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INDECOMPOSABLE FACTORIZATIONS OF MULTIGRAPHS

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A one-factorization of a complete multigraph is called *decomposable* if some proper subset of the factors also forms a one-factorization of a complete multigraph; otherwise it is indecomposable. Some results on the existence of indecomposable one-factorizations will be proven.

1. Introduction

Standard graph-theoretic notions are assumed. Our graphs will all be finite and undirected. To avoid an abundance of braces, we denote the edge joining vertex x to vertex y by (x, y), even though it is undirected, or by xy if possible.

The complete multigraph λK_v has v vertices and there are λ edges joining each pair of vertices. A one-factor of λK_v is a set of edges which between them contain every vertex precisely once; a one-factorization is a set of one-factors which precisely partitions the edges of λK_v.

Clearly v must be even for a one-factor to exist; say $v = 2n$. It is well known that K_{2n} has a one-factorization for every n (see, for example, [4, p. 439]). Taking λ copies of this yields a one-factorization of λK_{2n}. So every λK_{2n} has a one-factorization.

Given a one-factorization of λK_{2n}, it may be that there exists an integer λ_1 (less than λ) such that some $\lambda_1(2n-1)$ of the one-factors form a one-factorization of $\lambda_1 K_{2n}$. In that case the one-factorization of λK_{2n} is called *decomposable*; otherwise it is *indecomposable*. When $\lambda > 1$, the one-factorizations of λK_{2n} just exhibited are all decomposable. It is natural to ask for which values of λ and n do there exist indecomposable one-factorizations of λK_{2n}.

A one-factorization is called simple if it contains no repeated one-factor. There is no direct correspondence between simplicity and indecomposability. However, simple one-factorizations will be useful in the sequel.

In this paper we present some results on the existence of indecomposable one-factorizations which are not simple. Some of these involve so-far unrepresented parameters λ and $2n$. Others have parameters which were previously

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constructed; but as most known results involve simple factorizations, we prove the existence of non-isomorphic indecomposable one-factorizations in infinitely many cases.

Some notation will be convenient. We write N_k for the set of the first k positive integers. If $\mathcal{F}$ is a one factorization of the K_{2n} based on vertex-set N_{2n}, and U is any ordered $2n$-set, then $\mathcal{F}(U)$ is constructed by replacing i by the ith member of U in every factor of $\mathcal{F}$, for every i. The factor derived from the factor F of $\mathcal{F}$ is denoted $F(U)$.

We also use one-factorizations of the complete bipartite graph $K_{n,n}$. If $\mathcal{L}$ is a one-factorization of the $K_{n,n}$ based on the two vertex-sets N_n and $N_{2n}\backslash N_n$, and U and V are ordered n-sets, then $\mathcal{L}(U, V)$ is the one-factorization formed from $\mathcal{L}$ by the substitutions

$$(1, 2, \ldots, n) \mapsto U$$

$$(n+1, n+2, \ldots, 2n) \mapsto V.$$

We say a one-factorization $\mathcal{F}$ of K_{2n} is *standardized* if the K_{2n} is based on N_{2n} and the ith factor contains $(i, 2n)$. A one-factorization of $K_{n,n}$ is standardized if the vertex-sets are N_n and $N_{2n}\backslash N_n$ and the first factor is

$$\{(i, n+1), (2, n+2), \ldots, (n, 2n)\}.$$

2. Known results

The following two general results appear in [2].

Theorem 1. *If $2n-1$ is prime then there is a simple indecomposable one-factorization of $(n-1)K_{2n}$.*

Theorem 2. *If there is an indecomposable one-factorization of λK_{2n}, where $\lambda < 2n$, then there is a simple indecomposable one-factorization of λK_{2s} whenever $s \geqslant 2n$.*

These results were used in [2] together with some ad hoc constructions to obtain a number of results on the existence of simple indecomposable one-factorizations for small λ:

Theorem 3. *A simple indecomposable one-factorization of λK_{2n} exists as follows:*

$\lambda = 2$: *if and only if* $2n \geqslant 6$;

$\lambda = 3$: *if and only if* $2n \geqslant 8$;

$\lambda = 4$: *if and only if* $2n \geqslant 8$;

$\lambda = 5$: *if* $2n = 10$, 12, 14 *or* $2n \geqslant 20$;

$\lambda = 6$: *if* $2n \geqslant 12$;

$\lambda = 8$: *if* $2n = 12$ *or* 14 *or* $2n \geqslant 24$;

$\lambda = 9$: *if* $2n = 12$ *or* 14 *or* $2n \geqslant 24$;

$\lambda = 10$: *if* $2n = 14$ *or* $2n \geqslant 28$;

$\lambda = 12$: *if* $2n \geqslant 32$.

In the course of proving this theorem, indecomposable (but non-simple) one-factorizations of $6K_8$ and $12K_{16}$ were found. It is known [2] that no indecomposable one-factorizations of λK_4 ($\lambda > 1$) or $3K_6$ exist. Apart from these two nonexistence results, the only known result about the enumeration of one-factorizations of λK_{2n} is the fact that there are precisely three non-isomorphic one-factorizations of $2K_6$, of which exactly one is indecomposable [5].

3. An upper bound

Since there are exactly $1 \cdot 3 \cdots (2n-1)$ one-factors of K_{2n}, the largest λ such that λK_{2n} has a simple factorization is

$$\lambda = 1 \cdot 3 \cdots (2n-3),$$

and for a simple indecomposable factorization we must have

$$\lambda < 1 \cdot 3 \cdots (2n-3).$$

However, this bound does not apply to indecomposable factorizations when simplicity is not required. We shall now derive a bound (which is probably very coarse) in that more general case.

By an *exact cover of depth d* on a set S we mean a collection of subsets of S, called blocks, such that each member of S belongs to exactly d blocks. (Repeated blocks are allowed.) If all the blocks are k-sets, the exact cover is called *regular* of degree k. *An exact cover in S* is *decomposable* if some proper subcollection of its blocks forms an exact cover on S. It is known (see [3]) that every sufficiently deep exact cover is decomposable: given s, there exists a positive integer $D[s]$ such that any exact cover of depth greater than $D[s]$ on an s-set is decomposable. It follows that there is also a maximum depth for a regular exact cover of degree k on an s-set: we denote it $D[s, k]$.

Lemma 4 [1]. *Whenever* $s \geqslant k \geqslant 1$,

$$D[s, k] < s^k \cdot \binom{sk + s + 1}{s}.$$

Theorem 5. *If there is an indecomposable factorization of λK_{2n}, then*

$$\lambda < [n(2n-1)]^{n(2n-1)} \binom{2n^3+n^2-n+1}{2n^2-n}.$$

Proof. Suppose there is an indecomposable factorization $\mathcal{F}$ of λK_{2n}. Denote by S the set of all edges of K_{2n}: S is a set of size $n(2n-1)$. The factors in $\mathcal{F}$, interpreted as subsets of S, form an n-regular exact cover of depth d on S. So

$$\lambda \leqslant D[n(2n-1), n],$$

giving the result. □

4. The case $2n = 6$

In this section we prove that no indecomposable one-factorization of λK_6 can exist for $\lambda \geqslant 3$. We assume there is an indecomposable one-factorization $\mathcal{F}$ of λK_6 for some $\lambda \geqslant 3$ and derive a contradiction. (Recall that the result was already known in the case $\lambda = 3$.)

For notational convenience we assume K_6 to have vertices 0, 1, 2, 3, 4, 5. Since the fifteen one-factors of K_6 form a one-factorization of $3K_6$, not all of them can appear in $\mathcal{F}$: say $\{01, 23, 45\}$ is not represented. We denote the other possible one-factors as follows; say A occurs a times in $\mathcal{F}$, and so on.

$A = \{01, 24, 35\}$	$H = \{03, 15, 24\}$
$B = \{01, 25, 34\}$	$I = \{04, 12, 35\}$
$C = \{02, 13, 45\}$	$J = \{04, 13, 25\}$
$D = \{02, 14, 35\}$	$K = \{04, 15, 23\}$
$E = \{02, 15, 34\}$	$L = \{05, 12, 34\}$
$F = \{03, 12, 45\}$	$M = \{05, 13, 24\}$
$G = \{03, 13, 25\}$	$N = \{05, 14, 23\}$

Since edge 01 must appear in λ factors, we have

$$a + b = \lambda. \tag{1}$$

One could derive fourteen more equations in this way. In particular, considering 24, 02, 14 and 34 we get

$$k + n = \lambda \tag{2}$$

$$c + d + e = \lambda \tag{3}$$

$$d + g + n = \lambda \tag{4}$$

$$b + e + l = \lambda \tag{5}$$

and (1) + (2) + (3) − (4) − (5) is

$$a + c - g + k - l = \lambda \qquad (6)$$

We can assume that $a \geq \frac{1}{2}\lambda$ and $k \geq \frac{1}{2}\lambda$: if $a < \frac{1}{2}\lambda$ and $k < \frac{1}{2}\lambda$, then carry out the permutation (01)(45) on all members of $\mathcal{F}$ – it exchanges A with B and K with N and leaves {01, 23, 45} unchanged; if $a < \frac{1}{2}\lambda$ and $k \geq \frac{1}{2}\lambda$ then (01) is the relevant permutation; if $a \geq \frac{1}{2}\lambda$ and $k < \frac{1}{2}\lambda$ then use (45).

The factors $\{A, C, G, K, L\}$ form a one-factorization of K_6, so a, c, g, k and l cannot all be non-zero. The permutation (01)(23)(45) exchanges G and L, and leaves A, C and {01, 23, 45} unchanged; so without loss of generality we can assume $g \leq l$.

Since a and k are positive, this means we can assume either c or g to be zero. But the equations derived from considering edges 03 and 45 are

$$f + g + h = \lambda, \qquad (7)$$

$$c + f = \lambda, \qquad (8)$$

whence $h = c - g$ and $c \geq g$. So $g = 0$. Substituting this into equation (6), and recalling that $a \geq \frac{1}{2}\lambda$ and $k \geq \frac{1}{2}\lambda$ we obtain $c - l \leq 0$. Counting occurrences of 12 we see that

$$f + i + l = \lambda;$$

from (8) we get $i = c - l$, and as i cannot be negative we have $c = l$ and $a = k = \frac{1}{2}\lambda$. Equation (5) tells us now that $e = a - c$, so $c \leq a \leq \frac{1}{2}\lambda$, and therefore from (8) f is non-zero. Since not all the members of the one-factorization $\{A, E, F, J, N\}$ can be represented, $e = 0$, whence c must equal $\frac{1}{2}\lambda$ also.

It is now easy to see that $e = g = i = m = 0$, and that the other ten factors each occur $\frac{1}{2}\lambda$ times. (The equations derived from edges 04 and 05 give the information about i and m.) If λ is odd, we have a contradiction. Otherwise we have $\frac{1}{2}\lambda$ duplicates of the one-factorization $\{A, B, C, D, F, H, J, K, L, N\}$ of $2K_6$, and $\mathcal{F}$ is decomposable. So we have proven the following Theorem.

Theorem 6. *There is no indecomposable one-factorization of λK_6 when $\lambda \geq 3$.*

5. Doubling constructions

Theorem 7. *Suppose there exists an indecomposable one-factorization of λK_{2n} for some $\lambda > 1$. Then there exists an indecomposable one-factorization of λK_{4n} which is not simple.*

Proof. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_{\lambda(2n-1)}\}$ is an indecomposable one-factorization of the λK_{2n} based on N_{2n}. Select two ordered $2n$-sets U and V, and a standardized one-factorization $\mathcal{L}$ of $K_{2n,2n}$. Then the factors in the one-factorization $\mathscr{L}(U, V)$, together with the $\lambda(2n-1)$ factors $F_i(U) \cup F_i(V)$, $1 \leq i \leq \lambda(2n-1)$, form a non-simple one-factorization of K_{4n}.

Suppose this factorization were decomposable: say $\{H_1, H_2, \ldots, H_s\}$ were a one-factorization of μK_{4n} where the H_i are among the factors listed. Write H_i' for the intersection of H_i with the K_{2n} based on U. Then $\{H_1', H_2', \ldots, H_s'\}$ is a one-factorization of the μK_{2n} based on U, and $\mathscr{F}(U)$ is indecomposable – a contradiction. □

Theorem 8. *Suppose there exists an indecomposable one-factorization of λK_{2n}, for some $\lambda > 1$. Then there exists an indecomposable one-factorization of λK_{4n-2} which is not simple.*

Proof. Suppose $\mathscr{F}$ is an indecomposable one-factorization of λK_{2n}. Select two ordered $2n$-sets $U = (u_1, u_2, \ldots, u_{2n})$ and $(v_1, v_2, \ldots, v_{2n})$ and write $U^* = U \setminus \{u_{2n}\}$, $V^* = V \setminus \{v_{2n}\}$. If the factor F of $\mathscr{F}$ contains the edge $(i, 2n)$ then $F(U)$ contains (u_i, u_{2n}); define F^* to be $F(U) \cup F(V)$ with (u_i, u_{2n}) and (v_i, v_{2n}) deleted and (u_i, v_i) appended. Also select a standardized one-factorization $\mathscr{L}$ of $K_{2n-1,2n-1}$, and define $\mathscr{L}^*$ to consist of λ copies of the factorization $\mathscr{L}(U^*, V^*)$, with all λ copies of the factor

$$\{(u_1, v_1), (u_2, v_2), \ldots, (u_{2n-1}, v_{2n-1})\}$$

removed. Then

$$\mathscr{L}^* \cup \{F_1^*, F_2^*, \ldots, F_{\lambda(2n-1)}^*\}$$

is the required one-factorization of λK_{4n-2}. □

6. Further directions

It must be pointed out that the results here and in [2] barely scratch the surface of a hard problem.

The main constructions in both papers are recursive, and in particular yield a factorization with the same λ-value as some known factorization. No indecomposable factorization of $7K_{2n}$ is known for any n, and no case of λK_{2n} has been solved for any λ greater than 12 where $2\lambda + 1$ is not prime. Further direct constructions are needed. In particular, ad hoc constructions for $5K_{16}$, $6K_{10}$ or any $7K_{2n}$ would be very useful.

In Section 4 we obtained a constraint corresponding to each edge of K_6; as K_6 has 15 edges and 15 one-factors, enough information existed for a proof. As $2n$ increases, the number of one-factors of K_{2n} goes up much faster than the number of edges, so no generalization of this method can be expected. It is even conceivable that indecomposable one-factorizations of λK_8 exist for any λ less than $28^{28} \cdot \binom{141}{28}$, the bound from Theorem 5 (although, if pressed, we would guess that this is not so). Any light on this problem would be significant.

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