Journal of Algebra 322 (2009) 2150–2179

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Demazure crystals of generalized Verma modules and a flagged RSK correspondence ☆

Jae-Hoon Kwon

Department of Mathematics, University of Seoul, Seoul 130-743, Republic of Korea

ARTICLE INFO

Article history:
Received 11 December 2008
Available online 12 May 2009
Communicated by Peter Littelmann

Keywords:
Quantum groups
Crystal graphs
Demazure crystals
RSK correspondence

ABSTRACT

We prove that the Robinson–Schensted–Knuth correspondence is a $\mathfrak{gl}_\infty$-crystal isomorphism between two realizations of the crystal graph of a generalized Verma module with respect to a maximal parabolic subalgebra of $\mathfrak{gl}_\infty$. A flagged version of the RSK correspondence is derived in a natural way by computing a Demazure crystal graph of a generalized Verma module. As an application, we discuss a relation between a Demazure crystal and plane partitions with a bounded condition.

1. Introduction

The Robinson–Schensted–Knuth (simply RSK) algorithm [15] has been playing fundamental roles in combinatorics and representation theory with generalizations in various directions. It gives a bijection between the set $\mathcal{M}$ of $\mathbb{N}\times\mathbb{N}$ matrices with non-negative integral entries of finite support and the set $\mathcal{T}$ of pairs of semistandard tableaux of the same shape, and explains in a bijective way the expansion of the Cauchy kernel into Schur functions, called the Cauchy identity:

$$\frac{1}{\prod_{i,j\geqslant 1}(1-x_i y_j)} = \sum_{\lambda} s_\lambda(X)s_\lambda(Y),$$

where the sum is over all partitions λ and $s_\lambda(X)$ (or $s_\lambda(Y)$) is the Schur function in $X=\{x_1,x_2,\ldots\}$ (or $Y=\{y_1,y_2,\ldots\}$). A representation theoretic interpretation of the Cauchy identity can be given by a general principle called Howe duality [7], that is, $S(\mathbb{C}^{>0}\otimes\mathbb{C}^{>0})$, the symmetric algebra generated by

☆ This work was supported by KRF Grant #2008-314-C00004.
E-mail address: jhkwon@uos.ac.kr.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.04.034

$\mathbb{C}^{>0} \otimes \mathbb{C}^{>0}$ has a multiplicity-free decomposition into irreducible $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bimodules parameterized by partitions, where $\mathbb{C}^{>0}$ is the complex vector space with a basis $\{v_i \mid i \in \mathbb{N}\}$ and $\mathfrak{gl}_{>0} = \mathfrak{gl}(\mathbb{C}^{>0})$ is the corresponding general linear Lie algebra.

We have a more direct interpretation of the RSK map with the help of the Kashiwara's crystal base theory of the quantum group $U_q(\mathfrak{gl}_{>0})$ [10,12,14]. That is, both $\mathcal{M}$ and $\mathcal{T}$ have two $\mathfrak{gl}_{>0}$-crystal structures commuting with each other, which are called $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bicrystals or double crystals [20], and the RSK map is an isomorphism of bicrystals. The decomposition as a $(\mathfrak{gl}_{>0}, \mathfrak{gl}_{>0})$-bimodule follows immediately by considering highest weight crystal elements in $\mathcal{M}$. We refer the readers to [4, 17,20–23] for more results on bicrystal, its application and generalization to other types of Lie algebras.

The main purpose of this paper is to give a new representation theoretic interpretation of the RSK correspondence and its applications. Let $\mathfrak{gl}_\infty$ be the general linear Lie algebra, which is spanned by the elementary matrices E_{ij} $(i, j \in \mathbb{Z} \setminus \{0\})$. Let $\mathfrak{l} = \mathfrak{gl}_{<0} \oplus \mathfrak{gl}_{>0}$ be a Levi subalgebra, where $\mathfrak{gl}_{\gtrless 0}$ is a subalgebra spanned by E_{ij} $(i, j \gtrless 0)$, respectively, and let $\mathfrak{u}_\pm$ be the nilradical spanned by E_{ij} for $i > 0$, $j < 0$ (resp. $i < 0$, $j > 0$). We may identify $S(\mathbb{C}^{>0} \otimes \mathbb{C}^{>0})$ with the enveloping algebra $U(\mathfrak{u}_-)$, which is a generalized Verma module induced from the maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+$ (see [3] for a quantized version of this fact and its relation with canonical basis). Motivated by this observation, we introduce $\mathfrak{gl}_\infty$-crystal structures on $\mathcal{M}$ and $\mathcal{T}$ extending the $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal structures (note that $\mathfrak{gl}_{<0} \simeq \mathfrak{gl}_{>0}$), and then show that the RSK map is an isomorphism of $\mathfrak{gl}_\infty$-crystals (Theorem 3.6). Indeed, these are obtained by finding the missing Kashiwara operators compatible with the RSK map, which correspond to the simple root α_0 connecting the Dynkin diagrams of $\mathfrak{gl}_{<0}$ and $\mathfrak{gl}_{>0}$.

The RSK map also enables us to define a natural embedding of $\mathbf{B}(n\Lambda_0)$ into $\mathcal{M}$, where Λ_0 is the 0th fundamental weight of $\mathfrak{gl}_\infty$ and $\mathbf{B}(n\Lambda_0)$ is the crystal graph of the irreducible representation of the quantum group $U_q(\mathfrak{gl}_\infty)$ with highest weight $n\Lambda_0$ (Proposition 4.3). Hence we may regard $\mathcal{M}$ as a crystal graph of the quantum group $U_q(\mathfrak{u}_-)$ since it can be realized as a limit of $\mathbf{B}(n\Lambda_0)$. In general, we describe a crystal graph of a generalized Verma module with arbitrary $\mathfrak{l}$-dominant highest weight.

Next, we define Demazure crystals $\mathcal{M}_w$ and $\mathcal{T}_w$ for $w \in W$ following [11], where W is the Weyl group for $\mathfrak{gl}_\infty$, and give explicit combinatorial descriptions of them (Theorems 5.4 and 5.7). As an interesting corollary, the resulting flagged RSK correspondence between $\mathcal{M}_w$ and $\mathcal{T}_w$ (Corollary 5.9) explains a nice relation between the support of a matrix in $\mathcal{M}$ and the flag conditions of the corresponding tableaux in $\mathcal{T}$, which was observed earlier in a purely combinatorial way (cf. [31]). In terms of characters, this can be summarized as follows; for each $w \in W$ we have

$$\sum_{\substack{S \subset \mathbb{N}^2 \\ w(S) \leqslant w}} \prod_{(i,j) \in S} \frac{x_i y_j}{1 - x_i y_j} = \sum_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant d(w)}} \widehat{s}_\nu(X_{\alpha(w)}) \widehat{s}_\nu(Y_{\beta(w)}),$$

where the left-hand side is the sum over supports in $\mathcal{M}$ dominated by w with respect to the Bruhat order, and the right-hand side is the sum over products of flagged Schur functions with flag conditions $\alpha(w), \beta(w)$ determined by w (see Section 5 for the precise definitions of these notations). We present variations by considering symmetric matrices in $\mathcal{M}$ as crystal graphs for affine Lie subalgebras $\mathfrak{b}_\infty$ and $\mathfrak{c}_\infty$ of $\mathfrak{gl}_\infty$.

Finally, we discuss an application to plane partitions. We show that a Demazure crystal associated with w corresponds to a set of (symmetric) plane partitions whose shapes are bounded by a partition λ corresponding to w, and obtain its trace generating function as a Demazure character.

The paper is organized as follows. In Section 2, we recall necessary background on crystal graphs. In Section 3, we define $\mathfrak{gl}_\infty$-crystal structures on $\mathcal{M}$ and $\mathcal{T}$. In Section 4, we describe a crystal graph of a generalized Verma module with arbitrary $\mathfrak{l}$-dominant highest weights including $\mathcal{M}$. In Section 5, we define and compute the Demazure crystals $\mathcal{M}_w$ and $\mathcal{T}_w$ explicitly. In Section 6, we discuss an application of Demazure crystals to plane partitions.

2. Preliminaries

2.1. Lie algebra $\mathfrak{gl}_\infty$

Let $\mathbb{Z}^\times$ denote the set of non-zero integers. Let $\mathfrak{gl}_\infty$ denote the Lie algebra of $\mathbb{Z}^\times \times \mathbb{Z}^\times$ complex matrices with finitely many non-zero entries. Let E_{ij} be the elementary matrix with 1 at the ith row and the jth column and zero elsewhere.

The Cartan subalgebra is given by $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}^\times} \mathbb{C}E_{ii}$. Denote by $\langle\cdot,\cdot\rangle$ the natural pairing on $\mathfrak{h}^* \times \mathfrak{h}$. Let $\Pi^\vee = \{h_{-i} = E_{-i-1,-i-1} - E_{-i,-i},\ h_i = E_{ii} - E_{i+1,i+1}\ (i \in \mathbb{Z}_{>0}),\ h_0 = E_{-1-1} - E_{11}\}$ be the set of simple coroots, $\Pi = \{\alpha_{-i} = \epsilon_{-i-1} - \epsilon_{-i},\ \alpha_i = \epsilon_i - \epsilon_{i+1}\ (i \in \mathbb{Z}_{>0}),\ \alpha_0 = \epsilon_{-1} - \epsilon_1\}$ the set of simple roots, and $\Delta^+ = \{\epsilon_i - \epsilon_j \mid i, j \in \mathbb{Z}^\times,\ i < j\}$ the set of positive roots, where $\epsilon_i \in \mathfrak{h}^*$ is determined by $\langle\epsilon_i, E_{jj}\rangle = \delta_{ij}$. The Dynkin diagram associated with the Cartan matrix $(\langle\alpha_j, h_i\rangle)_{i,j\in\mathbb{Z}}$ is

$$\cdots \ \text{—}\underset{\alpha_{-n}}{\bigcirc}\text{—}\ \cdots\ \text{—}\underset{\alpha_{-1}}{\bigcirc}\text{—}\underset{\alpha_{0}}{\bigcirc}\text{—}\underset{\alpha_{1}}{\bigcirc}\text{—}\ \cdots\ \text{—}\underset{\alpha_{n}}{\bigcirc}\text{—}\ \cdots.$$

Let $P = \bigoplus_{i\in\mathbb{Z}^\times} \mathbb{Z}\epsilon_i \oplus \mathbb{Z}\Lambda_0$ be the weight lattice of $\mathfrak{gl}_\infty$, where Λ_0 is given by $\langle\Lambda_0, E_{-i,-i}\rangle = -\langle\Lambda_0, E_{ii}\rangle = \frac{1}{2}$ for $i \in \mathbb{Z}_{>0}$. There is a partial ordering $\geqslant$ on P, where $\lambda \geqslant \mu$ if and only if $\lambda - \mu$ is a non-negative integral linear combination of α_i's ($i \in \mathbb{Z}$). Let $P^+ = \{\Lambda \in P \mid \Lambda(h_i) \geqslant 0,\ i \in \mathbb{Z}\}$, the set of dominant integral weights. For $i \in \mathbb{Z}^\times$, let

$$\Lambda_i = \begin{cases} \Lambda_0 - \sum_{k=i}^{-1} \epsilon_k, & \text{if } i < 0,\\ \Lambda_0 + \sum_{k=1}^{i} \epsilon_k, & \text{if } i > 0. \end{cases}$$

We call $\Lambda_i \in P^+$ ($i \in \mathbb{Z}$) the ith fundamental weight of $\mathfrak{gl}_\infty$.

Remark 2.1. A usual definition of $\mathfrak{gl}_\infty$ is given as the space of $\mathbb{Z}\times\mathbb{Z}$ matrices or sometimes $(\mathbb{Z}+\frac{1}{2}) \times (\mathbb{Z}+\frac{1}{2})$ matrices. But our different convention here allows us to describe more easily a symmetry between $\mathfrak{gl}_{<0}$ and $\mathfrak{gl}_{>0}$, and their associated crystal graphs in terms of semistandard tableaux with integral entries, which will be introduced in later sections.

2.2. Crystal graphs

Let us recall the notion of crystal graphs for the Lie algebra $\mathfrak{gl}_\infty$ (cf. [10–12]).

Definition 2.2. A $\mathfrak{gl}_\infty$-*crystal* is a set B together with the maps $\mathrm{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \to B \cup \{\mathbf{0}\}$ for $i \in \mathbb{Z}$, satisfying the following conditions:

(1) for $b \in B$, we have

$$\varphi_i(b) = \langle \mathrm{wt}(b), h_i \rangle + \varepsilon_i(b),$$

(2) if $\tilde{e}_i b \in B$ for $b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \qquad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1, \qquad \mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i,$$

(3) if $\tilde{f}_i b \in B$ for $b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \qquad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \qquad \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i,$$

(4) $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$ for $b, b' \in B$,
(5) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = \mathbf{0}$,

where $\mathbf{0}$ is a formal symbol and $-\infty$ is the smallest element in $\mathbb{Z} \cup \{-\infty\}$ such that $-\infty + n = -\infty$ for all $n \in \mathbb{Z}$.

A $\mathfrak{gl}_\infty$-crystal B becomes a colored oriented graph, where $b \xrightarrow{i} b'$ if and only if $b' = \tilde{f}_i b$ $(i \in \mathbb{Z})$, and it is called a *crystal graph for* $\mathfrak{gl}_\infty$. Let $\mathbb{C}[P]$ be the group algebra of P with basis $\{e^\lambda \mid \lambda \in P\}$. We define the *character of* B by $\mathrm{ch}\, B = \sum_{b \in B} e^{\mathrm{wt}(b)}$.

Let B_1 and B_2 be $\mathfrak{gl}_\infty$-crystals. A *morphism* $\psi : B_1 \to B_2$ is a map from $B_1 \cup \{\mathbf{0}\}$ to $B_2 \cup \{\mathbf{0}\}$ such that

(1) $\psi(\mathbf{0}) = \mathbf{0}$,
(2) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, and $\varphi_i(\psi(b)) = \varphi_i(b)$ whenever $\psi(b) \neq \mathbf{0}$,
(3) $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$ for $b \in B_1$ such that $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{e}_i b) \neq \mathbf{0}$,
(4) $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ for $b \in B_1$ such that $\psi(b) \neq \mathbf{0}$ and $\psi(\tilde{f}_i b) \neq \mathbf{0}$.

We call ψ an *embedding* and B_1 a *subcrystal of* B_2 when ψ is injective, and *strict* if $\psi : B_1 \cup \{\mathbf{0}\} \to B_2 \cup \{\mathbf{0}\}$ commutes with $\tilde{e}_i$ and $\tilde{f}_i$ $(i \in \mathbb{Z})$, where we assume that $\tilde{e}_i \mathbf{0} = \tilde{f}_i \mathbf{0} = \mathbf{0}$.

We define the *tensor product of* B_1 and B_2 to be the set $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_i \in B_i\ (i = 1, 2)\}$ with

$$\begin{aligned}
\mathrm{wt}(b_1 \otimes b_2) &= \mathrm{wt}(b_1) + \mathrm{wt}(b_2),\\
\varepsilon_i(b_1 \otimes b_2) &= \max\bigl(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \mathrm{wt}(b_1), h_i \rangle\bigr),\\
\varphi_i(b_1 \otimes b_2) &= \max\bigl(\varphi_i(b_1) + \langle \mathrm{wt}(b_2), h_i \rangle, \varphi_i(b_2)\bigr),\\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geqslant \varepsilon_i(b_2),\\ b_1 \otimes \tilde{e}_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}\\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2),\\ b_1 \otimes \tilde{f}_i b_2, & \text{if } \varphi_i(b_1) \leqslant \varepsilon_i(b_2), \end{cases}
\end{aligned}$$

for $i \in \mathbb{Z}$, where we assume that $\mathbf{0} \otimes b_2 = b_1 \otimes \mathbf{0} = \mathbf{0}$.

For $b_i \in B_i$ $(i = 1, 2)$, let $C(b_i)$ denote the connected component of b_i in B_i as a $\mathbb{Z}$-colored oriented graph. We say that b_1 *is equivalent to* b_2 if there is an isomorphism of crystals $C(b_1) \to C(b_2)$ sending b_1 to b_2.

Let $\mathbf{B}$ be a $\mathfrak{gl}_\infty$-crystal given by

$$\cdots \xrightarrow{-2} -2 \xrightarrow{-1} -1 \xrightarrow{0} 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots,$$

where $\mathrm{wt}(k) = \epsilon_k$, and $\varepsilon_i(k)$ (resp. $\varphi_i(k)$) is the number of i-colored arrows coming into k (resp. going out of k) for $k \in \mathbf{B}$.

Let $\mathfrak{gl}_{<0}$ and $\mathfrak{gl}_{>0}$ be the subalgebras of $\mathfrak{gl}_\infty$ spanned by $\{E_{ij} \mid i, j \in \mathbb{Z}_{<0}\}$ and $\{E_{ij} \mid i, j \in \mathbb{Z}_{>0}\}$, respectively. We can define $\mathfrak{gl}_{<0}$-crystals (resp. $\mathfrak{gl}_{>0}$-crystals) as in Definition 2.2 with respect to $\tilde{e}_i$'s and $\tilde{f}_i$'s for $i \in \mathbb{Z}_{<0}$ (resp. $i \in \mathbb{Z}_{>0}$), and view $\mathbf{B}_{<0} = \{k \in \mathbf{B} \mid k < 0\}$ (resp. $\mathbf{B}_{>0} = \{k \in \mathbf{B} \mid k > 0\}$) as a $\mathfrak{gl}_{<0}$-crystal (resp. $\mathfrak{gl}_{>0}$-crystal). In addition, let us consider a $\mathfrak{gl}_{<0}$-crystal $\mathbf{B}^\vee_{<0}$ given as follows:

$$-1^\vee \xrightarrow{-1} -2^\vee \xrightarrow{-2} -3^\vee \xrightarrow{-3} \cdots,$$

where $\mathrm{wt}(-k^\vee) = -\epsilon_{-k}$ for $k > 0$. Note that $\mathbf{B}^\vee_{<0}$ is the dual crystal of $\mathbf{B}_{<0}$ (cf. [12]).

For $\lambda \in P$, let $T_\lambda = \{t_\lambda\}$ be a $\mathfrak{gl}_\infty$-crystal with $\mathrm{wt}(t_\lambda) = \lambda$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = \mathbf{0}$, and $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$ for $i \in \mathbb{Z}$.

2.3. Semistandard tableaux

Let $\mathscr{P}$ be the set of partitions. We identify a partition $\lambda = (\lambda_i)_{i \geqslant 1}$ with a *Young diagram* or a subset $\{(i, j) \mid 1 \leqslant j \leqslant \lambda_i\}$ of $\mathbb{N} \times \mathbb{N}$ [25]. The number of non-zero parts of λ is denoted by $\ell(\lambda)$ and called the *length of* λ. For $\mu \in \mathscr{P}$ with $\mu \subset \lambda$, λ/μ denotes the *skew Young diagram*, which is given by $\lambda - \mu$ as a subset in $\mathbb{N} \times \mathbb{N}$, and $|\lambda/\mu|$ the number of boxes in the diagram. We denote by $\lambda' = (\lambda'_i)_{i \geqslant 1}$ the *conjugate of* λ, and λ^π the skew Young diagram obtained by 180°-rotation of λ, which is called an *anti-normal* shape.

Let $\mathcal{A}$ be a linearly ordered set. For a skew Young diagram λ/μ, we call a tableau T obtained by filling λ/μ with entries in $\mathcal{A}$ a *semistandard tableau of shape* λ/μ if the entries in each row are weakly increasing from left to right, and the entries in each column are strictly increasing from top to bottom, and write $\mathrm{sh}(T) = \lambda/\mu$. We denote by $SST_{\mathcal{A}}(\lambda/\mu)$ the set of all semistandard tableaux of shape λ/μ with entries in $\mathcal{A}$. Let $\mathcal{W}_{\mathcal{A}}$ be the set of finite words in $\mathcal{A}$. We associate to each $T \in SST_{\mathcal{A}}(\lambda/\mu)$ a word $w(T) \in \mathcal{W}_{\mathcal{A}}$ which is obtained by reading the entries of T row by row from top to bottom, and from right to left in each row.

Suppose that $\mathcal{A} = \mathbf{B}$, $\mathbf{B}_{>0}$, $\mathbf{B}_{<0}$, and $\mathbf{B}^\vee_{<0}$, with the linear ordering $<$ induced from the partial ordering on P. Then $\mathcal{W}_{\mathbf{B}}$ is a $\mathfrak{gl}_\infty$-crystal since we may view each non-empty finite word $w = w_1 \cdots w_r$ as $w_1 \otimes \cdots \otimes w_r \in \mathbf{B}^{\otimes r}$. Similarly, $\mathcal{W}_{\mathbf{B}_{>0}}$ (resp. $\mathcal{W}_{\mathbf{B}_{<0}}$ or $\mathcal{W}_{\mathbf{B}^\vee_{<0}}$) becomes a $\mathfrak{gl}_{>0}$-crystal (resp. $\mathfrak{gl}_{<0}$-crystal). Sending T to $w(T)$ gives an injective map from $SST_{\mathcal{A}}(\lambda/\mu)$ to $\mathcal{W}_{\mathcal{A}}$, and the image of $SST_{\mathcal{A}}(\lambda/\mu)$ together with $\{\mathbf{0}\}$ is stable under the operators $\tilde{e}_i$, $\tilde{f}_i$. Hence it is a crystal graph [14].

Suppose that $\lambda/\mu = \tau$ or τ^π for some $\tau \in \mathscr{P}$. Then $SST_{\mathcal{A}}(\lambda/\mu)$ is connected. In particular, if $\mathcal{A} = \mathbf{B}_{>0}$ (resp. $\mathbf{B}^\vee_{<0}$), then $SST_{\mathcal{A}}(\lambda/\mu)$ contains a highest weight element $H_{\lambda/\mu}$, where in each column of $H_{\lambda/\mu}$, the lth entry from the top position is filled with l (resp. $-l^\vee$). Note that any $T \in SST_{\mathcal{A}}(\lambda/\mu)$ is obtained by applying finitely many $\tilde{f}_i$'s to $H_{\lambda/\mu}$.

3. $\mathfrak{gl}_\infty$-Crystals and the RSK correspondence

In this section, we define $\mathfrak{gl}_\infty$-crystal structures on the set $\mathcal{M}$ of matrices with non-negative integral entries of finite support and the set $\mathcal{T}$ of pairs of semistandard tableaux of the same shape. We show that the RSK correspondence, which is a bijection from $\mathcal{M}$ to $\mathcal{T}$, is an isomorphism of $\mathfrak{gl}_\infty$-crystals.

Since it is already known that the RSK correspondence is a morphism of $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystals [4, 20], our main result in this section is to extend it as a $\mathfrak{gl}_\infty$-crystal morphism by defining the missing operators $\tilde{e}_0$, $\tilde{f}_0$ on $\mathcal{M}$ and $\mathcal{T}$, which are compatible with the RSK algorithm.

3.1. Crystal of integral matrices

Let

$$\begin{aligned} \Omega = \big\{ (\mathbf{i}, \mathbf{j}) \in \mathcal{W}_{\mathbf{B}^\vee_{<0}} \times \mathcal{W}_{\mathbf{B}_{>0}} \ \big| \\ &(1)\ \mathbf{i} = i_1 \cdots i_r \text{ and } \mathbf{j} = j_1 \cdots j_r \text{ for some } r \geqslant 0, \\ &(2)\ (i_1, j_1) \leqslant \cdots \leqslant (i_r, j_r) \big\}, \end{aligned} \tag{3.1}$$

where for (i, j) and $(k, l) \in \mathbf{B}^\vee_{<0} \times \mathbf{B}_{>0}$,

$$(i, j) < (k, l) \quad \Leftrightarrow \quad \begin{cases} j < l, & \text{or} \\ j = l \text{ and } i > k. \end{cases}$$

Similarly, let Ω' be the set of pairs $(\mathbf{k}, \mathbf{l}) \in \mathcal{W}_{\mathbf{B}^\vee_{<0}} \times \mathcal{W}_{\mathbf{B}_{>0}}$ such that (1) $\mathbf{k} = k_1 \cdots k_r$ and $\mathbf{l} = l_1 \cdots l_r$ for some $r \geqslant 0$ and (2) $(k_1, l_1) \leqslant' \cdots \leqslant' (k_r, l_r)$, where

$$(i,j) <' (k,l) \quad \Leftrightarrow \quad \begin{cases} i<k, & \text{or} \\ i=k \text{ and } j>l. \end{cases}$$

Then Ω is a $\mathfrak{gl}_{<0}$-crystal, where $\tilde{x}_i(\mathbf{i},\mathbf{j}) = (\tilde{x}_i\mathbf{i},\mathbf{j})$ for $(\mathbf{i},\mathbf{j}) \in \Omega$, $x=e,f$ and $i \in \mathbb{Z}_{<0}$. Here, we assume that $\tilde{x}_i(\mathbf{i},\mathbf{j}) = \mathbf{0}$ if $\tilde{x}_i\mathbf{i} = \mathbf{0}$. Similarly, Ω' is a $\mathfrak{gl}_{>0}$-crystal, where $\tilde{x}_j(\mathbf{k},\mathbf{l}) = (\mathbf{k},\tilde{x}_j\mathbf{l})$ for $(\mathbf{k},\mathbf{l}) \in \Omega'$, $x=e,f$ and $j \in \mathbb{Z}_{>0}$.

Consider the following set of $\mathbb{N}\times\mathbb{N}$ matrices with non-negative integers of finite support;

$$\mathcal{M} = \left\{ A = (a_{-i^\vee,j})_{i,j\geqslant 1} \,\middle|\, a_{-i^\vee,j} \in \mathbb{Z}_{\geqslant 0},\ \sum_{i,j\geqslant 1} a_{-i^\vee,j} < \infty \right\}. \tag{3.2}$$

For $(\mathbf{i},\mathbf{j}) \in \Omega$, define $A(\mathbf{i},\mathbf{j}) = (a_{-i^\vee,j})$ to be the matrix in $\mathcal{M}$, where $a_{-i^\vee,j}$ is the number of k's such that $(i_k, j_k) = (-i^\vee, j)$. Then the map $(\mathbf{i},\mathbf{j}) \mapsto A(\mathbf{i},\mathbf{j})$ is a bijection between Ω and $\mathcal{M}$, where the pair of empty words $(\emptyset,\emptyset)$ corresponds to zero matrix, say $\mathbb{O}$. Similarly, we have a bijection $(\mathbf{k},\mathbf{l}) \mapsto A(\mathbf{k},\mathbf{l})$ from Ω' to $\mathcal{M}$.

With these bijections, $\mathcal{M}$ becomes a crystal graph for both $\mathfrak{gl}_{<0}$ and $\mathfrak{gl}_{>0}$. Moreover, the operators $\tilde{e}_i, \tilde{f}_i$ on $\mathcal{M}\cup\{\mathbf{0}\}$ commute with $\tilde{e}_j, \tilde{f}_j$ for $i \in \mathbb{Z}_{<0}$ and $j \in \mathbb{Z}_{>0}$, where we assume $\tilde{x}_i\mathbf{0} = \mathbf{0}$ for $x=e,f$ and $i \in \mathbb{Z}$. Hence $\mathcal{M}$ becomes a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal (cf. [4,20]).

Now, for $A = (a_{-i^\vee,j}) \in \mathcal{M}$, we define

$$\tilde{e}_0 A = \begin{cases} A - E_{-1^\vee,1}, & \text{if } a_{-1^\vee,1} \neq 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\tilde{f}_0 A = A + E_{-1^\vee,1},$$

where $E_{-1^\vee,1} \in \mathcal{M}$ denotes the elementary matrix with 1 at the position $(-1^\vee,1)$ and 0 elsewhere. Put $\mathrm{wt}(A) = \sum_{i,j>0} a_{-i^\vee j}(-\epsilon_{-i}+\epsilon_j)$, $\varepsilon_0(A) = a_{-1^\vee,1}$, and $\varphi_0(A) = \langle \mathrm{wt}(A), h_0\rangle + \varepsilon_0(A)$. Then we have the following.

Proposition 3.1. *$\mathcal{M}$ is a $\mathfrak{gl}_\infty$-crystal, and*

$$\mathcal{M} = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}\mathbb{O} \mid r\geqslant 0,\ i_1,\ldots,i_r \in \mathbb{Z}\} \setminus \{\mathbf{0}\}.$$

In particular, $\mathcal{M}$ is connected with a unique highest weight element $\mathbb{O}$.

Proof. It is easy to see that $\mathcal{M}$ is a $\mathfrak{gl}_\infty$-crystal. Let $A \in \mathcal{M}$ be given. We claim that $A = \tilde{f}_{i_1}\cdots\tilde{f}_{i_r}\mathbb{O}$ for some $r\geqslant 0$ and $i_1,\ldots,i_r \in \mathbb{Z}$. We use induction on $s(A) = \sum_{i,j} a_{-i^\vee,j}$. If $s(A)=0$, then it is clear. Let $s(A)$ be positive. First, A is connected to a diagonal matrix $A^\circ = (a^\circ_{-i^\vee,j})$ such that $a^\circ_{-1^\vee,1} \geqslant a^\circ_{-2^\vee,2} \geqslant a^\circ_{-3^\vee,3} \geqslant \cdots$ since $\mathcal{M}$ is a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal (cf. [4,17]). That is, $\tilde{e}_{j_1}\cdots\tilde{e}_{j_r}A = A^\circ$ for some $r\geqslant 0$ and $j_1,\ldots,j_r \in \mathbb{Z}^\times$ and $\tilde{e}_i A^\circ = \mathbf{0}$ for all $i \in \mathbb{Z}^\times$. If $a^\circ_{-1^\vee,1} = 0$, then $A^\circ = \mathbb{O}$. If not, then $\tilde{e}_0 A^\circ \neq \mathbf{0}$ and $s(A^\circ) = s(A) - 1$. Hence, the proof completes by induction hypothesis. $\square$

3.2. *Crystal of bitableaux*

By the RSK algorithm, each $A \in \mathcal{M}$ is in one-to-one correspondence with $(P(A), Q(A))$ in $SST_{\mathbf{B}^\vee_{<0}}(\lambda) \times SST_{\mathbf{B}_{>0}}(\lambda)$ for some $\lambda \in \mathscr{P}$ [15]. In this paper, we need a variation of this correspondence with anti-normal shaped tableaux. Let us describe it in the following.

Let $\nu \in \mathscr{P}$ and $T \in SST_{\mathcal{A}}(\nu^\pi)$ be given. For $a \in \mathcal{A}$, we define $T \leftarrow a$ to be the tableau of an anti-normal shape obtained from T by applying the following procedure: (1) let a' be the largest entry in the right-most column which is smaller than or equal to a, (2) replace a' by a. If there is no such a', put a at the top of the column and stop the procedure, (3) repeat (1) and (2) on the next column with a'. For $w = w_1 \cdots w_r \in \mathcal{W}_{\mathcal{A}}$, we define $\mathbf{P}(w)$ to be $(\cdots((w_1 \leftarrow w_2) \leftarrow w_3)\cdots) \leftarrow w_r$. Note that $w_{\mathrm{rev}} = w_r \cdots w_1$ is equivalent to $\mathbf{P}(w)$ as elements of crystals.

Let $A \in \mathcal{M}$ be given with $A = A(\mathbf{i},\mathbf{j}) = A(\mathbf{k},\mathbf{l})$ for $(\mathbf{i},\mathbf{j}) \in \Omega$ and $(\mathbf{k},\mathbf{l}) \in \Omega'$. Let $\mathbf{i}_{\mathrm{rev}}$ and $\mathbf{l}_{\mathrm{rev}}$ be the reverse words of $\mathbf{i}$ and $\mathbf{l}$, respectively. We define

$$\mathbf{P}(A) = \mathbf{P}(\mathbf{i}_{\mathrm{rev}}), \qquad \mathbf{Q}(A) = \mathbf{P}(\mathbf{l}_{\mathrm{rev}}).$$

Let $\mathbf{i} = i_1 \cdots i_r$. For $1 \leqslant k \leqslant r$, let us fill a box in $\mathrm{sh}\,\mathbf{P}(A)$ with c if it is created when i_k is inserted into $((i_r \leftarrow i_{r-1})\cdots) \leftarrow i_{k+1}$ and $j_k = c$. Then we have a tableau $Q \in SST_{\mathbf{B}_{>0}}(\nu^\pi)$ with $\nu^\pi = \mathrm{sh}\,\mathbf{P}(\mathbf{i}_{\mathrm{rev}})$ for some $\nu \in \mathscr{P}$. By the symmetry of the RSK correspondence, we have $Q = \mathbf{P}(\mathbf{l}_{\mathrm{rev}}) = \mathbf{Q}(A)$. Put

$$\mathcal{T} = \bigsqcup_{\nu \in \mathscr{P}} SST_{\mathbf{B}^\vee_{<0}}\left(\nu^\pi\right) \times SST_{\mathbf{B}_{>0}}\left(\nu^\pi\right). \tag{3.3}$$

Hence we have a bijection

$$\kappa : \mathcal{M} \to \mathcal{T}, \tag{3.4}$$

where $\kappa(A) = (\mathbf{P}(A), \mathbf{Q}(A))$.

Example 3.2. Let

$$A = (a_{-i^\vee, j})_{1 \leqslant i, j \leqslant 3} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \in \mathcal{M},$$

where we assume that $a_{-i^\vee, j} = 0$ unless $1 \leqslant i, j \leqslant 3$. Then

$$\mathbf{i} = -2^\vee - 2^\vee - 1^\vee - 3^\vee - 3^\vee - 2^\vee - 1^\vee, \qquad \mathbf{l} = 3\,1\,2\,1\,1\,2\,2,$$

and

$$\mathbf{P}(A) = \begin{array}{cccc} & -1^\vee & -2^\vee & -2^\vee \\ -1^\vee & -2^\vee & -3^\vee & -3^\vee \end{array}, \qquad \mathbf{Q}(A) = \begin{array}{cccc} & 1 & 1 & 1 \\ 2 & 2 & 2 & 3 \end{array}.$$

Clearly $\mathcal{T}$ is a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal, and for $A \in \mathcal{M}$, $\mathbf{P}(A)$ (resp. $\mathbf{Q}(A)$) is equivalent to A as elements of $\mathfrak{gl}_{<0}$ (resp. $\mathfrak{gl}_{>0}$)-crystals. Summarizing, we have the following.

Proposition 3.3. *κ is a $(\mathfrak{gl}_{<0}, \mathfrak{gl}_{>0})$-bicrystal isomorphism.*

Now, let us describe a $\mathfrak{gl}_\infty$-crystal structure on $\mathcal{T}$. Suppose that $(S, T) \in \mathcal{T}$ is given. For each kth column of ν enumerated from the right, let s_k and t_k be the smallest entries in the kth column of S and T, respectively. We assign

$$\sigma_k = \begin{cases} +, & \text{if the } k\text{th column is empty}, \\ +, & \text{if } s_k > -1^\vee \text{ and } t_k > 1, \\ -, & \text{if } s_k = -1^\vee \text{ and } t_k = 1, \\ \cdot, & \text{otherwise}. \end{cases} \tag{3.5}$$

In the sequence $\sigma = (\ldots, \sigma_2, \sigma_1)$, we replace a pair $(\sigma_{s'}, \sigma_s) = (-, +)$, where $s' > s$ and $\sigma_t = \cdot$ for $s < t < s'$, by $(\cdot, \cdot)$, and repeat this process as far as possible until we get a sequence with no $-$ placed to the left of $+$. We call this sequence the 0-*signature* of (S, T).

We call the left-most $-$ in the 0-signature of (S, T) the 0-*good* $-$ *sign*, and define $\tilde{e}_0(S, T)$ to be the bitableaux obtained from (S, T) by removing $-1^\vee$ and 1 in the columns corresponding to the 0-good $-$ sign. If there is no 0-good $-$ sign, then we define $\tilde{e}_0(S, T) = \mathbf{0}$. We call the right-most $+$ in the 0-signature of (S, T) the 0-*good* $+$ *sign*, and define $\tilde{f}_0(S, T)$ to be the bitableaux obtained from (S, T) by adding $-1^\vee$ and 1 on top of the columns corresponding to the 0-good $+$ sign. If there is no 0-good $+$ sign, then we define $\tilde{f}_0(S, T) = \mathbf{0}$.

Example 3.4. Let $(S, T) \in \mathcal{T}$ be given with σ as follows.

$$S = \begin{matrix} & & -1^\vee & -3^\vee \\ -1^\vee & -2^\vee & -4^\vee & -4^\vee \end{matrix}, \qquad T = \begin{matrix} & & 1 & 3 \\ 1 & 1 & 3 & 4 \end{matrix},$$
$$\sigma = (\ldots, \sigma_6, \sigma_5, \sigma_4, \sigma_3, \sigma_2, \sigma_1) = (\ldots, +, +, -, \cdot, -, +).$$

Then the 0-good $-$ sign is σ_4, and 0-good $+$ sign is σ_5. Hence,

$$\tilde{e}_0(S, T) = \left(\begin{matrix} & -1^\vee & -3^\vee \\ -2^\vee & -4^\vee & -4^\vee \end{matrix}, \begin{matrix} & 1 & 3 \\ 1 & 3 & 4 \end{matrix} \right),$$
$$\tilde{f}_0(S, T) = \left(\begin{matrix} & & & -1^\vee & -3^\vee \\ -1^\vee & -1^\vee & -2^\vee & -4^\vee & -4^\vee \end{matrix}, \begin{matrix} & & & 1 & 3 \\ 1 & 1 & 1 & 3 & 4 \end{matrix} \right).$$

Proposition 3.5. *$\mathcal{T}$ is a $\mathfrak{gl}_\infty$-crystal, and*

$$\mathcal{T} = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(\emptyset, \emptyset) \;\middle|\; r \geqslant 0,\ i_1, \ldots, i_r \in \mathbb{Z} \right\} \setminus \{\mathbf{0}\}.$$

In particular, $\mathcal{T}$ is connected with a unique highest weight element $(\emptyset, \emptyset)$.

Proof. It is straightforward to check that $\tilde{e}_0(S, T), \tilde{f}_0(S, T) \in \mathcal{T} \cup \{\mathbf{0}\}$. Let (S, T) be given with $\mathrm{sh}(S) = \mathrm{sh}(T)$ non-empty. We assume that $\tilde{e}_i(S, T) = \mathbf{0}$ for all $i \in \mathbb{Z}^\times$. Then S (resp. T) is a highest weight element of a $\mathfrak{gl}_{<0}$-crystal (resp. $\mathfrak{gl}_{>0}$-crystal), where in each column of S (resp. T), the lth entry from the top position is filled with $-l^\vee$ (resp. l). Hence $\tilde{e}_0(S, T) \neq \mathbf{0}$. If we use induction on $|\mathrm{sh}(S)|$, then we conclude that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r}(\emptyset, \emptyset) = (S, T)$ for some $r \geqslant 0$ and $i_1, \ldots, i_r \in \mathbb{Z}$. □

3.3. Isomorphism

Now we are in a position to state the main result in this section.

Theorem 3.6. *The map $\kappa : \mathcal{M} \to \mathcal{T}$ is a $\mathfrak{gl}_\infty$-crystal isomorphism.*

Proof. By Proposition 3.3, it suffices to show that κ commutes with $\tilde{e}_0$ and $\tilde{f}_0$. More precisely, we claim that for $A \in \mathcal{M}$ and $k \geqslant 1$,

$$\kappa\left(\tilde{f}_0^k A\right) = \kappa(kE_{-1^\vee,1} + A) = \tilde{f}_0^k \kappa(A). \tag{3.6}$$

We use induction on $t(A) = \sum_{k \geqslant 1} a_{-k^\vee,1}$. We may assume that $a_{-1^\vee,1} = 0$.

If $t(A) = 0$, then it is not difficult to see that (3.6) holds. Suppose that $t(A) > 0$. Let $-p^\vee$ be the largest one such that $a_{-p^\vee,1} \neq 0$. Let $B = A - E_{-p^\vee,1}$. By induction hypothesis, we have for $k \geqslant 1$

$$\kappa\left(\tilde{f}_0^k B\right) = \kappa(kE_{-1^\vee,1} + B) = \tilde{f}_0^k \kappa(B).$$

For $k \geqslant 1$, let $\kappa(kE_{-1^\vee,1} + X) = (\mathbf{P}^{(k)}(X), \mathbf{Q}^{(k)}(X))$ with $X = A, B$. Then $\mathbf{P}^{(k)}(A) = \mathbf{P}^{(k)}(B) \leftarrow -p^\vee$ by definition of $\mathbf{P}(\cdot)$ and $\mathbf{Q}^{(k)}(A)$ is obtained from $\mathbf{Q}^{(k)}(B)$ by filling the corresponding box, say c, in $\mathrm{sh}(\mathbf{P}^{(k)}(A))/\mathrm{sh}(\mathbf{P}^{(k)}(B))$ with 1. For notational convenience, let us write $\kappa(kE_{-1^\vee,1} + A) = \kappa(kE_{-1^\vee,1} + B) \leftarrow (-p^\vee, 1)$. Let

$$\sigma = (\ldots, \sigma_2, \sigma_1), \qquad \sigma' = \left(\ldots, \sigma'_2, \sigma'_1\right)$$

be the sequences of signs associated with $\kappa(kE_{-1^\vee,1} + B)$ and $\kappa(kE_{-1^\vee,1} + A)$, respectively (see (3.5)), and let

$$\tilde{\sigma} = (\ldots, \tilde{\sigma}_2, \tilde{\sigma}_1), \qquad \tilde{\sigma}' = \left(\ldots, \tilde{\sigma}'_2, \tilde{\sigma}'_1\right)$$

be the 0-signatures of $\kappa(kE_{-1^\vee,1} + B)$ and $\kappa(kE_{-1^\vee,1} + A)$, respectively.

Suppose that by the insertion of $-p^\vee$ into $\mathbf{P}^{(k)}(B)$, c is filled with $-q^\vee$ for some $q \leqslant p$, and it is located at the tth column enumerated from the rightmost one.

Case 1. $q > 1$. Let $kE_{-1^\vee,1} + B = A(\mathbf{i}, \mathbf{j})$ with $(\mathbf{i}, \mathbf{j}) \in \Omega$. Consider the horizontal strip made by inserting the subwords of $\mathbf{i}_{\mathrm{rev}}$ corresponding to the first column of $kE_{-1^\vee,1} + B$.

Then we observe the following facts.

(1) No $-1^\vee$ has been bumped out in the bumping path for $\mathbf{P}^{(k)}(B) \leftarrow -p^\vee$.
(2) By induction hypothesis all $-1^\vee$'s which have been added on $\mathbf{P}(B)$ by applying $\tilde{f}_0^k$ to $\kappa(B)$ are placed to the right of $-q^\vee$ in the tth column, and they do not intersect with the bumping path for $\mathbf{P}^{(k)}(B) \leftarrow -p^\vee$.
(3) The insertion of $-p^\vee$ into $\mathbf{P}^{(k)}(B)$ does not change the sign σ_k for $1 \leqslant k \leqslant t-1$, and hence $\sigma_k = \sigma'_k$ for $1 \leqslant k \leqslant t-1$.

Hence we have

$$\tilde{e}_0^k\left[\kappa(kE_{-1^\vee,1} + B) \leftarrow \left(-p^\vee, 1\right)\right] = \kappa(B) \leftarrow \left(-p^\vee, 1\right) = \kappa(A)$$

and

$$\tilde{f}_0^k \kappa(A) = \kappa(kE_{-1^\vee,1} + B) \leftarrow \left(-p^\vee, 1\right) = \kappa(kE_{-1^\vee,1} + A).$$

Case 2. $q = 1$. Consider the bumping path for $\mathbf{P}^{(k)}(B) \leftarrow -p^\vee$. Then there exists $1 \leqslant s \leqslant t$ such that

(1) $-x^\vee$ ($x \geqslant 2$) has been bumped out from the $(k-1)$th column and placed at the kth column for $2 \leqslant k \leqslant s$,
(2) $-1^\vee$ has been bumped out from the $(k-1)$th column and placed at the kth column for $s+1 \leqslant k \leqslant t$.

As in Case 1, it follows that all $-1^\vee$'s which have been added to $\mathbf{P}(B)$ by applying $\tilde{f}_0^k$ to $\kappa(B)$ are placed to the right of the tth column, and $\sigma_r = \sigma'_r$ for $1 \leqslant r \leqslant s$.

Since all $-1^\vee$'s in the rth column of $\mathbf{P}^{(k)}(B)$ for $s \leqslant r \leqslant t-1$ have been shifted to the left by one column by the insertion of $-p^\vee$ to $\mathbf{P}^{(k)}(B)$, we have $\sigma_r = \sigma'_r$ for $s+1 \leqslant r \leqslant t-1$. Note that $\sigma_t = +$ and $\sigma'_t = -$.

Let u be the top entry of the sth column in $\mathbf{Q}^{(k)}(B)$. If $u = 1$, then we have $\sigma_s = -$ and $\sigma'_s = \cdot$. If $u > 1$, then we have $\sigma_s = \cdot$ and $\sigma'_s = +$. Now, comparing σ and σ' (hence $\tilde{\sigma}$ and $\tilde{\sigma}'$), it is not difficult to see that

$$\tilde{e}_0^k\kappa(kE_{-1^\vee,1}+A)=\tilde{e}_0^k\big[\kappa(kE_{-1^\vee,1}+B)\leftarrow\big(-p^\vee,1\big)\big]=\kappa(B)\leftarrow\big(-p^\vee,1\big)=\kappa(A).$$

This completes the proof. □

Example 3.7. Consider

$$\kappa\begin{pmatrix}1&0&1&0\\0&0&0&1\\1&0&0&0\\1&0&1&0\end{pmatrix}=\left(\begin{matrix}&&-1^\vee&-3^\vee\\-1^\vee&-2^\vee&-4^\vee&-4^\vee\end{matrix},\begin{matrix}&&1&3\\1&1&3&4\end{matrix}\right).$$

Applying $\tilde{e}_0$ and $\tilde{f}_0$ on both sides, we get

$$\kappa\begin{pmatrix}\mathbf{0}&0&1&0\\0&0&0&1\\1&0&0&0\\1&0&1&0\end{pmatrix}=\left(\begin{matrix}&-1^\vee&-3^\vee\\-2^\vee&-4^\vee&-4^\vee\end{matrix},\begin{matrix}&1&3\\1&3&4\end{matrix}\right)$$

and

$$\kappa\begin{pmatrix}\mathbf{2}&0&1&0\\0&0&0&1\\1&0&0&0\\1&0&1&0\end{pmatrix}=\left(\begin{matrix}&&&-1^\vee&-3^\vee\\-1^\vee&-1^\vee&-2^\vee&-4^\vee&-4^\vee\end{matrix},\begin{matrix}&&&1&3\\1&1&1&3&4\end{matrix}\right),$$

respectively (see Example 3.4).

4. Crystal graphs of generalized Verma modules

Let $\mathfrak{u}_\pm$ be the subalgebra of $\mathfrak{gl}_\infty$ spanned by E_{ij} for $i<0$, $j>0$ (resp. $i>0$, $j<0$). Let $\mathfrak{p}=\mathfrak{gl}_{<0}\oplus\mathfrak{gl}_{>0}\oplus\mathfrak{u}_+$ be a maximal parabolic subalgebra. Then we have $\mathfrak{gl}_\infty=\mathfrak{u}_-\oplus\mathfrak{p}$. The set of roots for the nilradical $\mathfrak{u}_-$ is given by $\Delta(\mathfrak{u}_-)=\{-\epsilon_i+\epsilon_j \mid i>0,\ j<0\}$. Let $U(\mathfrak{u}_-)$ be the universal enveloping algebra of $\mathfrak{u}_-$. By PBW theorem, $U(\mathfrak{u}_-)$ has a basis parameterized by $\mathcal{M}$.

In this section, we prove that the $\mathfrak{gl}_\infty$-crystal $\mathcal{M}$ is a crystal graph of the generalized Verma module $U(\mathfrak{u}_-)$ in the sense that it is the limit of the crystal graph of the irreducible highest weight $\mathfrak{gl}_\infty$-module with highest weight $n\Lambda_0$ as $n\to\infty$.

4.1. Crystal $\mathbf{B}(n\Lambda_0)$

Let $\mathcal{F}$ be the set of *semi-infinite words*

$$w=\cdots w_{-3}w_{-2}w_{-1}$$

with letters in $\mathbf{B}$ such that

(1) $w_{i-1}<w_i$ for all $i<0$,
(2) $w_{i-1}=w_i-1$ for all $i\ll 0$.

$$\mathbf{w}=\begin{array}{cccc} w^{(4)} & w^{(3)} & w^{(2)} & w^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ -4 & -4 & -4 & -2 \\ -3 & -2 & 1 & 2 \\ -2 & -1 & 3 & 3 \\ -1 & 1 & & \\ 1 & 2 & & \\ 2 & 3 & & \\ 3 & & & \\ 5 & & & \end{array}, \qquad H_{\Lambda_\lambda}=\begin{array}{cccc} w^{(4)} & w^{(3)} & w^{(2)} & w^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ -4 & -4 & -4 & -4 \\ -3 & -3 & -3 & -3 \\ -2 & -2 & -2 & -2 \\ -1 & -1 & & \\ 1 & 1 & & \\ 2 & 2 & & \\ 3 & & & \\ 4 & & & \end{array}.$$

Fig. 1. Semi-infinite semistandard tableaux of shape $\lambda=(4,2,-1,-1)$.

For $w\in\mathcal{F}$, we define $\mathrm{wt}(w)=\Lambda_0+\sum_{k\in\mathbf{B}}m_k\epsilon_k\in P$, where $m_k=|\{i\mid w_i=k\}|-\delta_{-k,|k|}$. It is well defined since $m_k=0$ for almost all $k\in\mathbf{B}$. For each $i\in\mathbb{Z}$, we define the operators $\tilde{e}_i,\tilde{f}_i:\mathcal{F}\to\mathcal{F}\cup\{\mathbf{0}\}$ by the same way as we do on $\mathcal{W}_{\mathbf{B}}$. Then $\tilde{e}_i$ and $\tilde{f}_i$ are well defined, and $\mathcal{F}$ is a $\mathfrak{gl}_\infty$-crystal, where $\varepsilon_i(w)$ (resp. $\varphi_i(w)$) is the number of i-colored arrows coming into w (resp. going out of w) for $w\in\mathcal{F}$. For $i\leqslant 0$, let $H_{\Lambda_i}=\cdots\ i-3\ i-2\ i-1$, and for $i>0$, let $H_{\Lambda_i}=\cdots\ -2\ -1\ 1\ \cdots\ i-1\ i$. We have the following decomposition as a $\mathfrak{gl}_\infty$-crystal

$$\mathcal{F}=\bigsqcup_{i\in\mathbb{Z}}\mathbf{B}(\Lambda_i),$$

where $\mathbf{B}(\Lambda_i)$ is the connected component of H_{Λ_i} with $\mathrm{wt}(H_{\Lambda_i})=\Lambda_i$. Recall that $\mathcal{F}$ is the crystal graph of the Fock space representation, which can be realized as the space of semi-infinite wedge vectors, and $\mathbf{B}(\Lambda_i)$ is the crystal graph of the irreducible highest weight $\mathfrak{gl}_\infty$-module with highest weight Λ_i [27].

Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ be a sequence of non-increasing n integers, called a *generalized partition of length* n. We call an n-tuple of semi-infinite words $\mathbf{w}=(w^{(1)},\ldots,w^{(n)})$ a *semi-infinite semistandard tableau of shape* λ if

(1) $w^{(i)}=\cdots w^{(i)}_{-3}w^{(i)}_{-2}w^{(i)}_{-1}\in\mathbf{B}(\Lambda_{\lambda_{n-i+1}})$ for $1\leqslant i\leqslant n$,
(2) $w^{(i+1)}_{k+d_i}\leqslant w^{(i)}_k$ for $1\leqslant i<n$ and $k<0$, where $d_i=\lambda_{n-i+1}-\lambda_{n-i}$.

We may identify $\mathbf{w}$ with a semistandard tableau with infinitely many rows and n columns, where each row of $\mathbf{w}$ reads (from left to right) as follows:

$$w^{(n)}_{k+d_1+\cdots+d_{n-1}}\leqslant\cdots\leqslant w^{(3)}_{k+d_1+d_2}\leqslant w^{(2)}_{k+d_1}\leqslant w^{(1)}_k\quad(k\in\mathbb{Z}).$$

Here we assume that $w^{(i)}_k$ is empty if there is no corresponding entry (see Fig. 1).

Let $\Lambda_\lambda=\sum_{k=1}^n\Lambda_{\lambda_k}\in P^+$ and let $\mathbf{B}(\Lambda_\lambda)$ be the set of all semi-infinite semistandard tableaux of shape λ. We may assume that $\mathbf{w}=w^{(1)}\otimes\cdots\otimes w^{(n)}\in\mathbf{B}(\Lambda_{\lambda_n})\otimes\cdots\otimes\mathbf{B}(\Lambda_{\lambda_1})$. By similar arguments as in the case of usual semistandard tableaux [14], we can check the following.

Proposition 4.1. $\mathbf{B}(\Lambda_\lambda)$ *together with* $\mathbf{0}$ *is stable under* $\tilde{e}_i$ *and* $\tilde{f}_i$ $(i\in\mathbb{Z})$, *and*

$$\mathbf{B}(\Lambda_\lambda)=\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_r}H_{\Lambda_\lambda}\mid r\geqslant 0,\ i_1,\ldots,i_r\in\mathbb{Z}\}\setminus\{\mathbf{0}\},$$

where $H_{\Lambda_\lambda}=H_{\Lambda_{\lambda_n}}\otimes\cdots\otimes H_{\Lambda_{\lambda_1}}$.

Hence $\mathbf{B}(\Lambda_\lambda)$ is a $\mathfrak{gl}_\infty$-crystal, and it is isomorphic to the crystal graph of the irreducible highest weight $\mathfrak{gl}_\infty$-module with highest weight Λ_λ since it is the connected component in $\mathbf{B}(\Lambda_{\lambda_n}) \otimes \cdots \otimes \mathbf{B}(\Lambda_{\lambda_1})$ including H_{Λ_λ} of integral dominant weight.

Now, let us consider $\mathbf{B}(n\Lambda_0)$ for $n \in \mathbb{N}$. Given $\mathbf{w} = (w^{(1)}, \ldots, w^{(n)}) \in \mathbf{B}(n\Lambda_0)$, let $\mathbf{w}_{>0}$ and $\mathbf{w}_{<0}$ be the subtableaux of $\mathbf{w}$ consisting of positive and negative entries, respectively. Note that $\mathbf{w}_{>0} \in SST_{\mathbf{B}_{>0}}(\mu^\pi)$ for some $\mu \in \mathscr{P}$ with $\mu_1 \leqslant n$, and $\mathbf{w}_{<0}$ is a semi-infinite semistandard tableau of shape $(-\mu'_n, \ldots, -\mu'_1)$.

Suppose that $\mathbf{w}_{<0} = (w^{(1)}_{<0}, \ldots, w^{(n)}_{<0})$. For each $1 \leqslant k \leqslant n$, we have $w^{(k)}_{<0} \in \mathbf{B}(\Lambda_{-\mu'_k})$ and $\mathrm{wt}(w^{(k)}_{<0}) = \Lambda_0 - \sum_{i \in I_k} \epsilon_i$ for a unique $I_k = \{-i_{k,1} > \cdots > -i_{k,\mu'_k}\} \subset \mathbf{B}_{<0}$. Let $\mathbf{w}^\vee_{<0}$ be the tableau of shape μ^π, whose kth column (from the right) is filled with $\{-i^\vee_{k,1} < \cdots < -i^\vee_{k,\mu'_k}\} \subset \mathbf{B}^\vee_{<0}$. Then we have $\mathbf{w}^\vee_{<0} \in SST_{\mathbf{B}^\vee_{<0}}(\mu^\pi)$. Now, define

$$\Psi_n(\mathbf{w} \otimes t_{-n\Lambda_0}) = \kappa^{-1}\big(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big) \in \mathcal{M}. \tag{4.1}$$

Since $(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0})$ is uniquely determined by $\mathbf{w}$, Ψ_n is injective.

Example 4.2. Let $\mathbf{w} \in \mathbf{B}(4\Lambda_0)$ be as follows:

$$\mathbf{w} = \begin{matrix} \vdots & \vdots & \vdots & \vdots \\ -5 & -5 & -5 & -5 \\ -4 & -4 & -3 & -2 \\ -3 & -3 & -2 & -1 \\ -2 & -1 & 1 & 3 \\ 1 & 1 & 3 & 4 \end{matrix}.$$

Then

$$\mathbf{w}^\vee_{<0} = \begin{matrix} & & -1^\vee & -3^\vee \\ -1^\vee & -2^\vee & -4^\vee & -4^\vee \end{matrix}, \qquad \mathbf{w}_{>0} = \begin{matrix} & & 1 & 3 \\ 1 & 1 & 3 & 4 \end{matrix}.$$

Therefore,

$$\kappa^{-1}\big(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Proposition 4.3. *For $n \geqslant 1$, the map*

$$\Psi_n : \mathbf{B}(n\Lambda_0) \otimes T_{-n\Lambda_0} \to \mathcal{M}$$

is a $\mathfrak{gl}_\infty$-crystal embedding, which commutes with $\tilde{e}_i$ ($i \in \mathbb{Z}$).

Proof. Let $\mathbf{w}$ be given. For $i \in \mathbb{Z}_{>0}$ and $x = e, f$, if $\tilde{x}_i \mathbf{w} \neq \mathbf{0}$, then

$$\begin{aligned}
\Psi_n\big(\tilde{x}_i(\mathbf{w} \otimes t_{-n\Lambda_0})\big) &= \Psi_n\big((\tilde{x}_i \mathbf{w}) \otimes t_{-n\Lambda_0}\big) \\
&= \kappa^{-1}\big(\mathbf{w}^\vee_{<0}, \tilde{x}_i \mathbf{w}_{>0}\big) \\
&= \tilde{x}_i \kappa^{-1}\big(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big) \quad \text{by Proposition 3.3} \\
&= \tilde{x}_i \Psi_n(\mathbf{w} \otimes t_{-n\Lambda_0}).
\end{aligned}$$

Similarly, for $i \in \mathbb{Z}_{<0}$ and $x = e, f$, if $\tilde{x}_i \mathbf{w} \neq \mathbf{0}$, then

$$\begin{aligned}
\Psi_n\big(\tilde{x}_i(\mathbf{w} \otimes t_{-n\Lambda_0})\big) &= \Psi_n\big((\tilde{x}_i \mathbf{w}) \otimes t_{-n\Lambda_0}\big) \\
&= \kappa^{-1}\big(\tilde{x}_i \mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big) \quad \text{by [17, Lemma 5.8]} \\
&= \tilde{x}_i \kappa^{-1}\big(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big) \quad \text{by Proposition 3.3} \\
&= \tilde{x}_i \Psi_n(\mathbf{w} \otimes t_{-n\Lambda_0}).
\end{aligned}$$

Finally, comparing the definitions of $\tilde{x}_0$ $(x = e, f)$ on $\mathbf{B}(n\Lambda_0)$ and $\mathcal{T}$, it is straightforward to see that

$$\Psi_n\big(\tilde{x}_0(\mathbf{w} \otimes t_{-n\Lambda_0})\big) = \kappa^{-1}\big(\tilde{x}_0\big(\mathbf{w}^\vee_{<0}, \mathbf{w}_{>0}\big)\big).$$

Since κ commutes with $\tilde{e}_0$ and $\tilde{f}_0$ by Theorem 3.6, we have $\Psi_n(\tilde{x}_0(\mathbf{w} \otimes t_{-n\Lambda_0})) = \tilde{x}_0 \Psi_n(\mathbf{w} \otimes t_{-n\Lambda_0})$. The other conditions for Ψ_n to be a morphism can be verified directly. Finally, Ψ_n commutes with $\tilde{e}_i$ $(i \in \mathbb{Z})$ since $\operatorname{Im} \Psi_n \subset \mathcal{M}$ together with $\{\mathbf{0}\}$ is stable under $\tilde{e}_i$. $\square$

Remark 4.4. We have

$$\begin{aligned}
&\operatorname{Im} \Psi_n = \kappa^{-1}\Bigg(\bigsqcup_{\mu \in \mathscr{P},\, \mu_1 \leqslant n} SST_{\mathbf{B}^\vee_{<0}}\big(\mu^\pi\big) \times SST_{\mathbf{B}_{>0}}\big(\mu^\pi\big)\Bigg), \\
&\operatorname{Im} \Psi_n \subset \operatorname{Im} \Psi_{n+1} \quad (n \geqslant 1), \\
&\quad \mathcal{M} = \bigcup_{n \geqslant 1} \operatorname{Im} \Psi_n.
\end{aligned}$$

Note that there exists a strict morphism $\Phi_n : \mathcal{M} \otimes T_{n\Lambda_0} \to \mathbf{B}(n\Lambda_0)$ sending $\mathbb{O} \otimes t_{n\Lambda_0}$ to $H_{n\Lambda_0}$ such that $\Phi_n(A \otimes t_{n\Lambda_0}) \neq \mathbf{0}$ if and only if $A \in \operatorname{Im} \Psi_n$.

4.2. Crystal graphs of generalized Verma modules

Given $\mu, \nu \in \mathscr{P}$, we put

$$\begin{aligned}
\mathcal{M}_{\mu,\nu} &= \mathcal{M} \times SST_{\mathbf{B}^\vee_{<0}}\big(\mu^\pi\big) \times SST_{\mathbf{B}_{>0}}(\nu), \\
\mathbb{O}_{\mu,\nu} &= (\mathbb{O}, H_{\mu^\pi}, H_\nu) \in \mathcal{M}_{\mu,\nu}.
\end{aligned} \tag{4.2}$$

For $(A, S_{<0}, S_{>0}) \in \mathcal{M}_{\mu,\nu}$, $i \in \mathbb{Z}$ and $x = e, f$, we define $\tilde{x}_i(A, S_{<0}, S_{>0})$ as follows:

(1) If $i \in \mathbb{Z}_{<0}$ and $\tilde{x}_i(A \otimes S_{<0}) = A' \otimes S'_{<0}$, then

$$\tilde{x}_i(A, S_{<0}, S_{>0}) = \big(A', S'_{<0}, S_{>0}\big).$$

(2) If $i \in \mathbb{Z}_{>0}$ and $\tilde{x}_i(A \otimes S_{>0}) = A'' \otimes S''_{>0}$, then

$$\tilde{x}_i(A, S_{<0}, S_{>0}) = \big(A'', S_{<0}, S''_{>0}\big).$$

(3) $\tilde{x}_0(A, S_{<0}, S_{>0}) = (\tilde{x}_0 A, S_{<0}, S_{>0})$.

Here, we assume $\tilde{x}_i(A, S_{<0}, S_{>0}) = \mathbf{0}$ if any of its components is $\mathbf{0}$.

Proposition 4.5. *For $\mu, \nu \in \mathscr{P}$, $\mathcal{M}_{\mu,\nu}$ is a $\mathfrak{gl}_\infty$-crystal and*

$$\mathcal{M}_{\mu,\nu} = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbb{O}_{\mu,\nu} \mid r \geqslant 0,\ i_1, \ldots, i_r \in \mathbb{Z}\} \setminus \{\mathbf{0}\}.$$

Proof. It is easy to see that $\mathcal{M}_{\mu,\nu}$ is a $\mathfrak{gl}_\infty$-crystal. Let $(A, S_{<0}, S_{>0}) \in \mathcal{M}_{\mu,\nu}$ be given with $\tilde{e}_i(A, S_{<0}, S_{>0}) = \mathbf{0}$ for all $i \in \mathbb{Z}$. First, we have $\tilde{e}_i A = \mathbf{0}$ for all $i \in \mathbb{Z}^\times$, which implies that A is a diagonal matrix with entries $a_{-1^\vee,1} \geqslant a_{-2^\vee,2} \geqslant \cdots$. Since $\tilde{e}_0 A = \mathbf{0}$, we have $a_{-1^\vee,1} = 0$ and hence $A = \mathbb{O}$. This implies that $S_{<0} = H_{\mu^\pi}$ and $S_{>0} = H_\nu$ since $\tilde{e}_i S_{<0} = \mathbf{0}$ for $i \in \mathbb{Z}_{<0}$ and $\tilde{e}_i S_{>0} = \mathbf{0}$ for $i \in \mathbb{Z}_{>0}$, respectively. $\square$

For $n \geqslant \mu_1 + \nu_1$, we put

$$\Lambda_{\mu,\nu;n} = \Lambda_\lambda \in P^+,$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a generalized partition of length n such that

$$\big(\max(\lambda_1, 0), \ldots, \max(\lambda_n, 0)\big) = \nu',$$
$$\big(\max(-\lambda_n, 0), \ldots, \max(-\lambda_1, 0)\big) = \mu'.$$

Proposition 4.6. *Let $\mu, \nu \in \mathscr{P}$ be given. Then for $n \geqslant \mu_1 + \nu_1$, there exists a $\mathfrak{gl}_\infty$-crystal embedding*

$$\Psi_{\mu,\nu;n} : \mathbf{B}(\Lambda_{\mu,\nu;n}) \otimes T_{-n\Lambda_0} \to \mathcal{M}_{\mu,\nu},$$

sending $H_{\Lambda_{\mu,\nu;n}} \otimes t_{-n\Lambda_0}$ to $\mathbb{O}_{\mu,\nu}$ and commuting with $\tilde{e}_i$ ($i \in \mathbb{Z}$).

Proof. Given $\mathbf{w} = (w^{(1)}, \ldots, w^{(n)}) \in \mathbf{B}(\Lambda_{\mu,\nu;n})$, consider the subtableau of $\mathbf{w}$ consisting of positive entries, say $\mathbf{w}_{>0}$. Let $\mathbf{w}^+_{>0}$ be the subtableau of $\mathbf{w}_{>0}$ corresponding to the positions where $H_{\Lambda_{\mu,\nu;n}}$ has positive entries, and let $\mathbf{w}^-_{>0}$ be the compliment of $\mathbf{w}^+_{>0}$ in $\mathbf{w}_{>0}$. Note that $\mathbf{w}^+_{>0} \in SST_{\mathbf{B}_{>0}}(\nu)$ and $\mathbf{w}^-_{>0} \in SST_{\mathbf{B}_{>0}}((\eta/\mu)^\pi)$ for some $\eta \supset \mu$ with $\eta_1 \leqslant n$. Here, we understand $(\eta/\mu)^\pi$ as the skew diagram obtained by 180°-rotation of η/μ.

Let $\mathbf{w}_{<0}$ be the subtableau of $\mathbf{w}$ consisting of negative entries, which is also a semi-infinite semistandard tableau of shape $(-\eta'_n, \ldots, -\eta'_1)$. By the same method as in (4.1), we obtain $\mathbf{w}^\vee_{<0} \in SST_{\mathbf{B}^\vee_{<0}}(\eta^\pi)$ from $\mathbf{w}_{<0}$.

Given $A = A(\mathbf{i}, \mathbf{j}) \in \mathcal{M}$ and $S \in SST_{\mathbf{B}^\vee_{<0}}(\mu^\pi)$, we define $\mathbf{P}(S \leftarrow A)$ to be the tableau obtained by inserting $\mathbf{i}_{\mathrm{rev}} = i_r \cdots i_1$ to S, that is,

$$\mathbf{P}(S \leftarrow A) = \big(\cdots\big((S \leftarrow i_r) \leftarrow i_{r-1}\big)\cdots\big) \leftarrow i_1.$$

Suppose that $\mathrm{sh}\,\mathbf{P}(S \leftarrow A) = \tau^\pi$ for some $\tau \supset \mu$. For $1 \leqslant k \leqslant r$, let us fill a box in $(\tau/\mu)^\pi$ with c if it is created when i_k is inserted into $((S \leftarrow i_r)\cdots) \leftarrow i_{k+1}$ and $j_k = c$. This defines the recording tableau $\mathbf{Q}(S \leftarrow A)$ of shape $(\tau/\mu)^\pi$.

Now, we let $A \in \mathcal{M}$ and $S_{<0} \in SST_{\mathbf{B}^\vee_{<0}}(\mu^\pi)$ be the unique pair such that $\mathbf{P}(S_{<0} \leftarrow A) = \mathbf{w}^\vee_{<0}$ and $\mathbf{Q}(S_{<0} \leftarrow A) = \mathbf{w}^-_{>0}$, and let $S_{>0} = \mathbf{w}^+_{>0}$. This defines an injective map $\Psi_{\mu,\nu;n} : \mathbf{B}(\Lambda_{\mu,\nu;n}) \otimes T_{-n\Lambda_0} \to \mathcal{M}_{\mu,\nu}$ by

$$\Psi_{\mu,\nu;n}(\mathbf{w} \otimes t_{-n\Lambda_0}) = (A, S_{<0}, S_{>0}).$$

Modifying the arguments in Theorem 3.6 and Proposition 4.3, we can check that $\Psi_{\mu,\nu;n}$ is a $\mathfrak{gl}_\infty$-crystal embedding, which commutes with $\tilde{e}_i$ $(i \in \mathbb{Z})$. $\square$

Example 4.7. Let $\mathbf{w} \in \mathbf{B}(\Lambda_{(4,2,-1,-1)})$ be as in Fig. 1. Note that $\mu = (2)$, $\nu = (2,2,1,1)$ and $n = 4$. We have

$$\mathbf{w}^\vee_{<0} = \begin{array}{ccc} & -1^\vee & -1^\vee \\ & -2^\vee & -3^\vee \\ -3^\vee & -3^\vee & -4^\vee \end{array}, \qquad \mathbf{w}^-_{>0} = \begin{array}{ccc} & 1 & 2 \\ & 3 & 3 \\ 1 & \square & \square \end{array}, \qquad \mathbf{w}^+_{>0} = \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \\ 5 & \end{array}.$$

Then we can check that $\mathbf{P}(S \leftarrow A) = \mathbf{w}^\vee_{<0}$ and $\mathbf{Q}(S \leftarrow A) = \mathbf{w}^-_{>0}$, where

$$S = \begin{array}{ccc} & \square & \square \\ & \square & \square \\ \square & -3^\vee & -4^\vee \end{array}, \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Hence

$$\Psi_{(2),(2,2,1,1);4}(\mathbf{w} \otimes t_{-4\Lambda_0}) = \left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, -3^\vee\ -4^\vee, \begin{array}{cc} 1 & 2 \\ 2 & 3 \\ 3 & \\ 5 & \end{array} \right) \in \mathcal{M}_{(2),(2,2,1,1)}.$$

Remark 4.8. (1) In case of $\nu = \emptyset$, the map sending $(\mathbf{w}^\vee_{<0}, \mathbf{w}^-_{>0})$ to $(A, S_{<0})$ is a skew version of the RSK correspondence introduced by Sagan and Stanley [32].

(2) Since $\mathcal{M}_{\mu,\nu} = \bigcup_{n \geqslant \mu_1+\nu_1} \mathrm{Im}\,\Psi_{\mu,\nu;n}$ and $\mathrm{Im}\,\Psi_{\mu,\nu;n} \subset \mathrm{Im}\,\Psi_{\mu,\nu;n+1}$, $\mathcal{M}_{\mu,\nu}$ is the limit of $\mathbf{B}(\Lambda_{\mu,\nu;n})$ as $n \to \infty$ and hence we may view it as a crystal graph of the generalized Verma module induced from an irreducible $\mathfrak{l}$-module with $\mathfrak{l}$-dominant highest weight $-\sum_{i<0} \mu_{-i}\epsilon_i + \sum_{j>0} \nu_j \epsilon_j$. We also have a strict morphism $\Phi_{\mu,\nu;n} : \mathcal{M}_{\mu,\nu} \otimes T_{n\Lambda_0} \to \mathbf{B}(\Lambda_{\mu,\nu;n})$ sending $\mathbb{O}_{\mu,\nu} \otimes t_{n\Lambda_0}$ to $H_{\Lambda_{\mu,\nu;n}}$ such that $\Phi_{\mu,\nu;n}((A, S_{<0}, S_{>0}) \otimes t_{n\Lambda_0}) \neq \mathbf{0}$ if and only if $(A, S_{<0}, S_{>0}) \in \mathrm{Im}\,\Psi_{\mu,\nu;n}$.

(3) Bitableaux realizations of irreducible highest weight representations of Lie (super) algebras including $\mathbf{B}(\Lambda_\lambda)$ and their combinatorics can be found in [18].

5. Demazure crystals and a flagged RSK correspondence

5.1. Demazure crystals

For $i \in \mathbb{Z}$, let $s_i \in GL(\mathfrak{h}^*)$ be the simple reflection with respect to α_i defined by

$$s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i \quad \big(\lambda \in \mathfrak{h}^*\big).$$

Then s_i acts as the transposition on $\{\epsilon_i \mid i \in \mathbb{Z}^\times\}$ (hence on $\mathbb{Z}^\times$) given by

$$s_i = \begin{cases} (i\ i+1), & \text{if } i > 0, \\ (i-1\ i), & \text{if } i < 0, \\ (-1\ 1), & \text{if } i = 0. \end{cases}$$

Let W be the Weyl group of $\mathfrak{gl}_\infty$, which is generated by $\{s_i \mid i \in \mathbb{Z}\}$, and let $\ell(w)$ denote the length of $w \in W$. For $\Lambda \in P^+$, let W_Λ be the stabilizer of Λ, and let $W^\Lambda = \{w \mid \ell(ws_i) > \ell(w) \text{ for } s_i \in W_\Lambda\}$. Let $<$ denote the Bruhat order on W. It induces the Bruhat order on W^Λ, which is also denoted by $<$.

Let $w \in W^\Lambda$ be given and $w = s_{i_1} \cdots s_{i_r}$ its reduced expression. We define the *Demazure crystal of* $\mathbf{B}(\Lambda)$ *associated with* w [11] by

$$\mathbf{B}_w(\Lambda) = \left\{ \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_r}^{m_r} H_\Lambda \;\middle|\; m_1, \ldots, m_r \geqslant 0 \right\} \setminus \{\mathbf{0}\}. \tag{5.1}$$

For $i \in \mathbb{Z}$, an i-string S in $\mathbf{B}(\Lambda)$, a connected component with only i-arrows, satisfies one of the following three conditions;

$$S \subset \mathbf{B}_w(\Lambda),$$

$$S \cap \mathbf{B}_w(\Lambda) = \emptyset,$$

$$S \cap \mathbf{B}_w(\Lambda) \text{ is a highest weight element of } S. \tag{5.2}$$

Now, given $\mu, \nu \in \mathscr{P}$ and $w \in W^\Lambda$ with $\Lambda = \Lambda_{\mu,\nu;N}$ for some $N > \mu_1 + \nu_1$, we define

$$\mathcal{M}_{\mu,\nu,w} = \left\{ \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_r}^{m_r} \mathbb{O}_{\mu,\nu} \;\middle|\; m_1, \ldots, m_r \geqslant 0 \right\} \setminus \{\mathbf{0}\}. \tag{5.3}$$

Since each element in $\mathcal{M}_{\mu,\nu,w}$ is contained in $\Psi_{\mu,\nu;n}(\mathbf{B}_w(\Lambda_{\mu,\nu;n}) \otimes T_{-n\Lambda_0})$ for some sufficiently large n, $\mathcal{M}_{\mu,\nu,w}$ does not depend on N and the choice of a reduced expression of w, and hence it is well defined. Note that for $w, w' \in W^\Lambda$, $\mathcal{M}_{\mu,\nu,w} \subset \mathcal{M}_{\mu,\nu,w'}$ if and only if $w \leqslant w'$.

5.2. Grassmannian permutations

Let λ be a partition. The *residue* of a box $(i,j) \in \lambda$ is given by $j - i$. A *standard tableau of shape* λ is a tableau obtained by filling λ with $\{1, \ldots, |\lambda|\}$ in such a way that the entries in each column (resp. row) are increasing from top to bottom (resp. left to right).

Consider $W_{\Lambda_0} = \langle s_i \mid i \in \mathbb{Z}^\times \rangle$ and let $w \in W^{\Lambda_0}$ be given. Let $D(w) = \{(i,j) \in \mathbb{Z}^\times \times \mathbb{Z}^\times \mid i < w^{-1}(j),\ j < w(i)\}$ be the *diagram of* w and let $\lambda(w) = (\lambda(w)_i)_{i \geqslant 1}$ be the *shape of* w, where $\lambda(w)_i = |\{j \mid (-i, j) \in D(w)\}|$. Since $w(i) < w(i+1)$ for $i \in \mathbb{Z}^\times \setminus \{-1\}$, and $w(-1) > w(1)$, $\lambda(w)$ is a partition. Conversely, a partition λ determines a unique permutation $w \in W^{\Lambda_0}$ such that $\lambda(w) = \lambda$. Hence, we have a bijection from W^{Λ_0} to $\mathscr{P}$ sending w to $\lambda(w)$, where $|\lambda(w)| = \ell(w)$. If T is a standard tableau of shape $\lambda(w)$ and a_i is the residue of the box corresponding to i in T $(1 \leqslant i \leqslant \ell(w))$, then $w = s_{a_{\ell(w)}} s_{a_{\ell(w)-1}} \cdots s_{a_1}$ gives a reduced expression of w. For $w, w' \in W^{\Lambda_0}$, we have $w \leqslant w'$ if and only if $\lambda(w) \subseteq \lambda(w')$ [24].

Example 5.1. Let $w \in W^{\Lambda_0}$ be given by

$$w = \begin{bmatrix} \cdots & -6 & -5 & -4 & -3 & -2 & -1 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ \cdots & -6 & -4 & -2 & 2 & 5 & 6 & -5 & -3 & -1 & 1 & 3 & 4 & \cdots \end{bmatrix},$$

where $w(i) = i$ for $|i| \geqslant 7$. Then $\lambda(w) = (6, 6, 4, 2, 1)$, where the residue on each box is given by

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 1 & 2 & 3 & 4 \\ -2 & -1 & 0 & 1 & & \\ -3 & -2 & & & & \\ -4 & & & & & \end{array}.$$

5.3. Flagged skew Schur functions

Let $X = \{x_1, x_2, \ldots\}$ be a set of formal commuting variables and $X_k = \{x_1, \ldots, x_k\}$ for $k \geqslant 1$. Let $\phi = (\phi_1, \ldots, \phi_d)$ be a sequence of weakly increasing positive integers of length d, which is called a *flag of length d*. Given a skew Young diagram λ/μ with $\ell(\lambda), \ell(\mu) \leqslant d$, we define the *flagged Schur function $s_{\lambda/\mu}(X_\phi)$* by

$$s_{\lambda/\mu}(X_\phi) = \det\bigl(h_{\lambda_i - \mu_j - i + j}(X_{\phi_i})\bigr)_{1 \leqslant i, j \leqslant d},$$

where $h_k(X_{\phi_i})$ is the kth complete symmetric function in X_{ϕ_i} [24]. An equivalent definition is that

$$s_{\lambda/\mu}(X_\phi) = \sum_T x^T,$$

where the sum ranges over all semistandard tableaux of shape λ/μ with entries in $\mathbb{N}$ such that the entries in the ith row are no more than ϕ_i for $1 \leqslant i \leqslant d$. Here, $x^T = \prod_i x_i^{m_i}$, where m_i is the number of occurrences of i in T.

Let $SST_{\mathbf{B}^\vee_{<0}}(\lambda/\mu)_\phi$ (resp. $SST_{\mathbf{B}_{>0}}(\lambda/\mu)_\phi$) be the set of semistandard tableaux of shape λ/μ such that the entries in the ith row are no more than $-\phi_i^\vee$ (resp. ϕ_i) for $1 \leqslant i \leqslant d$. Let $Y = \{y_1, y_2, \ldots\}$ be another set of formal commuting variables and $Y_k = \{y_1, \ldots, y_k\}$ for $k \geqslant 1$. Then we have

$$\operatorname{ch} SST_{\mathbf{B}^\vee_{<0}}(\lambda/\mu)_\phi = s_{\lambda/\mu}(X_\phi), \qquad \operatorname{ch} SST_{\mathbf{B}_{>0}}(\lambda/\mu)_\phi = s_{\lambda/\mu}(Y_\phi),$$

where we put

$$x_i = e^{\mathrm{wt}(-i^\vee)} = e^{-\epsilon_{-i}}, \qquad y_j = e^{\mathrm{wt}(j)} = e^{\epsilon_j}$$

for $-i^\vee \in \mathbf{B}^\vee_{<0}$ and $j \in \mathbf{B}_{>0}$. When $S \in SST_{\mathbf{B}^\vee_{<0}}(\nu^\pi)_\phi$ (resp. $S \in SST_{\mathbf{B}_{>0}}(\nu^\pi)_\phi$) is given for $\nu \in \mathscr{P}$ with $\ell(\nu) \leqslant d$, we view $\nu^\pi = (n^d)/(n - \nu_d, \ldots, n - \nu_1)$ for some $n \geqslant \nu_1$, and understand that the entries of S in each ith row from the bottom are no more than $-\phi^\vee_{d-i+1}$ (resp. ϕ_{d-i+1}) for $1 \leqslant i \leqslant d$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a sequence of weakly decreasing positive integers of length d. Let

$$\widehat{s}_{\lambda/\mu}(X_\alpha) = \det\bigl(h_{\lambda_i - \mu_j - i + j}(X_{\alpha_j})\bigr)_{1 \leqslant i, j \leqslant d}.$$

It is easy to check that $\widehat{s}_{\lambda/\mu}(X_\alpha) = s_{\widehat{\mu}/\widehat{\lambda}}(X_\phi)$, where $\widehat{\lambda} = (n - \lambda_{d-i+1})_{1 \leqslant i \leqslant d}$, $\widehat{\mu} = (n - \mu_{d-i+1})_{1 \leqslant i \leqslant d}$ for some $n \geqslant \lambda_1, \mu_1$, and ϕ is the reverse sequence of α. In particular, we have for $\nu \in \mathscr{P}$ with $\ell(\nu) \leqslant d$,

$$s_{\nu^\pi}(X_\phi) = \widehat{s}_\nu(X_\alpha).$$

5.4. A flagged RSK correspondence

In the sequel, we assume that S is a finite subset of $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. We define $\theta(S)$ to be the border strip of the smallest partition λ such that $S \subset \lambda$. Recall that a *border strip* of a partition λ is a skew diagram λ/μ, where $\lambda = (\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d)$ and $\mu = (\alpha_2, \ldots, \alpha_d | \beta_2, \ldots, \beta_d)$ following Frobenius notation. We put $c(S) = (\alpha_1, \beta_1)$.

Example 5.2. Let $S = \{(1,1), (1,4), (2,2), (3,1), (3,3), (4,3)\}$. Then

$$S = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & \cdots \\ \hline 1 & \bullet & & & \bullet & \\ 2 & & \bullet & & & \\ 3 & \bullet & & \bullet & & \\ 4 & & & \bullet & & \\ \vdots & & & & & \end{array}, \qquad \theta(S) = (4,3,3,3)/(2,2,2) = \begin{array}{cccc} \square & \square & \blacksquare & \blacksquare \\ \square & \square & \blacksquare & \\ \square & \square & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \ast \end{array},$$

where the skew diagram consisting of the black boxes is the border strip $\theta(S)$ and $*$ indicates the point $c(S) = (4,4)$.

We define inductively a finite sequence of points $c_1 = (\alpha_1, \beta_1), \ldots, c_d = (\alpha_d, \beta_d)$ as follows:

(1) let $S^{(1)} = S$ and put $c_1 = c(S^{(1)})$,
(2) for $1 \leqslant k \leqslant d-1$, let $S^{(k+1)} = S^{(k)} \setminus \theta(S^{(k)})$, and put $c_{k+1} = c(S^{(k+1)})$,

where d is the smallest one such that $S^{(d+1)} = \emptyset$. Note that c_{k+1} is located to the northwest of c_k, that is, $\alpha_k > \alpha_{k+1}$ and $\beta_k > \beta_{k+1}$. Now, we define

$$\lambda(S) = (\alpha_1, \ldots, \alpha_d | \beta_1, \ldots, \beta_d) \in \mathscr{P} \tag{5.4}$$

following Frobenius notation, where $(\alpha_1, \ldots, \alpha_d)$ (resp. $(\beta_1, \ldots, \beta_d)$) corresponds to the lower (resp. upper) flag of $\lambda(S)$ with respect to its diagonal. We define $w(S)$ to be the permutation in W^{Λ_0} corresponding to $\lambda(S)$, that is,

$$\lambda\big(w(S)\big) = \lambda(S). \tag{5.5}$$

Example 5.3. Let S be as in Example 5.2. Then

$$S^{(1)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \bullet & & & \bullet \\ 2 & & \bullet & & \\ 3 & \bullet & & \bullet & \\ 4 & & & \bullet & \end{array}, \qquad \theta\big(S^{(1)}\big) = (4,3,3,3)/(2,2,2) = \begin{array}{cccc} \square & \square & \blacksquare & \blacksquare \\ \square & \square & \blacksquare & \\ \square & \square & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \ast \end{array},$$

$$S^{(2)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \bullet & & & \\ 2 & & \bullet & & \\ 3 & \bullet & & & \\ 4 & & & & \end{array}, \qquad \theta\big(S^{(2)}\big) = (2,2,1)/(1) = \begin{array}{cccc} \square & \blacksquare & \cdot & \cdot \\ \blacksquare & \blacksquare & \cdot & \\ \blacksquare & \ast & \cdot & \\ \cdot & \cdot & & \end{array},$$

$$S^{(3)} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & \bullet & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}, \qquad \theta\bigl(S^{(3)}\bigr) = (1) = \begin{array}{cc} \blacksquare & \cdot \\ \cdot & \cdot \\ \cdot & \end{array},$$

where we have $c_1 = (4, 4)$, $c_2 = (3, 2)$, $c_3 = (1, 1)$. Hence

$$\lambda(S) = (4, 3, 1|4, 2, 1) = (4, 3, 3, 2).$$

For $w \in W^{\Lambda_0}$ with a reduced expression $w = s_{i_1} \cdots s_{i_r}$, we put

$$\mathcal{M}_w = \bigl\{ \tilde{f}_{i_1}^{m_1} \cdots \tilde{f}_{i_r}^{m_r} \mathbb{O} \bigm| m_1, \ldots, m_r \geqslant 0 \bigr\} \setminus \{\mathbf{0}\}, \tag{5.6}$$

that is, $\mathcal{M}_w = \mathcal{M}_{\emptyset,\emptyset,w}$ (see (5.3)). For $A \in \mathcal{M}$, let $\operatorname{supp}(A) = \{(i, j) \mid a_{-i^\vee, j} \neq 0\} \subset \mathbb{N}^2$ be the support of A.

Theorem 5.4. *For $w \in W^{\Lambda_0}$, we have*

$$\mathcal{M}_w = \bigl\{ A \bigm| \lambda\bigl(\operatorname{supp}(A)\bigr) \subset \lambda(w) \bigr\} = \bigl\{ A \bigm| w\bigl(\operatorname{supp}(A)\bigr) \leqslant w \bigr\}.$$

Proof. It suffices to prove the first identity. We use induction on $\ell(w) = |\lambda(w)|$.

If $|\lambda(w)| = 1$, then $w = s_0$ and it is clear. We assume that $|\lambda(w)| \geqslant 2$. Choose $w' \in W^{\Lambda_0}$ such that $\ell(w') = \ell(w) - 1$, equivalently $\lambda(w') \subset \lambda(w)$ with $|\lambda(w)/\lambda(w')| = 1$. Let r be the residue of $\lambda(w)/\lambda(w')$.

Choose a standard tableau T of shape $\lambda(w)$ such that the largest entry occurs at $\lambda(w)/\lambda(w')$. Let a_i be the residue of the box corresponding to i in T ($1 \leqslant i \leqslant \ell(w)$). Then we have reduced expressions $w = s_{a_{\ell(w)}} s_{a_{\ell(w)-1}} \cdots s_{a_1}$ and $w' = s_{a_{\ell(w)-1}} \cdots s_{a_1}$ with $a_{\ell(w)} = r$. Note that

$$\mathcal{M}_w = \bigcup_{k \geqslant 0} \tilde{f}_r^k \mathcal{M}_{w'} \setminus \{\mathbf{0}\}. \tag{5.7}$$

Let $\mathcal{N}_w = \{A \mid \lambda(\operatorname{supp}(A)) \subset \lambda(w)\}$. By induction hypothesis, it suffices to show that $\mathcal{N}_w = \bigcup_{k \geqslant 0} \tilde{f}_r^k \mathcal{N}_{w'} \setminus \{\mathbf{0}\}$.

Let $A \in \mathcal{N}_w$ be given. We first claim that $A \in \tilde{f}_r^k \mathcal{N}_{w'}$ for some $k \geqslant 0$. We will keep the previous notations $\lambda(S)$, $\theta(S^{(k)})$, and $c_k = (\alpha_k, \beta_k)$ ($1 \leqslant k \leqslant d$) with $S = \operatorname{supp}(A)$. If $\lambda(S)$ does not contain the box $c = \lambda(w)/\lambda(w')$, then $\lambda(S) \subset \lambda(w')$ and $A \in \mathcal{N}_{w'}$. So we may assume that $c \in \lambda(S)$.

Case 1. Suppose that $r = 0$. By definition of $\lambda(S)$, we have $\theta(S^{(d)}) = (1, 1)$. If we choose $k \geqslant 1$ such that the entry of $\tilde{e}_0^k A$ at $(1, 1)$ is 0, then $\operatorname{supp}(\tilde{e}_0^k A) = \operatorname{supp}(A) \setminus \{(1, 1)\}$. Hence $\lambda(\operatorname{supp}(\tilde{e}_0^k A)) \subset \lambda(w')$ and $\tilde{e}_0^k A \in \mathcal{N}_{w'}$.

Case 2. Suppose that $r \neq 0$. We may assume that $r > 0$ since the argument for $r < 0$ is almost the same. In this case, we have $c_s = (\alpha_s, \beta_s)$ with $\beta_s = r + 1$ for some $1 \leqslant s \leqslant d$, and $c = (s, r + 1)$. There exists $1 \leqslant m \leqslant \alpha_s$ such that $a_{-m^\vee, r+1} \neq 0$ with $(m, r + 1) \in S^{(s)}$ and $(i, r + 1) \notin S^{(s)}$ for $m < i \leqslant \alpha_s$. Since c is a removable corner of $\lambda(w)$ and hence of $\lambda(S)$, we have $a_{-i^\vee, r} = 0$ for $1 \leqslant i < m$ (see Fig. 2). Otherwise, we have $\beta_{s-1} = r$, which implies that c is not removable. Let $A = A(\mathbf{k}, \mathbf{l})$ and consider the subword of $\mathbf{l}$ consisting of r and $r+1$. Then the $r+1$'s corresponding to the non-zero entries $a_{-i^\vee, r+1}$ for $1 \leqslant i \leqslant m$ can be replaced by r applying $\tilde{e}_r^k$ for some $k \geqslant 1$. Equivalently, applying $\tilde{e}_r^k$, the entries $a_{-i^\vee, r}$ (resp. $a_{-i^\vee, r+1}$) are replaced by $a_{-i^\vee, r} + a_{-i^\vee, r+1}$ (resp. 0) for $1 \leqslant i \leqslant m$. On the other hand, we

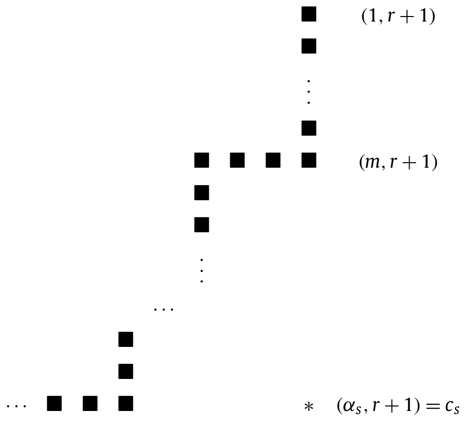


Fig. 2. The border strip $\theta(S^{(s)})$.

can check that (α_t,β_t) are invariant under $\tilde{e}_r^k$ for $1\leqslant t\leqslant d$ with $t\neq s$. Hence $\lambda(\mathrm{supp}(\tilde{e}_r^k A))\subset\lambda(w')$ and $\tilde{e}_r^k A\in\mathcal{N}_{w'}$.

Conversely, let $A\in\mathcal{N}_{w'}$ be given. Using similar arguments, it is not difficult to check that either $\lambda(\mathrm{supp}(\tilde{f}_r^k A))=\lambda(\mathrm{supp}(A))$ or $\lambda(\mathrm{supp}(\tilde{f}_r^k A))/\lambda(\mathrm{supp}(A))=c$ whenever $\tilde{f}_r^k A\neq\mathbf{0}$. Hence $\lambda(\mathrm{supp}(\tilde{f}_r^k A))\subset\lambda(w)$. This completes our induction. $\square$

Corollary 5.5. *For $w\in W^{\Lambda_0}$, we have*

$$\mathrm{ch}\,\mathcal{M}_w=\sum_{\substack{S\subset\mathbb{N}^2\\ w(S)\leqslant w}}\prod_{(i,j)\in S}\frac{x_i y_j}{1-x_i y_j}.$$

Remark 5.6. Identifying $(i,j)\in\mathbb{N}^2$ with $-\epsilon_{-i}+\epsilon_j\in\Delta(\mathfrak{u}_-)$, one may write

$$\mathrm{ch}\,\mathcal{M}_w=\sum_{\substack{S\subset\Delta(\mathfrak{u}_-)\\ w(S)\leqslant w}}\prod_{\alpha\in S}\frac{e^{\alpha}}{1-e^{\alpha}}.$$

Now, put

$$\mathcal{T}_w=\kappa(\mathcal{M}_w)=\left\{\tilde{f}_{i_1}^{m_1}\cdots\tilde{f}_{i_r}^{m_r}(\emptyset,\emptyset)\;\middle|\;m_1,\dots,m_r\geqslant 0\right\}\setminus\{\mathbf{0}\}\tag{5.8}$$

for $w\in W^{\Lambda_0}$ with a reduced expression $w=s_{i_1}\cdots s_{i_r}$. We define $\alpha(w)=(\alpha_1,\dots,\alpha_d)$ and $\beta(w)=(\beta_1,\dots,\beta_d)$ to be strict partitions of length d such that

$$\lambda(w)=\big(\alpha(w)|\beta(w)\big).\tag{5.9}$$

We put $d(w)=d$, the diagonal length of $\lambda(w)$ and

$$\phi(w) = (\phi_1, \ldots, \phi_d) = (\alpha_d, \ldots, \alpha_1),$$
$$\psi(w) = (\psi_1, \ldots, \psi_d) = (\beta_d, \ldots, \beta_1), \tag{5.10}$$

which are flags of length d.

Theorem 5.7. *For $w \in W^{\Lambda_0}$, we have*

$$\mathcal{T}_w = \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant d(w)}} SST_{\mathbf{B}^\vee_{<0}}\left(\nu^\pi\right)_{\phi(w)} \times SST_{\mathbf{B}_{>0}}\left(\nu^\pi\right)_{\psi(w)}.$$

Proof. Let $\mathcal{S}_w$ be the right-hand side of the above identity. We will use induction on $\ell(w) = |\lambda(w)|$. When $\ell(w) = 1$, i.e. $w = s_0$, it is clear. We assume that $\ell(w) \geqslant 2$.

Choose $w' \in W^{\Lambda_0}$ such that $\ell(w') = \ell(w) - 1$, or $\lambda(w') \subset \lambda(w)$ with $|\lambda(w)/\lambda(w')| = 1$. Let r be the residue of $\lambda(w)/\lambda(w')$. By (5.7), (5.8) and the induction hypothesis, we have only to show that $\mathcal{S}_w = \bigcup_{k \geqslant 0} \tilde{f}_r^k \mathcal{S}_{w'} \setminus \{\mathbf{0}\}$. We assume that $\lambda(w)$ is as in (5.9) with $d = d(w)$.

Case 1. Suppose that $r = 0$. Then we have $d \geqslant 2$, $\alpha_d = \beta_d = 1$ and

$$\lambda(w') = (\alpha_1, \ldots, \alpha_{d-1} | \beta_1, \ldots, \beta_{d-1}).$$

Let $(S, T) \in \mathcal{S}_{w'}$ be given, where $\mathrm{sh}(S) = \mathrm{sh}(T) = \eta^\pi$ for some $\eta \in \mathscr{P}$ with $\ell(\eta) \leqslant d(w')$. For $k \geqslant 1$, suppose that $(S', T') = \tilde{f}_0^k(S, T) \neq \mathbf{0}$ and $\mathrm{sh}(S') = \mathrm{sh}(T') = \tau^\pi$ for some $\tau \in \mathscr{P}$. By definition of $\tilde{f}_0$, we have $\ell(\tau) \leqslant \ell(\eta) + 1 \leqslant d(w)$. If $\ell(\tau) < d(w)$, then it is clear that $(S', T') \in \mathcal{S}_{w'} \subset \mathcal{S}_w$. Assume that $\ell(\tau) = d(w)$, that is, $\ell(\tau) = \ell(\eta) + 1 = d(w)$. Then the first rows of S' and T' are filled only with $-1^\vee$ and 1, respectively, and the entries in the other rows of S' and T' still satisfy the flag conditions given by $\phi(w')$ and $\psi(w')$, respectively. Since $\phi(w) = (1, \phi(w'))$ and $\psi(w) = (1, \psi(w'))$, we have $(S', T') \in \mathcal{S}_w$.

Conversely, let $(S, T) \in \mathcal{S}_w$ be given with $\mathrm{sh}(S) = \mathrm{sh}(T) = \tau^\pi$ for some $\tau \in \mathscr{P}$. If $\ell(\tau) < d(w)$, then $(S, T) \in \mathcal{S}_{w'}$. If $\ell(\tau) = d(w)$, then the first rows of S and T are filled only with $-1^\vee$ and 1, respectively, and they are all removable under successive application of $\tilde{e}_0$. Hence, the highest weight element (S', T') in the 0-string of (S, T) belongs to $\mathcal{S}_{w'}$.

Case 2. Suppose that $r \neq 0$. We may assume that $r > 0$ since the argument for $r < 0$ is almost the same. In this case, we have $d(w') = d(w)$, and for some $1 \leqslant i < d$

$$\lambda(w') = (\alpha_1, \ldots, \alpha_i, \ldots, \alpha_d | \beta_1, \ldots, \beta_i - 1, \ldots, \beta_d),$$

where $\beta_i = r + 1$.

Let $(S, T) \in \mathcal{S}_{w'}$ be given. Note that the entries of T in the ith row from the bottom are no more than $\beta_i - 1 = r$, and no r appears in the above rows. This implies that $\tilde{f}_r^k(S, T) \in \mathcal{S}_w \cup \{\mathbf{0}\}$ for $k \geqslant 0$.

Conversely, let $(S, T) \in \mathcal{S}_w$ be given. The entries of T in the $(i + 1)$th row from the bottom are no more than $\beta_{i+1} < \beta_i - 1 = r$. So any $r + 1$ in the ith row of T from the bottom, if exists, can be replaced by r applying $\tilde{e}_r^k$ for some $k \geqslant 0$. This implies that $\tilde{e}_r^k(S, T) \in \mathcal{S}_{w'}$. $\square$

Corollary 5.8. *For $w \in W^{\Lambda_0}$, we have*

$$\mathrm{ch}\, \mathcal{T}_w = \sum_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant d(w)}} \widehat{s}_\nu(X_{\alpha(w)}) \widehat{s}_\nu(Y_{\beta(w)}).$$

Combining Theorems 5.4 and 5.7, we obtain a flagged version of the RSK correspondence and the Cauchy identity as follows.

Corollary 5.9. *For $w \in W^{\Lambda_0}$, the map κ in* (3.4) *gives a bijection*

$$\{A \in \mathcal{M} \mid \lambda(\mathrm{supp}(A)) \subset \lambda(w)\} \to \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant d(w)}} SST_{\mathbf{B}^{\vee}_{<0}}(\nu^{\pi})_{\phi(w)} \times SST_{\mathbf{B}_{>0}}(\nu^{\pi})_{\psi(w)},$$

when restricted to $\mathcal{M}_w$, and it commutes with $\tilde{e}_i$ $(i \in \mathbb{Z})$. In particular, we have

$$\sum_{\substack{S \subset \mathbb{N}^2 \\ w(S) \leqslant w}} \prod_{(i,j) \in S} \frac{x_i y_j}{1 - x_i y_j} = \sum_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant d(w)}} \widehat{s}_{\nu}(X_{\alpha(w)}) \widehat{s}_{\nu}(Y_{\beta(w)}).$$

Remark 5.10. (1) For $m, n \geqslant 1$, let $w_{m,n}$ be the element in W^{Λ_0} such that $\lambda(w_{m,n}) = (n^m)$. In this case, we recover the usual RSK correspondence with $m \times n$ matrices and the Cauchy identity with variables x_i, y_j $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$.

(2) For $S \subset \mathbb{N}^2$, let $d(S) = d(w(S))$. Then for $n \geqslant 1$, we have

$$\sum_{\substack{S \subset \mathbb{N}^2 \\ d(S) \leqslant n}} \prod_{(i,j) \in S} \frac{x_i y_j}{1 - x_i y_j} = \sum_{\substack{\nu \in \mathscr{P} \\ \ell(\nu) \leqslant n}} s_{\nu}(X) s_{\nu}(Y).$$

When multiplied by $e^{-n\Lambda_0}$ the right-hand side of the identity is equal to the character of the irreducible highest weight representation of $\mathfrak{gl}_{\infty}$ with highest weight $-n\Lambda_0$, which is not integrable [9]. Hence the left-hand side gives another character formula for this highest weight module. Note that the right-hand side has a Jacobi–Trudi type formula (see [31, Ch. 7, Ex. 7.16d] and [18] for its generalization to irreducible $\mathfrak{gl}_{\infty}$-modules with negative integral charges) and a Weyl–Kac type formula [19].

(3) From the correspondence between $\mathcal{M}_w$ and $\mathcal{T}_w$, we see that the entries in the first columns of bitableaux in $\mathcal{T}$ is determined only by the support of the corresponding matrix in $\mathcal{M}$. This fact was also observed by Stanley [31, Ch. 7, Ex. 7.100] in a purely combinatorial way.

5.5. Demazure crystal $\mathbf{B}_w(n\Lambda_0)$

Proposition 5.11. *For $w \in W^{\Lambda_0}$ and $n \geqslant 1$, the map $\kappa \circ \Psi_n$ gives a weight preserving bijection*

$$\mathbf{B}_w(n\Lambda_0) \otimes T_{-n\Lambda_0} \to \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \nu \subset (n^{d(w)})}} SST_{\mathbf{B}^{\vee}_{<0}}(\nu^{\pi})_{\phi(w)} \times SST_{\mathbf{B}_{>0}}(\nu^{\pi})_{\psi(w)}.$$

Proof. It follows from Remark 4.4 and Theorem 5.7. □

Remark 5.12. Given $A = A(\mathbf{i}, \mathbf{j}) \in \mathcal{M}$ with $(\mathbf{i}, \mathbf{j}) \in \Omega$, let $c(A)$ be the maximal length of decreasing subwords of $\mathbf{i}$. It is well known that $c(A)$ is equal to the number of columns in $\mathbf{P}(A)$ or $\mathbf{Q}(A)$ [15]. By Remark 4.4 and Theorem 5.4, the embedding Ψ_n gives a bijection

$$\mathbf{B}_w(n\Lambda_0) \to \{A \mid \lambda(\mathrm{supp}(A)) \subset \lambda(w),\ c(A) \leqslant n\}.$$

For $i \in \mathbb{Z}$ and $\lambda \in P$, let D_i be the linear operator on $\mathbb{C}[P]$ defined by

$$D_i\big(e^{\lambda}\big) = e^{\lambda} \cdot \frac{1 - e^{-(1+\langle \lambda, h_i \rangle)\alpha_i}}{1 - e^{-\alpha_i}}.$$

The operators D_i satisfy the braid relations, and hence for a reduced expression of $w = s_{i_1} \cdots s_{i_r} \in W$, the operator $D_w = D_{i_1} \cdots D_{i_r}$ is well defined. By (5.2), we have

$$\operatorname{ch} \mathbf{B}_w(n\Lambda_0) = D_w\big(e^{n\Lambda_0}\big).$$

Combining with Proposition 5.11, we obtain the following combinatorial identity.

Corollary 5.13. *Let $n, d \geqslant 1$, and let α, β be two strict partitions of length d. Then*

$$D_w\big(e^{n\Lambda_0}\big)e^{-n\Lambda_0} = \sum_{\substack{\nu \in \mathscr{P} \\ \nu \subset (n^d)}} \widehat{s}_\nu(X_\alpha)\widehat{s}_\nu(Y_\beta),$$

where w is the unique element in W^{Λ_0} such that $\lambda(w) = (\alpha|\beta)$.

5.6. Crystals of symmetric matrices

From now on, let ϵ denote either 1 or 2. We put

$$\widehat{\mathcal{M}}^{\epsilon} = \{A \in \mathcal{M} \mid a_{-i^\vee, j} = a_{-j^\vee, i} \text{ for } i, j \geqslant 1,\ \epsilon \text{ divides } a_{-i^\vee, i} \text{ for } i \geqslant 1\}. \tag{5.11}$$

For $i \in \mathbb{Z}_{\geqslant 0}$, let

$$\begin{aligned} \tilde{E}_0 &= (\tilde{e}_0)^{\epsilon}, & \tilde{F}_0 &= (\tilde{f}_0)^{\epsilon}, \\ \tilde{E}_i &= \tilde{e}_i\tilde{e}_{-i}, & \tilde{F}_i &= \tilde{f}_i\tilde{f}_{-i} \quad (i \in \mathbb{Z}_{>0}). \end{aligned} \tag{5.12}$$

By similar arguments as in Proposition 3.1, we can check the following.

Proposition 5.14.

(1) *$\widehat{\mathcal{M}}^{\epsilon} \cup \{\mathbf{0}\}$ is invariant under $\tilde{E}_i$ and $\tilde{F}_i$ for $i \in \mathbb{Z}_{\geqslant 0}$.*
(2) *$\widehat{\mathcal{M}}^{\epsilon} = \{\tilde{F}_{i_1} \cdots \tilde{F}_{i_r}\mathbb{O} \mid r \geqslant 0,\ i_1, \ldots, i_r \in \mathbb{Z}_{\geqslant 0}\} \setminus \{\mathbf{0}\}$.*

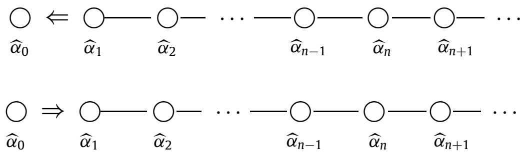
Put

$$\begin{aligned} \widehat{P} &= \left\{\lambda \in P \;\middle|\; \frac{1}{\epsilon}\langle \lambda, h_0 \rangle \in \mathbb{Z},\ \langle \lambda, h_i \rangle = \langle \lambda, h_{-i} \rangle \ (i \in \mathbb{Z}_{>0})\right\}, \\ \widehat{\Pi} &= \big\{\widehat{\alpha}_0 = \epsilon\alpha_0,\ \widehat{\alpha}_i = \alpha_i + \alpha_{-i}\ (i \in \mathbb{Z}_{>0})\big\} \subset \widehat{P}. \end{aligned}$$

Then $\widehat{\Pi}$ is a set of simple roots for the root system associated to the affine Lie algebra $\mathfrak{b}_\infty$ (resp. $\mathfrak{c}_\infty$) when $\epsilon = 1$ (resp. $\epsilon = 2$) [8], where $\widehat{P}$ is a weight lattice. The associated Dynkin diagrams are as follows.

$\mathfrak{b}_\infty$:

$\mathfrak{c}_\infty$:



For $i \in \mathbb{Z}_{\geqslant 0}$, let $\widehat{h}_i \in \widehat{P}^*$ be determined by $\langle \lambda, \widehat{h}_i \rangle = \langle \lambda, h_i \rangle = \langle \lambda, h_{-i} \rangle$ if $i > 0$, and $\langle \lambda, \widehat{h}_0 \rangle = \frac{1}{\epsilon} \langle \lambda, h_0 \rangle$ for $\lambda \in \widehat{P}$. Then $\widehat{\Pi}^\vee = \{\widehat{h}_i \mid i \in \mathbb{Z}_{\geqslant 0}\}$ is a set of simple coroots. As in Definition 2.2, one may define an x_∞-crystal ($x = \mathfrak{b}, \mathfrak{c}$) with respect to $\tilde{E}_i, \tilde{F}_i, \widehat{\varepsilon}_i, \widehat{\varphi}_i$ ($i \in \mathbb{Z}_{\geqslant 0}$) and $\widehat{\mathrm{wt}}$. By Proposition 5.14, $\widehat{\mathfrak{M}}^\epsilon$ is an x_∞-crystal with highest weight element $\mathbb{O}$. Here, for $A \in \widehat{\mathfrak{M}}^\epsilon$ $\widehat{\mathrm{wt}}(A) = \mathrm{wt}(A) \in \widehat{P}$, $\widehat{\varepsilon}_i(A) = \varepsilon_i(A)$, $\widehat{\varphi}_i(A) = \varphi_i(A)$ ($i > 0$) and $\widehat{\varepsilon}_0(A) = \frac{1}{\epsilon}\varepsilon_0(A)$, $\widehat{\varphi}_0(A) = \frac{1}{\epsilon}\varphi_0(A)$.

Since each A in $\widehat{\mathfrak{M}}^\epsilon$ is symmetric, we have $\kappa(A) = (-S^\vee, S)$, where $S \in SST_{\mathbf{B}_{>0}}(\nu^\pi)$ for some $\nu \in \mathscr{P}$ with $\epsilon|\nu$, that is, $\epsilon|\nu_i$ for $i \geqslant 1$ [15,17], and $-S^\vee$ is the semistandard tableau obtained by replacing each entry i in S with $-i^\vee$. Hence the map $\widehat{\kappa} : A \mapsto S$ gives a bijection

$$\widehat{\kappa} : \widehat{\mathfrak{M}}^\epsilon \to \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \epsilon|\nu}} SST_{\mathbf{B}_{>0}}\big(\nu^\pi\big). \tag{5.13}$$

Proposition 5.15. *Put $\widehat{\Lambda}_0 = \epsilon \Lambda_0$. For $n \geqslant 1$, let*

$$\widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0) = \{\tilde{F}_{i_1} \cdots \tilde{F}_{i_r} H_{n\widehat{\Lambda}_0} \mid r \geqslant 0,\ i_1, \ldots, i_r \in \mathbb{Z}_{\geqslant 0}\} \setminus \{\mathbf{0}\}.$$

Then $\Psi_{\epsilon n}(\widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0) \otimes T_{-n\widehat{\Lambda}_0}) = \widehat{\mathfrak{M}}^\epsilon \cap \mathrm{Im}\, \Psi_{\epsilon n}$.

Proof. Let $\mathbf{w} \in \mathbf{B}(n\widehat{\Lambda}_0)$ be given. By Propositions 4.3 and 5.14, we have

$$\mathbf{w} \in \widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0) \quad \Leftrightarrow \quad \Psi_{\epsilon n}(\mathbf{w} \otimes t_{-n\widehat{\Lambda}_0}) \in \widehat{\mathfrak{M}}^\epsilon.$$

Hence, we obtain the required identity. $\square$

Remark 5.16. By Remark 4.4, (5.13) and Proposition 5.15, the map $\widehat{\kappa} \circ \Psi_{\epsilon n}$ gives a weight preserving bijection

$$\widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0) \otimes T_{-n\widehat{\Lambda}_0} \to \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \epsilon|\nu,\ \nu_1 \leqslant \epsilon n}} SST_{\mathbf{B}_{>0}}\big(\nu^\pi\big). \tag{5.14}$$

By [13, Theorem 5.1], $\widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0)$ is isomorphic to the crystal graph of the irreducible highest weight x_∞-module with highest weight $n\widehat{\Lambda}_0$, and Proposition 5.15 implies that $\widehat{\mathfrak{M}}^\epsilon$ is the limit of $\widehat{\mathbf{B}}^\epsilon(n\widehat{\Lambda}_0)$.

Let σ be the linear automorphism on P defined by $\sigma(\Lambda_0) = \Lambda_0$ and $\sigma(\epsilon_i) = -\epsilon_{-i}$ for $i \in \mathbb{Z}$. Then $\sigma(\alpha_i) = \alpha_{-i}$ for $i \in \mathbb{Z}$. Let

$$\widehat{W} = \{w \in W \mid w\sigma = \sigma w\}.$$

Put $\widehat{s}_0 = s_0$ and $\widehat{s}_i = s_i s_{-i}$ for $i \in \mathbb{Z}_{>0}$. Then $\widehat{W}$ is the Coxeter group generated by $\widehat{s}_i$ ($i \in \mathbb{Z}_{\geqslant 0}$), which is isomorphic to the Weyl group of x_∞ (see [5, Section 5.2]). Let $\widehat{W}^{\widehat{\Lambda}_0}$ be the set of minimal length left coset representatives of $\widehat{W}_{\widehat{\Lambda}_0}$. We have $\widehat{W}^{\widehat{\Lambda}_0} = \widehat{W} \cap W^{\Lambda_0}$, and $\lambda(w) = \lambda(w)'$ or $\phi(w) = \psi(w)$ for $w \in \widehat{W}^{\widehat{\Lambda}_0}$.

For $w \in \widehat{W}^{\widehat{\Lambda}_0}$ with a reduced expression $w = \widehat{s}_{i_1} \cdots \widehat{s}_{i_r}$ ($i_1, \ldots, i_r \in \mathbb{Z}_{\geqslant 0}$), let

$$\widehat{\mathcal{M}}^{\epsilon}_w = \left\{ \tilde{F}^{m_1}_{i_1} \cdots \tilde{F}^{m_r}_{i_r} \mathbb{O} \,\middle|\, m_1, \ldots, m_r \geqslant 0 \right\} \setminus \{\mathbf{0}\},$$
$$\widehat{\mathbf{B}}^{\epsilon}_w(n\widehat{\Lambda}_0) = \left\{ \tilde{F}^{m_1}_{i_1} \cdots \tilde{F}^{m_r}_{i_r} H_{n\widehat{\Lambda}_0} \,\middle|\, m_1, \ldots, m_r \geqslant 0 \right\} \setminus \{\mathbf{0}\}. \tag{5.15}$$

Since $\widehat{\mathbf{B}}^{\epsilon}(n\widehat{\Lambda}_0)$ is the crystal graph of an integrable highest weight x_∞-module, the Demazure crystal $\widehat{\mathbf{B}}^{\epsilon}_w(n\widehat{\Lambda}_0)$ is well defined and so is $\widehat{\mathcal{M}}^{\epsilon}_w$ by Proposition 5.15.

Proposition 5.17. *For $w \in \widehat{W}^{\widehat{\Lambda}_0}$, we have*

$$\widehat{\mathcal{M}}^{\epsilon}_w = \widehat{\mathcal{M}}^{\epsilon} \cap \mathcal{M}_w, \qquad \widehat{\kappa}\big(\widehat{\mathcal{M}}^{\epsilon}_w\big) = \bigsqcup_{\substack{\nu \in \mathscr{P} \\ \epsilon | \nu}} SST_{\mathbf{B}_{>0}}\big(\nu^{\pi}\big)_{\phi(w)}.$$

Proof. Let us prove the first identity. Then the second one follows from Proposition 5.7 and (5.13). First, it is clear by definition that $\widehat{\mathcal{M}}^{\epsilon}_w \subset \widehat{\mathcal{M}}^{\epsilon} \cap \mathcal{M}_w$. Suppose that $A \in \widehat{\mathcal{M}}^{\epsilon} \cap \mathcal{M}_w$ is given. Let $w' = \widehat{s}_{i_2} \cdots \widehat{s}_{i_r}$, where $w = \widehat{s}_{i_1} \cdots \widehat{s}_{i_r}$ is a reduced expression. If $i_1 > 0$, then we have

$$\tilde{e}_{i_1} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{e}_{-i_1} A \neq \mathbf{0} \quad \Leftrightarrow \quad \tilde{E}_{i_1} A \neq \mathbf{0},$$

since A is symmetric. If $i_1 = 0$, then we have $\tilde{e}_0 A \neq \mathbf{0}$ if and only if $\tilde{E}_0 A \neq \mathbf{0}$ by definition. From (5.7), it follows that $\tilde{E}^k_{i_1} A \in \mathcal{M}_{w'}$ for some $k \geqslant 0$. If we use induction on $\ell(w)$, then we have $\tilde{E}^k_{i_1} A \in \widehat{\mathcal{M}}^{\epsilon} \cap \mathcal{M}_{w'} = \widehat{\mathcal{M}}^{\epsilon}_{w'}$, and hence $A \in \widehat{\mathcal{M}}^{\epsilon}_w$. $\square$

Put $Z = \{z_i = x_i y_i \mid i \geqslant 1\}$. For $S \subset \mathbb{N}^2$, let $S' = \{(i,j) \mid (j,i) \in S\}$. Then Proposition 5.17 gives the following identity.

Corollary 5.18. *For $w \in \widehat{W}^{\widehat{\Lambda}_0}$, we have*

$$\sum_{\substack{S = S' \subset \mathbb{N}^2 \\ w(S) \leqslant w}} \prod_{(i,i) \in S} \frac{z_i^{\epsilon}}{1 - z_i^{\epsilon}} \prod_{\substack{(i,j) \in S \\ i < j}} \frac{z_i z_j}{1 - z_i z_j} = \sum_{\substack{\nu \in \mathscr{P} \\ \epsilon | \nu \\ \ell(\nu) \leqslant d(w)}} \widehat{s}_{\nu}(Z_{\alpha(w)}).$$

Corollary 5.19. *For $w \in \widehat{W}^{\widehat{\Lambda}_0}$ and $n \geqslant 1$, the map $\widehat{\kappa} \circ \Psi_{\epsilon n}$ gives a weight preserving bijection*

$$\widehat{\mathbf{B}}^{\epsilon}_w(n\widehat{\Lambda}_0) \otimes T_{-n\widehat{\Lambda}_0} \to \bigsqcup_{\substack{\nu \in \mathscr{P},\, \epsilon | \nu \\ \nu \subset (\epsilon n^{d(w)})}} SST_{\mathbf{B}_{>0}}\big(\nu^{\pi}\big)_{\phi(w)}.$$

Proof. It follows from Proposition 5.11 and (5.14). $\square$

For $i \in \mathbb{Z}_{\geqslant 0}$ and $\lambda \in \widehat{P}$, let $\widehat{D}_i$ be the linear operator on $\mathbb{C}[\widehat{P}] \subset \mathbb{C}[P]$ defined by

$$\widehat{D}_i\big(e^{\lambda}\big) = e^{\lambda} \frac{1 - e^{-(1 + \langle \lambda, \widehat{h}_i \rangle)\widehat{\alpha}_i}}{1 - e^{-\widehat{\alpha}_i}}.$$

For a reduced expression of $w = \widehat{s}_{i_1} \cdots \widehat{s}_{i_r} \in \widehat{W}$, put $\widehat{D}_w = \widehat{D}_{i_1} \cdots \widehat{D}_{i_r}$. Combining the Demazure character formula $\mathrm{ch}\, \widehat{\mathbf{B}}^{\epsilon}_w(n\widehat{\Lambda}_0) = \widehat{D}_w(e^{n\widehat{\Lambda}_0})$ with Corollary 5.19, we obtain the following identity.

Corollary 5.20. *Let $n, d \geqslant 1$, and let α be a strict partition of length d. Then*

$$\widehat{D}_w\big(e^{n\widehat{\Lambda}_0}\big)e^{-n\widehat{\Lambda}_0} = \sum_{\substack{\nu\in\mathscr{P},\,\epsilon|\nu \\ \nu\subset(\epsilon n^d)}} \widehat{s}_\nu(Z_\alpha),$$

where w is the unique element in $\widehat{W}^{\widehat{\Lambda}_0}$ such that $\lambda(w) = (\alpha|\alpha)$.

6. Plane partitions

A *plane partition* is a collection of non-negative integers $\pi = (\pi_{ij})_{i,j\geqslant 1}$ such that $\pi_{ij} \neq 0$ for only finitely many i, j, and $\pi_{ij} \geqslant \pi_{i+1j}$ and $\pi_{ij} \geqslant \pi_{ij+1}$ for $i, j \geqslant 1$. The *shape of* π denoted by $\mathrm{sh}(\pi)$ is a Young diagram determined by the support of π, i.e. $\{(i,j) \mid \pi_{ij} \neq 0\}$. We may identify π with a tableau of shape $\mathrm{sh}(\pi)$ with entries in $\mathbb{N}$ weakly decreasing in each row and column from left to right and top to bottom, respectively. Let $\mathcal{P}$ denote the set of plane partitions.

Let us recall the correspondence between $\mathcal{M}$ and $\mathcal{P}$ [2]. Let $A \in \mathcal{M}$ be given with $\kappa(A) = (\mathbf{P}(A), \mathbf{Q}(A))$. For $k \geqslant 1$, let $\lambda^{(k)} = (\alpha(k)|\beta(k))$ be a partition where $\alpha(k)$ and $\beta(k)$ are strict partitions given by reading the entries of the kth columns of $\mathbf{P}(A)$ and $\mathbf{Q}(A)$ from bottom to top (ignoring $-$ and $\vee$ in $\mathbf{P}(A)$), respectively. Note that $\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots$. We define $\pi(A) = (\pi(A)_{ij})_{i,j\geqslant 1}$ by $\pi(A)_{ij} = |\{k \mid (i,j) \in \lambda^{(k)}\}|$. It is easy to check that $\pi(A)$ is a plane partition, and the mapping $A \mapsto \pi(A)$ yields a bijection from $\mathcal{M}$ to $\mathcal{P}$. One may identify a plane partition π with a set of unit cubes, where π_{ij} cubes are stacked vertically at each position $(i,j) \in \mathrm{sh}(\pi)$. Then $\lambda^{(k)}$ is the kth layer of $\pi(A)$ from the bottom.

For $\pi \in \mathcal{P}$, let $|\pi| = \sum_{i,j}\pi_{ij}$ and for $r \in \mathbb{Z}$, let $\mathrm{tr}_r(\pi) = \sum_{i\geqslant 1}\pi_{ii+r}$, which is called the *$r$-trace of* π [6,30]. Let q and v_r $(r \in \mathbb{Z})$ be formal variables. For a subset X of $\mathcal{P}$, the *norm* (resp. *trace*) *generating function of* X is defined to be

$$\sum_{\pi\in X} q^{|\pi|} \quad \text{and} \quad \sum_{\pi\in X}\prod_{r\in\mathbb{Z}} v_r^{\mathrm{tr}_r(\pi)},$$

respectively. Note that a norm generating function can be obtained from a trace generating function by specializing $v_r = q$ for $r \in \mathbb{Z}$.

For $n \geqslant 1$ and $\lambda \in \mathscr{P}$, we put

$$\begin{aligned} \mathcal{P}_{\leqslant n} &= \{\pi \in \mathcal{P} \mid \pi_{11} \leqslant n\}, \\ \mathcal{P}(\lambda) &= \big\{\pi \in \mathcal{P} \,\big|\, \mathrm{sh}(\pi) \subset \lambda\big\}, \\ \mathcal{P}(\lambda)_{\leqslant n} &= \mathcal{P}_{\leqslant n} \cap \mathcal{P}(\lambda). \end{aligned}$$

Note that $\mathcal{P}_{\leqslant 1}$ is the set of ordinary partitions $\mathscr{P}$, and hence a $\mathfrak{gl}_\infty$-crystal [27], where for $\lambda \in \mathscr{P}$ and $r \in \mathbb{Z}$, $\tilde{f}_r\lambda$ is defined to be a partition μ such that μ/λ is a single box with residue r, or $\mathbf{0}$ if such μ does not exist. If we define the weight of the empty partition to be 0, then it is easy to see that $\mathcal{P}_{\leqslant 1}$ is isomorphic to $\mathbf{B}(\Lambda_0) \otimes T_{-\Lambda_0}$. For $n \geqslant 2$, we define a $\mathfrak{gl}_\infty$-crystal structure on $\mathcal{P}_{\leqslant n}$ by identifying $\pi = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}) \in \mathcal{P}_{\leqslant n}$ with an element of $\mathbf{B}(\Lambda_0)^{\otimes n} \otimes T_{-n\Lambda_0}$, where $\lambda^{(k)}$ is the kth layer of π. Now, we define a $\mathfrak{gl}_\infty$-crystal structure on $\mathcal{P}$ by identifying each $\pi \in \mathcal{P}$ with an element of $\mathbf{B}(\Lambda_0)^{\otimes n} \otimes T_{-n\Lambda_0}$ for a sufficiently large n. Then $\mathcal{P}$ is a well-defined $\mathfrak{gl}_\infty$-crystal and the inclusion map of $\mathcal{P}_{\leqslant n}$ into $\mathcal{P}$ is a $\mathfrak{gl}_\infty$-crystal embedding.

Proposition 6.1. *As $\mathfrak{gl}_\infty$-crystals, we have*

(1) $\mathcal{P} \simeq \mathcal{M}$,
(2) $\mathcal{P}_{\leqslant n} \simeq \mathbf{B}(n\Lambda_0) \otimes T_{-n\Lambda_0}$ *for* $n \geqslant 1$.

Proof. Suppose that $\pi = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}) \in \mathcal{P}_{\leqslant n}$ is given. Recall that we identify π with an element in $\mathbf{B}(\Lambda_0)^{\otimes n} \otimes T_{-n\Lambda_0}$. Since $\lambda^{(1)} \supset \cdots \supset \lambda^{(n)}$, π is an element in $\mathbf{B}(n\Lambda_0) \otimes T_{-n\Lambda_0}$, where $\lambda^{(k)}$ corresponds to the kth column from the right of the semi-infinite semistandard tableau associated with π. Hence we have a bijection from $\mathcal{P}_{\leqslant n}$ to $\mathbf{B}(n\Lambda_0) \otimes T_{-n\Lambda_0}$, say ϕ_n. By definition of the $\mathfrak{gl}_\infty$-crystal structure on $\mathcal{P}_{\leqslant n}$, it is straightforward to check that ϕ_n commutes with $\tilde{x}_i$ ($x = e, f$, $i \in \mathbb{Z}$). This proves (2).

Now, let us prove (1). Suppose that $\pi \in \mathcal{P}$ is given. Choose a sufficiently large n such that $\pi \in \mathcal{P}_{\leqslant n}$. Put $A = \Psi_n \circ \phi_n(\pi)$. By definition of Ψ_n and ϕ_n, A does not depend on the choice of n, and $\pi(A) = \pi$. This implies that the map $\pi \mapsto A$ is a morphism of $\mathfrak{gl}_\infty$-crystals, and hence an isomorphism since it is a bijection. $\square$

Remark 6.2. We have shown in the proof of Proposition 6.1 that the correspondence $A \mapsto \pi(A)$ is a $\mathfrak{gl}_\infty$-crystal isomorphism from $\mathcal{M}$ to $\mathcal{P}$, where the subcrystal $\operatorname{Im}\Psi_n \simeq \mathbf{B}(n\Lambda_0) \otimes T_{-n\Lambda_0}$ is mapped to $\mathcal{P}_{\leqslant n}$.

Corollary 6.3.

(1) $\operatorname{ch}\mathcal{M}$ *is the trace generating function of* $\mathcal{P}$.
(2) $e^{-n\Lambda_0}\operatorname{ch}\mathbf{B}(n\Lambda_0)$ *is the trace generating function of* $\mathcal{P}_{\leqslant n}$ *for* $n \geqslant 1$.

Proof. For $\pi \in \mathcal{P}$, we have $\mathrm{wt}(\pi) = -\sum_{r\in\mathbb{Z}} k_r\alpha_r$, where k_r is equal to the r-trace of π. Identifying $e^{-\alpha_r}$ with v_r in $\operatorname{ch}\mathcal{M}$ and $e^{-n\Lambda_0}\operatorname{ch}\mathbf{B}(n\Lambda_0)$, we obtain the trace generating functions for $\mathcal{P}$ and $\mathcal{P}_{\leqslant n}$, respectively. $\square$

Remark 6.4. By the celebrated Weyl–Kac character formula [8], the trace generating function of $\mathcal{P}_{\leqslant n}$ is

$$\frac{\sum_{w\in W} e^{w(n\Lambda_0+\rho)-n\Lambda_0-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})},$$

where $\rho \in \mathfrak{h}^*$ is given by $\langle \rho, h_i \rangle = 1$ for all $i \in \mathbb{Z}$. The norm generating function for $\mathcal{P}$ and $\mathcal{P}_{\leqslant n}$ are the corresponding principal q-characters, which are obtained by putting $e^{-\alpha_i} = q$ for $i \in \mathbb{Z}$. Then we recover

$$\operatorname{ch}_q \mathcal{M} = \frac{1}{\prod_{i\geqslant 1}(1-q^i)^i} \quad \text{and} \quad \operatorname{ch}_q \mathbf{B}(n\Lambda_0) = \frac{1}{\prod_{i\geqslant 1}(1-q^i)^{\min(i,n)}}$$

(see [8] for evaluating q-characters), which are originally due to MacMahon [26].

Now, consider $\mathcal{P}(\lambda)$ and $\mathcal{P}(\lambda)_{\leqslant n}$, which are subcrystals of $\mathcal{P}$ and $\mathcal{P}_{\leqslant n}$, respectively. Then the results in the previous section give the following representation theoretic interpretations on them.

Proposition 6.5. *Let* $\lambda \in \mathscr{P}$ *be given, and let* $w \in W^{\Lambda_0}$ *be such that* $\lambda(w) = \lambda$. *As* $\mathfrak{gl}_\infty$-*crystals, we have*

(1) $\mathcal{P}(\lambda) \simeq \mathcal{M}_w$,
(2) $\mathcal{P}(\lambda)_{\leqslant n} \simeq \mathbf{B}_w(n\Lambda_0) \otimes T_{-n\Lambda_0}$ *for* $n \geqslant 1$.

Proof. (1) Let $A \in \mathcal{M}$ be given. By Theorem 5.7, we see that $A \in \mathcal{M}_w$ if and only if $\lambda^{(1)} \subset \lambda(w)$, where $\lambda^{(1)}$ is the first layer of $\pi(A)$. Hence $\pi(\mathcal{M}_w) = \mathcal{P}(\lambda)$. By Proposition 6.1, we have $\mathcal{P}(\lambda) \simeq \mathcal{M}_w$ (see Remark 6.2).

(2) Since $\pi(\mathcal{M}_w) = \mathcal{P}(\lambda)$ and $\pi(\operatorname{Im}\Psi_n) = \mathcal{P}_{\leqslant n}$, we have

$$\mathcal{P}(\lambda)_{\leqslant n} = \pi(\mathcal{M}_w \cap \operatorname{Im}\Psi_n) = \pi \circ \Psi_n\big(\mathbf{B}_w(n\Lambda_0) \otimes T_{-n\Lambda_0}\big) \simeq \mathbf{B}_w(n\Lambda_0) \otimes T_{-n\Lambda_0}$$

by Proposition 5.11. □

By Corollary 6.3 and Proposition 6.5, we obtain the generating functions for plane partitions bounded by a given shape as follows.

Corollary 6.6. *Let $\lambda \in \mathscr{P}$ be given, and let $w \in W^{\Lambda_0}$ be such that $\lambda(w) = \lambda$.*

(1) *The trace generating function of $\mathcal{P}(\lambda)$ is*

$$\sum_{\substack{S \subset \Delta(\mathfrak{u}_-) \\ \lambda(S) \subset \lambda}} \prod_{\alpha \in S} \frac{e^{\alpha}}{1 - e^{\alpha}}.$$

(2) *The trace generating function of $\mathcal{P}(\lambda)_{\leqslant n}$ is $e^{-n\Lambda_0} D_w(e^{n\Lambda_0})$ for $n \geqslant 1$.*

Remark 6.7. (1) There are determinantal formulas for the norm and trace generating functions of various classes of plane partitions including $\mathcal{P}(\lambda)$ and $\mathcal{P}(\lambda)_{\leqslant n}$ (see [16] for a most general form and the references therein for the previous works by other people). Also there are evaluations of those determinants into nice product forms for some special classes of plane partitions. But there is no such formula for $\mathcal{P}(\lambda)$ and $\mathcal{P}(\lambda)_{\leqslant n}$ as far as we know.

(2) A representation theoretic approach to plane partitions was first introduced by Proctor [28], where the norm generating functions for $\mathcal{P}((u^v))_{\leqslant n}$ was proved to be the q-dimension of the irreducible $\mathfrak{sl}_{u+v}$-module with highest weight $n\omega$ (ω is the uth fundamental weight).

A plane partition $\pi = (\pi_{ij})$ is called *symmetric* if $\pi_{ij} = \pi_{ji}$ for all $i, j \geqslant 1$. Similarly for $\epsilon = 1, 2$, $n \geqslant 1$ and $\lambda \in \mathscr{P}$ with $\lambda = \lambda'$, we put

$$\begin{aligned}
\widehat{\mathcal{P}}^{\epsilon} &= \{\pi \in \mathcal{P} \mid \pi \text{ is symmetric and } \epsilon \text{ divides } \pi_{ii} \text{ for all } i \geqslant 1\}, \\
\widehat{\mathcal{P}}^{\epsilon}_{\leqslant n} &= \mathcal{P}_{\leqslant n} \cap \widehat{\mathcal{P}}^{\epsilon}, \\
\widehat{\mathcal{P}}(\lambda)^{\epsilon} &= \mathcal{P}(\lambda) \cap \widehat{\mathcal{P}}^{\epsilon}, \\
\widehat{\mathcal{P}}(\lambda)^{\epsilon}_{\leqslant n} &= \mathcal{P}_{\leqslant n}(\lambda) \cap \widehat{\mathcal{P}}^{\epsilon}.
\end{aligned}$$

Note that $\widehat{\mathcal{M}}^{\epsilon}$ is in one-to-one correspondence with $\widehat{\mathcal{P}}^{\epsilon}$ by (5.13). As in Section 5.6, we assume that $x = \mathfrak{b}$ if $\epsilon = 1$ and $x = \mathfrak{c}$ if $\epsilon = 2$. We define an x_{∞}-crystal structure on $\widehat{\mathcal{P}}^{\epsilon} \subset \mathcal{P}$ with $\tilde{E}_i$, $\tilde{F}_i$ for $i \geqslant 0$ (5.12). Similar to Propositions 6.1 and 6.5, we can prove the following.

Proposition 6.8. *Let $\lambda \in \mathscr{P}$ with $\lambda = \lambda'$ be given, and let $w \in \widehat{W}^{\widehat{\Lambda}_0}$ be such that $\lambda(w) = \lambda$. As x_{∞}-crystals, we have*

(1) $\widehat{\mathcal{P}}^{\epsilon} \simeq \widehat{\mathcal{M}}^{\epsilon}$,
(2) $\widehat{\mathcal{P}}^{\epsilon}_{\leqslant n} \simeq \widehat{\mathbf{B}}^{\epsilon}(n\widehat{\Lambda}_0) \otimes T_{-n\widehat{\Lambda}_0}$ *for $n \geqslant 1$,*
(3) $\widehat{\mathcal{P}}^{\epsilon}(\lambda) \simeq \widehat{\mathcal{M}}^{\epsilon}_w$,
(4) $\widehat{\mathcal{P}}(\lambda)^{\epsilon}_{\leqslant n} \simeq \widehat{\mathbf{B}}^{\epsilon}_w(n\widehat{\Lambda}_0) \otimes T_{-n\widehat{\Lambda}_0}$ *for $n \geqslant 1$.*

Remark 6.9. (1) We define for a subset X of $\widehat{\mathcal{P}}^{\epsilon}$ the norm (resp. trace) generating function of X by

$$\sum_{\pi\in X} q^{|\pi|} \quad \text{and} \quad \sum_{\pi\in X}\prod_{r\geqslant 0} v_r^{\mathrm{tr}'_r(\pi)},$$

respectively, where $\mathrm{tr}'_r(\pi)=\mathrm{tr}_r(\pi)$ for $r\geqslant 1$ and $\mathrm{tr}'_0(\pi)=\epsilon^{-1}\,\mathrm{tr}_0(\pi)$. For $\pi\in\widehat{\mathcal{P}}^{\epsilon}$, we have $\mathrm{wt}(\pi)=-\sum_{r\geqslant 0}k_r\widehat{\alpha}_r$, where $k_r=\mathrm{tr}'_r(\pi)$. Hence, identifying $e^{-\widehat{\alpha}_r}$ with v_r, we obtain the trace generating functions for symmetric plane partitions in Proposition 6.8 as the characters of the corresponding x_∞-crystals. The norm generating functions can be obtained by specializing $e^{-\widehat{\alpha}_r}=q^2$ for $r\geqslant 1$ and $e^{-\widehat{\alpha}_0}=q^{\epsilon}$.

(2) The norm generating function of $\widehat{\mathcal{P}}(m^m)^1_{\leqslant n}$ was conjectured by MacMahon [26], and it was proved by Andrews [1] and Macdonald [25]. It was observed by Proctor [28,29] that the norm generating function of $\widehat{\mathcal{P}}(m^m)^{\epsilon}_{\leqslant n}$ is the q-dimension of the irreducible representation of the complex simple Lie algebra $\mathfrak{so}(2m+1)\subset\mathfrak{b}_\infty$ or $\mathfrak{sp}(2m)\subset\mathfrak{c}_\infty$ with highest weight corresponding to $n\widehat{\Lambda}_0$ (following our notation).

(3) Recently Tingley [33] gave a nice representation theoretic interpretation of cylindric plane partitions in terms of crystal graphs for affine Lie algebra $\widehat{\mathfrak{sl}}_n$ and its generating function. It would be interesting to find an application of affine Demazure crystals to cylindric plane partitions.

Acknowledgments

The author would like to thank A. Lascoux for helpful comments and discussion on this work, and the referee for careful reading and pointing out some mistakes in the previous version of this article.

References

[1] G.E. Andrews, Plane partitions. I. The MacMahon conjecture, in: Studies in Foundations and Combinatorics, in: Adv. in Math. Suppl. Stud., vol. 1, Academic Press, New York, 1978, pp. 131–150.
[2] E.A. Bender, D.E. Knuth, Enumeration of plane partitions, J. Combin. Theory Ser. A 13 (1972) 40–54.
[3] J. Brundan, Dual canonical bases and Kazhdan–Lusztig polynomials, J. Algebra 306 (2006) 17–46.
[4] V.I. Danilov, G.A. Koshevoy, Bi-crystals and crystal $(GL(V), GL(W))$ duality, RIMS preprint 1458 (2004).
[5] J. Fuchs, A.N. Schellekens, C. Schweigert, From Dynkin diagram symmetries to fixed point structures, Comm. Math. Phys. 180 (1996) 39–97.
[6] E.R. Gansner, The Hillman–Grassl correspondence and the enumeration of reverse plane partitions, J. Combin. Theory Ser. A 30 (1981) 71–89.
[7] R. Howe, Remarks on classical invariant theory, Trans. Amer. Math. Soc. 313 (1989) 539–570.
[8] V.G. Kac, Infinite-Dimensional Lie Algebras, third ed., Cambridge Univ. Press, Cambridge, 1990.
[9] V.G. Kac, A. Radul, Representation theory of the vertex algebra $W_{1+\infty}$, Transform. Groups 1 (1996) 41–70.
[10] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, Comm. Math. Phys. 133 (1990) 249–260.
[11] M. Kashiwara, Crystal bases and Littelmann's refined Demazure character formula, Duke Math. J. 71 (1993) 839–858.
[12] M. Kashiwara, On crystal bases, in: Representations of Groups, in: CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.
[13] M. Kashiwara, Similarity of crystal bases, in: Contemp. Math., vol. 194, 1996, pp. 177–186.
[14] M. Kashiwara, T. Nakashima, Crystal graphs for representations of the q-analogue of classical Lie algebras, J. Algebra 165 (1994) 295–345.
[15] D.E. Knuth, Permutations, matrices, and generalized Young tableaux, Pacific J. Math. 34 (1970) 709–727.
[16] C. Krattenthaler, Generating functions for plane partitions of a given shape, Manuscripta Math. 69 (1990) 173–201.
[17] J.-H. Kwon, Crystal graphs for Lie superalgebras and Cauchy decomposition, J. Algebraic Combin. 25 (2007) 57–100.
[18] J.-H. Kwon, Rational semistandard tableaux and character formula for the Lie superalgebra $\widehat{\mathfrak{gl}}_{\infty|\infty}$, Adv. Math. 217 (2008) 713–739.
[19] J.-H. Kwon, A combinatorial proof of a Weyl-type formula for hook Schur polynomials, J. Algebraic Combin. 28 (2008) 439–459.
[20] A. Lascoux, Double crystal graphs, in: Studies in Memory of Issai Schur, in: Progr. Math., vol. 210, Birkhäuser, 2003, pp. 95–114.
[21] C. Lecouvey, Crystal bases and combinatorics of infinite rank quantum groups, Trans. Amer. Math. Soc. 361 (2009) 297–329.
[22] C. Lenart, A unified approach to combinatorial formulas for Schubert polynomials, J. Algebraic Combin. 20 (2004) 263–299.
[23] M.A.A. van Leeuwen, Double crystals of binary and integral matrices, Electron. J. Combin. 13 (2006).
[24] I.G. Macdonald, Notes on Schubert Polynomials, Laboratoire de combinatoire et d'informatique mathématique (LACIM), Université du Québec à Montrèal, Montreal, 1991.

[25] I.G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Univ. Press, Oxford, 1995.
[26] P.A. MacMahon, Combinatory Analysis, vols. 1–2, Cambridge Univ. Press, Cambridge, 1915–1916; reprinted in one volume by Chelsea, New York, 1960.
[27] K. Misra, T. Miwa, Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$, Comm. Math. Phys. 134 (1990) 79–88.
[28] R.A. Proctor, Bruhat lattices, plane partition generating functions, and minuscule representations, European J. Combin. 5 (1984) 331–350.
[29] R.A. Proctor, New symmetric plane partition identities from invariant theory work of De Concini and Procesi, European J. Combin. 11 (1990) 289–300.
[30] R.P. Stanley, The conjugate trace and trace of a plane partition, J. Combin. Theory Ser. A 14 (1973) 53–65.
[31] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge Univ. Press, Cambridge, 1998.
[32] B.E. Sagan, R. Stanley, Robinson–Schensted algorithms for skew tableaux, J. Combin. Theory Ser. A 55 (1990) 161–193.
[33] P. Tingley, Three combinatorial models for affine $\mathfrak{sl}_n$ crystals, with applications to cylindric plane partitions, Int. Math. Res. Not. 143 (2007).