Meromorphic functions sharing two small functions with its derivative

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Abstract

In this paper, we find all the forms of meromorphic functions $f(z)$ that share the value 0 CM\textsuperscript{∗}, and share $b(z)$ IM\textsuperscript{∗} with $g(z) = a_1(z)f(z) + a_2(z)f'(z)$. And $a_1(z)$, $a_2(z)$ and $b(z)$ ($a_2(z), b(z) \neq 0$) be small functions with respect to $f(z)$. As an application, we show that some of nonlinear differential equations have no transcendental meromorphic solution.

Keywords: Meromorphic function; Value sharing; Small function

1. Introduction and results

Let $f$ be a nonconstant meromorphic function in the complex domain. We shall adopt the standard notations in Nevanlinna’s value distribution theory of meromorphic functions such as the characteristic function $T(r,f)$, the counting function of the poles $N(r,f)$, and the proximity function $m(r,f)$ (see, e.g., [5]). We also denote $N_k(r,f)$ the counting function of the poles of $f$ with multiplicities less than or equal to $k$, and $N_k(r,f)$ the counting function of the poles of $f$ with multiplicities greater than or equal to $k$. The notation $S(r,f)$ is used to define any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of $r$ of finite linear measure
A meromorphic function $a \neq \infty$ is called a small function with respect to $f$ provided that $T(r,a) = S(r,f)$. Note that the set of all small functions of $f$ is a field. Let $f$ and $g$ be nonconstant meromorphic functions, and $b$ a small function with respect to $f$ and $g$. We say that $f$ and $g$ share $b$ IM (CM) provided that $f - b$ and $g - b$ have the same zeros ignoring (counting) multiplicities. Denote by $\tilde{N}(r,f = b = g)$ the reduced counting function of the common zeros of $f - b$ and $g - b$ ignoring the multiplicities, and $\tilde{N}_E(r,f = b = g)$ the reduced counting function of the common zeros of $f - b$ and $g - b$ with the same multiplicities. We say that $f$ and $g$ share $b$ IM* provided that

$$\tilde{N}\left(r, \frac{1}{f-b}\right) - \tilde{N}(r,f = b = g) = S(r,f)$$

and

$$\tilde{N}\left(r, \frac{1}{g-b}\right) - \tilde{N}(r,f = b = g) = S(r,g).$$

Similarly, we say that $f$ and $g$ share $b$ CM* provided that

$$\tilde{N}\left(r, \frac{1}{f-b}\right) - \tilde{N}_E(r,f = b = g) = S(r,f)$$

and

$$\tilde{N}\left(r, \frac{1}{g-b}\right) - \tilde{N}_E(r,f = b = g) = S(r,g).$$

Obviously, if $f$ and $g$ share $b$ IM (CM), then they share $b$ IM* (CM*).

In 1976, Rubel and Yang [10] proved that if $f$ is an entire function and shares two finite values CM with $f'$, then $f \equiv f'$. Mues and Steinmetz [9], and Gundersen [4] improved this result and proved the following theorem.

**Theorem A.** Let $f$ be a nonconstant meromorphic function, $a$ and $b$ be two distinct finite values. If $f$ and $f'$ share the values $a$ and $b$ CM, then $f = f'$.

Frank and Weissenborn [1] further improved Theorem A and obtained the following result.

**Theorem B.** Let $f$ be a nonconstant meromorphic function. If $f$ and $f^{(k)}$ share two distinct values $a$ and $b$ CM, then $f = f^{(k)}$.

An example given in [8] shows that the “CM” in Theorem B cannot be replaced by “IM.” However, if 0 is a Picard exceptional value of $f$ and $f^{(k)}$, Zheng and Wang [12] proved the following theorem.

**Theorem C.** Let $f(z)$ be a nonconstant meromorphic function, and $k \geq 2$ be an integer. If 0 is a Picard exceptional value of both $f$ and $f^{(k)}$, and in addition, $f$ and $f^{(k)}$ share a nonzero finite value IM, then $f(z) = e^{Az + B}$, where $A$ and $B$ are constants satisfying $A^k = 1$.

Gundersen [3] gave an example as follows, which shows that the condition $k \geq 2$ in Theorem C cannot be replaced by $k \geq 1$, i.e., $k \neq 1$. 

Example (I). Let \( f(z) = 2A/(1 - Be^{-2z}) \), where \( A \neq 0 \) is a constant. It is easy to verify that 0 is the Picard value of \( f \) and \( f' \). \( A \) is a shared value of \( f \) and \( f' \) IM, and \( f \neq f' \).

The following result indicates that Gundersen’s example is unique in some sense.

**Theorem D.** [11] Let \( f \) be a nonconstant meromorphic function, and \( b \) be a nonzero finite value. If \( f \) and \( f' \) share the value 0 CM, and share \( b \) IM, then \( f = f' \), or \( f(z) = 2b/(1 - ce^{-2z}) \), where \( c \) is a nonzero constant.

In the present paper, we shall prove the following results.

**Theorem 1.** Let \( f(z) \) be a nonconstant meromorphic function, and \( a_1(z), a_2(z) \) and \( b(z) \) \((a_2(z), b(z) \neq 0)\) be small functions with respect to \( f(z) \). If \( f(z) \) and \( g(z) = a_1(z)f(z) + a_2(z)f'(z) \) share 0 CM*, and share \( b(z) \) IM*, then \( f(z) = g(z) \) or \( f(z) \) takes one of the following two forms:

(i) \( f = b/(h - 1) \) and \( a_1b + a_2b' = -b \), where \( h \) satisfies \( h'/h = -1/a_2 \);

(ii) \( f = 2b/(1 - h) \) and \( a_1b + a_2b' = 0 \), where \( h \) satisfies \( h'/h = -2/a_2 \).

**Example (II).** Let \( f(z) = A/(Be^{-z} - 1) \), where \( A \neq 0 \) is a constant. It is easy to verify that 0 is the Picard value of \( f \) and \( f' \). \( A \) is a shared value of \( f \) and \( f' \) IM, and \( f \neq f' \).

Note: (I) is an example of case (ii) in Theorem 1, while (II) is an example of case (i) in Theorem 1.

**Corollary 1.** Suppose that \( a_i(z) \) \((i = 1, 2, 3)\), and \( b(z) \) are meromorphic functions, and \( a_2a_3b \neq 0 \). Then any of the following three equations:

\[
b(a_1f + a_2f' - f)^2 = f(a_1f + a_2f')(f - b),
\]

\[
(a_1f + a_2f')^2(f - b) = a_3f^3(a_1f + a_2f' - b),
\]

\[
(a_1f + a_2f')(f - b)^3 = a_3f(a_1f + a_2f' - b)^2
\]

has no nonconstant meromorphic solution \( f \) satisfying \( T(r, a_i) = S(r, f) \) \((i = 1, 2, 3)\), and \( T(r, b) = S(r, f) \).

Hence, the above three equations have no transcendental meromorphic solutions provided that \( a_i(z) \) \((i = 1, 2, 3)\) and \( b(z) \) are rational functions with \( a_2a_3b \neq 0 \).

**Corollary 2.** Let \( f \) be a nonconstant meromorphic function, and \( b (\neq 0, \infty) \) be a small function of \( f \). If \( f \) and \( f' \) share 0 CM*, and share \( b \) IM*, then \( f = f' \) or \( b \) is constant and \( f(z) = 2b/(1 - ce^{-2z}) \), where \( c \) is a nonzero constant.

**Theorem 2.** Let \( f \) be a nonconstant meromorphic function, and \( b(z) (\neq 0, \infty) \) be a small function of \( f \). If \( f \) and \( f^{(k)} \) \((k \geq 2)\) share \( b(z) \) IM*, and in addition,

\[
N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S(r, f),
\]

then \( f(z) = ce^{\lambda z} \), where \( c \) and \( \lambda \) are nonzero constants and \( \lambda^k = 1 \).
Obviously, Corollary 2 is a generalization of Theorem D, and Theorem 2 is a generalization of Theorem C.

2. Lemmas

Before proving the main results, we present some existing results in the following three lemmas, which will be used in the proofs of Theorems 1 and 2.

**Lemma 1.** [6] Let $f$ be a nonconstant meromorphic function, and $f_1, \ldots, f_n$ ($n \geq 2$) be nonzero meromorphic functions satisfying $T(r, f_i) \leq O(T(r, f))$, and suppose

$$\bar{N}(r, f_i) + \bar{N}(r, \frac{1}{f_i}) = S(r, f)$$

for $i = 1, 2, \ldots, n$. Further, if $f_i \neq 1$ for $i = 1, \ldots, n$, and $f_1 + f_2 + \cdots + f_n = 1$, then at least two (all, if $n \leq 3$) $f_i$ satisfy $T(r, f_i) = S(r, f)$.

**Lemma 2.** [7] Let $f_1$ and $f_2$ be two nonconstant meromorphic functions satisfying

$$\bar{N}(r, f_i) + \bar{N}(r, \frac{1}{f_i}) = S(r), \quad i = 1, 2.$$

If $f_1^s f_2^t - 1$ is not identically zero for all integers $s$ and $t$ $(|s| + |t| > 0)$, then for any positive number $\varepsilon$, we have

$$N_0(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r),$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of $f_1$ and $f_2$ related to the common 1-points and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))$ as $r \to \infty$, except for a set of $r$ of finite linear measure.

**Lemma 3.** [2] If $f$ is a transcendental meromorphic function, and $k$ is a positive integer, then the following inequality

$$(k - 1)\bar{N}(r, f) \leq (1 + \varepsilon)N\left(r, \frac{1}{f^{(k)}}\right) + (1 + \varepsilon)(N(r, f) - \bar{N}(r, f)) + S(r, f)$$

holds for any positive number $\varepsilon$.

3. Proof of the results

Now we shall use a generalized version of Nevanlinna’s Second Fundamental Theorem (see, e.g., [5, p. 47]) to prove our main results, Theorems 1 and 2.

**Proof of Theorem 1.** Since $f$ and $g$ share $0, b, \infty$ IM*, it is easily seen from the Second Fundamental Theorem that

$$T(r, f) \leq 3T(r, g) + S(r, f), \quad (2)$$

$$T(r, g) \leq 3T(r, f) + S(r, g). \quad (3)$$
Hence, \( S(r, g) = S(r, f) := S(r) \). In particular, since \( f \) and \( g \) share 0 CM*, and \( g = a_1 f + a_2 f' \), we obtain \( \tilde{N}(r, f = g = 0) \leq \tilde{N}(r, 1/a_2) = S(r, f) \). Thus we have

\[
\tilde{N}\left(r, \frac{1}{f}\right) + \tilde{N}\left(r, \frac{1}{g}\right) = S(r).
\] (4)

Suppose \( f \neq g \). Otherwise, nothing needs to be proved. Since \( f \) and \( g \) share \( b \) IM*, it follows from the second fundamental theorem, the theorem on the logarithmic derivative and (4) that

\[
\begin{align*}
T(r, g) &\leq \tilde{N}(r, g) + \tilde{N}\left(r, \frac{1}{g}\right) + \tilde{N}\left(r, \frac{1}{g-b}\right) + S(r) \\
&= \tilde{N}(r, f) + \tilde{N}\left(r, \frac{1}{g-b}\right) + S(r) \\
&\leq \tilde{N}(r, f) + \tilde{N}\left(r, \frac{1}{g/f - 1}\right) + S(r) \\
&\leq \tilde{N}(r, f) + \tilde{N}\left(r, \frac{1}{a_2 f'/f + a_1 - 1}\right) + S(r) \\
&\leq \tilde{N}(r, f) + T\left(r, \frac{f'}{f}\right) + S(r) \\
&\leq \tilde{N}(r, f) + m\left(r, \frac{f'}{f}\right) + \tilde{N}(r, f) + \tilde{N}\left(r, \frac{1}{f}\right) + S(r) \\
&= 2\tilde{N}(r, f) + S(r) \\
&\leq N(r, g) + S(r) \\
&\leq T(r, g) + S(r).
\end{align*}
\]

Therefore, we get

\[
\begin{align*}
T(r, g) &= 2\tilde{N}(r, f) + S(r) = N(r, g) + S(r), \quad \text{(5)} \\
m(r, g) &= S(r). \quad \text{(6)}
\end{align*}
\]

Noting that \( N(r, g) = N(r, f) + \tilde{N}(r, f) \), we have

\[
N(r, f) = \tilde{N}(r, f) + S(r), \quad \text{(7)}
\]

which clearly shows that \( \tilde{N}(r, f) = S(r) \), and thus

\[
\tilde{N}\left(r, \frac{1}{g-b}\right) = \tilde{N}\left(r, \frac{1}{f-b}\right) + S(r) = \tilde{N}(r, f) + S(r). \quad \text{(8)}
\]

By the inequalities before (5), we have \( T(r, f'/f) = \tilde{N}(r, f) + S(r) \). Then (4) and (8) together imply that

\[
\begin{align*}
\tilde{N}\left(r, \frac{1}{g-f}\right) &= \tilde{N}\left(r, \frac{1}{g/f - 1}\right) + S(r) \\
&\leq T\left(r, \frac{f}{f}\right) + S(r) \\
&= \tilde{N}(r, f) + S(r) \\
&= \tilde{N}\left(r, \frac{1}{g-b}\right) + S(r).
\end{align*}
\]
Since \( f \) and \( g \) share \( b \) \( \text{IM}^* \), we have
\[
\tilde{N}(r, \frac{1}{g-b}) \leq \tilde{N}(r, \frac{1}{g-f}) + S(r).
\]
Therefore,
\[
\tilde{N}(r, \frac{1}{g-f}) = \tilde{N}(r, \frac{1}{g-b}) + S(r). \tag{9}
\]

Let
\[
\alpha = \frac{g'}{g} - 2 \frac{f'}{f}. \tag{10}
\]
Consider the poles of \( f \). Simple poles of \( f \) are not poles of \( \alpha \), and multiple poles of \( f \) can be neglected in view of (7). It follows from (4) and the lemma of logarithmic derivative that
\[
T(r, \alpha) = S(r). \tag{11}
\]
Since \( f \) and \( g \) share \( b \) \( \text{IM}^* \), we see from (9) that \( f-b, g-b \) and \( g-f \) share 0 \( \text{IM}^* \).

Suppose that \( z_0 \) is a double zero of \( g-f \), which is a zero of both \( f-b \) and \( g-b \), but not a zero or pole of \( a_1, a_2 \) or \( b \). Then we have \( \alpha(z_0) = \alpha_1(z_0) = (a_1(z_0) - 1)/a_2(z_0) \). If \( \alpha = (a_1 - 1)/a_2 \), then it follows from (10) that any common zero \( z \) of \( f-b \) and \( g-b \) must be multiple zero of \( g-f \) provided that \( z \) is not a zero or pole of \( a_1, a_2, b \). Therefore, by (9) and (4), we have
\[
\tilde{N}(r, \frac{1}{g-b}) = \tilde{N}(r, \frac{1}{g-f}) + S(r) \leq \frac{1}{2} N(r, \frac{1}{g-f}) + S(r)
\]
\[
= \frac{1}{2} N(r, \frac{f}{g-f}) + S(r) \leq \frac{1}{2} T(r, \frac{g}{f}) + S(r)
\]
\[
= \frac{1}{2} N(r, \frac{f'}{f}) + S(r) = \frac{1}{2} \tilde{N}(r, f) + S(r),
\]
and thus the estimates (8) and (9) imply \( \tilde{N}(r, f) = S(r) \). However, on the other hand, (2) and (5) yield \( T(r, f) = S(r) \), a contradiction. Hence \( \alpha \neq (a_1 - 1)/a_2 \), leading to
\[
\tilde{N}(r, \frac{1}{g-f}) \leq \tilde{N}(r, \frac{1}{\alpha - (a_1 - 1)/a_2}) + S(r) = S(r). \tag{12}
\]

If \( bg = f^2 \), then \( b(g-b) = (f-b)(f+b) \). Note that \( f \) and \( g \) share \( b \) \( \text{IM}^* \). We get \( \tilde{N}(r, 1/(f+b)) = S(r) \). Let \( h_1 = (f+b)/f \). Then \( \tilde{N}(r, h_1) + \tilde{N}(r, 1/h_1) = S(r) \). Therefore, \( T(r, h_1')/h_1 = S(r) \). From \( bg = f^2 \) and the definition of \( g \), we have \( f' = f(f-a_1b)/(b_2a_2) \), which is inserted into \( h_1' = (b'f - b f')/(f(f+b)) \) yields \( h_1' = (a_1b + a_2b' + b)/(a_2(f+b)) = -1/a_2 \). If \( a_1b + a_2b' + b \neq 0 \), then we get \( T(r, f) = S(r) \), a contradiction. Thus \( a_1b + a_2b' + b = 0 \), then \( h_1' = -1/a_2 \), and \( f = b/(h_1 - 1) \). Hence \( f \) assumes the first form in Theorem 1.

In the following, we assume
\[
bg \neq f^2. \tag{13}
\]
If \( -b(g-f)^2 = f(f-b)(g-b) \), then
\[
\frac{b^2}{f} = -g + 3b \frac{g}{f} - b \left( \frac{g}{f} \right)^2.
\]
Since every zero of \( f \) is a double pole of the right-hand side of the above equation, except for the zeros and poles of \( b, a_1, a_2 \), we have \( N(r, 1/f) = 2\bar{N}(r, 1/f) + S(r) = S(r) \).

From the above equation and the lemma of logarithmic derivative, together by using \( m(r, g) = S(r) \) from (6), we get
\[
m(r, \frac{1}{f}) = S(r).
\]
Therefore, \( T(r, f) = S(r) \), a contradiction. Hence
\[
-b(g - f)^2 \neq f(f - b)(g - b).
\] (14)

If \( g(f - b)^2 = f^2(g - b) \), then \( 2fg = bg + f^2 \), which implies \( \bar{N}(r, f) = S(r) \). It follows from (2) and (5) that \( T(r, f) = S(r) \), a contradiction. Hence
\[
g(f - b)^2 \neq f^2(g - b).
\] (15)

If
\[
-b(g - f)^2 = f^2(g - b),
\] (16)
then \( -bg = f(f - 2b) \). Since \( \bar{N}(r, 1/g) = S(r) \), we have \( \bar{N}(r, 1/(f - 2b)) = S(r) \). Let \( h_2 = (f - 2b)/f \). It is easy to see that \( T(r, h_2'/h_2) = S(r) \). Since \( g = a_1 f + a_2 f' \), we get
\[
-b(a_1 f + a_2 f') = f(f - 2b),
\]
from which we obtain
\[
\frac{f'}{f} = -\frac{a_1}{a_2} - \frac{f - 2b}{a_2 b}.
\]
Hence,
\[
\frac{h_2'}{h_2} = \frac{-2b' f + 2bf'}{f(f - 2b)} = -\frac{2}{a_2} - \frac{2(a_1 b + a_2 b')}{a_2 (f - 2b)}.
\]
If \( a_1 b + a_2 b' \neq 0 \), then the above equation leads to \( T(r, f) = S(r) \), a contradiction. Thus \( a_1 b + a_2 b' = 0 \), then \( h_2'/h_2 = -2/a_2 \), and so \( f \) can be expressed as \( 2b/(1 - h_2) \), which is the second form in Theorem 1.

To complete the proof of Theorem 1, we need to show that
\[
-b(g - f)^2 \neq f^2(g - b)
\] (17)
always leads to a contradiction by distinguishing two cases below.

**Case 1.** Suppose that the following condition holds:
\[
\alpha = \frac{b'}{b} - 2\frac{1 - a_1}{a_2}.
\] (18)

If \( a_1 b + a_2 b' = b \), then by (10) and (18), we have
\[
\frac{g'}{g} - 2\frac{f'}{f} + \frac{b'}{b} = 0.
\]
By integrating the above equation, we know that \( bg/f^2 \) is a nonzero constant. From (5), (8), we have
\[
\bar{N}\left(r, \frac{1}{g - b}\right) = \frac{1}{2} T(r, g) + S(r),
\] (19)
and note that \( b(z)g(z)/f^2(z) = 1 \) holds for any common zero of \( f - b \) and \( g - b \) provided that it is not any zero or pole of \( b \). If \( f - b \) and \( g - b \) have a common zero (which is not a zero or
pole of \( b \) then we can conclude \( bg = b^2 \). If there does not exist a common zero, from the fact that \( f \) and \( g \) share \( b \) IM* we would deduce \( \tilde{N}(r, 1/(g - b)) = S(r) \) which together with (19) and (2) implies \( T(r, f) = S(r) \), a contradiction. Consequently, \( a_1 b + a_2 b' \neq b \). Suppose that \( z \) is a common zero of \( f - b \) and \( g - b \), but not any zero or pole of \( a_1, a_2 \) or \( b \). We have \( a_1(z) b(z) + a_2(z) b'(z) = b(z) \) provided that \( z \) is a multiple zero of \( f - b \). Therefore,

\[
\tilde{N}(r, \frac{1}{f - b}) \leq \tilde{N}(r, \frac{1}{a_1 b + a_2 b' - b}) + S(r) = S(r). \tag{20}
\]

Suppose that \( z_1 \) is a common zero of \( f - b \) and \( g - b \), but not any zero or pole of \( a_1, a_2 \) or \( b \). By (10), (18) and \( g = a_1 f + a_2 f' \), we get \( g'(z_1) - b'(z_1) = 0 \), which implies that \( z_1 \) is a multiple zero of \( g - b \). Hence,

\[
\tilde{N}_{11}(r, \frac{1}{g - b}) = S(r). \tag{21}
\]

From Eq. (21), together with (5) and (8), we deduce that

\[
\tilde{N}(r, f) = \tilde{N}(r, \frac{1}{g - b}) + S(r) \leq \frac{1}{2} N(r, \frac{1}{g - b}) + S(r) \\
\leq \frac{1}{2} T(r, g) + S(r) = \tilde{N}(r, f) + S(r).
\]

Therefore, \( N(r, 1/(g - b)) = 2 \tilde{N}(r, 1/(g - b)) + S(r) \), and thus, with the aid of (21),

\[
N_{3}(r, \frac{1}{g - b}) = S(r). \tag{22}
\]

Let

\[
f_1 = \frac{g(f - b)^2}{f^2(g - b)}, \quad f_2 = \frac{bg}{f^2}, \quad f_3 = -\frac{b(g-f)^2}{f^2(g - b)}.
\]

Then we have \( f_1 + f_2 + f_3 = 1 \). By (4), (7), (12), (21), (22) and (20), we obtain

\[
\tilde{N}(r, f_i) + \tilde{N}(r, \frac{1}{f_i}) = S(r), \quad i = 1, 2, 3.
\]

Further, by (13), (15) and (17), we get \( f_i \neq 1, i = 1, 2, 3 \). Therefore, by Lemma 1, we have \( T(r, f_i) = S(r) \). Note that \( f_1(z) = 1 \) holds for any pole of \( f \) provided that it is not any zero or pole of \( a_1, a_2 \) or \( b \). Hence we get

\[
\tilde{N}(r, f) \leq \tilde{N}(r, \frac{1}{f_1 - 1}) + S(r) \leq T(r, f_1) + S(r) = S(r).
\]

However, it follows from (2) and (5) that \( T(r, f) = S(r) \), which indicates that the above conclusion is not possible. Hence, Case 1 is ruled out.

**Case 2.** Suppose that (18) is not true, i.e.,

\[
\alpha \neq \frac{b'}{b} - 2\frac{1-a_1}{a_2}. \tag{23}
\]

If \( z_2 \) is a multiple zero of \( g - b \), which is a zero of \( f - b \) but not any zero or pole of \( a_1, a_2 \) or \( b \), then by a simple manipulation, we get

\[
\alpha(z_2) = \left(\frac{b'}{b} - 2\frac{1-a_1}{a_2}\right)(z_2).
\]
Therefore,
\[
\tilde{N}(2, \frac{r}{g - b}) \leq \tilde{N}(r, \frac{1}{\alpha - b'/b + 2(1 - a_1)/a_2}) + S(r)
\leq T(r, \alpha) + S(r) = S(r). \tag{24}
\]

Let
\[
\beta = \frac{g' - f'}{g - f} - \frac{g' - b'}{g - b}. \tag{25}
\]

Then by noting (7), a simple calculation shows that \(\beta(z) = 1/a_2(z)\) holds for “almost all” poles of \(f\). Since \(\tilde{N}(r, f) \neq S(r)\), we have \(\beta = 1/a_2\), i.e.,
\[
\frac{g' - f'}{g - f} - \frac{g' - b'}{g - b} = \frac{1}{a_2}. \tag{26}
\]

Again, a simple computation shows that any multiple zero of \(f - b\) must be zero of \(a_1b + a_2b' - b\) provided that it is also a zero of \(g - b\), but not any zero or pole of \(a_1, a_2\) or \(b\). If \(b \neq a_1b + a_2b'\), then we have
\[
\tilde{N}(2, \frac{1}{f - b}) = S(r). \tag{27}
\]

Similarly, let
\[
g_1 = \frac{g(f - b)}{f(g - b)}, \quad g_2 = \frac{b(g - f)}{f(f - b)}, \quad g_3 = -\frac{b(g - f)^2}{f(f - b)(g - b)}.
\]

We have \(g_1 + g_2 + g_3 = 1\). From (4), (7), (12), (24) and (27), we get
\[
\tilde{N}(r, g_i) + \tilde{N}(r, \frac{1}{g_i}) = S(r), \quad i = 1, 2, 3.
\]

Since \(f \neq g\), we have \(g_1 \neq 1\). Further, from (13) and (14), we obtain \(g_2 \neq 1\) and \(g_3 \neq 1\), respectively. Then it follows from Lemma 1 that \(T(r, g_1) = S(r)\). Note that \(g_1(z) = 1\) holds for “almost all” poles of \(f\). Hence \(\tilde{N}(r, f) = S(r)\), which together with (2) and (5) implies \(T(r, f) = S(r)\). This is impossible.

In the following, we assume that
\[
a_1b + a_2b' = b, \tag{28}
\]

which, together with \(g = a_1f + a_2f'\), shows that any common zero of \(f - b\) and \(g - b\) must be multiple zero of \(f - b\) provided that it is not any zero or pole of \(a_1, a_2\) or \(b\). Therefore,
\[
\tilde{N}(1, \frac{1}{f - b}) = S(r). \tag{29}
\]

From (29), (4), (8) and the second fundamental theorem, we can deduce that
\[
2\tilde{N}(r, f) = 2\tilde{N}(r, \frac{1}{f - b}) + S(r) \leq N(r, \frac{1}{f - b}) + S(r)
\leq T(r, f) + S(r) \leq \tilde{N}(r, f) + \tilde{N}(r, \frac{1}{f}) + \tilde{N}(r, \frac{1}{f - b}) + S(r)
\leq 2\tilde{N}(r, f) + S(r).
\]
Hence,

\[ T(r, f) = 2\tilde{N}(r, f) + S(r) = N\left(r, \frac{1}{f - b}\right) + S(r) = 2\tilde{N}\left(r, \frac{1}{f - b}\right) + S(r). \]  

(30)

Next, let

\[ F_1 = \frac{g - b}{g - f} \quad \text{and} \quad F_2 = \frac{bg}{f^2}. \]

Then by (4), (7), (12) and (24), we have \( \tilde{N}(r, F_i) + \tilde{N}(r, 1/F_i) = S(r), \ i = 1, 2. \) Therefore, \( T(r, F'_i/F_1) = S(r), \) and

\[
m\left(r, \frac{1}{F_1 - 1}\right) = m\left(r, \frac{1}{F_1 - 1} + 1\right) + S(r) = m\left(r, \frac{F_1}{F_1 - 1}\right) + S(r) \leq m\left(r, \frac{F'_1}{F_1 - 1}\right) + m\left(r, \frac{F_1}{F'_1}\right) + S(r) = S(r).
\]

If \( z_0 \) is a zero of \( f - b \) with multiplicity \( k \geq 1 \) (but not a zero or a pole of \( a_1, a_2, b \), then it is a zero of \( g - b \) with multiplicity \( k - 1 \) (i.e., for \( k = 1 \) it is not a zero of \( g - b \) at all). Hence by (30), we get

\[
T(r, F_1) = N\left(r, \frac{1}{F_1 - 1}\right) + S(r) = N\left(r, \frac{g - f}{f - b}\right) + S(r) = \tilde{N}(r, f) + \tilde{N}\left(r, \frac{1}{f - b}\right) + S(r) = T(r, f) + S(r).
\]

Then from (26), (28) and \( g = a_1 f + a_2 f', \) eliminating \( f' \) yields

\[((a_1 + 1)g + a_2 g' - 2b)f = 2g^2 - (a_2 b' + 2b)g + a_2 bg'.\]

(31)

Noting (4), we have

\[ N\left(r, \frac{1}{g}\right) = N\left(r, \frac{1}{f}\right) + \tilde{N}\left(r, \frac{1}{g}\right) + S(r) = N\left(r, \frac{1}{f}\right) + S(r). \]

(32)

On the other hand, from \( N(r, F_2) = N(r, 1/f) + \tilde{N}(r, 1/f) + S(r) \) and (4), we obtain \( N(r, F_2) = N(r, 1/f) + S(r). \) Thus \( m(r, F_2) \leq m(r, 1/f) + S(r) \) due to \( g = a_1 f + a_2 f'. \) Now rewriting (31) as

\[
\frac{2b}{f} = \left(a_1 + 1 + a_2 g'\frac{g'}{g}\right)\frac{g}{f} - 2\left(\frac{g'}{f}\right)^2 + \left(a_2 \frac{b'}{b} + 2 - a_2 \frac{g'}{g}\right)F_2,
\]

we can see that \( m(r, 1/f) \leq m(r, F_2) + S(r). \) Hence, \( m(r, F_2) = m(r, 1/f) + S(r). \) Therefore, we have \( T(r, F_2) = T(r, f) + S(r) = T(r, F_1) + S(r). \) Note that \( \tilde{N}(r, 1/(f - b)) = \frac{1}{2}T(r, f) + S(r), \) and “almost all” zeros of \( f - b \) are common 1-points of \( F_1 \) and \( F_2. \) By Lemma 2, there exist two integers \( s \) and \( t \) such that \( F'_i F'_2 = 1. \) It follows that \( |s|T(r, F_1) = |t|T(r, F_2) + O(1). \) Further, note that \( T(r, F_1) = T(r, F_2) + S(r), \) which implies that \( s = \pm t. \) Therefore, \( F_1 F_2 = c \)
or $F_1 = cF_2$, where $c$ is a constant satisfying $c^{\mid s \mid} = 1$. The equation $\tilde{N}(r, 1/(f - b)) = \frac{1}{2}T(r, f) + S(r)$ implies that $F_1$ and $F_2$ have many common 1-points. Consequently, $c = 1$. And thus $F_1F_2 = 1$ or $F_1 = F_2$.

If $F_1F_2 = 1$, then $bg(g - b) = f^2(g - f) = f^2g - f^3$, i.e., $f^2 = fg = b(g - b)g/f$. Therefore, by (6), we get

$$2m(r, f) = m\left(r, fg - \frac{b(g - b)g}{f}\right) + S(r) = m(r, fg) + S(r) \leq m(r, f) + S(r).$$

Hence, $m(r, f) = S(r)$. By (7) and (30), we get $T(r, f) = S(r)$, a contradiction.

If $F_1 = F_2$, then $g = b(g/f - r_1)(g/f - r_2)$, where $r_1$ and $r_2$ are two roots of $z^2 - z + 1 = 0$.

By (4), we get

$$\tilde{N}\left(r, \frac{1}{g/f - r_i}\right) = S(r), \quad i = 1, 2.$$

It is easy to see that $g/f$ is not a constant. Otherwise, if $g/f$ was constant, then from $g = b(g/f - r_1)(g/f - r_2)$, we would obtain $T(r, g) = S(r)$, hence in view of (2) $T(r, f) = S(r)$.

By the second fundamental theorem, we have

$$T(r, g/f) \leq \tilde{N}\left(r, \frac{1}{g/f}\right) + \tilde{N}\left(r, \frac{1}{g/f - r_1}\right) + \tilde{N}\left(r, \frac{1}{g/f - r_2}\right) + S\left(r, \frac{g}{f}\right).$$

Since $f$ and $g/f$ have the same poles (except for the zeros and poles of $a_1, a_2$), we have $\tilde{N}(r, f) \leq \tilde{N}(r, g/f) + S(r) \leq T(r, g/f) + S(r) = S(r) + S(r, g/f) = S(r)$. Then by noting that $g = a_1f + a_2f'$ and $\tilde{N}(r, 1/g) = S(r)$, we get $\tilde{N}(r, f) = S(r)$. Further by (30), we obtain $T(r, f) = S(r)$, again a contradiction. Hence Case 2 is also ruled out. The proof of Theorem 1 is complete. □

Proof of Corollary 1. First, note that the cases of $f \equiv g$ and $g \equiv 0$ in applying Theorem 1 can be ruled out by the assumptions on $f$ and $a_1, a_2, a_3, b$. If $f$ is a nonconstant meromorphic solution of one of the equations in Corollary 1, and $T(r, a_i) + T(r, b) = S(r, f)$, $i = 1, 2, 3$, then it is easy to verify that $f$ and $g = a_1f + a_2f'$ share 0 CM* and share $b$ IM*. By Theorem 1, $f$ assumes one of the forms in Theorem 1. If $f$ takes the first form, then $a_1f + a_2f' = b/(h - 1)^2$. If $f$ assumes the second form, then $a_1f + a_2f' = -4bh/(h - 1)^2$. Therefore, $f$ cannot be the solution of any equation in Corollary 1. This completes the proof of Corollary 1. □

Proof of Corollary 2. Since $a_1 = 0$ and $a_2 = 1$, we have $h' = -h$ and $b' = -b$ provided that $f$ assumes the first form in Theorem 1. Hence, $b$ cannot be a small function of $h$, and thus cannot be a small function of $f$. This is impossible. If $f$ assumes the second form in Theorem 1, then $b' = 0$ and $h' = -2h$. Hence $b$ is a constant and $h \equiv ce^{-2z}$. The proof of Corollary 2 is finished. □

Proof of Theorem 2. Suppose that $f$ and $f^{(k)}$ share $b$ IM* and

$$\tilde{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S(r, f).$$

(33)

If $f = f^{(k)}$, then there exist constants $c_1, \ldots, c_k$ such that

$$f(z) = c_1e^{\lambda_1z} + \cdots + c_ke^{\lambda_kz},$$

where \( \lambda_i, i = 1, \ldots, k \), are the roots of equation \( z^k = 1 \). Let \( \beta = f'/f \). Then \( \beta \neq 0 \). Note that \( N(r, f) = 0 \) and \( \tilde{N}(r, 1/f) = S(r, f) \). By the lemma on the logarithmic derivative, we have \( T(r, \beta) = S(r, f) \). Therefore, \( \beta \) is a small function with respect to \( e^{\lambda_j z}, j = 1, \ldots, k \). From the above equation, we have

\[
\lambda_1 c_1 e^{\lambda_1 z} + \cdots + \lambda_k c_k e^{\lambda_k z} = \beta \left( c_1 e^{\lambda_1 z} + \cdots + c_k e^{\lambda_k z} \right).
\]

By Lemma 1, it is easy to prove that \( e^{\lambda_1 z}, \ldots, e^{\lambda_k z} \) are linearly independent over the field of small functions with respect to \( e^{\lambda_j z}, j = 1, \ldots, k \). Therefore, \( c_j (\lambda_j - \beta) \equiv 0, j = 1, \ldots, k \). It follows that only one of the constants in \( \{c_1, \ldots, c_k\} \) is not zero. Hence \( f(z) = c e^{\lambda z} \), where \( c \) and \( \lambda \) are nonzero constants and \( \lambda^k = 1 \).

If \( f \neq f^{(k)} \), then \( f^{(k)}/f \neq 1 \). Suppose that \( z_0 \) is a common zero of \( f^{(k)} - b \) and \( f - b \) ignoring the multiplicities, but not the zero of \( b \). Then, we have \( f^{(k)}(z_0)/f(z_0) = 1 \). Since \( f \) and \( f^{(k)} \) share \( b \) IM*,, we have

\[
\tilde{N} \left( r, \frac{f^{(k)}}{f} - b \right) \leq \tilde{N} \left( r, \frac{1}{f^{(k)/f} - 1} \right) + S(r, f) \leq T \left( r, \frac{f^{(k)}}{f} \right) + S(r, f)
\]

\[
\leq k \tilde{N} \left( r, \frac{1}{f} \right) + k \tilde{N}(r, f) + S(r, f)
\]

\[
= k \tilde{N}(r, f) + S(r, f).
\]

By the second fundamental theorem, we have

\[
T(r, f^{(k)}) \leq \tilde{N}(r, f^{(k)}) + \tilde{N}(r, 1/f^{(k)}) + \tilde{N} \left( r, \frac{1}{f^{(k)} - b} \right) + S(r, f^{(k)})
\]

\[
= \tilde{N}(r, f) + \tilde{N} \left( r, \frac{1}{f^{(k)} - b} \right) + S(r, f)
\]

\[
\leq \tilde{N}(r, f) + k \tilde{N}(r, f) + S(r, f)
\]

\[
\leq N(r, f) + k \tilde{N}(r, f) + S(r, f)
\]

\[
= T(r, f^{(k)}) + S(r, f).
\]

Therefore, we obtain

\[
N(r, f) = \tilde{N}(r, f) + S(r, f), \quad \text{(34)}
\]

\[
\tilde{N} \left( r, \frac{1}{f^{(k)} - b} \right) = k \tilde{N}(r, f) + S(r, f). \quad \text{(35)}
\]

Then again by the second fundamental theorem and note that \( f \) and \( f^{(k)} \) share \( b \) IM*, we get

\[
T(r, f) \leq \tilde{N}(r, f) + \tilde{N} \left( r, \frac{1}{f} \right) + \tilde{N} \left( r, \frac{1}{f - b} \right) + S(r, f)
\]

\[
= (k + 1) \tilde{N}(r, f) + S(r, f). \quad \text{(36)}
\]

Let

\[
\alpha = \frac{f^{(k+1)}}{f^{(k)}} - (k + 1) \frac{f'}{f}.
\]

Obviously, \( m(r, \alpha) = S(r, f) \) by the lemma on the logarithmic derivative. In view of (34), “almost all” poles of \( f \) are simple. But these simple poles of \( f \) are removable singularities of \( \alpha \). Therefore, \( N(r, \alpha) = S(r, f) \). Hence we have \( T(r, \alpha) = S(r, f) \).
If \( f \) is a rational function, then \( \alpha \) must be a constant. Therefore, \( f^{(k)}/f^{k+1} = ce^{\alpha z} \), where \( c \) is a nonzero constant. If \( \alpha \neq 0 \), then \( f^{(k)}/f^{k+1} \) is not rational. This contradicts the assumption. Hence \( \alpha = 0 \), and thus \( f^{(k)} = cf^{k+1} \). Since \( T(r, b) = S(r, f) \), \( b \) must be a nonzero constant. Since \( f^{(k)} \) and \( f \) share \( b \), we deduce that \( cf^{k+1} \) and \( f \) share \( b \). The equation \( w^{k+1} - b/c = 0 \) has \( k + 1 \) different roots. We can select a root \( w_0 \) of this equation such that \( w_0 \neq b \) and \( f \) assumes the value \( w_0 \), which is possible since \( k + 1 \geq 3 \) and \( f \) is rational. If \( z_0 \) is a zero of \( f(z) - w_0 \), then \( cf^{k+1}(z_0) = b \), and thus \( f(z_0) = b \). Therefore, \( w_0 = b \). This is impossible.

If \( f \) is a transcendental meromorphic function, then by (33), (34) and Lemma 3, we get \( \bar{N}(r, f) = S(r, f) \). From this and (36), we get \( T(r, f) = S(r, f) \), a contradiction. This completes the proof of Theorem 2.

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References