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On pulse vaccination strategy in the SIR epidemic model with vertical transmission

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Abstract

The aim of this short paper is to improve a result recently given by Lu et al. on the global asymptotic stability of the eradication solution of the PVS applied to diseases with vertical transmission, by demonstrating that the condition for local stability guarantees also the global stability.

Keywords: Deterministic epidemic models; Stability theory; Impulsive differential equations; Vaccinations; Vertical transmission

1. Introduction

Pulse Vaccination Strategy (PVS) consists of periodical repetitions of impulsive vaccinations in a population, on all the age cohorts. At each vaccination time a constant fraction p of susceptible people is vaccinated. This kind of vaccination is called impulsive since all the vaccine doses are applied in a time which is very short with respect to the dynamics of the target disease. Its theoretical study was started by Agur and coworkers in [3]. PVS allows one to reach the eradication of a disease with some practical advantages, as discussed in [3,4,2].

In [4,2] Agur and coworkers formulated a model for including PVS in the SIR epidemic model with constant total population size. They found a solution such that the number of infectious individuals tends to zero, i.e. an eradication solution (ES), and the number of susceptible individuals varies periodically

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with the same period as the pulsed vaccination. They studied the properties of this solution, establishing a criterion for its local asymptotic stability. In [5] it has been demonstrated analytically that this local asymptotic stability (LAS) criterion also implies the global asymptotic stability (GAS) of the DFS in the meaningful domain of the model variables. Similar results for PVS have been found also for the SEIR epidemic model [6] and for models with co-presence of traditional vaccination schedules and non-exponentially distributed infectious and latent periods [7]. All these models refers to horizontally transmitted diseases, but a work [1] has recently been published on the application of PVS to vertically transmitted diseases, i.e. communicable pathologies in which there is "direct transfer of a disease from an infective parent to an unborn or newborn offspring" [8]. Some examples of such diseases are AIDS, Rubella, Hepatitis, etc. In it, a criterion is given for the LAS of the ES and another, slightly different, for its GAS. In our note, after pointing out that this second criterion is correct but is more restrictive than the one on LAS, we demonstrate that when there is LAS, then there is also GAS.

2. PVS in SIR model with vertical transmission

In the classical SIR model [9], the population that is involved in the spread of an infection is split into three epidemiological classes: The susceptibles (S): The subjects who can contract the disease; the removed (R): Those who, having been S and then I, are now no more involved in the spread of the disease; and, finally, the infectious (I): Those who are contagious. When there is also vertical contagion, the newborns of infectious may be born or susceptible or infectious. Furthermore, the birth rate b and the death rate m are constant and equal (which implies that the population size remains constant). Including in this model the PVS, we have:

$$\begin{aligned}
&S' = m(S+R+V) + m\rho I - \beta(t)SI, \qquad S(0) = S_0\\
&S(nT^+) = (1-p)S(nT^-), \qquad n \in \mathbb{N}_+\\
&I' = m(1-\rho)I + \beta(t)SI - (g+m)I, \qquad I(0) = I_0\\
&R' = gI - mR, \qquad R(0) = R_0\\
&V = 1 - S - I - R
\end{aligned} \tag{1}$$

which is a system of impulsive differential equations (IDEs), where:

- the total population size is normalized to one and the variables S, I and R model the fractions of susceptible, infected and removed subjects in the population;
- ρ is the probability that a child who is born from and infectious mother is born susceptible;
- g is the inverse of the infectious period;
- $\beta(t)$ is the Contact Rate (CR). We assume that it is constant or periodical with period one year;
- the vaccine gives, after its application, permanent immunity and p is the fraction of susceptible subjects to whom the vaccine is inoculated at $t = nT$, $n \in \mathbb{N}_+$. The size of susceptibles immediately after the n-th pulse vaccination at the time nT is equal to the size of susceptibles immediately before nT minus the fraction which received the vaccination. When the CR is constant, as in [1], T is allowed to be a positive real number; when the CR shows seasonal oscillation, we consider it as an integer number [2].

The model is well-posed, in the sense that, in their temporal evolution, the state variables (S, I, R) remain in the meaningful set $A = \{(S, I, R, V) \mid (S, I, R) \in [0, 1]^3, S + I + R + V = 1\}$ if the initial state $(S_0, I_0, R_0, V_0) \in A$. The model admits a disease-free periodic solution (i.e. a solution "free"

from infectious subjects) DFS $= (S^*(t; m, T, p), 0, 0, 1 - S^*(t; m, T, p))$, where $S^*(t; m, T, p)$ is a T-periodic function given by the periodical extension of the function:

$$j^*(t; m, T, p) = 1 - \frac{p\mathrm{Exp}(m(T-t))}{\mathrm{Exp}(mT) - 1 + p}, \qquad 0 \le t < T \tag{2}$$

that is $S^*(t; m, T, p) = j^*(\mathrm{Mod}(t, T); m, T, p)$.

In [1], Lu et al. demonstrated that if the CR is constant ($\beta(t) = \beta_o$) the DFS is LAS under this condition:

$$R_2(T) := \frac{\beta_o}{(m\rho + g)T} \int_0^T S^*(\tau; m, T, p)\mathrm{d}\tau < 1. \tag{3}$$

In the case of periodically variable CR, it is straightforward to demonstrate that the condition becomes:

$$\hat{R}_2(T) := \frac{1}{(m\rho + g)T} \int_0^T \beta(\tau) S^*(\tau; m, T, p)\mathrm{d}\tau < 1. \tag{4}$$

For what regards the GAS, in [1], it is demonstrated that the following constraint guarantees the GAS of the DFS:

$$R_2^\rho(T) := \frac{\beta_o((m(1+\rho)T - p)(\mathrm{E}^{m(1+\rho)T} - 1) + m(1+\rho)T\rho)}{mT(g + m\rho)(1+\rho)(\mathrm{E}^{m(1+\rho)T} - 1 + p)} < 1. \tag{5}$$

3. New results on GAS

In [1], it is demonstrated that:

- $R_2^\rho(T)$ is a decreasing function of p;
- $R_2(T) < R_2^\rho(T)$;
- when $p = 0$ it is $R_2(T) = R_2^\rho(T) = \mathfrak{R}$, where $\mathfrak{R}$ is the basic reproductive number for the SIR model.

Because of these properties, we have that if $p > p_{cr}$ is the solution of (3) and if $p > p_2$ is the solution of the (5) it is:

$$p_2 > p_{cr}$$

so the GAS constraint is more restrictive than the LAS constraint. This has a practical consequence: to an higher p corresponds a higher number of people being vaccinated per pulse, since this is expressed by the function

$$N_{\mathrm{vacc}}(p) = pS(nT^-)$$

which is an increasing function of p [6]. Therefore, it is important to verify if, as it happens to other PVS models, the LAS condition guarantees also the GAS. In fact, it holds the following:

Proposition 3.1. *If there holds the inequality* (4) *then the DFS solution is GAS in A.*

Proof. Let us start by writing the equation for S in the following form:

$$S' = -mI + mI + m(S + R + V) + m\rho I - mS - \beta(t)SI = m(1 - S) - \beta(t)SI - m(1 - \rho)I$$

so that we may use the simple techniques shown in [5]. In fact, since $S' \leq m(1-S)$, as a consequence it holds that:

$$S(t) \leq X(t) \to S^*(t; m, T, p)$$

where $X(t)$ is the solution of the following impulsive differential equation:

$$X' = m(1-X),\ X(nT^+) = (1-p)X(nT^-),\ X(0) = S(0)$$

which implies that:

$$I' = I(\beta(t)S - (m\rho + g)) \leq I(\beta(t)X - (m\rho + g)) \to I(\beta(t)S^*(t) - (m\rho + g)).$$

As a consequence:

$$\hat{R}_2(T) < 1 \Rightarrow I(t) \to 0^+$$

and, in turn, also the removed fraction will tend to 0^+. □

Therefore, it is possible to eradicate the disease in a stable way by inoculating only $N_V(p_{cr})$ doses of vaccine, instead of $N_{\text{vacc}}(p_2) > N_{\text{vacc}}(p_{cr})$.

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