

For the “only if” part, consider a decomposition $X = Y \oplus Z$, and assume there is an inclusion $Z \subseteq \text{Ker } \varphi$. Let π be the endomorphism of X projecting onto Y , then $\varphi\pi = \varphi$. Since φ is a cover, π is an automorphism, whence $Z = 0$. \square

2. Approximations and reflexive subcategories

Stability of (pre-)covers under base change is delicate to track. To avoid this task, we develop a notion between precover and cover. The next definition is in line with that of CM-approximations [7]; for $\mathcal{G}(R)$ it broadens the notion used in [11].

(2.1) Definitions. Let \mathcal{B} be a subcategory of $\text{mod}(R)$ and set

$$\mathcal{B}^\perp = \{L \in \text{mod}(R) \mid \text{Ext}_R^i(B, L) = 0 \text{ for all } B \in \mathcal{B} \text{ and all } i > 0\}.$$

Let M be an R -module. A \mathcal{B} -approximation of M is a short exact sequence

$$0 \rightarrow L \rightarrow B \rightarrow M \rightarrow 0,$$

where B is in \mathcal{B} and L is in \mathcal{B}^\perp .

(2.2) Let \mathcal{B} be a subcategory of $\text{mod}(R)$ and M be an R -module.

- (a) If $0 \rightarrow \text{Ker } \varphi \rightarrow B \xrightarrow{\varphi} M \rightarrow 0$ is a \mathcal{B} -approximation of M , then φ is a *special* \mathcal{B} -precover of M ; see [31, Prop. 2.1.3].
- (b) If $B \xrightarrow{\varphi} M$ is a surjective \mathcal{B} -cover, and \mathcal{B} is closed under syzygies and extensions, then the sequence $0 \rightarrow \text{Ker } \varphi \rightarrow B \xrightarrow{\varphi} M \rightarrow 0$ is a \mathcal{B} -approximation of M by Wakamatsu’s lemma.
- (c) Assume $\text{mod}(R)$ has the Krull–Schmidt property (e.g., R is henselian) and \mathcal{B} is closed under direct summands. The module M has a \mathcal{B} -cover if and only if it has a \mathcal{B} -precover; see [29, Cor. 2.5].

The next two results study the behavior of approximations under base change.

Let $\vartheta : R \rightarrow S$ be a ring homomorphism. We say that ϑ is of finite flat dimension if S , viewed as an R -module through ϑ , has a bounded resolution by flat R -modules. We write $\text{Tor}_{i>0}^R(S, \mathcal{B}) = 0$ if for all $B \in \mathcal{B}$, and for all $i > 0$, the modules $\text{Tor}_i^R(S, B)$ vanish. We denote by $S \otimes \mathcal{B}$ the subcategory of S -modules $S \otimes_R B$ with $B \in \mathcal{B}$.

(2.3) Lemma. *Let $R \rightarrow S$ be a ring homomorphism of finite flat dimension. Let \mathcal{B} be a subcategory of $\text{mod}(R)$ such that $\text{Tor}_{i>0}^R(S, \mathcal{B}) = 0$. If $L \in \mathcal{B}^\perp$ and $\text{Tor}_{i>0}^R(S, L) = 0$, then for every $m \in \mathbb{Z}$ and every $B \in \mathcal{B}$ there is an isomorphism*

$$\text{Ext}_S^m(S \otimes_R B, S \otimes_R L) \cong \text{Tor}_{-m}^R(S, \text{Hom}_R(B, L)).$$

In particular, there are isomorphisms $\text{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \text{Hom}_R(B, L)$, and $S \otimes_R L$ is in $(S \otimes \mathcal{B})^\perp$.

Proof. Fix $B \in \mathcal{B}$. Take a free resolution $E \rightarrow B$ and a bounded flat resolution $F \rightarrow S$ over R . By the vanishing of (co)homology, the induced morphisms

$$S \otimes_R E \rightarrow S \otimes_R B, \quad F \otimes_R L \rightarrow S \otimes_R L, \quad \text{and} \quad \text{Hom}_R(B, L) \rightarrow \text{Hom}_R(E, L)$$

are homology isomorphisms. In particular, the first one is a free resolution of the S -module $S \otimes_R B$. The functors $\text{Hom}_R(E, -)$ and $F \otimes_R -$ preserve homology isomorphisms. This explains the first, third, and fifth isomorphisms below.

$$\begin{aligned} \text{Ext}_S^m(S \otimes_R B, S \otimes_R L) &\cong H^m(\text{Hom}_S(S \otimes_R E, S \otimes_R L)) \\ &\cong H^m(\text{Hom}_R(E, S \otimes_R L)) \end{aligned}$$

$$\begin{aligned} &\cong H^m(\text{Hom}_R(E, F \otimes_R L)) \\ &\cong H^m(F \otimes_R \text{Hom}_R(E, L)) \\ &\cong H^m(F \otimes_R \text{Hom}_R(B, L)) \\ &\cong \text{Tor}_{-m}^R(S, \text{Hom}_R(B, L)). \end{aligned}$$

The second isomorphism follows from Hom-tensor adjointness, and the fourth is tensor evaluation; see [17, Prop. II.5.14]. For $m = 0$ the composite isomorphism reads $\text{Hom}_S(S \otimes_R B, S \otimes_R L) \cong S \otimes_R \text{Hom}_R(B, L)$. That $S \otimes_R L$ is in $\langle S \otimes \mathcal{B} \rangle^\perp$ follows as Tor_i^R is zero for $i < 0$. \square

(2.4) Proposition. *Let $R \rightarrow S$ be a ring homomorphism of finite flat dimension and \mathcal{B} be a subcategory of $\text{mod}(R)$. Let M be an R -module with a \mathcal{B} -approximation $0 \rightarrow L \rightarrow B \rightarrow M \rightarrow 0$. If $\text{Tor}_{i>0}^R(S, \mathcal{B}) = 0$ and $\text{Tor}_{i>0}^R(S, M) = 0$, then*

$$0 \rightarrow S \otimes_R L \rightarrow S \otimes_R B \rightarrow S \otimes_R M \rightarrow 0$$

is an $\langle S \otimes \mathcal{B} \rangle$ -approximation.

Proof. By the assumptions on \mathcal{B} and M , application of the functor $S \otimes_R -$ to the \mathcal{B} -approximation of M yields the desired short exact sequence and also equalities $\text{Tor}_{i>0}^R(S, L) = 0$. Now Lemma (2.3) gives that $S \otimes_R L$ is in $\langle S \otimes \mathcal{B} \rangle^\perp$. \square

(2.5) Let \mathcal{B} be a subcategory of $\text{mod}(R)$ with $R \in \mathcal{B}^\perp$. For every $B \in \mathcal{B}$ and every R -module N , dimension shifting yields

$$\text{Ext}_R^i(B, N) \cong \text{Ext}_R^{i+h}(B, N_h) \quad \text{for } i > 0 \text{ and } h \geq 0.$$

Moreover, for $h \geq 0$ the algebraic dual B^* is an h th syzygy of $(B_h)^*$, so

$$\text{Ext}_R^i(B^*, N) \cong \text{Ext}_R^{i+h}((B_h)^*, N) \quad \text{for } i > 0 \text{ and } h \geq 0.$$

If, furthermore, \mathcal{B} is closed under syzygies and algebraic duality, then these isomorphisms combine to yield

$$\text{Ext}_R^i(B^*, N_j) \cong \text{Ext}_R^i((B_h)^*, N_{j-h}) \quad \text{for } i > 0 \text{ and } j \geq h \geq 0. \tag{2.5.1}$$

In particular, (2.5.1) holds when \mathcal{B} is a category satisfying the next definition.

(2.6) Definition. A subcategory \mathcal{B} of $\text{mod}(R)$ is *reflexive* if R is in $\mathcal{B} \cap \mathcal{B}^\perp$ and \mathcal{B} is closed under

- (1) direct sums and direct summands,
- (2) syzygies, and
- (3) algebraic duality.

It is standard that the category $\mathcal{G}(R)$ of totally reflexive R -modules is a reflexive subcategory of $\text{mod}(R)$. Moreover, using the characterization of $\mathcal{G}(R)$ provided by [13, (1.1.2) and (4.1.4)], it

is straightforward to verify that every reflexive subcategory of $\text{mod}(R)$ is, in fact, a subcategory of $\mathcal{G}(R)$.

(2.7) In the rest of the paper, $\mathcal{F}(R)$ denotes the category of finitely generated free R -modules. Let \mathcal{B} be a reflexive subcategory of $\text{mod}(R)$. There are containments

$$\mathcal{F}(R) \subseteq \mathcal{B} \subseteq \mathcal{G}(R).$$

Further, let $R \rightarrow S$ be a ring homomorphism of finite flat dimension, then

$$\text{Tor}_{i>0}^R(S, \mathcal{B}) = 0,$$

as every module in \mathcal{B} is an infinite syzygy.

The next observation is crucial for our proofs of the main theorems.

(2.8) Assume $\text{mod}(R)$ has the Krull–Schmidt property (e.g., R is henselian) and let \mathcal{B} be a reflexive subcategory of $\text{mod}(R)$ closed under extensions. We claim that an R -module M has a \mathcal{B} -precover if and only if it has a \mathcal{B} -approximation. Indeed, let $\varphi : B \rightarrow M$ be a \mathcal{B} -precover; by (2.2)(c) the module M also has a \mathcal{B} -cover. Decompose B as $B' \oplus B''$, where B'' is the largest direct summand of B contained in $\text{Ker } \varphi$. By Lemma (1.6) the factorization $\varphi' : B' \rightarrow M$ is a cover, and by (2.2)(b) the sequence $0 \rightarrow \text{Ker } \varphi' \rightarrow B' \rightarrow M \rightarrow 0$ is a \mathcal{B} -approximation.

(2.9) **Lemma.** *Let \mathcal{B} be a reflexive subcategory of $\text{mod}(R)$ and M be an R -module. If M has a \mathcal{B} -approximation, then every syzygy of M has a \mathcal{B} -approximation.*

Proof. Let $0 \rightarrow L \rightarrow B \rightarrow M \rightarrow 0$ be a \mathcal{B} -approximation. It is sufficient to prove that every first syzygy M_1 has a \mathcal{B} -approximation. By the horseshoe construction, there is a short exact sequence $0 \rightarrow L_1 \rightarrow B_1 \rightarrow M_1 \rightarrow 0$, and the syzygy B_1 is in \mathcal{B} by assumption. Let X be in \mathcal{B} . Since \mathcal{B} is reflexive, there is an isomorphism $X \cong X^{**}$, and also the module $((X^*)_1)^*$ is in \mathcal{B} . Now (2.5.1) yields the second isomorphism in the chain

$$\text{Ext}_R^i(X, L_1) \cong \text{Ext}_R^i(X^{**}, L_1) \cong \text{Ext}_R^i(((X^*)_1)^*, L) = 0. \quad \square$$

(2.10) **Proposition.** *Let $R \rightarrow S$ be a ring homomorphism of finite flat dimension. If \mathcal{B} is a reflexive subcategory of $\text{mod}(R)$, then $\langle S \otimes \mathcal{B} \rangle$ is a reflexive subcategory of $\text{mod}(S)$. In particular, $\langle S \otimes \mathcal{G}(R) \rangle$ is reflexive.*

Proof. The S -module S is in $\langle S \otimes \mathcal{B} \rangle$. As $R \in \mathcal{B}^\perp$, it follows from (2.7) and Lemma (2.3) that S is in $\langle S \otimes \mathcal{B} \rangle^\perp$. By definition, $\langle S \otimes \mathcal{B} \rangle$ is closed under direct sums and direct summands; this leaves (2) and (3) in Definition (2.6) to verify.

First we prove closure under syzygies. Take $B \in \mathcal{B}$ and consider a short exact sequence $0 \rightarrow B_1 \rightarrow F \rightarrow B \rightarrow 0$, where F is a free R -module. By assumption, the syzygy B_1 is in \mathcal{B} . By (2.7) the sequence

$$0 \rightarrow S \otimes_R B_1 \rightarrow S \otimes_R F \rightarrow S \otimes_R B \rightarrow 0$$

is exact. It shows that the syzygy $S \otimes_R B_1$ of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, it follows that any summand of $S \otimes_R B$ has a first syzygy in $\text{add}(S \otimes \mathcal{B})$, in particular, in $\langle S \otimes \mathcal{B} \rangle$. By Schanuel’s lemma, a module in $\langle S \otimes \mathcal{B} \rangle$ with some first syzygy in $\langle S \otimes \mathcal{B} \rangle$ has every first syzygy in $\langle S \otimes \mathcal{B} \rangle$. Finally, given a short exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$, where M, N , and their first syzygies are in $\langle S \otimes \mathcal{B} \rangle$, we claim that also a first syzygy of X is in $\langle S \otimes \mathcal{B} \rangle$. Indeed, take presentations of M and N . Since $\langle S \otimes \mathcal{B} \rangle$ is closed under extensions, it follows from the horseshoe construction that a first syzygy of X is in $\langle S \otimes \mathcal{B} \rangle$.

Next we prove closure under algebraic duality. Take $B \in \mathcal{B}$ and note that by (2.7), Lemma (2.3) applies (with $L = R$) to yield the isomorphism

$$\text{Hom}_S(S \otimes_R B, S) \cong S \otimes_R \text{Hom}_R(B, R).$$

Thus, the algebraic dual of $S \otimes_R B$ is in $S \otimes \mathcal{B}$. Moreover, the algebraic dual of any summand of $S \otimes_R B$ is in $\text{add}(S \otimes \mathcal{B})$, in particular, in $\langle S \otimes \mathcal{B} \rangle$. It is now sufficient to prove that for every short exact sequence $0 \rightarrow M \rightarrow X \rightarrow N \rightarrow 0$, where M, N , and the duals M^* and N^* are in $\langle S \otimes \mathcal{B} \rangle$, also the dual X^* is in $\langle S \otimes \mathcal{B} \rangle$. Since $\langle S \otimes \mathcal{B} \rangle$ is closed under extensions, this is immediate from the exact sequence

$$0 \rightarrow N^* \rightarrow X^* \rightarrow M^* \rightarrow \text{Ext}_S^1(N, S),$$

where $\text{Ext}_S^1(N, S) = 0$ as S is in $\langle S \otimes \mathcal{B} \rangle^\perp$. \square

3. Approximations detect the Gorenstein property

The main result of this section is Theorem C from the introduction. Lemma (3.2) furnishes the base case; for that we study a standard homomorphism.

(3.1) For modules X and N over a ring S there is a natural map

$$\theta_{XN} : X \otimes_S N \rightarrow \text{Hom}_S(X^*, N),$$

given by evaluation $\theta(x \otimes n)(\zeta) = \zeta(x)n$. Auslander computed the kernel and cokernel of this map in [3, Prop. 6.3]. Because the map is pivotal for our proof of the next lemma, we include a computation for the case where X is totally reflexive.

Consider a short exact sequence $0 \rightarrow N_1 \rightarrow F \rightarrow N \rightarrow 0$, where F is a free S -module. For any totally reflexive S -module X , the evaluation homomorphism θ_{XF} is an isomorphism, and the commutative diagram

$$\begin{array}{ccccccc} X \otimes_S N_1 & \longrightarrow & X \otimes_S F & \longrightarrow & X \otimes_S N & \longrightarrow & 0 \\ \downarrow \theta_{XN_1} & & \cong \downarrow \theta_{XF} & & \downarrow \theta_{XN} & & \\ 0 \longrightarrow & \text{Hom}_S(X^*, N_1) & \longrightarrow & \text{Hom}_S(X^*, F) & \longrightarrow & \text{Hom}_S(X^*, N) & \longrightarrow \text{Ext}_S^1(X^*, N_1) \longrightarrow 0 \end{array}$$

shows that there is an isomorphism $\text{Coker } \theta_{XN} \cong \text{Ext}_S^1(X^*, N_1)$. The snake lemma applies to yield $\text{Ker } \theta_{XN} \cong \text{Coker } \theta_{XN_1} \cong \text{Ext}_S^1(X^*, N_2)$, and then (2.5.1) gives

$$\text{Ker } \theta_{XN} \cong \text{Ext}_S^1((X_2)^*, N) \quad \text{and} \quad \text{Coker } \theta_{XN} \cong \text{Ext}_S^1((X_1)^*, N). \tag{3.1.1}$$

(3.2) Lemma. *Let (S, \mathfrak{n}, ℓ) be a complete local ring of depth 0. Let \mathcal{C} be a reflexive subcategory of $\text{mod}(S)$. If ℓ has a \mathcal{C} -approximation and ℓ is not in \mathcal{C} , then $\mathcal{C} = \mathcal{F}(S)$.*

Proof. Consider a \mathcal{C} -approximation $0 \rightarrow L \xrightarrow{\alpha} C \rightarrow \ell \rightarrow 0$, and dualize to get $0 \rightarrow \ell^* \rightarrow C^* \xrightarrow{\alpha^*} L^*$. Let I be the image of α^* , and let φ be the factorization of α^* through the inclusion $I \hookrightarrow L^*$.

First we prove that the surjection φ is a $\langle \mathcal{C} \rangle$ -precover of I . Let X be a module in $\langle \mathcal{C} \rangle$. If X is a free S -module, then any homomorphism $X \rightarrow I$ lifts through φ . We may now assume that X is indecomposable and not free. Because $\text{Hom}_S(X, I)$ is a submodule of $\text{Hom}_S(X, L^*)$, it suffices to prove surjectivity of

$$\text{Hom}_S(X, \alpha^*) : \text{Hom}_S(X, C^*) \rightarrow \text{Hom}_S(X, L^*),$$

which we do next.

The vertical maps in the commutative diagram below are evaluation homomorphisms, see (3.1).

$$\begin{array}{ccccccc} X \otimes_R L & \xrightarrow{\iota} & X \otimes_R C & \longrightarrow & X \otimes_R \ell & \longrightarrow & 0 \\ \theta_{XL} \downarrow & & \theta_{XC} \downarrow & & \theta_{X\ell} \downarrow & & \\ 0 & \longrightarrow & \text{Hom}_S(X^*, L) & \longrightarrow & \text{Hom}_S(X^*, C) & \longrightarrow & \text{Hom}_S(X^*, \ell) \longrightarrow \text{Ext}_S^1(X^*, L) \end{array}$$

First we argue that the rows of this diagram are short exact sequences. The module X is in $\langle \mathcal{C} \rangle$ and hence in $\mathcal{G}(S)$, see (2.7), so $\text{Ext}_S^1(X^*, L) = 0$. Moreover, θ_{XL} is an isomorphism by (3.1.1), hence ι is injective. Next note that for every $\zeta \in X^*$ the image of $\zeta : X \rightarrow S$ is in \mathfrak{n} as X is indecomposable and not free. Thus, for all $x \in X$ and $u \in \ell$, we have $\theta_{X\ell}(x \otimes u)(\zeta) = \zeta(x)u = 0$. Finally, apply $\text{Hom}_S(-, S)$ to the diagram above and use Hom-tensor adjointness to get

$$\begin{array}{ccccccc} \text{Hom}_S(X^*, \ell)^* & \longrightarrow & \text{Hom}_S(X^*, C)^* & \longrightarrow & \text{Hom}_S(X^*, L)^* & \longrightarrow & \text{Ext}_S^1(\text{Hom}_S(X^*, \ell), S) \\ 0 \downarrow & & \theta_{XC}^* \downarrow & & \theta_{XL}^* \downarrow \cong & & 0 \downarrow \\ \text{Hom}_S(X, \ell^*) & \longrightarrow & \text{Hom}_S(X, C^*) & \xrightarrow{\text{Hom}_S(X, \alpha^*)} & \text{Hom}_S(X, L^*) & \longrightarrow & \text{Ext}_S^1(X \otimes_R \ell, S). \end{array}$$

The diagram shows that $\text{Hom}_S(X, \alpha^*)$ is surjective, as desired.

Now $\varphi : C^* \rightarrow I$ is a $\langle \mathcal{C} \rangle$ -precover, so by completeness of S , the module I has a $\langle \mathcal{C} \rangle$ -cover; see (2.2)(c). The ring has depth 0, so ℓ^* is a non-zero ℓ -vector space. By the assumptions on \mathcal{C} , the residue field ℓ cannot be a direct summand of C^* . As $\text{Ker } \varphi = \ell^*$, it follows from Lemma (1.6) that φ is a $\langle \mathcal{C} \rangle$ -cover. For every $X \in \langle \mathcal{C} \rangle$ Wakamatsu’s lemma gives $\text{Ext}_S^1(X, \ell^*) = 0$. Consequently, every module in \mathcal{C} is projective and hence free, since S is local. \square

(3.3) Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local ring and denote by $\mathcal{M}(R)$ the category of maximal Cohen–Macaulay R -modules.

(a) If R is Cohen–Macaulay, then $\mathcal{G}(R) \subseteq \mathcal{M}(R)$ by the Auslander–Bridger formula [4, §3.2, Prop. 3]. Conversely, if $\mathcal{G}(R) \subseteq \mathcal{M}(R)$, then R is Cohen–Macaulay.

- (b) If R is Gorenstein, then the categories $\mathcal{G}(R)$ and $\mathcal{M}(R)$ coincide by [4, §3.2, Thm. 3] and the Auslander–Bridger formula. Conversely, if $\mathcal{G}(R) = \mathcal{M}(R)$, then R is Gorenstein. Indeed, R is Cohen–Macaulay by (a), so $\Omega_{\dim R}^R(k)$ is in $\mathcal{M}(R)$, hence in $\mathcal{G}(R)$, and therefore R is Gorenstein by [4, §3.2, Rmk. after Thm. 3].
- (c) If R is Gorenstein, then a short exact sequence $0 \rightarrow L \rightarrow G \rightarrow M \rightarrow 0$ is a CM-approximation if and only if it is a $\mathcal{G}(R)$ -approximation. This follows from (b) and the fact that L is in $\mathcal{M}(R)^\perp$ if and only if L has finite injective dimension.

If R is Gorenstein, then every R -module has a CM-approximation by [7, Thm. A]. In view of (3.3)(c) the next result contains a converse, cf. Theorem C.

(3.4) Theorem. *Let (R, \mathfrak{m}, k) be a local ring and \mathcal{B} be a reflexive subcategory of $\text{mod}(R)$. If k has a \mathcal{B} -approximation, then R is Gorenstein or $\mathcal{B} = \mathcal{F}(R)$.*

In our proof of this theorem we use the next lemma. We do not know a reference giving a direct argument, so one is supplied here.

(3.5) Lemma. *Let (R, \mathfrak{m}, k) be a local ring, and let $\mathbf{x} = x_1, \dots, x_n$ be a sequence in $\mathfrak{m} \setminus \mathfrak{m}^2$. If \mathbf{x} is linearly independent modulo \mathfrak{m}^2 , then k is a direct summand of the module $\Omega_n^R(k)/\mathbf{x}\Omega_n^R(k)$.*

Proof. Let $(K(\mathbf{x}), d)$ be the Koszul complex on \mathbf{x} . If necessary, supplement \mathbf{x} to a minimal generating sequence \mathbf{x}, \mathbf{y} for \mathfrak{m} . Let (F, ∂) be a minimal free resolution of k . The identification $R/(\mathbf{x}, \mathbf{y}) = k$ lifts to a morphism of complexes $\sigma : K(\mathbf{x}, \mathbf{y}) \rightarrow F$. Serre proves in [24, Appendix I.2] that σ is injective and degreewise split. The natural inclusion $\iota : K(\mathbf{x}) \hookrightarrow K(\mathbf{x}, \mathbf{y})$ is also degreewise split, so the composite $\rho = \sigma \iota$ is an injective morphism of complexes and degreewise split.

From the short exact sequence $0 \rightarrow \Omega_n^R(k) \xrightarrow{\iota} F_{n-1} \rightarrow \Omega_{n-1}^R(k) \rightarrow 0$, we get an exact sequence in homology that reads in part

$$\text{Tor}_1^R(R/(\mathbf{x}), \Omega_{n-1}^R(k)) \rightarrow R/(\mathbf{x}) \otimes_R \Omega_n^R(k) \xrightarrow{R/(\mathbf{x}) \otimes_R \iota} R/(\mathbf{x}) \otimes_R F_{n-1}. \tag{*}$$

The module $\text{Tor}_1^R(R/(\mathbf{x}), \Omega_{n-1}^R(k)) \cong \text{Tor}_n^R(R/(\mathbf{x}), k)$ is annihilated by \mathfrak{m} .

Let e be a generator of $K(\mathbf{x})_n$. The image $\rho_n(e)$ in F_n is a minimal generator as ρ_n is split. Set $\varepsilon = \partial_n \rho_n(e) \in \Omega_n^R(k)$; since F is minimal, ε is a minimal generator of the syzygy $\Omega_n^R(k)$. The minimal generator $1 \otimes \varepsilon$ of $R/(\mathbf{x}) \otimes_R \Omega_n^R(k)$ is in the kernel of $(R/(\mathbf{x}) \otimes_R \iota)$, as the element $\varepsilon = \partial_n \rho_n(e) = \rho_{n-1} d_n(e)$ is in $\mathbf{x}F_{n-1}$. By exactness of (*) the element $1 \otimes \varepsilon$ is annihilated by \mathfrak{m} , hence it generates a 1-dimensional k -vector space that is a direct summand of $\Omega_n^R(k)/\mathbf{x}\Omega_n^R(k)$. \square

Proof of (3.4). We aim to apply Lemma (3.2). By Propositions (2.4) and (2.10), and by faithful flatness of \widehat{R} , we may assume R is complete. Set $d = \text{depth } R$; by Lemma (2.9) the d th syzygy $\Omega_d^R(k)$ has a \mathcal{B} -approximation:

$$0 \rightarrow L \rightarrow B \rightarrow \Omega_d^R(k) \rightarrow 0.$$

Let $\mathbf{x} = x_1, \dots, x_d$ be an R -regular sequence in $\mathfrak{m} \setminus \mathfrak{m}^2$ linearly independent modulo \mathfrak{m}^2 . The Koszul homology modules

$$H_i(\mathbf{K}(\mathbf{x}) \otimes_R \Omega_d^R(\mathfrak{k})) \cong \text{Tor}_i^R(R/(\mathbf{x}), \Omega_d^R(\mathfrak{k})) \cong \text{Tor}_{i+d}^R(R/(\mathbf{x}), \mathfrak{k})$$

vanish for $i > 0$, so \mathbf{x} is also $\Omega_d^R(\mathfrak{k})$ -regular.

Set $S = R/(\mathbf{x})$; by (2.7) and Proposition (2.4) the sequence

$$0 \rightarrow S \otimes_R L \rightarrow S \otimes_R B \xrightarrow{\varphi} S \otimes_R \Omega_d^R(\mathfrak{k}) \rightarrow 0$$

is an $\langle S \otimes \mathcal{B} \rangle$ -approximation. Moreover, the category $\langle S \otimes \mathcal{B} \rangle$ is reflexive by Proposition (2.10). By Lemma (3.5) the residue field \mathfrak{k} is a direct summand of $S \otimes_R \Omega_d^R(\mathfrak{k})$, so by (1.4) there is an $\langle S \otimes \mathcal{B} \rangle$ -precover of \mathfrak{k} . Since S is complete, it follows from (2.8) that \mathfrak{k} has an $\langle S \otimes \mathcal{B} \rangle$ -approximation.

Assume R is not Gorenstein. Then S is not Gorenstein, so the residue field \mathfrak{k} is not in $\mathcal{G}(S)$ and hence not in $\langle S \otimes \mathcal{B} \rangle$; see [4, §3.2, Rmk. after Thm. 3] or [13, Thm. (1.4.9)]. By Lemma (3.2) every module in $\langle S \otimes \mathcal{B} \rangle$ is now free, so for every $B \in \mathcal{B}$ the module $S \otimes_R B$ is free over S . By (2.7) the sequence \mathbf{x} is B -regular; therefore, B is a free R -module by Nakayama’s lemma. \square

An approximation of a module M is *minimal* if the map onto M is a cover. When R is Gorenstein, every R -module has a minimal CM-approximation by unpublished work of Auslander; see [8, Sec. 4] and [14, Thm. 5.5]. Hence we have

(3.6) Corollary. *Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local ring and assume there is a non-free module in $\mathcal{G}(R)$. The following are then equivalent:*

- (i) R is Gorenstein.
- (ii) \mathfrak{k} has a $\mathcal{G}(R)$ -approximation.
- (iii) Every finitely generated R -module has a minimal $\mathcal{G}(R)$ -approximation.

(3.7) If R has a dualizing complex, cf. [17, V.§2], then \mathfrak{k} has a Gorenstein projective precover $X \rightarrow \mathfrak{k}$ by [20, Thm. 2.11]. Assume X is finitely generated, i.e., X is in $\mathcal{G}(R)$ and, further, that R is henselian. If X is free, then it follows from (2.8) that \mathfrak{k} has a $\mathcal{G}(R)$ -approximation $0 \rightarrow L \rightarrow X' \rightarrow \mathfrak{k} \rightarrow 0$, where X' is free. Hence, \mathfrak{k} is in $\mathcal{G}(R)^\perp$ and then $\mathcal{G}(R) = \mathcal{F}(R)$. If X is not free, then R is Gorenstein by (3.6).

(3.8) Questions. Let $(R, \mathfrak{m}, \mathfrak{k})$ be a local ring. If \mathfrak{k} has a $\mathcal{G}(R)$ -precover, is then $\mathcal{G}(R)$ precovering? If $\mathcal{G}(R)$ is precovering and contains a non-free module, is then R Gorenstein?

4. On the number of totally reflexive modules

In this section we prove Theorems A and B. Note that by (1.3) the latter would follow immediately from a positive answer to the second question in (3.8).

(4.1) Lemma. *Let R be a local ring and M and N be finitely generated R -modules. If only finitely many isomorphism classes of R -modules X can fit in a short exact sequence $0 \rightarrow N \rightarrow X \rightarrow M \rightarrow 0$, then the R -module $\text{Ext}_R^1(M, N)$ has finite length.*

Proof. Given an R -module X , we denote by $[X]$ the subset of $\text{Ext}_R^1(M, N)$ whose elements have representatives of the form $0 \rightarrow N \rightarrow Y \rightarrow M \rightarrow 0$, where $Y \cong X$. By assumption, there exist non-isomorphic R -modules X_0, \dots, X_n such that $\text{Ext}_R^1(M, N)$ is the disjoint union of the sets $[X_i]$. We may take $X_0 = M \oplus N$, so $[X_0]$ is the zero submodule of $\text{Ext}_R^1(M, N)$. We must prove that there is an integer $q > 0$ such that $\mathfrak{m}^q \text{Ext}_R^1(M, N)$ is contained in $[X_0]$.

By [16, Cor. 1] there are integers p_i such that if $M/\mathfrak{m}^p M \oplus N/\mathfrak{m}^p N \cong X_i/\mathfrak{m}^p X_i$ for some $p \geq p_i$, then $X_i \cong M \oplus N$. Set $q = \max\{p_1, \dots, p_n\}$. Take a short exact sequence ξ in $\mathfrak{m}^q \text{Ext}_R^1(M, N)$; it belongs to some set $[X_i]$. By [26, Thm. 1.1] the sequence $\xi \otimes_R R/\mathfrak{m}^q$ splits, so $M/\mathfrak{m}^q M \oplus N/\mathfrak{m}^q N \cong X_i/\mathfrak{m}^q X_i$. By the choice of q this implies $X_i \cong M \oplus N$, so $i = 0$, i.e. ξ is in the zero submodule $[X_0]$. \square

Let $R \rightarrow S$ be a flat ring homomorphism. It does not follow from the natural isomorphism $S \otimes_R \text{Ext}_R^1(M, N) \cong \text{Ext}_S^1(S \otimes_R M, S \otimes_R N)$ that every extension of the S -modules $S \otimes_R N$ and $S \otimes_R M$ has the form $S \otimes_R X$ for some R -module X . In a seminar, Roger Wiegand alerted us to the next result.

(4.2) Lemma. *Let $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ be a flat ring homomorphism with $\mathfrak{m}S = \mathfrak{n}$ and $R/\mathfrak{m} \cong S/\mathfrak{n}$. Let M and N be finitely generated R -modules and ξ be an element of the S -module $\text{Ext}_S^1(S \otimes_R M, S \otimes_R N)$. If the R -module $\text{Ext}_R^1(M, N)$ has finite length, then there is an element χ in $\text{Ext}_R^1(M, N)$ such that $\xi = S \otimes_R \chi$.*

Proof. The functor $S \otimes_R -$ from the category $\text{mod}(R)$ to itself induces a natural isomorphism $K \rightarrow S \otimes_R K$ on R -modules of finite length. Applied to $\text{Ext}_R^1(M, N)$ this yields the first isomorphism below

$$\text{Ext}_R^1(M, N) \xrightarrow{\cong} S \otimes_R \text{Ext}_R^1(M, N) \xrightarrow{\cong} \text{Ext}_S^1(S \otimes_R M, S \otimes_R N).$$

The composite sends an exact sequence χ to $S \otimes_R \chi$. \square

The next result is Theorem B from the introduction.

(4.3) Theorem. *Let R be a local ring. If the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R)$ is finite, then R is Gorenstein or $\mathcal{G}(R) = \mathcal{F}(R)$.*

Proof. Assume there are only finitely many isomorphism classes of indecomposable modules in $\mathcal{G}(R)$. By (1.3) the residue field k then has a $\mathcal{G}(R)$ -precover $\varphi: B \rightarrow k$. We claim that $\widehat{R} \otimes_R \varphi$ is an $(\widehat{R} \otimes \mathcal{G}(R))$ -precover of k . Since \widehat{R} is complete, this implies the existence of an $(\widehat{R} \otimes \mathcal{G}(R))$ -approximation of k , see (2.8), and the desired conclusion follows from Theorem (3.4) and faithful flatness of \widehat{R} .

To prove the claim, we must show that

$$\text{Hom}_{\widehat{R}}(H', \widehat{R} \otimes_R \varphi) : \text{Hom}_{\widehat{R}}(H', \widehat{R} \otimes_R B) \rightarrow \text{Hom}_{\widehat{R}}(H', k)$$

is surjective for every module $H' \in \langle \widehat{R} \otimes \mathcal{G}(R) \rangle$. By flatness of \widehat{R} , surjectivity holds for modules in $\widehat{R} \otimes \mathcal{G}(R)$ and hence for every module in $\text{add}(\widehat{R} \otimes \mathcal{G}(R))$. It is now sufficient to prove that the category $\text{add}(\widehat{R} \otimes \mathcal{G}(R))$ is closed under extensions, because then $\langle \widehat{R} \otimes \mathcal{G}(R) \rangle$ is $\text{add}(\widehat{R} \otimes \mathcal{G}(R))$.

First we show that $\widehat{R} \otimes \mathcal{G}(R)$ is closed under extensions. Fix modules G and K in $\mathcal{G}(R)$, and consider short exact sequences $0 \rightarrow G \rightarrow H \rightarrow K \rightarrow 0$. Each H is in $\mathcal{G}(R)$, and the minimal number of generators of each H is bounded by the sum of the numbers of minimal generators for G and K . Since the number of indecomposable modules in $\mathcal{G}(R)$ is finite, there are, up to isomorphism, only finitely many such modules H . By Lemma (4.1) the module $\text{Ext}_R^1(K, G)$ has finite length, and by (4.2) every element of $\text{Ext}_R^1(\widehat{R} \otimes_R K, \widehat{R} \otimes_R G)$ is extended from $\text{Ext}_R^1(K, G)$.

To prove that $\text{add}(\widehat{R} \otimes \mathcal{G}(R))$ is closed under extensions, let G' and K' be summands of extended modules, i.e., $G' \oplus G'' \cong \widehat{R} \otimes_R G$ and $K' \oplus K'' \cong \widehat{R} \otimes_R K$ for modules $G, K \in \mathcal{G}(R)$. Consider a short exact sequence $0 \rightarrow G' \rightarrow H' \rightarrow K' \rightarrow 0$. Then a sequence

$$0 \rightarrow G' \oplus G'' \rightarrow H' \oplus G'' \oplus K'' \rightarrow K' \oplus K'' \rightarrow 0,$$

is exact, so by what has already been proved, the middle term $H' \oplus G'' \oplus K''$ is in $\widehat{R} \otimes \mathcal{G}(R)$; whence H' is in $\text{add}(\widehat{R} \otimes \mathcal{G}(R))$. \square

In view of (3.3)(a) we have

(4.4) Corollary. *Let R be a Cohen–Macaulay local ring. If R is of finite CM representation type, then R is Gorenstein or $\mathcal{G}(R) = \mathcal{F}(R)$.*

The next result contains Theorem A from the introduction.

(4.5) Theorem. *Let R be a local ring and assume the set of isomorphism classes of indecomposable modules in $\mathcal{G}(R) \setminus \mathcal{F}(R)$ is finite and not empty. Then R is Gorenstein and an isolated singularity. Further, \widehat{R} is a hypersurface singularity; if finite CM representation type ascends from R to \widehat{R} , then \widehat{R} is even a simple singularity.*

Proof. By Theorem 4.3 the ring R is Gorenstein. From (3.3)(b) it follows that R is of finite CM representation type and hence an isolated singularity by [19, Cor. 2]. By [18, Satz 1.2] the completion \widehat{R} is a hypersurface singularity and, assuming that also \widehat{R} is of finite CM representation type, it follows from [32, Cor. (8.16)] that \widehat{R} is a simple singularity. \square

(4.6) Remark. In [23] Schreyer conjectured that a Cohen–Macaulay local k -algebra R is of finite CM representation type if and only if \widehat{R} is of finite CM representation type. In [30] R. Wiegand proved descent of finite CM representation type from \widehat{R} to R for any local ring R . Ascent is verified in [30] when R is Cohen–Macaulay and either \widehat{R} is an isolated singularity or $\dim R \leq 1$. Ascent also holds for excellent Cohen–Macaulay local rings by work of Leuschke and R. Wiegand [22].

(4.7) Remarks. Constructing rings with infinitely many totally reflexive modules is easy using Theorem 4.3. Indeed, let Q be a local ring of positive dimension and set $R = Q[[X]]/(X^2)$. As R is not reduced, it is not an isolated singularity. The R -module $R/(X)$ is in $\mathcal{G}(R)$ and is not free, cf. [13, exa. (4.1.5)], so by (4.3) there are infinitely many non-isomorphic indecomposable modules in $\mathcal{G}(R)$.

More generally, Avramov, Gasharov, and Peeva [9] construct a non-free totally reflexive module³ G over any ring of the form $R \cong Q/(\mathbf{x})$, where (Q, \mathfrak{q}) is local and $\mathbf{x} \in \mathfrak{q}^2$ is a Q -regular sequence. Such a ring R is said to have an embedded deformation of codimension c , where c is the length of \mathbf{x} . Again (4.3) implies the existence of infinitely many non-isomorphic indecomposable modules in $\mathcal{G}(R)$. If \hat{R} has an embedded deformation of codimension $c \geq 2$, a recent argument of Avramov and Iyengar builds from G an infinite family of non-isomorphic indecomposable modules in $\mathcal{G}(R)$; see [10, Thm. 6.8 and proof of 6.4.(1)]. For such R , this gives a constructive proof of the abundance of modules in $\mathcal{G}(R)$.

(4.8) Question. Let R be a local ring that is not Gorenstein. Given an indecomposable totally reflexive R -module $G \not\cong R$, are there constructions that produce infinite families of non-isomorphic indecomposable modules in $\mathcal{G}(R)$?

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³ Actually, even a module of CI-dimension 0 as defined in [9, (1.2)].

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