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Bounds on the largest eigenvalues of trees with a given size of matching☆

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Abstract

Very little is known about upper bound for the largest eigenvalue of a tree with a given size of matching. In this paper, we find some upper bounds for the largest eigenvalue of a tree in terms of the number of vertices and the size of matchings, which improve some known results.

AMS classification: 05C20; 05C50

Keywords: Tree; Matching; The largest eigenvalue of a graph; Upper bound

1. Introduction

Let G be a connected graph with n vertices and $A(G)$ the adjacency matrix of G. Then $A(G)$ is irreducible and symmetric. All eigenvalues of G are real, and the largest eigenvalue of G is one multiplicity. Without loss of generality, we can assume that $\lambda_1(G) > \lambda_2(G) \geqslant \lambda_3(G) \geqslant \cdots \geqslant \lambda_n(G)$ are all eigenvalues of G. When G is a bipartite graph, its eigenvalues have physical interpretations in the quantum chemical theory, so it is significant and necessary to investigate the relations between the

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graph-theoretic properties of G and its eigenvalues. Up to now, the eigenvalues of a tree T with a perfect matching have been studied by several authors (see [2,7,8]). However, when a tree has no perfect matching but has an m-matching M, namely, M consists of m mutually independent edges, very little is known about the eigenvalues of a tree T with an m-matching. The purpose of this paper is to find some upper bounds for the largest eigenvalues of trees in terms of the number of vertices and the size of matchings.

Let T be a tree with n vertices. The classical upper bound of $\lambda_1(T)$ is

$$\lambda_1(T) \leqslant \sqrt{n-1} \tag{1.1}$$

with equality if and only if T is the star graph S_n. Star graph S_n with n vertices can be characterized within the set of all trees with n vertices by the property: each matching consists of only one edge. Hence in order to improve (1.1) for trees, it is natural to impose some upper bounds on the size of a matching of trees. In this paper, we will refine (1.1) for the trees with an m-matching.

We denote by S_n, K_n, and P_n the star graph, the complete graph, and the path graph with n vertices, respectively, and denote by rK_s the disjoint union of r copies of K_s. We denote by $G \cup H$ the graph whose components are G and H. Other graph-theoretic notations may refer to [1].

2. Some lemmas

Denote the characteristic polynomial of a graph G by $p(G; x)$, and recall that the largest eigenvalue of G is just the largest root of the equation $p(G; x) = 0$. Therefore,

$$p(G; x) > 0 \quad \text{for all } x > \lambda_1(G). \tag{2.1}$$

As an immediate consequence of (2.1), we have the following elementary but useful statement.

Lemma 2.1 [3,4]. *Let F and H be two graphs. If $p(F; x) < p(H; x)$ for $x \geqslant \lambda_1(H)$, then $\lambda_1(F) > \lambda_1(H)$.*

The following result is often used to calculate the characteristic polynomials of trees.

Lemma 2.2 [3]. *Let T be a tree and $e = uv$ be an edge of T. Then*

$$p(T; x) = p(T - e; x) - p(T - u - v; x). \tag{2.2}$$

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A graph G' is a subgraph of G if $V(G') \subseteq V(G)$, and $E(G') \subseteq E(G)$. A subgraph G' of G is called proper if $G' \neq G$. A spanning subgraph of G is a subgraph G' with

$V(G') = V(G)$. Let G be a connected graph, and G' be a proper spanning subgraph of G. By the well-known Frobenius theorem, we have $\lambda_1(G) > \lambda_1(G')$. Moreover, the following lemma holds.

Lemma 2.3 [6].

(i) *Let G be a connected graph, and G' be a proper spanning subgraph of G. Then*

$$p(G'; x) > p(G; x) \quad \textit{for all } x \geqslant \lambda_1(G).$$

(ii) *Let G', H' be spanning subgraphs of connected graphs G and H, respectively, and $\lambda_1(G) \geqslant \lambda_1(H)$, and G' is a proper subgraph of G. Then*

$$p(G' \cup H'; x) > p(G \cup H; x) \quad \textit{for all } \lambda \geqslant \lambda_1(G).$$

Two edges of a graph are said to be independent if they are not incident with a common vertex. An m-matching of a graph G is a set of m mutually independent edges. It is clear that every m-matching is a subgraph mK_2 of G. In this paper, we say a tree T with an m-matching means that T has at least an m-matching, and T may or may not have a matching whose size is more than m. A matching M saturates a vertex v, and v is said to be M-saturated if some edge of M is incident with v; otherwise, v is M-unsaturated. A matching M is said to be perfect if every vertex of G is M-saturated. It is easy to prove by induction that a perfect matching of a tree is unique when it exists. The following three lemmas are often used to prove our main results in the following section.

Lemma 2.4. *Let T be a tree with n $(n > 2)$ vertices and with a perfect matching. Then T has at least two pendant vertices such that they are adjacent to vertices of degree 2, respectively.*

Proof. First, we root T at a vertex r and choose a pendant vertex v furthest from r. Let $e = vw$ be a pendant edge. If the degree of w is not 2, there would be a pendant vertex $u \neq v$ joined to w and T cannot have a perfect matching. Second we root T at the vertex v and choose a pendant vertex x furthest from v. As the above proof, x is also adjacent to a vertex of degree 2. □

By Lemma 2.4 we have:

Lemma 2.5. *Let T be an n-vertex tree with an m-matching, and $n = 2m + 1$. Then T has a pendant vertex which is adjacent to a vertex of degree 2.*

Lemma 2.6. *Let T be an n-vertex tree with an m-matching where $n > 2m$. Then there is an m-matching M and a pendant vertex v such that M does not saturate v.*

Proof. For $n \leqslant 3$ the result clearly holds. We assume that $n > 3$ and proceed by induction. Consider an m-matching $\bar{M}$ of T. Root T at a vertex r and let v be a pendant vertex furthest from r. Let vw be the pendant edge which is incident with v. If the edge vw does not belong to $\bar{M}$, then the conclusion follows. So we may assume that the edge vw belongs to $\bar{M}$. If the degree of w is not 2, then there is a pendant vertex $\bar{v} \neq v$ joined to w which is $\bar{M}$-unsaturated. Thus we may assume the degree of w is 2. Let ww' be the edge with $w' \neq v$, and let T' be the tree obtained from T by removing vertices v and w and edges vw and ww'. Then T' has $n-2=n'$ vertices and an m'-matching, where $m'=m-1$. Since $n'>2m'$, it follows by induction that T' has an m'-matching M' and a pendant vertex v' which is M'-unsaturated. If $v' \neq w'$, then $M' \cup \{vw\}$ is an m-matching of T not saturating the pendant vertex v' of T. If $v'=w'$, then $M' \cup \{v'w\}$ is an m-matching of T not saturating the pendant vertex v. Hence the lemma holds by induction. □

3. The largest eigenvalues of trees with a given size of matching

Let n and m be positive integers and $n \geqslant 2m$. We define a tree $A(n,m)$ with n vertices as follows: $A(n,m)$ is obtained from the star graph S_{n-m+1} with $n-m+1$ vertices by attaching a pendant edge to each of certain $m-1$ non-central vertices of S_{n-m+1}. We call $A(n,m)$ a spur and note that it has an m-matching. The center of $A(n,m)$ is the center of the star S_{n-m+1}. For $n>2m$, let $B(n,m)$ be the graph obtained from the spur $A(n-1,m)$ by attaching a pendant edge to one vertex of degree 2. Then $B(n,m)$ has an m-matching. The center of $B(n,m)$ is the center of the spur $A(n-1,m)$. For $m \geqslant 3$, let $C(n,m)$ be the graph obtained from the spur $A(n-2,m-1)$ by attaching a path of length 2 to one vertex of degree 2. Then $C(n,m)$ has an m-matching. The center of $C(n,m)$ is the center of the spur $A(n-2,m-1)$. In Fig. 1 we have drawn $A(14,6)$, $B(14,6)$ and $C(14,6)$.

We now compute the characteristic polynomials of graphs $A(n,m)$, $B(n,m)$, and $C(n,m)$, we need the following lemma [3, p. 60].

Lemma 3.1. *Let H be a graph obtained from the graph G with vertex-set $\{x_1, x_2, \ldots, x_l\}$ in the following way*:
(i) *To each vertex x_i of G a set $\mathscr{V}_i$ of k new isolated vertices is added*; *and*
(ii) *x_i is joined by an edge to each of the k vertices of $\mathscr{V}_i$ $(i=1,2,\ldots,l)$.*
Then

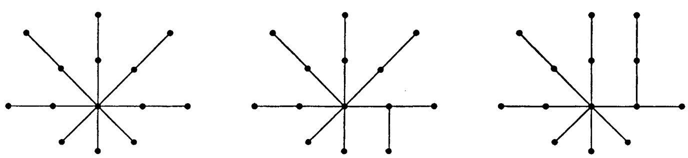

Fig. 1. Trees $A(14,6)$, $B(14,6)$ and $C(14,6)$.

$$p(H;x) = x^{lk} p\left(G; x - \frac{k}{x}\right). \tag{3.1}$$

Proposition 3.2.

$$\begin{aligned} p(A(n,m);x) = {} & x^{n-2m}(x^2-1)^{m-2} \\ & \times\left[x^4 - (n-m+1)x^2 + n - 2m + 1\right], \end{aligned} \tag{3.2}$$

$$\begin{aligned} p(B(n,m);x) = {} & x^{n-2m}(x^2-1)^{m-3}\left[x^6 - (n-m+2)x^4 + (3n-4m-1)x^2 \right. \\ & \left. -2(n-2m)\right], \end{aligned} \tag{3.3}$$

$$\begin{aligned} p(C(n,m);x) = {} & x^{n-2m}(x^2-1)^{m-4} \\ & \times\left[x^8 - (n-m+3)x^6 + (4n-5m+1)x^4 \right. \\ & \left. -(4n-7m+3)x^2 + n - 2m + 1\right]. \end{aligned} \tag{3.4}$$

Proof. If $n = 2m$, using the above lemma by taking $G = S_m$, $l = m$, and $k = 1$, then

$$p(A(2m,m);x) = (x^2-1)^{m-2}\left[x^4 - (m+1)x^2 + 1\right].$$

If $n > 2m$, using Lemma 2.2 repeatedly, then

$$\begin{aligned} p(A(n,\ m);x) &= xp(A(n-1,m);x) - x^{n-2m}(x^2-1)^{m-1} \\ &= x^{n-2m}p(A(2m,m);x) - (n-2m)x^{n-2m}(x^2-1)^{m-1} \\ &= x^{n-2m}(x^2-1)^{m-2}\left[x^4 - (n-m+1)x^2 + n - 2m + 1\right]. \end{aligned}$$

Eqs. (3.3) and (3.4) can be derived using Lemma 2.2 and Eq. (3.2) by noting that

$$p(B(n,m);x) = xp(A(n-1,m);x) - xp(A(n-3,m-1);x),$$

$$\begin{aligned} p(C(n,m);x) = {} & (x^2-1)p(A(n-2,m-1);x) \\ & -x^2 p(A(n-4,m-2);x). \qquad \square \end{aligned}$$

From the above proposition, it is easy to obtain

$$\begin{aligned} \lambda_1(A(n,m)) &= \frac{1}{2}\sqrt{2(n-m+1) + 2\sqrt{(n-m-1)^2 + 4m - 4}} \\ &= \frac{1}{2}\left(\sqrt{n-m+1-2\sqrt{n-2m+1}} \right. \\ & \qquad \left. + \sqrt{n-m+1+2\sqrt{n-2m+1}}\right), \end{aligned} \tag{3.5}$$

and for $m = 2$,

$$p(B(n,2);x) = x^{n-4}(x^4 - (n-1)x^2 + 2n - 8),$$

$$\lambda_1(B(n,2)) = \frac{1}{2}\sqrt{2(n-1) + 2\sqrt{n^2 - 10n + 33}}. \tag{3.6}$$

Theorem 3.3. *Let T be an n-vertex tree with an m-matching, $n \geqslant 2m$, and $T \neq A(n,m)$. Then*

$$p(T;x) > p(A(n,m);x) \quad \text{for all } x \geqslant \lambda_1(T). \tag{3.7}$$

Therefore

$$\lambda_1(T) < \frac{1}{2}\left(\sqrt{n - m + 1 - 2\sqrt{n - 2m + 1}} + \sqrt{n - m + 1 + 2\sqrt{n - 2m + 1}}\right). \tag{3.8}$$

Proof. It is sufficient to prove (3.7) by Lemma 2.1. We prove the theorem by induction on n. First suppose $n = 2m$. We prove that the theorem holds in the case of $n = 2m$ by induction on m. If $m = 1, 2, 3$, then the theorem holds clearly by the facts that there are at most two trees with $n = 2m$ vertices and an m-matching for $m = 1, 2, 3$.

We now suppose $m \geqslant 4$ and proceed by induction. Let T be any tree with $2m$ vertices and with an m-matching. By Lemma 2.4, T has a pendant vertex v which is adjacent to a vertex w of degree 2. Thus vw is an edge of T and there is a unique vertex $u \neq v$ such that uw is also an edge of T. Let T' be the tree obtained from T by removing vertices v and w and edges vw and uw, namely, $T' = T - v - w$. Then T' is a tree with $2(m-1)$ vertices and with an $(m-1)$-matching. By the induction assumption,

$$p(T';x) \geqslant p(A(2(m-1), m-1);x) \quad \text{for all } x \geqslant \lambda_1(T'). \tag{3.9}$$

By Lemma 2.2, we have

$$\begin{aligned} p(T;x) &= p(T - uw;x) - p(T - w - u;x) \\ &= (x^2 - 1)p(T';x) - xp(T - v - w - u;x), \end{aligned}$$

$$\begin{aligned} p(A(2m,m);x) = (x^2 - 1)p(A(2(m-1), m-1);x) \\ -xp(K_1 \cup (m-2)K_2;x), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &p(T;x) - p(A(2m,m);x) \\ &\quad = (x^2 - 1)\big[p(T';x) - p(A(2(m-1), m-1);x)\big] \\ &\qquad + x\big[p(K_1 \cup (m-1)K_2;x) - p(T - v - w - u;x)\big]. \end{aligned}$$

Since $T - v - w - u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T - v - w - u) < \lambda_1(T') < \lambda_1(T)$. Since T' has an $(m-1)$-matching and

$2n-3$ vertices, $T-v-w-u=T'-u$ has an $(m-2)$-matching and $K_1\cup(m-2)K_2$ is a proper spanning subgraph of $T-v-w-u$ when $T\neq A(2m,m)$. By Lemma 2.3 we have $\lambda_1(T-v-w-u)>\lambda_1(K_1\cup(m-2)K_2)$ and

$$p(K_1\cup(m-2)K_2;x)>p(T-v-w-u;x)\\ \text{for all } x>\lambda_1(T-v-w-u).$$

Hence by Eqs (3.9) and (3.10), we have

$$p(T;x)>p(A(2m,m);x)\quad\text{for all } x\geqslant\lambda_1(T).$$

This completes the induction on m and proves the theorem when $n=2m$.

We now suppose $n>2m$ and proceed by induction on n. Let T be any tree with n vertices and with an m-matching. By Lemma 2.6, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let u be the unique vertex such that vu is a pendant edge of T. Let T' be the tree obtained from T by removing vertex v and edge vu, namely, $T'=T-v$. Then T' is a tree with $n-1$ vertices and with an m-matching. By the induction assumption,

$$p(T';x)\geqslant p(A(n-1,m);x)\quad\text{for all } x\geqslant\lambda_1(T'). \tag{3.11}$$

By Lemma 2.2, we have

$$\begin{aligned} p(T;x)&=p(T-vu;x)-p(T-v-u;x)\\ &=xp(T';x)-p(T-v-u;x),\\ p(A(n,m);x)&=xp(A(n-1,m);x)\\ &\qquad-p((n-2m)K_1\cup(m-1)K_2;x), \qquad (3.12)\\ p(T;x)&-p(A(n,m);x)\\ &=x\big[p(T';x)-p(A(n-1,m);x)\big]\\ &\quad+\big[p((n-2m)K_1\cup(m-1)K_2;x)-p(T-v-u;x)\big]. \end{aligned}$$

Since $T-v-u=T'-u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T-v-u)<\lambda_1(T')<\lambda_1(T)$. Since T' has an m-matching, $T-v-u=T'-u$ has an $(m-1)$-matching and $(n-2m)K_1\cup(m-1)K_2$ is a proper spanning subgraph of $T-v-u$ when $T\neq A(n,m)$. By Lemma 2.3, we have $\lambda_1(T-v-u)>\lambda_1((n-2m)K_1\cup(m-1)K_2)$ and

$$p((n-2m)K_1\cup(m-1)K_2;x)>p(T-w-u;x)\\ \text{for all } x\geqslant\lambda_1(T-v-u).$$

Hence by Eqs (3.11) and (3.12), we have

$$p(T;x)\geqslant p(A(n,m);x)\quad\text{for all } x\geqslant\lambda_1(T).$$

This completes the proof of Theorem 3.3 by induction. □

Let T be a tree which is not the star graph. Then T has an 2-matching. Taking $m = 2$, we obtain:

Corollary 3.4 [5]. *Let T be an n-vertex tree and $T \neq S_n$. Then*

$$\lambda_1(T) \leqslant \frac{1}{2}\sqrt{2(n-1) + 2\sqrt{n^2 - 6n + 13}}, \tag{3.13}$$

and equality holds if and only if $T = A(n, 2)$.

Corollary 3.5 [8]. *Let T be an n-vertex ($n = 2m$) tree with a perfect matching. Then*

$$\lambda_1(T) \leqslant \frac{1}{2}\left(\sqrt{m-1} + \sqrt{m+3}\right), \tag{3.14}$$

and equality holds if and only if $T = A(2m, m)$.

Theorem 3.6. *Let T be an n-vertex tree with an m-matching, $n > 2m$, $T \neq A(n, m)$ and $T \neq B(n, m)$. Then*

$$p(T; x) > p(B(n, m); x) \quad \textit{for all } x \geqslant \lambda_1(T), \tag{3.15}$$

and $\lambda_1(T) < \lambda_1(B(n, m))$.

Proof. It is sufficient to prove (3.15) by Lemma 2.1. We prove the theorem by induction on n. First we suppose $n = 2m + 1$. We prove that theorem holds in the case of $n = 2m + 1$ by induction on m. If $m = 1, 2$, then the theorem holds clearly. If $m = 3$, there are six graphs with seven vertices and with a 3-matching, and the theorem holds (see Fig. 2).

We now suppose $m \geqslant 4$ and proceed by induction. Let T be any tree with $2m + 1$ vertices and with an m-matching. By Lemma 2.5, T has a pendant vertex v which is adjacent to a vertex w of degree 2. Thus vw is an edge of T and there is a unique vertex $u \neq v$ such that uw is also an edge of T. Let T' be the tree obtained from T by removing vertices v and w and edges vw and uw, namely, $T' = T - v - w$. Then T' is a tree with $2(m - 1) + 1$ vertices and with an $(m - 1)$-matching.

If $T' = A(2m - 1, m - 1)$, then T must be isomorphic to any of the graphs in Fig. 3, because $T \neq A(2m + 1, m)$, $B(2m + 1, m)$. Therefore we may choose other vertices v', w' instead of v, w such that $T - v' - w' = T' \neq A(2m - 1, m - 1)$. Thus we may always assume that $T - v - w = T' \neq A(2m - 1, m - 1)$.

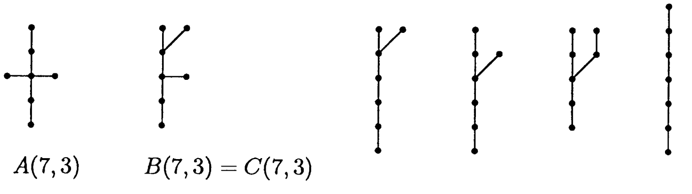

Fig. 2. Trees with seven vertices and a 3-matching.

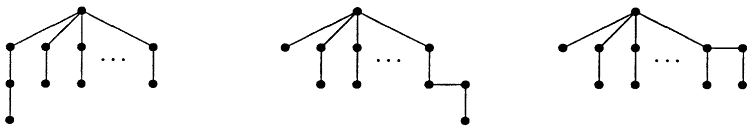

Fig. 3.

By the induction assumption,

$$p(T'; x) \geqslant p(B(2(m-1)+1, m-1); x) \quad \text{for all } x \geqslant \lambda_1(T'). \tag{3.16}$$

By Lemma 2.2 we have

$$\begin{aligned} p(T; x) &= p(T - wu; x) - p(T - w - u; x) \\ &= (x^2 - 1)p(T'; x) - xp(T - v - w - u; x), \end{aligned} \tag{3.17}$$

$$\begin{aligned} p(B(2m+1, m); x) &= (x^2 - 1)p(B(2m-1, m-1); x) \\ &\quad -xp((K_1 \cup P_3 \cup (m-3)K_2; x). \end{aligned} \tag{3.18}$$

Since $T - v - w - u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T - v - w - u) < \lambda_1(T') < \lambda_1(T)$. Since T' has an $(m-1)$-matching and $2m-1$ vertices, $T - v - w - u = T' - u$ has an $(m-2)$-matching and $2m-2$ vertices. If $K_1 \cup P_3 \cup (m-3)K_2$ is a proper spanning subgraph of $T - v - w - u$, then for $x \geqslant \lambda_1(T) > \lambda_1(T - v - w - u)$, we have

$$p(K_1 \cup P_3 \cup (m-3)K_2; x) > p(T - v - w - u; x).$$

Hence

$$\begin{aligned} &p(T; x) - p(B(2m+1, m); x) \\ &\quad = (x^2 - 1)\big[p(T'; x) - p(B(2(m-1)+1, m-1); x)\big] \\ &\qquad + x\big[p(K_1 \cup P_3 \cup (m-3)K_2; x) - p(T - v - w - u; x)\big] > 0. \end{aligned}$$

If $K_1 \cup P_3 \cup (m-3)K_2$ is not a proper spanning subgraph of $T - v - w - u$, then $T - v - w - u$ must be isomorphic to any of the graphs in Fig 4. Here T'' is a forest with perfect matching and at least one connected component C has more than four vertices. By Lemma 2.4, C has at least two pendant vertices which are adjacent to vertices of degree 2. Therefore T must be isomorphic to $A(2m+1, m)$, $B(2m+1, m)$ or any of the graphs in Fig. 5.

For the graph in Fig. 5(a), we have $T - v - w - u = K_1 \cup P_3 \cup (m-3)K_2$. Thus for $x \geqslant \lambda_1(T)$, we have

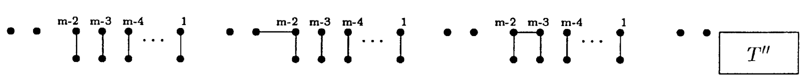

Fig. 4.

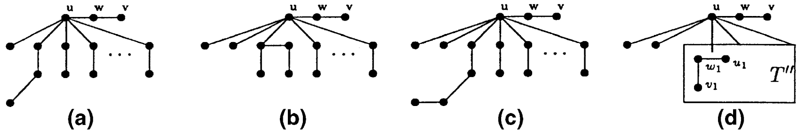

Fig. 5.

$$p(T;x) - p(B(2m+1,m);x)$$
$$= (x^2-1)\big[p(T';x) - p(B(2(m-1)+1,m-1);x)\big] > 0.$$

For the graphs in Fig. 5(b) and (c), $T - v - w - u = 2K_2 \cup P_4 \cup (m-4)K_2$. Since $p(P_4;x) = xp(P_3;x) - p(P_2;x)$ and $p(P_2;x) = x^2 - 1$,

$$p(K_1 \cup P_3 \cup (m-3)K_2;x) - p(T-v-w-u;x)$$
$$= xp(P_3;x)(x^2-1)^{m-3} - x^2p(P_4;x)(x^2-1)^{m-4}$$
$$= (x^2-1)^{m-4}\big[(x^2-1)^2 - p(P_4;x)\big].$$

Thus for $x \geqslant \lambda_1(T)$, we have

$$p(T;x) - p(B(2m+1,m);x)$$
$$= (x^2-1)\big[p(T';x) - p(B(2(m-1)+1,m-1);x)\big]$$
$$+x(x^2-1)^{m-4}\big[(x^2-1)^2 - p(P_4;x)\big] > 0.$$

For the graph in Fig. 5(d), since the component C of T'' has two pendant vertices which are adjacent to vertices of degree 2, one may replace vertices v, u, w by v_1, w_1, u_1, then $K_1 \cup P_3 \cup (m-3)K_2$ is a proper spanning subgraph of $T - v_1 - w_1 - u_1$ and the result holds from the previous proof. Thus, we have proven $p(T;x) > p(B(2m+1,m);x)$ holds for all $x \geqslant \lambda_1(T)$.

This completes the induction on m and proves the theorem when $n = 2m+1$.

We now suppose $n > 2m+1$ and proceed by induction on n. Let T be any tree with n vertices and with an m-matching. By Lemma 2.6, T has an m-matching M and a pendant vertex v such that M does not saturate v. Let u be the unique vertex such that vu is a pendant edge. Let T' be the tree obtained from T by removing vertex v and edge vu, namely, $T' = T - v$. Then T' is a tree with $n-1$ vertices and with an m-matching.

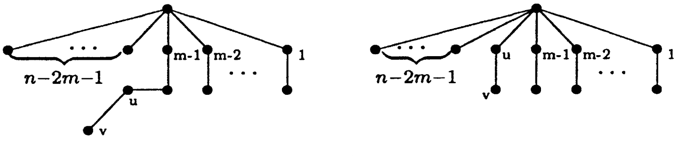

Fig. 6.

Case 1. If T' is isomorphic to $A(n-1,m)$, then T must be isomorphic to one of the graphs in Fig. 6 since $T \neq A(n,m)$, $B(n,m)$.

Thus $T-v-u$ has a proper spanning subgraph $K_1 \cup A(n-3,m-1)$. By Lemma 2.3, we have $\lambda_1(T-v-u) > \lambda_1(K_1 \cup A(n-3,m-1))$ and

$$p(K_1 \cup A(n-3,m-1);x) > p(T-v-u;x) \quad \text{for all } x \geqslant \lambda_1(T-v-u).$$

By Lemma 2.2, we have

$$\begin{aligned} p(T;x) &= p(T-vu;x) - p(T-v-u;x) \\ &= xp(A(n-1,m);x) - p(T-v-u;x). \\ p(B(n,m);x) &= xp(A(n-1,m);x) - p(K_1 \cup A(n-3,m-1);x). \end{aligned}$$

Since $T-v-u = T'-u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T-v-u) < \lambda_1(T') < \lambda_1(T)$. Hence, by the above three equalities, we have

$$p(B(n,m);x) \leqslant p(T;x) \quad \text{for all } x \geqslant \lambda_1(T).$$

Case 2. If $T' = T-v$ is not isomorphic to $A(n-1,m)$, then by the induction assumption we have

$$p(T';x) \geqslant p(B(n-1,m);x) \quad \text{for all } x \geqslant \lambda_1(T'). \tag{3.19}$$

By Lemma 2.2, we have

$$\begin{aligned} p(T;x) &= p(T-vu;x) - p(T-v-u;x) \\ &= xp(T';x) - p(T-v-u;x). \\ p(B(n,m);x) &= xp(B(n-1,m);x) \\ &\quad -p((n-2m-1)K_1 \cup P_3 \cup (m-2)K_2;x). \end{aligned} \tag{3.20}$$

$$\begin{aligned} &p(T;x) - p(B(n,m);x) \\ &\quad = x\big[p(T';x) - p(B(n-1,m);x)\big] \\ &\qquad +\big[p((n-2m-1)K_1 \cup P_3 \cup (m-1)K_2;x) - p(T-v-u;x)\big]. \end{aligned}$$

Since $T-v-u = T'-u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T-v-u) < \lambda_1(T') < \lambda_1(T)$. Since T' has an m-matching, $T-v-u = T'-u$ has an $(m-1)$-matching and $n-2$ vertices. Note that $(n-2m-1)K_1 \cup P_3 \cup (m-2)K_2$ is not a proper spanning subgraph of $T-v-u$ if and only if $T-v-u$

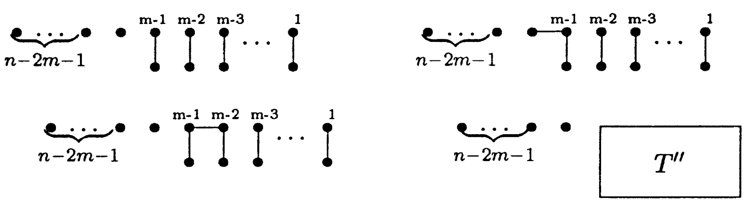

Fig. 7.

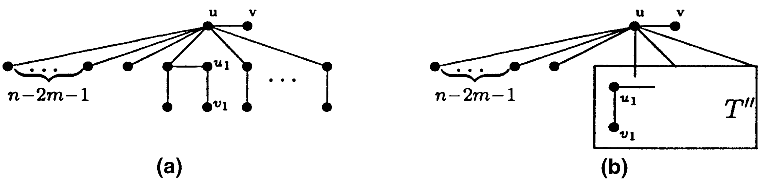

Fig. 8.

is isomorphic to any of the graphs in Fig. 7. Here T'' is a forest with perfect matching and at least one connected component C has more than two vertices. By Lemma 2.4, C has at least two pendant vertices which are adjacent to vertices of degree 2. Hence T must be isomorphic to $A(n, m)$, $B(n, m)$ or one of the graphs in Fig. 8.

The component C of T'' has at least two pendant vertices which are adjacent to vertices of degree 2. In both cases, Fig. 8(a) and (b), we may replace v, u by v_1, u_1. Then $(n-2m-1)K_1 \cup P_3 \cup (m-2)K_2$ is a proper spanning subgraph of $T-v-u$.

Therefore, if $T \neq A(n, m), B(n, m)$, then using Lemma 2.3, we have $\lambda_1(T-v-u) > \lambda_1((n-2m-1)K_1 \cup P_3 \cup (m-2)K_2)$ and

$$p((n-2m-1)K_1 \cup P_3 \cup (m-2)K_2; x) > p(T-v-u; x) \quad \text{for all } x \geqslant \lambda_1(T-v-u). \tag{3.21}$$

Hence, by Eqs (3.19)–(3.21), we have

$$p(T; x) > p(B(n, m); x) \quad \text{for all } x \geqslant \lambda_1(T).$$

This completes the proof of Theorem 3.6 by induction. □

In the above theorem, we obtain the following corollary by taking $m = 2$.

Corollary 3.7 [5]. *Let T be an n-vertex tree ($n > 4$), and $T \neq S_n, A(n, 2)$. Then*

$$\lambda_1(T) \leqslant \frac{1}{2}\sqrt{2(n-1) + 2\sqrt{n^2 - 10n + 33}}, \tag{3.22}$$

and equality holds if and only if $T = B(n, 2)$.

The following result is concerned in trees with perfect matchings.

Theorem 3.8. *Let T be a tree with $n = 2m$ ($m \geqslant 3$) vertices and with a perfect matching, and $T \neq A(2m, m), C(2m, m)$. Then*

$$p(T; x) > p(C(2m, m); x) \quad \textit{for all } x \geqslant \lambda_1(T), \tag{3.23}$$

and $\lambda_1(T) < \lambda_1(C(2m, m))$.

Proof. It is sufficient to prove (3.23). We prove the theorem by induction on m. If $m = 3$, then the unique tree $T \neq A(6, 3) = C(6, 3)$ with six vertices and with a perfect matching is the path P_6, and the theorem holds. We now suppose $m \geqslant 4$ and proceed by induction. Let T be any tree with $2m$ vertices and with a perfect matching. By Lemma 2.4, T has a pendant vertex v which is adjacent to a vertex w of degree 2. Thus vw is an edge and there is a unique vertex $u \neq v$ such that uw is also an edge of T. Let T' be the tree obtained from T by removing vertices v and w and edges vw and uw, namely, $T' = T - v - w$. Then T' is a tree with $2(m-1)$ vertices and with a perfect matching.

Case 1. If there exist vertices v, w, u satisfying the above-mentioned property and $T' = T - v - w$ is isomorphic to $A(2(m-1), m-1)$, then by Lemma 2.2 we have

$$\begin{aligned} p(T;x) &= p(T - wu, x) - p(T - w - u; x) \\ &= (x^2 - 1)p(A(2(m-1), m-1); x) \\ &\quad -xp(T - v - w - u; x), \end{aligned} \tag{3.24}$$

$$\begin{aligned} p(C(2m, m); x) &= (x^2 - 1)p(A(2(m-1), m-1); x) \\ &\quad -xp((K_1 \cup A(2(m-2), m-2); x). \end{aligned} \tag{3.25}$$

Since $T - v - w - u = T' - u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T - v - w - u) < \lambda_1(T') < \lambda_1(T)$. Since $T' = A(2(m-1), m-1)$, and $T \neq A(2m, m)$, $C(2m, m)$, the vertex u is neither the center of $T' = A(2(m-1), m-1)$ nor a vertex of degree 2 of $T' = A(2(m-1), m-1)$. Hence u is a vertex of degree 1 in T', and $T' - u$ is connected and it has a proper spanning subgraph $K_1 \cup A(2(m-2), m-2)$ (see Fig. 9).

By Lemma 2.3 we have $\lambda_1(T - v - w - u) > \lambda_1(K_1 \cup A(2(m-2), m-2))$ and

$$\begin{aligned} &p(K_1 \cup A(2(m-2), m-2); x) > p(T - v - w - u; x) \\ &\quad \text{for all } x \geqslant \lambda_1(T - v - w - u). \end{aligned} \tag{3.26}$$

Hence, by Eqs. (3.24)–(3.26), we have

$$p(T;x) > p(C(2m, m); x) \quad \text{for all } x \geqslant \lambda_1(T).$$

Case 2. If $T' = T - v - w$ is not isomorphic to $A(2(m-1), m-1)$ for any vertices v, w, u in which v is a pendant vertex and w has only two neighbors v, u, then by the induction assumption, we have

$$p(T'; x) \geqslant p(C(2(m-1); m-1); x) \quad \text{for all } x \geqslant \lambda_1(T'). \tag{3.27}$$

By Lemma 2.2, we have

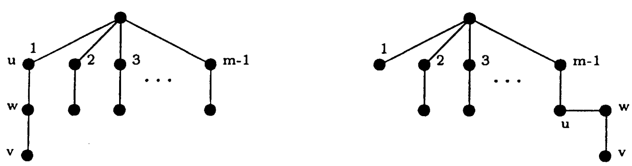

Fig. 9.

$$p(T;x) = p(T-wu;x) - p(T-w-u;x)$$
$$= (x^2-1)p(T';x) - xp(T-v-w-u;x),$$
$$p(C(2m,m);x) = (x^2-1)p(C(2(m-1),m-1);x)$$
$$-xp(K_1 \cup P_4 \cup (m-4)K_2;x), \tag{3.28}$$
$$p(T;x) - p(C(2m,m);x)$$
$$= (x^2-1)\big[p(T';x) - p(C(2(m-1),m-1);x)\big]$$
$$+x\big[p(K_1 \cup P_4 \cup (m-4)K_2;x) - p(T-v-w-u;x)\big].$$

Since $T-v-w-u = T'-u$ is a proper subgraph of T' and T' is a proper subgraph of T, $\lambda_1(T-v-w-u) < \lambda_1(T') < \lambda_1(T)$. Since T' has an $(m-1)$-matching, $T-v-w-u = T'-u$ has an $(m-2)$-matching and $n-3 = 2(m-2)+1$ vertices. Thus $K_1 \cup (m-2)K_2$ is a spanning subgraph of $T-v-u-w$. Hence $K_1 \cup P_4 \cup (m-4)K_2$ is not a proper spanning subgraph of $T-v-w-u$ if and only if $T-v-w-u$ is isomorphic to any of the graphs in Fig 10.

Therefore T must be isomorphic to either of an $A(2m,m)$, $C(2m,m)$ or any of the graphs in Fig. 11.

Among graphs in Fig. 11, (a) is impossible since it has no m-matching, (b) and (c) are impossible because we may choose vertices v_1, w_1 instead of v, w such that $T-v_1-w_1$ is isomorphic to $A(2(m-1),m-1)$. Hence $K_1 \cup P_4 \cup (m-4)K_2$ is a proper spanning subgraph of $T-v-w-u$ when $T \neq A(2m,m)$, $C(2m,m)$. By Lemma 2.3 we have $\lambda_1(T-v-w-u) > \lambda_1(K_1 \cup P_4 \cup (m-4)K_2)$ and

$$p(K_1 \cup P_4 \cup (m-4)K_2;x) > p(T-v-w-u;x)$$
$$\text{for all } x \geqslant \lambda_1(T-v-w-u). \tag{3.29}$$

Hence, using Eqs. (3.27)–(3.29), we have

$$p(T;x) > p(C(2m,m);x) \quad \text{for all } x \geqslant \lambda_1(T).$$

This completes the proof of the theorem. □

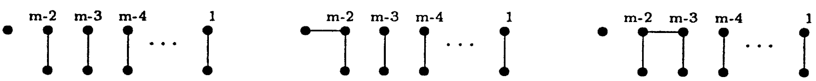

Fig. 10.

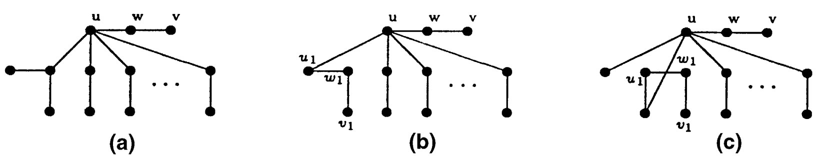

Fig. 11.

$T_1,\ m=3$ $T_2,\ m=5$

Fig. 12.

Table 1

	$\sqrt{n-1}$	Th. 3.3	Cor. 3.4	Cor. 3.5	Th. 3.6	Cor. 3.7	Th. 3.8	λ_1
T_1	3	2.52433	2.8530		2.433	2.71579		2.367
T_2	3	2.52433	2.8530	2.4142		2.71579	2.2850	2.250

We conclude this paper by the example shown in Fig. 12 which compares our new bounds with the old known bounds. Let $n=10$, and T_1 and T_2 be two trees with 10 vertices and with 3-matching and 5-matching, respectively.

Table 1 gives bounds in terms of our results and known results, and λ_1 is the factual value of $\lambda_1(T)$.

In general, the bound in terms of Theorem 3.3, that is,

$$\frac{1}{2}\sqrt{2(n-m+1)+2\sqrt{(n-m-1)^2+4m-4}}$$

is a decreasing function of m. So, for any tree $T \neq S_n$, and it is always better than known bounds $\frac{1}{2}\sqrt{2(n-1)+2\sqrt{n^2-6n+13}}$ (i.e., Corollary 3.4.)

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