JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **180**, 325–341 (1993)

Operations on the Class of Fuzzy Sets on a Universe Endowed with a Fuzzy Topology

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Submitted by Ulrich Höhle

Received February 12, 1991

1. INTRODUCTION

An operation on a topological space (X, T) has been defined by S. Kasahara [18] as a mapping φ from T into 2^X such that $A \subseteq A^{\varphi}$, $\forall A \in T$, where $A^{\varphi} = \varphi(A)$. Abd El-Monsef *et al.* [1, 2] extended Kasahara's operation by introducing an operation φ on the power set 2^X of X endowed with a topology T such that $int(A) \subseteq A^{\varphi}$, $\forall A \in 2^X$. In this paper we extend this concept to the class of all fuzzy sets on X endowed with a Chang fuzzy topology τ. This extension will make an extensive use of the notions of q-neighbourhood and q-coincidence due to Pu and Liu [26] and the notion of a fuzzy singleton as introduced in [19].

In Section 4, first for any operation φ we introduce the class of all φ-open fuzzy sets that generalizes the classes of all open, semi-open [6], pre-open [9], semi-pre-open [5], and feebly open [23] fuzzy sets. Second, starting with two operations φ_1, φ_2 we define the concepts of $\varphi_{1,2}$-closure ($\varphi_{1,2}$-interior) of fuzzy sets that generalizes fuzzy closure [26], fuzzy θ-closure [16, 25], fuzzy δ-closure [10], fuzzy semi-closure [12], fuzzy semi-θ-closure [28], and fuzzy semi-δ-closure. We show that the class of $\varphi_{1,2}$-open fuzzy sets plays a significant role in the context of fuzzy topology

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0022-247X/93 $5.00

in a way analogous to that of the $\varphi_{1,2}$-open sets in general topology [1, 2, 14, 18].

In Section 5, we apply this extension to generalize some types of fuzzy separation axioms due to Pu and Liu [26], Luo [22], Azad [6], and Kandil and El-Shaffi [17].

In Section 6, our discussion focuses on a special choice for one of the operations. More specifically, many interesting new results are deduced for the special choice $\varphi_1 = int$ and φ_2 arbitrary. In particular the interaction between operations in a topological space and its corresponding induced fuzzy topological space is described in detail.

We use F to denote fuzzy and fts to denote a Chang fuzzy topological space [8].

2. Preliminaries

The class of all fuzzy sets on a universe X is denoted by I^X. Fuzzy sets on X are denoted by Greek letters as μ, ρ, η, etc. Crisp subsets of X are denoted by capital letters as A, B, C, etc. Fuzzy singletons are denoted by x_ε, y_ν, z_ρ. The class of all fuzzy singletons in X is denoted by $S(X)$. For every $x_\varepsilon \in S(X)$ and $\mu \in I^X$, we define $x_\varepsilon \subseteq \mu$ iff $\varepsilon \leqslant \mu(x)$. A fuzzy set μ is called quasi-coincident with a fuzzy set ρ, denoted by $\mu q \rho$, iff there exists $x \in X$ such that $\mu(x) + \rho(x) > 1$. If μ is not quasi-coincident with ρ, then we write $\mu \bar{q} \rho$. Let $x_\varepsilon \in S(X)$, $\mu \in I^X$, and $A \subseteq X$. By $N(x_\varepsilon)$, $N_Q(x_\varepsilon)$, $int(\mu)$, $S.int(\mu)$, $\theta.int(\mu)$, $\delta.int(\mu)$, $S.\theta.int(\mu)$, $S.\delta.int(\mu)$, $cl(\mu)$, $S.cl(\mu)$, $\theta.cl(\mu)$, $\delta.cl(\mu)$, $S.\theta.cl(\mu)$, $S.\delta.cl(\mu)$, $co\mu$, and 1_A, we mean the neighbourhood (nbd, for short) system of x_ε, the Q-neighbourhood (q-nbd, for short) system of x_ε, the interior of μ, the semi-interior of μ, the θ-interior of μ, the δ-interior of μ, the semi-θ-interior of μ, the semi-δ-interior of μ, the closure of μ, the semi-closure of μ, the θ-closure, the δ-closure of μ, the semi-θ-closure of μ, the semi-δ-closure of μ, the complement of μ, and the characteristic mapping of A, where δ-interior (resp. θ-interior) is a special case of our new concept of φ_{12}-interior (Definition 4.6(2) below) when $\varphi_1 = int$ and $\varphi_2 = int \circ cl$ (resp. $\varphi_1 = int$ and $\varphi_2 = cl$).

Definition 2.1. For $\mu \in I^X$ we define

(i) $\mu_\alpha = \{x \mid x \in X \text{ and } \mu(x) \geqslant \alpha\}$ as the weak α-cut of μ, where $\alpha \in \,]0, 1]$. The weak 1-cut is called the kernel of μ and is denoted as $\ker(\mu)$.

(ii) $\mu_{\bar{\alpha}} = \{x \mid x \in X \text{ and } \mu(x) > \alpha\}$ as the strong α-cut of μ, where $\alpha \in [0, 1[$. The strong 0-cut of μ is called the support of μ and is denoted as $\operatorname{supp}(\mu)$.

(iii) $\operatorname{hgt}(\mu) = \sup_{x \in X} \mu(x)$ as the height of μ.

DEFINITION 2.2 [21]. Let (X, T) be an ordinary topological space. The set of all lower semicontinuous functions from (X, T) into the closed unit interval equipped with the usual topology constitutes a fuzzy topology on X that is called the induced fuzzy topology associated with (X, T) and is denoted as $(X, \omega(T))$.

The representation theorem [20] states that a fuzzy set can be decomposed into a family of ordinary subsets of the universe, namely its weak or strong α-cuts,

$$\mu = \bigcup_{\alpha \in]0,1]} (\underline{\alpha} \cap 1_{\mu_\alpha}) = \bigcup_{\alpha \in [0,1[} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}), \qquad \forall \mu \in I^X,$$

where $\underline{\alpha}$ means the constant mapping on X with value α.

3. OPERATIONS AND DUAL OPERATIONS ON FUZZY SETS

DEFINITION 3.1. Let (X, τ) be a fts. A mapping $\varphi: I^X \to I^X$ is called an operation on I^X iff $(\forall \mu \in I^X)$ $(int(\mu) \subseteq \mu^\varphi)$, where μ^φ denotes the value of φ in μ. The class of all operations on I^X is denoted by $O_{I(X,\tau)}$.

The operations φ, $\psi \in O_{I(X,\tau)}$ are said to be dual iff $(\forall \mu \in I^X)$ $(\mu^\psi = co((co\mu)^\varphi))$, or equivalently, $(\forall \mu \in I^X)$ $(\mu^\varphi = co((co\mu)^\psi))$. The dual operation of φ is denoted by $\tilde{\varphi}$.

DEFINITION 3.2. Let (X, τ) be a fts. A partial order "$\leqslant$" on $O_{I(X,\tau)}$ is defined as $\varphi_1 \leqslant \varphi_2 \Leftrightarrow (\forall \mu \in I^X)(\mu^{\varphi_1} \subseteq \mu^{\varphi_2})$, where φ_1, $\varphi_2 \in O_{I(X,\tau)}$. It is easy to prove that $(O_{I(X,\tau)}, \leqslant)$ is a completely distributive lattice.

EXAMPLES 3.3. Let (X, τ) be a fts. Then:

(i) The following operations belong to $O_{I(X,\tau)}$: int, *cl*, *cl* ∘ int, int ∘ *cl*, int ∘ *cl* ∘ int, $\theta.cl$, $S.\theta.cl$, $S.cl$, $(\theta.cl) \circ cl$.

(ii) The following are the dual operations of those defined in (i): *cl*, int, int ∘ *cl*, *cl* ∘ int, *cl* ∘ *int* ∘ *cl*, θ.int, $S.\theta$.int, S.int, $(\theta.int) \circ int$.

(iii) *Identity operation* $\iota : \mu^\iota = \mu$ and $\tilde{\iota} = \iota$;

(iv) *Constant operation* $1 : \mu^1 = X$.

(v) *Support operation* $\sigma : \mu^\sigma = 1_{\mathrm{supp}(\mu)}$.

(vi) *Fuzzy union operation* $\bigvee_\rho$ $(\rho \in I^X) : \mu^{\bigvee_\rho} = \mu \vee \rho$; the associated dual operation is denoted by $\bigwedge_{co\rho}$, since $\mu^{\tilde{\bigvee}_\rho} = \mu \wedge (co\rho)$ (de Morgan).

DEFINITION 3.4. An operation $\varphi \in O_{I(X,\tau)}$ is called:

(i) q-regular with respect to (w.r.t., for short) $\Sigma \subseteq I^X$ iff $(\forall x_\varepsilon \in S(X))$ $(\forall \mu, \rho \in N_Q(x_\varepsilon, \Sigma))(\exists \eta \in N_Q(x_\varepsilon, \Sigma))(\eta^\varphi \subseteq \varphi^\varphi \cap \rho^\varphi)$;

(ii) regular w.r.t. $\Sigma \subseteq I^X$ iff $(\forall x_\varepsilon \in S(X))(\forall \mu, \rho \in N(x_\varepsilon, \Sigma))$ $(\exists \eta \in N(x_\varepsilon, \Sigma))(\eta^\varphi \subseteq \mu^\varphi \cap \rho^\varphi)$;

(iii) monotonous iff $(\forall \mu, \rho \in I^X)(\mu \subseteq \rho \Rightarrow \mu^\varphi \subseteq \rho^\varphi)$;

(iv) weakly finite intersection preserving (WFIP, for short) w.r.t. $\Sigma \subseteq I^X$ iff $(\forall \eta \in \Sigma)(\forall \mu \in I^X)(\eta \cap \mu^\varphi \subseteq (\eta \cap \mu)^\varphi)$.

THEOREM 3.5. *Let (X, τ) be a fts, $\Sigma \subseteq I^X$, and $\varphi \in O_{I(X,\tau)}$. Then φ is monotonous $\Rightarrow \varphi$ is regular w.r.t. $\Sigma \Rightarrow \varphi$ is q-regular w.r.t. Σ.*

The following examples show that the converse is not true in general.

EXAMPLE 3.6. Let $\tau = \{\underline{\alpha} \mid \alpha \in [0, 1]\}$ and let $\varphi: I^X \to I^X$ be defined by

$$\mu^\varphi = \begin{cases} \mu, & \mu \neq \underline{0.5} \\ X, & \mu = \underline{0.5}. \end{cases}$$

Then φ is regular w.r.t. τ but not monotonous.

EXAMPLE 3.7. Let $X = \{x, y\}$ and consider the following fuzzy topology,

$$\tau = \{X, \varnothing\} \cup \{x_\varepsilon \mid \varepsilon \geqslant 0.5\} \cup \{x_\varepsilon \cup y_{0.5} \mid \varepsilon \geqslant 0.5\}.$$

(i) The operation $\varphi_1: I^X \to I^X$ defined by

$$\mu^{\varphi_1} = \begin{cases} x_1, & \mu = x_{2/3} \\ \mu, & \text{otherwise} \end{cases}$$

is q-regular w.r.t. τ but not regular w.r.t. τ. Indeed, for $x_{2/3}$, we have $x_{2/3} \cap (x_{2/3} \cup y_{0.5}) \in N(x_{0.5}, \tau)$ and $(x_{2/3})^{\varphi_1} \cap (x_{2/3} \cup y_{0.5})^{\varphi_1} = x_{2/3}$, but there is no $\eta \in N(x_{2/3}, \tau)$ such that $\eta^{\varphi_1} \subseteq x_{2/3}$. Thus φ_1 is not regular w.r.t. τ.

(ii) The operation $\varphi_2: I^X \to I^X$ defined by

$$\mu^{\varphi_2} = \begin{cases} x_1, & \mu = x_{0.5} \\ \mu, & \text{otherwise} \end{cases}$$

is not q-regular w.r.t. τ. Indeed, for x_ε, $\varepsilon > 0.5$, we have $x_{0.5} \cap (x_{0.5} \cup y_{0.5}) \in N_Q(x_\varepsilon, \tau)$ and $(x_{0.5})^{\varphi_2} \cap (x_{0.5} \cup y_{0.5})^{\varphi_2} = x_{0.5}$, but there is no $\eta \in N_Q(x_\varepsilon, \tau)$ such that $\eta^{\varphi_2} \subseteq x_{0.5}$.

EXAMPLES 3.8. Let (X, τ) be a fts. Then:

(i) All operations in Examples 3.3 are monotonous.

(ii) The operations int, $cl \circ int$, $int \circ cl$, $int \circ cl \circ int$, and cl are WFIP w.r.t. τ.

(iii) The operations $\theta.cl$ and $(\theta.cl) \circ cl$ are WFIP w.r.t. $\tau \cap \tau'$.

4. Generalizations of Basic Fuzzy Topological Concepts

Definition 4.1. Let (X, τ) be a fts, $\varphi \in O_{I(X,\tau)}$, $x_\varepsilon \in S(X)$, and $\mu \in I^X$. Then μ is called φ-open iff $\mu \subseteq \mu^\varphi$. We will denote the class of all φ-open fuzzy sets on X by $\varphi OF(X)$ and the class of all φ-open q-nbds of x_ε by $N_Q(x_\varepsilon, \varphi OF(X))$. The fuzzy set μ is called φ-closed iff $co\mu$ is φ-open.

Examples 4.2. Let (X, τ) be a fts and let $\varphi \in O_{I(X,\tau)}$.

(i) If $\varphi \leqslant \iota$, then $\varphi OF(X) = \{\mu \mid \mu \in I^X \text{ and } \mu^\varphi = \mu\}$. In particular, if $\varphi = int$, then $\varphi OF(X) = \tau$.

(ii) If $\varphi > \iota$, then $\varphi OF(X)$ coincides with I^X.

(iii) If $\varphi = cl \circ int$, then $\varphi OF(X)$ coincides with the class of all semi-open fuzzy sets on X denoted by $SOF(X)$.

(iv) If $\varphi = int \circ cl$, then $\varphi OF(X)$ coincides with the class of all pre-open fuzzy sets on X denoted by $POF(X)$.

(v) If $\varphi = cl \circ int \circ cl$, then $\varphi OF(X)$ coincides with the class of all semi-pre-open fuzzy sets on X denoted by $SPOF(X)$.

(vi) If $\varphi = S.cl \circ int$, then $\varphi OF(X)$ coincides with the class of all feebly open fuzzy sets, denoted by $FOF(X)$.

Lemma 4.3. *Let (X, τ) be a fts, φ, φ_1, $\varphi_2 \in O_{I(X,\tau)}$, and μ, $\eta \in I^X$. Then:*

(1) *Every open fuzzy set is a φ-open fuzzy set.*

(2) *If $\varphi_1 \leqslant \varphi_2$, then we have $\varphi_1 OF(X) \subseteq \varphi_2 OF(X)$.*

(3) *If φ is monotonous, then an arbitrary union of φ-open fuzzy sets is φ-open.*

(4) *If φ is WFIP w.r.t. $\varphi OF(X)$ and μ, $\rho \in \varphi OF(X)$, then $\mu \cap \rho \in \varphi OF(X)$.*

Theorem 4.4. *Let (X, τ) be a fts and $\varphi \in O_{I(X,\tau)}$.*

(i) *If φ is monotonous, then $\varphi OF(X)$ forms a fuzzy supratopology on X [24].*

(ii) *If φ is monotonous and WFIP w.r.t. $\varphi OF(X)$, then $\varphi OF(X)$ forms a fuzzy topology on X.*

(iii) *If $\tau = I^X$, then $\varphi OF(X) = I^X$.*

Proof. (i) By Definition 4.1, it is easy to note that $X, \varnothing \in \varphi OF(X)$ and by Lemma 4.3(3), $\varphi OF(X)$ is closed under arbitrary union and so $\varphi OF(X)$ is a fuzzy supratopology.

(ii) This follows from Definition 4.1 and Lemma 4.3(3) and (4).

(iii) This is obvious.

EXAMPLE 4.5. Let (X, τ) be the fts described in Example 3.7 and let $\varphi = cl \circ int$. Then the family $\varphi OF(X)$ is a fuzzy supratopology on X but not a fuzzy topology on X.

DEFINITION 4.6. Let (X, τ) be a fts, φ_1, $\varphi_2 \in O_{I(X, \tau)}$, $x_\varepsilon \in S(X)$, and $\mu \in I^X$.

(1) The $\varphi_{1,2}$-closure of μ, denoted by $\varphi_{1,2}.cl(\mu)$, is defined by $x_\varepsilon \subseteq \varphi_{1,2}.cl(\mu) \Leftrightarrow (\forall \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} q \mu)$.

(2) The $\varphi_{1,2}$-interior of μ, denoted by $\varphi_{1,2}.int(\mu)$, is defined by $x_\varepsilon q \varphi_{1,2}.int(\mu) \Leftrightarrow (\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$.

(3) μ is $\varphi_{1,2}$-open $\Leftrightarrow \mu \subseteq \varphi_{1,2}.int(\mu)$.

(4) μ is $\varphi_{1,2}$-closed $\Leftrightarrow \mu \supseteq \varphi_{1,2}.cl(\mu)$.

Obviously a fuzzy set μ is $\varphi_{1,2}$-open iff its complement is $\varphi_{1,2}$-closed.

EXAMPLES 4.7. Let (X, τ) be a fts, φ_1, φ_2, φ_3, $\varphi_4 \in O_{I(X, \tau)}$.

(1) For $\varphi_1 = int$, we have:

(i) if $int \leqslant \varphi_2 \leqslant \iota$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) coincides with F-closed (resp. F-closure operation);

(ii) if $\varphi_2 = cl$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) coincides with $F.\theta$-closed (resp. $F.\theta$-closure operation);

(iii) if $\varphi_2 = int \circ cl$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) coincides with $F.\delta$-closed (resp. $F.\delta$-closure operation);

(iv) if φ_2 is an arbitrary operation on I^X, then $cl \leqslant \varphi_{1,2}.cl$.

(2) For $\varphi_1 = cl \circ int$, we have:

(i) if $\varphi_2 = \iota$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) coincides with F.semi-closed (resp. F.semi-closure operation);

(ii) if $\varphi_2 = cl$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) coincides with $F.\theta$-semiclosed (resp. $F.\theta$-semi-closure);

(iii) if $\varphi_2 = int \circ cl$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure operation) will be called $F.\delta$-semiclosed (resp. $F.\delta$-semi-closure).

(3) If $\varphi_1 = int \circ cl$ and $\varphi_2 = \iota$, then $\varphi_{1,2}$-closed (resp. $\varphi_{1,2}$-closure) coincides with F.pre-closed (resp. F.pre-closure).

(4) If $\varphi_1 = \varphi_2 \leqslant \iota$, then $cl \leqslant \varphi_{1,2}.cl$.

(5) If $\varphi_1 = \varphi_2 \geqslant \iota$, then $cl = \varphi_{1,2}.cl$.

(6) If $\varphi_1 \geqslant \varphi_3$ and $\varphi_2 \leqslant \varphi_4$, then $\varphi_{1,2}.cl \leqslant \varphi_{3,4}.cl$.

LEMMA 4.8. *Let* (X, τ) *be a fts,* $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$, $\mu \in I^X$, *and* $\eta \in \varphi_1 OF(X)$. *Then* $\eta q \varphi_{1,2}.cl(\mu) \Rightarrow \eta^{\varphi_2} q \mu$.

Proof. Let $\eta q \varphi_{1,2}.cl(\mu)$. Then $(\exists x \in X)(\eta(x) + \varphi_{1,2}.cl(\mu)(x) > 1)$. Put $\varphi_{1,2}.cl(\mu)(x) = \varepsilon$. Then $x_\varepsilon \subseteq \varphi_{1,2}.cl(\mu)$ and $x_\varepsilon q \eta$. From $x_\varepsilon \subseteq \varphi_{1,2}.cl(\mu)$, we have $(\forall \rho \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\rho^{\varphi_2} q \mu)$ and so $\eta^{\varphi_2} q \mu$.

LEMMA 4.9. *Let* $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$ *and* $\mu, \eta \in I^X$. *Then*:

(i) $x_\varepsilon q \varphi_{1,2}.cl(\mu) \Leftrightarrow (\forall \eta \in N(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} q \mu)$.

(ii) $\varphi_{1,2}.cl(\mu) = \bigcup \{x_\varepsilon \in S(X) \mid (\forall \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} q \mu)\}$.

(iii) $\varphi_{1,2}.cl(\varnothing) = \varnothing$.

(iv) *If* $(\varphi_2 \geqslant \iota$ *or* $\varphi_2 \geqslant \varphi_1)$, *then* $\mu \subseteq \varphi_{1,2}.cl(\mu)$.

(v) *If* $\mu \subseteq \eta$, *then* $\varphi_{1,2}.cl(\mu) \subseteq \varphi_{1,2}.cl(\eta)$.

(vi) *An arbitrary intersection of* $\varphi_{1,2}$*-closed fuzzy sets is* $\varphi_{1,2}$*-closed.*

(vii) *If* φ_2 *is q-regular w.r.t.* $\varphi_1 OF(X)$, *then the finite union of* $\varphi_{1,2}$*-closed fuzzy sets is* $\varphi_{1,2}$*-closed.*

(viii) *Let* $(\varphi_2 \geqslant \iota$ *or* $\varphi_2 \geqslant \varphi_1)$ *and* φ_1 *is monotonous. If* μ *is* $\varphi_{1,2}$*-open, then* μ *is* φ_1*-open.*

(ix) *If* $(\varphi_2 \geqslant \iota$ *or* $\varphi_2 \geqslant \varphi_1)$, *then* $(\varphi_{1,2}.cl) \circ \varphi_2 \in O_{I(X,\tau)}$.

Proof. The straightforward proofs are omitted. We only prove:

(iv) Suppose $x_\varepsilon \subseteq \mu$ and $\eta \in N_Q(x_\varepsilon, \varphi_1 OF(X))$, then we have $x_\varepsilon q \eta$ iff $1 < \varepsilon + \eta(x) \leqslant \mu(x) + \eta(x) \leqslant \mu(x) + \eta^{\varphi_1}(x)$. By hypothesis, we have $\mu(x) + \eta^{\varphi_2}(x) > 1$. This means that $\eta^{\varphi_2} q \mu$ and hence $x_\varepsilon \subseteq \varphi_{1,2}.cl(\mu)$.

(v) Suppose $\mu \subseteq \eta$ and $x_\varepsilon \subseteq \varphi_{1,2}.cl(\mu)$. Then $(\forall \rho \in N_Q(x_\varepsilon, \varphi_1 OF(X)))$ $(\rho^{\varphi_2} q \mu)$, equivalently $1 < \mu(x) + \rho^{\varphi_2}(x) \leqslant \eta(x) + \rho^{\varphi_2}(x)$, equivalently $\rho^{\varphi_2} q \eta$ and hence $x_\varepsilon \subseteq \varphi_{1,2}.cl(\eta)$.

(vii) Let $\lambda = \bigcup_{j=1}^n \lambda_j$, where $(\forall j \in \{1, 2, ..., n\})(\varphi_{1,2}.cl(\lambda_j) \subseteq \lambda_j)$. Let $x_\varepsilon \nsubseteq \lambda$. Then $(\forall j \in \{1, 2, ..., n\})(x_\varepsilon \nsubseteq \lambda_j)$. Hence $(\exists \eta_j \in N_Q(x_\varepsilon, \varphi_1 OF(X)))$ $(\forall j \in \{1, 2, ..., n\})(\eta_j^{\varphi_2} \bar{q} \lambda_j)$. Now φ_2 is q-regular w.r.t. $\varphi_1 OF(X)$ so $(\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \bigcap_{j=1}^n \eta_j^{\varphi_2})$. Hence $\eta^{\varphi_2} \bar{q} \lambda$ and so $x_\varepsilon \nsubseteq \varphi_{1,2}.cl(\lambda)$. Thus $\varphi_{1,2}.cl(\lambda) \subseteq \lambda$.

Now it is easy to prove the following theorem.

THEOREM 4.10. *Let (X, τ) be a fts and $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \in O_{I(X,\tau)}$.*

(i) *The class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open fuzzy sets in X is a fuzzy supratopology on X.*

(ii) *If φ_2 is q-regular w.r.t. $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ is a fuzzy topology on X denoted by $\tau_{\varphi_{1,2}}$.*

(iii) *If ($\varphi_2 \geqslant \iota$ or $\varphi_2 \geqslant \varphi_1$), then the fuzzy $\varphi_{1,2}$-closure operation defines on X a fuzzy pretopological space [7].*

(iv) *If ($\varphi_2 \geqslant \iota$ or $\varphi_2 \geqslant \varphi_1$) and φ_2 is q-regular w.r.t. $\varphi_1 OF(X)$, then the fuzzy $\varphi_{1,2}$-closure operation defines on X a fuzzy closure space [24].*

(v) *If $\varphi_{1,2}.cl \leqslant \varphi_{3,4}.cl$, then $\tau_{\varphi_{3,4}} \subseteq \tau_{\varphi_{1,2}}$.*

Proof. The straightforward proofs are omitted. We only prove:

(i) This follows directly from Definition 4.6 and Lemma 4.9(iii) and (iv).

(ii) This follows directly from Definition 4.6 and Lemma 4.9(iii), (iv), (vi), and (vii).

(iii) By Lemma 4.9(iii) and (iv), we have $\varphi_{12}.cl(\emptyset) = \emptyset$ and $\mu \subseteq \varphi_{12}.cl(\mu)$, $\forall \mu \in I^X$. Hence φ_{12} is a fuzzy pretopology operator [7].

(iv) By Lemma 4.9(iii), (iv), and (vii), we have $\varphi_{12}.cl(\emptyset) = \emptyset$, $\mu \subseteq \varphi_{12}.cl(\mu)$, $\forall \mu \in I^X$ and $\varphi_{12}(\mu \cup \eta) = \varphi_{12}(\mu) \cup \varphi_{12}(\eta)$, $\forall \mu, \eta \in I^X$. Hence φ_{12} is a Cech fuzzy closure operator [24].

EXAMPLES 4.11. Let (X, τ) be a fts and let $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$.

(1) If $\varphi_1 = \varphi_2 = int$, then $\tau_{\varphi_{1,2}}$ coincides with τ.

(2) If $\varphi_1 = int$ and $\varphi_2 = cl$, then $\tau_{\varphi_{1,2}}$ coincides with τ_θ [16, 25].

(3) If $\varphi_1 = int$ and $\varphi_2 = int \circ cl$, then $\tau_{\varphi_{1,2}}$ coincides with τ_δ [10].

(4) $\tau_\theta \subseteq \tau_\delta \subseteq \tau$ [10, 16, 25].

5. GENERALIZED FUZZY SEPARATION AXIOMS

In this section we introduce and study the concepts of $\varphi_{1,2}$-separation axioms in fuzzy topological spaces which generalize the axioms FT_1, FT_2 introduced by Pu and Liu [26], the axiom $FT_{2(1/2)}$ introduced by Yalvac [29], the axiom FR_2 introduced by Luo [22], the axiom FSR_2 introduced by Azad [6], and the axioms F^*T_1, F^*T_2, $F^*T_{2(1/2)}$, and F^*R_2 introduced by Kandil and El-Shaffi [17]. Various properties of these new classes of fuzzy topological spaces have been studied.

DEFINITION 5.1. Let (X, τ) be a fts and $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. Then (X, τ) is called:

(i) $\varphi_{1,2}.FT_1 \Leftrightarrow (\forall x_\varepsilon \in S(X))(x_\varepsilon$ is $\varphi_{1,2}$-closed).

(ii) $\varphi_{1,2}.F^*T_1 \Leftrightarrow (\forall x_\varepsilon, y_\nu \in S(X))(x_\varepsilon \bar{q} y_\nu \Rightarrow ((\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(y_\nu \bar{q} \mu^{\varphi_2})$ and $(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(x_\varepsilon \bar{q} \eta^{\varphi_2})))$.

(iii) $\varphi_{1,2}.FT_2 \Leftrightarrow (\forall x_\varepsilon, y_\nu \in S(X))(x \neq y \Rightarrow (\exists \mu \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N_Q(y_\nu, \varphi_1 OF(X)))(\mu^{\varphi_2} \cap \eta^{\varphi_2} = \varnothing)$.

(iv) $\varphi_{1,2}.F^*T_2 \Leftrightarrow (x_\varepsilon \bar{q} y_\nu \Rightarrow (\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(\mu^{\varphi_2} \bar{q} \eta^{\varphi_2}))$.

(v) $\varphi_{1,2}.FR_2 \Leftrightarrow (\forall x_\varepsilon \in S(X))(\forall \mu \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$.

(vi) $\varphi_{1,2}.F^*R_2 \Leftrightarrow (\forall x_\varepsilon \in S(X))(\forall \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$.

EXAMPLES 5.2. Let (X, τ) be a fts and $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. Then:

(1) For $\varphi_1 = int$, we have:

(i) if $int \leqslant \varphi_2 \leqslant \iota$, then $\varphi_{1,2}.F^*T_1(\varphi_{1,2}.FT_1)$ and $\varphi_{1,2}.F^*T_2(\varphi_{1,2}.FT_2)$ coincide with $F^*T_1(FT_1)$ and $F^*T_2(FT_2)$;

(ii) if $\varphi_2 = cl$, then $\varphi_{1,2}.F^*T_1(\varphi_{1,2}.FT_1)$, $\varphi_{1,2}.F^*T_2(\varphi_{1,2}.FT_2)$, and $\varphi_{1,2}.F^*R_2(\varphi_{1,2}.FR_2)$ coincide with $F^*T_2(FT_2)$, $F^*T_{2(1/2)}(FT_{2(1/2)})$, and $F^*R_2(FR_2)$.

(iii) If $\varphi_2 = int \circ cl$, then $\varphi_{1,2}.FR_2$ coincides with fuzzy semi-regular (FSR_2) due to Azad [6]. The axiom $\varphi_{1,2}.F^*R_2$ will be called in this case F^*SR_2.

(2) If $\varphi_1 = int \circ cl$ and $\varphi_2 = \iota$, then

(i) $\varphi_{1,2}.F^*T_1$ and $\varphi_{1,2}.F^*T_2$ are called semi-F^*T_1 and semi-F^*T_2;

(ii) $\varphi_{1,2}.FT_1$ and $\varphi_{1,2}.FT_2$ are called semi-FT_1 and semi-FT_2.

(3) If $\varphi_1 = cl \circ int$ and $\varphi_2 = S.cl$, then the axiom $\varphi_{1,2}.FR_2$ will be called in this case $S.FR_2$.

The following theorem shows that $\varphi_{1,2}.FT_1$ and $\varphi_{1,2}.F^*T_1$ are equivalent.

THEOREM 5.3. *Let (X, τ) be a fts, $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. A fts (X, τ) is $\varphi_{1,2}.F^*T_1 \Leftrightarrow (\forall x_\varepsilon \in S(X))$ (x_ε is $\varphi_{1,2}$-closed).*

Proof. Let (X, τ) be $\varphi_{1,2}.F^*T_1$ and $x_\varepsilon, y_\nu \in S(X)$ such that $x_\varepsilon \bar{q} y_\nu$. Then $(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(x_\varepsilon \bar{q} \eta^{\varphi_2})$ which implies that $y_\nu \bar{q} \varphi_{1,2}.cl(x_\varepsilon)$.

Thus we have $\varphi_{1,2}.cl(x_\varepsilon) \subseteq x_\varepsilon$. Conversely, suppose that $(\forall x_\varepsilon \in S(X))$ $(x_\varepsilon \supseteq \varphi_{1,2}.cl(x_\varepsilon))$ and let $x_\varepsilon \bar{q} y_\nu$. Then $x_\varepsilon \bar{q} \varphi_{1,2}.cl(y_\nu)$ and $y_\nu \bar{q} \varphi_{1,2}.cl(x_\varepsilon)$. Hence $(\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(y_\nu \bar{q} \mu^{\varphi_2}$ and $x_\varepsilon \bar{q} \eta^{\varphi_2})$. Thus (X, τ) is $\varphi_{1,2}.F^*T_1$.

COROLLARY 5.4. *A fts (X, τ) is $\varphi_{1,2}.FT_1$ iff $(X, \tau_{\varphi_{1,2}})$ is FT_1.*

The situation is quite different for $\varphi_{1,2}.FT_2(\varphi_{1,2}.F^*T_2)$ and $\varphi_{1,2}.FR_2(\varphi_{1,2}.F^*R_2)$ as may seen from the next theorem.

THEOREM 5.5. *Let (X, τ) be a fts and $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. Then:*

(i) *$(X, \tau_{\varphi_{1,2}})$ is FT_2 (resp. F^*T_2) $\Rightarrow (X, \tau)$ is $\varphi_{1,2}.FT_2$ (resp. $\varphi_{1,2}.F^*T_2$).*

(ii) *$(X, \tau_{\varphi_{1,2}})$ is FR_2 (resp. F^*R_2) $\Rightarrow (X, \tau)$ is $\varphi_{1,2}.FR_2$ (resp. $\varphi_{1,2}.F^*R_2$).*

THEOREM 5.6. *Let (X, τ) be a fts and let $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. Then:*

(i) *$((X, \tau)$ is $\varphi_{1,2}.F^*T_1 \Rightarrow (X, \tau)$ is $\varphi_{1,2}.F^*T_2) \Leftrightarrow (\forall x_\varepsilon, y_\nu \in S(X))$ $(x_\varepsilon \bar{q} y_\nu \Rightarrow (\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)) \cup N(y_\nu, \varphi_1 OF(X)))$ $(\mu^{\varphi_2}$ is $\varphi_{1,2}$-closed$))$.*

(ii) *$((X, \tau)$ is $\varphi_{1,2}.FT_1 \Rightarrow (X, \tau)$ is $\varphi_{1,2}.FT_2) \Leftrightarrow (\forall x_\varepsilon, y_\nu \in S(X))$ $(x \neq y \Rightarrow (\exists \mu \in N_Q(x_\varepsilon, \varphi_1 OF(X)) \cup N_Q(y_\nu, \varphi_1 OF(X)))$ $(\mu^{\varphi_2}$ is $\varphi_{1,2}$-closed$))$.*

Proof. (i) Let (X, τ) be a fts, $\varphi_{1,2}.F^*T_1 \Rightarrow \varphi_{1,2}.F^*T_2$, and $x_\varepsilon \bar{q} y_\nu$. Since (X, τ) is $\varphi_{1,2}.F^*T_1$, we have $(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(x_\varepsilon \bar{q} \eta^{\varphi_2})$. Also, because (X, τ) is $\varphi_{1,2}.F^*T_2$, it follows $(\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(\mu^{\varphi_2} \bar{q} \eta^{\varphi_2})$ which implies that $x_\varepsilon \bar{q} \varphi_{1,2}.cl(\eta^{\varphi_2})$. Hence $\varphi_{1,2}.cl(\eta^{\varphi_2}) \subseteq \eta^{\varphi_2}$. Therefore η^{φ_2} is $\varphi_{1,2}$-closed.

Conversely, let (X, τ) be an $\varphi_{1,2}.F^*T_1$ and $x_\varepsilon, y_\nu \in S(X)$ such that $x_\varepsilon \bar{q} y_\nu$. Then $(\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(y_\nu \bar{q} \mu^{\varphi_2})$. Since μ^{φ_2} is $\varphi_{1,2}$-closed, we have $y_\nu \bar{q} \varphi_{1,2}.cl(\mu^{\varphi_2})$ and so $(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(\mu^{\varphi_2} \bar{q} \eta^{\varphi_2})$. Thus (X, τ) is $\varphi_{1,2}.F^*T_2$. A similar proof can be given for (ii).

THEOREM 5.7. *Let (X, τ) be a fts and $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. If $(\varphi_2 \geqslant \iota$ or $\varphi_2 \geqslant \varphi_1)$ and (X, τ) is $\varphi_{1,2}.F^*T_2$, then $(\forall x_\varepsilon \in S(X))(x_\varepsilon = \bigcap \{\varphi_{1,2}.cl(\mu^{\varphi_2}) \mid \mu \in N(x_\varepsilon, \varphi_1 OF(X))\})$.*

Proof. Suppose (X, τ) is $\varphi_{1,2}.F^*T_2$ and $x_\varepsilon \in S(X)$. Then $(\forall y_\nu \in S(X))$ $(y_\nu \bar{q} x_\varepsilon \Rightarrow (\exists \mu \in N(x_\varepsilon, \varphi_1 OF(X)))(\exists \eta \in N(y_\nu, \varphi_1 OF(X)))(\mu^{\varphi_2} \bar{q} \eta^{\varphi_2}))$. Hence $y_\nu \bar{q} \varphi_{1,2}.cl(\mu^{\varphi_2})$ and so we obtain $\bigcap \{\varphi_{1,2}.cl(\mu^{\varphi_2}) \mid \mu \in N(x_\varepsilon, \varphi_1 OF(X))\}$ $\subseteq x_\varepsilon$. Therefore, $x_\varepsilon = \bigcap \{\varphi_{1,2}.cl(\mu^{\varphi}) \mid \mu \in N(x_\varepsilon, \varphi_1 OF(X))\}$.

THEOREM 5.8. *Let (X, τ) be a fts and $\varphi_1, \varphi_2 \in O_{I(X,\tau)}$. Then (X, τ) is $\varphi_{1,2}.FR_2$ (resp. $\varphi_{1,2}.F^*R_2$) iff every φ_1-open fuzzy set is $\varphi_{1,2}$-open.*

Proof. Suppose that (X, τ) is $\varphi_{1,2}.FR_2$ and $\mu \in \varphi_1 OF(X)$; $x_\varepsilon q\mu$. Because (X, τ) is $\varphi_{1,2}.FR_2$, we obtain $(\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$ which implies that $x_\varepsilon q \varphi_{1,2}.int(\mu)$. Thus $\mu \subseteq \varphi_{1,2}.int(\mu)$ and hence μ is $\varphi_{1,2}$-open. Conversely, let $x_\varepsilon \in S(X)$ and let $\mu \in N_Q(x_\varepsilon, \varphi_1 OF(X))$. Then $\mu \subseteq \varphi_{1,2}.int(\mu)$ and hence $(\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$. Thus (X, τ) is $\varphi_{1,2}.FR_2$.

COROLLARY 5.9. *Let $(\varphi_2 \geqslant \iota$ or $\varphi_2 \geqslant \varphi_1)$ and φ_1 is monotonous. Then a fts (X, τ) is an $\varphi_{1,2}.FR_2$ (resp. $\varphi_{1,2}.F^*R_2$) iff $\tau_{\varphi_{1,2}} = \varphi_1 OF(X)$.*

The following examples show that $\varphi_{1,2}.FT_2$ and $\varphi_{1,2}.F^*T_2$ are independent.

EXAMPLE 5.10. Let $X = \{x, y\}$ and $\tau = \{X, \emptyset, x_1, y_1\}$. Let φ_1, $\varphi_2 \in O_{I(X,\tau)}$ such that $\varphi_1 = int$ and $\varphi_2 = \iota$. Then (X, τ) is $\varphi_{1,2}.FT_2$ but not $\varphi_{1,2}.F^*T_2$. Indeed, for $x_{0.3}$, $x_{0.7} \in S(X)$ and $(\forall \mu \in N(x_{0.3}, \tau))$ $(\forall \eta \in N(x_{0.7}, \tau))(\mu q \eta)$.

EXAMPLE 5.11. Let X be an infinite set and consider the ordinary cofinite topology $\tau_\infty = \{U \mid U \subseteq X$ and coU is finite$\} \cup \{\emptyset\}$. It is easy to see that the class $\lambda(\tau_\infty) = \{\mu \mid \mu \in I^X$ and $\text{supp}(\mu) \in \tau_\infty\}$ is a fuzzy topology on X. Let φ_1, $\varphi_2 \in O_{I(X, \lambda(\tau_\infty))}$ such that $\varphi_1 = int$ and $\varphi_2 = \iota$. Then $(X, \lambda(\tau_\infty))$ is $\varphi_{1,2}.F^*T_2$ but not $\varphi_{1,2}.FT_2$.

THEOREM 5.12. *Let (X, τ) be a fts and φ_1, $\varphi_2 \in O_{I(X,\tau)}$. Then (X, τ) is $\varphi_{1,2}.F^*R_2 \Rightarrow (X, \tau)$ is $\varphi_{1,2}.FR_2$.*

Proof. Let (X, τ) be $\varphi_{1,2}.F^*R_2$, $x_\varepsilon \in S(X)$ and $\mu \in N_Q(x_\varepsilon, \varphi_1 OF(X))$. Then $\mu(x) > 1 - \varepsilon$. Hence $(\exists v \in]0, 1[)(\mu(x) > v > 1 - \varepsilon)$. From $x_v \subseteq \mu$ and because (X, τ) is $\varphi_{1,2}.F^*R_2$, we obtain $(\exists \eta \in N(x_v, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$. Since $\eta(x) \geqslant v > 1 - \varepsilon$, we have $\eta \in N_Q(x_\varepsilon, \varphi_1 OF(X))$ and hence (X, τ) is $\varphi_{1,2}.FR_2$.

The following example shows that the converse of Theorem 5.13 is not true in general.

EXAMPLE 5.13. Let $X = \{x\}$ and $\tau = \{X, \emptyset\} \cup \{x_\varepsilon \mid 0.3 < \varepsilon \leqslant 0.7\}$. If $\varphi_1 = int$ and $\varphi_2 = cl$, then it is easy to see that (X, τ) is $\varphi_{1,2}.FR_2$ but not $\varphi_{1,2}F^*R_2$. Indeed, for $x_{0.7}$, we have $x_{0.7} \subseteq x_{0.7} \in \tau$ but there is no $\mu \in N(x_{0.7}, \tau)$ such that $cl(\mu) \subseteq x_{0.7}$.

THEOREM 5.14. *Let (X, τ) be a fts, φ_1, $\varphi_2 \in O_{I(X,\tau)}$, and let $(\varphi_2 \geqslant \iota$ or $\varphi_2 \geqslant \varphi_1)$. A fts (X, τ) is $\varphi_{1,2}.FR_2$ iff every φ_1-open fuzzy set μ is a union of φ_1-open fuzzy sets η_j such that $(\forall j \in J)(\eta_j^{\varphi_2} \subseteq \mu)$.*

Proof. Let $\mu \in \varphi_1 OF(X)$, $x \in \mathrm{supp}(\mu)$, and let n_x be a strictly positive real number such that $1/n_x \leqslant \mu(x)$. For any positive real number n with $n \geqslant n_x$, put $\varepsilon = 1 - \mu(x) + 1/n$. Then $0 < \varepsilon < 1$. Since, $x_\varepsilon q \mu$, we have $(\exists \eta_n \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\eta^{\varphi_2} \subseteq \mu)$. Now $\eta_n(x) + \varepsilon = \eta_n(x) + 1 - \mu(x) + 1/n > 1$ and hence $\eta_n(x) > \mu(x) - 1/n$. Thus $[\bigcup \{\eta_n \mid n \geqslant n_x\}](x) = \mu(x)$. Then $\rho_x = \{\eta_n \mid n \geqslant n_x\}$ is a collection of φ_1-open fuzzy sets in X such that $\bigcup \{\eta_n \mid \eta_n \in \rho_x\} \subseteq \mu$, $[\bigcup \{\eta_n \mid \eta_n \in \rho_x\}](x) = \mu(x)$ and $(\forall \eta_n \in \rho_x)(\eta_n^{\varphi_2} \subseteq \mu)$. Now $\rho = \bigcup \{\rho_x \mid x \in \mathrm{supp}(\mu)\}$ is a collection of φ_1-open fuzzy sets such that $\bigcup \{\eta \mid \eta \in \rho\} = \mu$ and $(\forall \eta \in \rho)(\eta^{\varphi_2} \subseteq \mu)$.

Conversely, let $x_\varepsilon \in S(X)$ and $\mu \in N_Q(x_\varepsilon, \varphi_1 OF(X))$. By the given conditions, we have that there exists a collection $\{\eta_j \mid j \in J\} \subseteq \varphi_1 OF(X)$ such that $\bigcup \{\eta_j \mid j \in J\} = \mu$ and $(\forall j \in J)(\eta_j^{\varphi_2} \subseteq \mu)$. Thus, $(\exists j_0 \in J)(x_\varepsilon q \eta_{j_0}$ and $\eta_{j_0}^{\varphi_2} \subseteq \mu)$. Hence (X, τ) is $\varphi_{1,2}.FR_2$.

Remark 5.15. The above theorem implies that the concepts of $\varphi_{1,2}.FR_2$ (with $\varphi_1 = int$ and $\varphi_2 = cl$) and F-regularity due to Hutton and Reilly [13] are equivalent.

THEOREM 5.16. *If (X, τ) is $\varphi_{1,2}.FR_2$ (resp. $\varphi_{1,2}.F^*R_2$), then $(\forall \mu \in I^X)$ $(\varphi_{1,2}.cl(\mu)$ is $\varphi_{1,2}$-closed).*

Proof. Let $x_\varepsilon \nsubseteq \varphi_{1,2}.cl(\mu)$. Then $(\exists \eta \in N_Q(x_\varepsilon, \varphi_1 OF(X))(\eta^{\varphi_2} \bar{q} \mu)$. By Lemma 4.9, we have $\eta \bar{q} \varphi_{1,2}.cl(\mu)$. Because (X, τ) is $\varphi_{1,2}.FR_2$, we obtain $(\exists \rho \in N_Q(x_\varepsilon, \varphi_1 OF(X)))(\rho^{\varphi_2} \subseteq \eta)$ and hence $\rho^{\varphi_2} \bar{q} \varphi_{1,2}.cl(\mu)$ implies that $x_\varepsilon \nsubseteq \varphi_{1,2}.cl(\varphi_{1,2}.cl(\mu))$. Thus $\varphi_{1,2}.cl(\varphi_{1,2}.cl(\mu)) \subseteq (\varphi_{1,2}.cl(\mu))$ and hence $\varphi_{1,2}.cl(\mu)$ is $\varphi_{1,2}$-closed.

6. FUZZY TOPOLOGIZING WITH q-REGULAR OPERATIONS

In this section we will focus on a special choice for one of the operations. More specifically we will dwell upon the choice $\varphi_1 = int$ and φ_2 denotes an arbitrary operation. For this special choice we will use the notations τ_φ, $\varphi.cl$, and $\varphi.int$, instead of $\tau_{\varphi_{12}}$, $\varphi_{1,2}.cl$, and $\varphi_{1,2}.int$. Of course all of the results of the previous sections still hold for this special choice. We will now outline some interesting new properties for this special choice.

LEMMA 6.1. *Let (X, τ) be a fts and $\mu \in I^X$. Then:*

(i) $\mu \in \tau'_\varphi$ *iff* $(\forall x_\varepsilon \in S(X))(\forall \eta \in N_Q(x_\varepsilon, \tau))(\eta^\varphi q \mu \Rightarrow x_\varepsilon \subseteq \mu)$.

(ii) $\mu \in \tau_\varphi$ *iff* $(\forall x_\varepsilon \in S(X))(x_\varepsilon q \mu \Rightarrow (\exists \eta \in N_Q(x_\varepsilon, \tau))(\eta^\varphi \subseteq \mu))$.

Proof. The statements are immediate consequences of Definition 4.5 and Theorem 4.11.

LEMMA 6.2. *Let* $\mu \in I^X$ *and* $\{\eta_j \mid j \in J\} \subseteq I^X$ *such that* $(\forall j \in J)(\eta_j \subseteq \mu)$. *Then* $\mu = \bigcup_{j \in J} \eta_j$ *iff* $(\forall x_\varepsilon q\mu)(\exists j \in J)(x_\varepsilon q\eta_j)$.

PROPOSITION 6.3. *Let* (X, τ) *be a fts and* $\mu \in I^X$. *Then*:

(i) $\mu \in \tau_\varphi$ *iff there exists a family* $\{\eta_j \mid j \in J\} \subseteq \tau$ *such that*

$$\mu = \bigcup_{j \in J} \eta_j = \bigcup_{j \in J} \eta_j^\varphi.$$

(ii) $\mu \in \tau_\varphi$ *iff* $\mu = \bigcup \{\eta \mid \eta \in \tau$ *and* $\eta^\varphi \subseteq \mu\}$.

Proof. (i) If $\mu = \emptyset$, take $J = \emptyset$. So let $\mu \in \tau \backslash \{\emptyset\}$; then $(\forall x_\varepsilon q\mu)$ $(\exists \eta_{x_\varepsilon} \in \tau)$ $(x_\varepsilon q\eta_{x_\varepsilon} \subseteq \eta^\varphi_{x_\varepsilon} \subseteq \mu)$. By construction the families $\{\eta_{x_\varepsilon} \mid x_\varepsilon q\mu\}$ and $\{\eta^\varphi_{x_\varepsilon} \mid x_\varepsilon q\mu\}$ satisfy the condition of Lemma 6.3; so $\mu = \bigcup_{x_\varepsilon q\mu} \eta_{x_\varepsilon} = \bigcup_{x_\varepsilon q\mu} (\eta_{x_\varepsilon})^\varphi$.
Conversely, if $J = \emptyset$, then $\mu = \emptyset \in \tau_\varphi$; so let $J \neq \emptyset$. If $x_\varepsilon q\mu$, then obviously $(\exists j \in J)(x_\varepsilon q\eta_j \subseteq \eta_j^\varphi \subseteq \mu)$ which implies that $\mu \in \tau_\varphi$.

(ii) This is an immediate consequence of (i).

COROLLARY 6.4. *The fixpoints of* φ *all belong to* τ_φ, *i.e.*, $(\forall \mu \in I^X)$ $(\mu^\varphi = \mu \Rightarrow \mu \in \tau_\varphi)$.

DEFINITION 6.5. Let (X, T) be an ordinary topological space and $\varphi \in O_{(X,T)}$. Associated with the induced fuzzy topology $\omega(T)$ we define the mapping

$$\varphi_\omega \colon \omega(T) \to I^X \text{ as } (\forall \mu \in \omega(T)) \left(\mu^{\varphi_\omega} = \bigcup_{0 \leqslant \alpha < hgt(x)} (\underline{\alpha} \cap 1_{(\mu_{\bar\alpha})^\varphi}) \right).$$

LEMMA 6.6. *The mapping* φ_ω *is an operation on* $(X, \omega(T))$.

LEMMA 6.7. *Let* (X, T) *be an ordinary topological space and* $(X, \omega(T))$ *its induced fts*, $\varphi \in O_{(X,T)}$ *and* $\varphi_\omega \in O_{(X,\omega(T))}$ *as defined in* 6.6. *Then*:

(i) $(\forall U \in T)(1_{(U^\varphi)} = (1_U)^{\varphi_\omega})$.

(ii) $(\forall U \in T)((\underline{\alpha} \cap 1_U)^{\varphi_\omega}) = \underline{\alpha} \cap 1_{(U^\varphi)}$.

(iii) φ *is regular iff* φ_ω *is q-regular.*

Proof. Parts (i) and (ii) follow immediately from Definition 6.5.

(iii) Let μ, $\rho \in \omega(T)$ such that $x_\varepsilon q(\mu \cap \rho)$. Then $(\forall \alpha \in [0, (\mu \cap \rho)(x)[)$ $(\mu_{\bar\alpha}, \rho_{\bar\alpha} \in T$ and $x \in \mu_{\bar\alpha} \cap \rho_{\bar\alpha})$. Since φ is regular, we obtain $(\exists U \in T)$ $(x \in U$ and $U^\varphi \subseteq (\mu_{\bar\alpha})^\varphi \cap (\rho_{\bar\alpha})^\varphi)$. Put $\eta = \underline{((1 - \varepsilon + \mu \cap \rho(x))/2)} \cap 1_U$. Then $\eta \in N_Q(x_\varepsilon) \cap \omega(T)$. Moreover, $\eta^{\varphi_\omega} = \underline{((1 - \varepsilon + \mu \cap \rho(x))/2)} \cap 1_{U^\varphi} \subseteq \mu^{\varphi_\omega} \cap \rho^{\varphi_\omega}$. Thus, φ_ω is q-regular.

Conversely, let A, $B \in T$ such that $x \in A \cap B$. Then 1_A, $1_B \in \omega(T)$ and $x_\varepsilon q(1_A \cap 1_B)$. Since φ_ω is q-regular, we have $(\exists \mu \in N_Q(x_\varepsilon, \omega(T)))$ such that $\mu^{\varphi_\omega} \subseteq (1_A)^{\varphi_\omega} \cap (1_B)^{\varphi_\omega}$. Hence $(\exists \alpha \in [0, \mu(x)[)(x \in \mu_{\bar{\alpha}} \subseteq (\mu_\alpha)^\varphi \subseteq A^\varphi \cap B^\varphi)$. Thus φ is regular.

THEOREM 6.8. *Let* (X, T) *be an ordinary topological space and* $\varphi \in O_{(X, T)}$. *Then* $\omega(T_\varphi) = (\omega(T))_{\varphi_\omega}$.

Proof. Let $\mu \in \omega(T_\varphi)$ and $x_\varepsilon q\mu$. Then $(\forall \alpha \in [0, \mu(x)[)(x \in \mu_{\bar{\alpha}} \in T_\varphi)$. Hence $(\exists U \in T)(x \in U \subseteq (U)^\varphi \subseteq \mu_{\bar{\alpha}})$. Put $\underline{((1 - \varepsilon + \mu(x))/2)} \cap 1_U$. Then $\eta \in N_Q(x_\varepsilon) \cap \omega(T)$. Moreover, $\eta^{\varphi_\omega} = \underline{((1 - \varepsilon + \mu(x))/2)} \cap 1_{U^\varphi} \subseteq \mu$. Thus $\mu \in (\omega(T))_{\varphi_\omega}$.

Conversely, let $\mu \in (\omega(T))_{\varphi_\omega}$. Then by Proposition 6.3(i), there exists a family $\{\eta_j \mid j \in J\} \subseteq \omega(T)$ such that $\mu = \bigcup_{j \in J} \eta_j = \bigcup_{j \in J} \eta_j^{\varphi_\omega}$. Hence, using the representation theorem and Definition 6.5, we may write

$$\mu = \bigcup_{j \in J} \left(\bigcup_{0 \leqslant \alpha < hgt(\eta_j)} (\underline{\alpha} \cap 1_{(\eta_j)_{\bar{\alpha}}}) \right) = \bigcup_{j \in J} \left(\bigcup_{0 \leqslant \alpha < hgt(\eta_j)} (\underline{\alpha} \cap 1_{((\eta_j)_{\bar{\alpha}})^\varphi}) \right).$$

Then by Lemma 4.2(i) of [11], we have

$$(\forall \alpha_0 \in [0, 1[) \left(\mu_{\bar{\alpha}_0} = \bigcup_{j \in J} \left(\bigcup_{\alpha < \alpha_0} (\eta_j)_{\bar{\alpha}} \right) = \bigcup_{j \in J} \left(\bigcup_{\alpha < \alpha_0} ((\eta_j)_{\bar{\alpha}})^\varphi \right) \right).$$

From $(\eta_j)_{\bar{\alpha}} \in T$, then $\mu_{\bar{\alpha}_0} \in T_\varphi$ and hence $\mu \in \omega(T_\varphi)$.

The following Lemmas 6.9 and 6.10 and Theorem 6.12 generalize Lemmas 4.1 and 4.2 and Theorem 4.3 of Geping and Lanfang [11], while Lemma 6.10 generalizes the corresponding result of Lowen [21].

LEMMA 6.9. *Let* (X, T) *be a topological space,* $\varphi \in O_{(X, T)}$, $\mu \in I^X$, *and* $A \subseteq X$. *The following statements hold*:

(i) $\mu \in (\omega(T))_{\varphi_\omega} \Leftrightarrow (\forall \alpha \in [0, 1[)(\mu_{\bar{\alpha}} \in T_\varphi)$.

(ii) $\mu \in ((\omega(T))_{\varphi_\omega})' \Leftrightarrow (\forall \alpha \in]0, 1])(\mu_\alpha \in (T_\varphi)')$.

(iii) $A \in T_\varphi \Leftrightarrow 1_A \in (\omega(T))_{\varphi_\omega}$.

(iv) $A \in (T_\varphi)' \Leftrightarrow 1_A \in ((\omega(T))_{\varphi_\omega})'$.

LEMMA 6.10. *Let* (X, T) *be an ordinary topological space,* $\varphi \in O_{(X, T)}$, $\mu \in I^X$, $A \subseteq X$, *and* $\alpha \in [0, 1[$. *Then*:

(i) $\varphi_\omega.cl(1_A) = 1_{\varphi.cl(A)}$.

(ii) $\varphi_\omega.int(1_A) = 1_{\varphi.int(A)}$.

(iii) $\varphi_\omega.cl(\underline{\alpha} \cap \mu) = \underline{\alpha} \cap \varphi_\omega.cl(\mu)$.

(iv) $\varphi_\omega.int(\underline{\alpha} \cap \mu) = \underline{\alpha} \cap \varphi_\omega.int(\mu)$.

Proof. As an example we prove (i) and (iii). Let $x_\varepsilon \subseteq \varphi_\omega.cl(1_A)$. Then $(\forall \eta \in N_Q(x_\varepsilon, \omega(T)))(\eta^{\varphi_\omega} q 1_A)$ and hence $(\forall \alpha \in [0, \mu(x)[)(x \in \eta_{\bar{\alpha}}$ and $\eta_{\bar{\alpha}}^{\varphi} \cap A \neq \emptyset)$. From $\eta_{\bar{\alpha}} \in T$, $x \in \varphi.cl(A)$ and hence $x_\varepsilon \subseteq 1_{\varphi.cl(A)}$.

Conversely, let $x_\varepsilon \subseteq 1_{\varphi.cl(A)}$. Then $x \in \varphi.cl(A)$ and hence $(\forall U \in N(x, T))(U^{\varphi} \cap A \neq \emptyset)$. Then $(\forall \varepsilon > 0)(1_U \in N_Q(x_\varepsilon, \omega(T))$ and $1_{U^\varphi} q 1_A)$. From $1_{U^\varphi} = (1_U)^{\varphi_\omega}$, we have $x_\varepsilon \subseteq \varphi_\omega.cl(1_A)$.

(iii) $x_\varepsilon \subseteq \varphi_\omega.cl(\underline{\alpha} \cap \mu) \Leftrightarrow (\forall \eta \in N_Q(x_\varepsilon, \omega(T)))(\eta^{\varphi_\omega} q(\underline{\alpha} \cap \mu)) \Leftrightarrow (\forall \eta \in N_Q(x_\varepsilon, \omega(T)))(\eta^{\varphi_\omega} q \underline{\alpha}$ and $\eta^{\varphi_\omega} q \mu) \Leftrightarrow (x_\varepsilon \subseteq \underline{\alpha}$ and $x_\varepsilon \subseteq \varphi_\omega.cl(\mu)) \Leftrightarrow (x_\varepsilon \subseteq \underline{\alpha} \cap \varphi_\omega.cl(\mu))$.

LEMMA 6.11. *Let (X, T) be an ordinary topological space, and $\mu \in I^X$. The following statements hold.*

(i) $\varphi_\omega.cl(\bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}})) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.cl(\mu_{\bar{\alpha}})})$.

(ii) $\varphi_\omega.int(\bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}})) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.int(\mu_{\bar{\alpha}})})$.

Proof. Let's prove the first equality. Since $\varphi_\omega.cl$ is monotonous, we have

$$\bigcup_{0 \leq \alpha < hgt(\mu)} \varphi_\omega.cl(\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}) \subseteq \varphi_\omega.cl\left(\bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}})\right).$$

So, let $x_\varepsilon \subseteq \varphi_\omega.cl(\bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}))$. Then $(\forall \eta \in N_Q(x_\varepsilon, \omega(T)))(\eta^{\varphi_\omega} q \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}))$. Then we obtain $(\exists \alpha \in [0, hgt(\mu)[)(\eta^{\varphi_\omega} q(\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}))$ and hence $(\exists \alpha \in [0, hgt(\mu)[)(x_\varepsilon \subseteq \varphi_\omega.cl(\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}}))$ which implies that $x_\varepsilon \subseteq \bigcup_{0 \leq \alpha < hgt(\mu)} \varphi_\omega.cl(\underline{\alpha} \cap 1_{\mu_{\bar{\alpha}}})$.

THEOREM 6.12. *Let $(X, \omega(T))$ be an induced fts and $\mu \in I^X$. The following relations hold*:

(i) $\varphi_\omega.cl(\mu) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.cl(\mu_{\bar{\alpha}})}) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.cl(\mu_\alpha)})$.

(ii) $\varphi_\omega.int(\mu) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.int(\mu_{\bar{\alpha}})}) = \bigcup_{0 \leq \alpha < hgt(\mu)} (\underline{\alpha} \cap 1_{\varphi.int(\mu_\alpha)})$.

Proof. (i) This is immediate from the decomposition theorem and Lemma 6.11(i).

(ii) This is immediate from the decomposition theorem and Lemma 6.11(ii).

Remark 6.13. If we put appropriate operations for φ and φ_ω in all of the above results, then we obtain the corresponding results for open, θ-open, and δ-open fuzzy sets [3, 4, 10–12, 15, 16, 21, 25, 27, 28].

THEOREM 6.14. *Let (X, T) be an ordinary topological space and $\varphi \in O_{(X,T)}$. Then (X, T) is $\varphi.T_1$ (resp. $\varphi.T_2$) iff $(X, \omega(T))$ is $\varphi_\omega.FT_1$ (resp. $\varphi_\omega.FT_2$).*

Proof. Let x_ε, $y_\nu \in S(X)$ and $x \neq y$. Then $(\exists U \in N(x, T))(\exists V \in N(y, T))$ $(U^\varphi \cap V^\varphi = \varnothing)$. This implies that $1_U \in N_Q(x_\varepsilon, \omega(T))$, $1_V \in N_Q(y_\nu, \omega(T))$, and $1_{U^\varphi} \cap 1_{V^\varphi} = \varnothing$. By Lemma 6.7(ii), we have $(1_U)^{\varphi_\omega} \cap (1_V)^{\varphi_\omega} = \varnothing$. Thus $(X, \omega(T))$ is $\varphi_\omega.FT_2$.

Conversely, let x, $y \in X$ and $x \neq y$. Then $(\exists \mu \in N_Q(x_1, \omega(T)))(\exists \eta \in N_Q(y_1, \omega(T)))(\mu^{\varphi_\omega} \cap \eta^{\varphi_\omega} = \varnothing)$. One can easily see that $(\forall \alpha_1 \in [0, \mu(x)[)$ $(\forall \alpha_2 \in [0, \eta(y)[)(x \in \mu_{\bar{\alpha}_1} \in T,\ y \in \eta_{\bar{\alpha}_2} \in T$ and $(\mu_{\bar{\alpha}_1})^\varphi \cap (\eta_{\bar{\alpha}_2})^\varphi = \varnothing)$. Thus (X, T) is $\varphi.T_2$.

THEOREM 6.15. *Let (X, T) be an ordinary topological space and $\varphi \in O_{(X, T)}$. Then (X, T) is $\varphi.R_2$ iff $(X, \omega(T))$ is $\varphi_\omega.FR_2$.*

Proof. Let $x_\varepsilon \in S(X)$ and $\mu \in N_Q(x_\varepsilon, \omega(T))$. Then $(\forall \alpha \in [0, \mu(x)[)$ $(x \in \mu_{\bar{\alpha}} \in T)$. Since (X, T) is $\varphi.R_2$, we have $(\exists U \in N(x, T))(U^\varphi \subseteq \mu_{\bar{\alpha}})$. Put $\eta = \underline{((1 - \varepsilon + \mu(x))/2)} \cap 1_U$. Then $\eta \in N_Q(x_\varepsilon, \omega(T))$. Moreover, $\eta^{\varphi_\omega} = \underline{((1 - \varepsilon + \mu(x))/2)} \cap 1_{U^\varphi} \subseteq \mu$. Thus we have $(X, \omega(T))$ is $\varphi_\omega.FR_2$.

Conversely, let $x \in X$ and $U \in T$ with $x \in U$. Then $x_1 q 1_U \in \omega(T)$. Hence $(\exists \eta \in N_Q(x_1, \omega(T)))(\eta^{\varphi_\omega} \subseteq 1_U)$. Then $(\forall \alpha \in [0, \mu(x)[)(x \in \eta_{\bar{\alpha}} \in T$ and $(\eta_{\bar{\alpha}})^\varphi \subseteq U)$. Thus (X, T) is $\varphi.R_2$.

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