A condition for a hamiltonian bipartite graph to be bipancyclic

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Abstract


Let $G$ be a hamiltonian bipartite graph of order $2n$ and let $C = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1)$ be a hamiltonian cycle of $G$. $G$ is said to be bipancyclic if it contains a cycle of length $2i$, for every $i, 2 < i < n$. Suppose the vertices $x_1$ and $x_2$ are such that $d(x_1) + d(x_2) \geq n + 1$. Then $G$ is either:

1. bipancyclic,
2. missing a 4-cycle (then $n$ is odd and the structure of $G$ is known),
3. missing a $(n + 1)$-cycle (then $n$ is odd and the structure of $G$ is known).

1. Introduction

Let $G$ be a finite graph of order $n$. Various sufficient conditions for a graph to be hamiltonian have been given in terms of vertex-degree and size of the graph [5–8]. Almost all these conditions imply the graph to be pancyclic, [2, 4, 12], following the meta-conjecture of Bondy. But if the graph is supposed to be hamiltonian, we need weaker conditions in order for the graph to be pancyclic [1, 3]. When $G$ is a balanced bipartite graph, we have analogous results [9–11]. In [13] Schmeichel and Hakimi proved the following theorem.

**Theorem.** If there exist two vertices $x_1$ and $x_2$, consecutive on a hamiltonian cycle of a graph $G$ of order $n$, such that $d(x_1) + d(x_2) \geq n$, then $G$ is either:

1. pancyclic,
2. bipartite,
3. missing only an $(n - 1)$-cycle.

We establish the analogous result for hamiltonian bipartite graphs.
**Theorem.** If there exist two vertices \( x_1 \) and \( x_2 \), the distance between which is two, on a Hamiltonian cycle of a bipartite graph \( G \) of order \( 2n \), such that:

\[
d(x_1) + d(x_2) \geq n + 1
\]

then \( G \) is bipancyclic except in two cases:

1. \( n \) is odd,
   
   \[
   N(x_1) = \{ y_j : j \text{ odd}, 1 \leq j \leq n \},
   \]
   
   \[
   N(x_2) = \{ y_j : j \text{ even}, 2 \leq j \leq n - 1 \} \cup \{ y_1 \},
   \]
   
   so that \( G \) contains no 4-cycle.

2. \( n \) is odd, \( n = 2l - 1 \),
   
   \[
   N(x_1) = N(x_2) = \{ y_1, y_2, \ldots, y_d \} \cup \{ y_{l+d}, \ldots, y_{2l-1} \},
   \]
   
   with \( d \leq l - 1 \).
   
   Then \( G \) contains no \( 2l \)-cycle (i.e. an \( (n + 1) \)-cycle).

**Notations.** Let \( G = (V(G), E(G)) \) with: \( V(G) = \{ x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \} \).

Let \( C = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1) \) be a Hamiltonian cycle of \( G \). For \( x_i \) and \( y_j \) in \( G \), let:

\[
\delta(x_i, y_j) = 1 \text{ if the edge } (x_i, y_j) \text{ is in } G,
\]

\[
\delta(x_i, y_j) = 0 \text{ if the edge } (x_i, y_j) \text{ is not in } G.
\]

For \( i = 1, 2 \), \( N(x_i) \) denotes the set of the neighbours of \( x_i \) and \( d(x_i) \) the degree of \( x_i \).

**2. Proof of the theorem**

We suppose that \( G \) does not contain a \( 2l \)-cycle for some \( l, 2 \leq l \leq n - 1 \). Then, \( G \) does not contain \( 2l \)-cycle with two edges incident with \( x_1 \) or \( x_2 \), and the other edges on the Hamiltonian cycle of \( G \).

**Lemma 1.** If \( G \) does not contain a \( 2l \) cycle:

for \( j, 1 \leq j \leq n - l + 1 \), \( \delta(x_1, y_j) \cdot \delta(x_1, y_{j+l-1}) = 0 \),

for \( j, 2 \leq j \leq n - l + 2 \), \( \delta(x_2, y_j) \cdot \delta(x_2, y_{j+l-1}) = 0 \)

(the indices are taken modulo \( n \)).

**2.1. Case \( l = 2 \)**

If \( G \) does not contain a 4-cycle, then \( x_1 \) and \( x_2 \) have no common neighbour but \( y_1 \) and for \( 2 \leq i \leq n - 1 \), \( x_1 \) or \( x_2 \) are not neighbours of both \( y_i \) and \( y_{i+1} \). The
only possible case is: n is odd, 
\[ N(x_1) = \{ y_j, j \text{ odd}\}, \]
\[ N(x_2) = \{ y_j, j = 1 \text{ or } j \text{ even}\}. \]

If there are no other edges in \( G \), it is easy to see that \( G \) contains no 4-cycle, but contains a 2\( l \)-cycle, for \( 3 \leq l \leq n \).

2.2. First conditions when \( 3 \leq l \leq n - 1 \)

For \( 3 \leq l \leq n - 1 \), \( G \) does not contain a 2\( l \)-cycle with two edges incident with \( x_1 \) or \( x_2 \), and the other edges on the hamiltonian cycle. If for some \( j \), \( 2 \leq j \leq n - l + 2 \), \( \delta(x_2, y_j) = \delta(x_1, y_{j+1-2}) \), \((x_1, y_1, x_2, y_j, \ldots, y_{j+1-2}, x_1)\) is a 2\( l \)-cycle in \( G \). If for some \( j \), \( n - l + 3 \leq j \leq n - 1 \), \( \delta(x_2, y_j) = \delta(x_1, y_{j+1-n}) \), \((x_1, y_{j+1-n}, \ldots, y_2, x_2, y_j, \ldots, y_n, x_1)\) is a 2\( l \)-cycle in \( G \).

Lemma 2. If \( G \) does not contain a 2\( l \)-cycle for some \( l \), \( 3 \leq l \leq n - 1 \),

\[ \begin{align*}
\delta(x_2, y_j) + \delta(x_1, y_{j+1-2}) &\leq 1 \quad \text{for } 2 \leq j \leq n - l + 2, \\
\delta(x_2, y_j) + \delta(x_1, y_{j+1-n}) &\leq 1 \quad \text{for } n - l + 3 \leq j \leq n - 1, \\
n &\in [2, n - l + 2] \iff j + l - 2 \in [l, n], \\
n &\in [n - l + 3, n - 1] \iff j + l - n \in [3, l - 1].
\end{align*} \]

We have then:

\[ d(x_1) + d(x_2) = \sum_{j=2}^{n-l+2} \delta(x_2, y_j) + \delta(x_1, y_{j+1-2}) \]
\[ + \sum_{j=n-l+3}^{n-1} \delta(x_2, y_j) + \delta(x_1, y_{j+1-n}) \]
\[ + \delta(x_2, y_n) + \delta(x_1, y_1) + \delta(x_1, y_2) \]

By Lemma 2: \( d(x_1) + d(x_2) \leq n + \delta(x_2, y_n) + \delta(x_1, y_2) \). The condition \( d(x_1) + d(x_2) \geq n + 1 \) implies: \( 1 \leq \delta(x_2, y_n) + \delta(x_1, y_2) \leq 2. \)

First case: \( \delta(x_2, y_n) = \delta(x_1, y_2) = 1. \)

By Lemma 1: \( \delta(x_1, y_{n+1}) = \delta(x_2, y_{n-l+1}). \)

By Lemma 2 and the condition \( d(x_1) + d(x_2) \geq n + 1: \)

\[ \delta(x_1, y_{n+1}) + \delta(x_2, y_3) + \delta(x_2, y_{n-l+1}) + \delta(x_1, y_{n-1}) \geq 1. \]

Then

\[ \delta(x_2, y_3) + \delta(x_1, y_{n-1}) \geq 1. \]

By an argument of symmetry, we can suppose \( \delta(x_2, y_3) = 1. \)
Second case: \( \delta(x_1, y_2) = 1, \delta(x_2, y_n) = 0. \)

By Lemma 1: \( \delta(x_1, y_{l+1}) = 0. \) The condition \( d(x_1) + d(x_2) \geq n + 1 \) implies:
\( \delta(x_1, y_{l+1}) + \delta(x_2, y_n) = 1. \) Then \( \delta(x_2, y_n) = 1. \)

Third case: \( \delta(x_1, y_2) = 0, \delta(x_2, y_n) = 1. \)

This case is symmetric to the second case. For the following, we can suppose:
\( \delta(x_1, y_2) = 1 = \delta(x_2, y_n). \)

2.3. New conditions when \( \delta(x_1, y_2) = \delta(x_2, y_3) = 1 \)

\( G \) contains cycles of lengths 4, 6 and \( 2n - 2. \) We suppose \( G \) does not contain a 2\( l \)-cycle, for some \( l, 4 \leq l \leq n - 2. \) If for one \( j, 4 \leq j \leq n - l + 2, \) \( \delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l+3}), \) \( (x_1, y_2, x_3, y_3, x_2, y_j, \ldots, y_{j+l-3}, x_1) \) is a 2\( l \)-cycle of \( G. \)

If for one \( j, n - l + 3 \leq j \leq n - 1, \) \( \delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l-n+1}), \) \( (x_1, y_n, \ldots, y_j, x_2, y_3, \ldots, y_{j+l-n+1}, x_1) \) is a 2\( l \)-cycle of \( G. \) Moreover, if \( \delta(x_2, y_j) = 1, \) \( (x_1, y_2, \ldots, y_j, x_2, y_1, x_1) \) is a 2\( l \)-cycle of \( G. \) Then, with Lemma 2, we can obtain the following.

Lemma 3. If \( G \) does not contain a 2\( l \)-cycle, for some \( l, 4 \leq l \leq n - 2, \) and if \( \delta(x_2, y_n) = 1 = \delta(x_1, y_2): \)

For \( 3 \leq j \leq n - l + 2, \) \( \delta(x_2, y_j) = 1 \Rightarrow \delta(x_1, y_{j+l-3}) = 0, \delta(x_1, y_{j+l-2}) = 0. \)

For \( n - l + 3 \leq j \leq n - 1, \) \( \delta(x_2, y_j) = 1 \Rightarrow \delta(x_1, y_{j+l-n}) = 0, \delta(x_1, y_{j+l-n+1}) = 0. \)

2.4. Proof of the theorem

We define ‘intervals of neighbours’ for \( x_2, \) and then, applying Lemma 3, we determine ‘forbidden intervals’ for the neighbours of \( x_1. \)

There exist integers \( r \geq 1, s \geq 0, \) intervals of \( \mathbb{N}, I_k, 1 \leq k \leq r, \) and \( J_h, 1 \leq h \leq s, \) such that:

For \( 1 \leq k \leq r, \) \( I_k \subset [1, n - l + 2] \) (resp. for \( 1 \leq h \leq s, J_h \subset [n - l + 3, n]). \)

For \( 1 \leq k \leq r - 1, \) \( \max(I_k) \leq \min(I_{k+1}) - 2 \) (resp. for \( 1 \leq h \leq s - 1, \) \( \max(J_h) \leq \min(J_{h+1}) - 2 \) with the property:

\[ N(x_2) = \left\{ y_j, j \in \left( \bigcup_{k=1}^{r} I_k \right) \cup \left( \bigcup_{h=1}^{s} J_h \right) \right\}. \]

Then:
\[ d(x_2) = \sum_{k=1}^{r} |I_k| + \sum_{h=1}^{s} |J_h|. \]

Let \( I_1 = [1, d], \) with \( d \geq 3, \) then for \( 2 \leq k \leq r, I_k \subset [d + 2, n - l + 2]. \) We define:
\[ T(I_1) = [l, d + l - 2] \quad \text{and for} \quad 2 \leq k \leq r, \]
\[ T(I_k) = ((l - 2) + I_k) \cup ((l - 3) + I_k), \quad \text{for} \quad 2 \leq k \leq r. \]
The bipancyclic hamiltonian bipartite graph

where \((l - 2) + I_k\) and \((l - 3) + I_k\) mean translations of the set \(I_k\), then for \(2 \leq k \leq r\):

\[
T(I_k) \subseteq [d + l - 1, n],
\]

\[
|T(I_k)| = |I_k| + 1; \quad \text{and} \quad |T(I_1)| = |I_1| - 1.
\]

If \(s > 0\), let \(\alpha\) and \(\beta\) such that: \(0 \leq \beta \leq \alpha\), and \(J_s = [n - \alpha, n - \beta]\), then for \(1 \leq h \leq s - 1\)

\[
J_h \subset [n - l + 3, n - \alpha - 2].
\]

We define

\[
T(J_s) = [l - \alpha, l - \beta + 1], \quad \text{if} \quad \beta > 1,
\]

\[
T(J_s) = [l - \alpha, l - 1], \quad \text{if} \quad \beta \leq 1 \leq \alpha,
\]

\[
T(J_s) = \emptyset \quad \text{if} \quad \beta = \alpha = 0,
\]

and for \(1 \leq h \leq s - 1\),

\[
T(J_h) = ((l - n) + J_h) \cup ((l - n + 1) + J_h),
\]

then for \(1 \leq h \leq s - 1\):

\[
T(J_h) \subseteq [3, l - \alpha - 1],
\]

\[
|T(J_h)| = |J_h| + 1.
\]

The relations (i), (ii), (iii), (iv) and the property of the non-adjacency of the intervals \(I_k\) (resp. \(J_h\)) allow us to conclude that, for \(1 \leq k \leq r\) and \(1 \leq h \leq s\), the intervals \(T(I_k)\) and \(T(J_h)\) are disjoint. Lemma 3 implies that \(N(x_i)\) is independent of any vertex in the set

\[
\left\{ y_j, j \in \left( \bigcup_{k=1}^{r} T(I_k) \right) \cup \left( \bigcup_{h=1}^{s} T(J_h) \right) \right\}.
\]

Then

\[
n - d(x_1) \geq \sum_{k=1}^{r} |T(I_k)| + \sum_{h=1}^{s} |T(J_h)|.
\]

For \(s = 0\) or \(s = 1\) and \(\beta > 1\): \(d(x_1) + d(x_2) \leq n - (r - 1) - (s - 1)\). The condition \(d(x_1) + d(x_2) \geq n + 1\) implies: \(r = 1, s = 0\).

For \(s \geq 1, \beta = 1\): \(d(x_1) + d(x_2) \leq n + 3 - r - s\) then \(r = 1 = s\).

For \(s \geq 1, \beta = 0\): \(d(x_1) + d(x_2) \leq n + 4 - r - s\) then \(2 \leq r + s \leq 3\), with \(r \geq 1, s \geq 1\).

We will study these three cases.

First case: \(r = 1, s = 0\).

\[
N(x_2) = \{ y_1, y_2, \ldots, y_d \}, \quad \text{with} \quad d \leq n - l + 2.
\]
By Lemma 3: $N(x_1) = \{y_k, k \notin [l, d + l - 2]\}$. As $d \leq l - 1$, then $\delta(x_1, y_k) = 1$ and $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ is a 2l-cycle.

\textbf{Second case: } $\beta = 1$, $r = s = 1$.

$N(x_2) = \{y_1, \ldots, y_d\} \cup \{y_{n-\alpha}, \ldots, y_{n-1}\}$, with $3 \leq d \leq l - 1, 1 \leq \alpha \leq l - 3$.

By Lemma 3: $N(x_1) = \{y_k, k \notin [l - \alpha, d + l - 2]\}$.

If $d + l - 1 \leq n - l + 1$: $(x_1, y_{n-l+1}, \ldots, y_n, x_1)$ is a 2l-cycle.

If $d + l - 1 > n - l + 1$:

(i) For $n \geq 2l$, $n - l + d - 1 \geq d + l - 1$, then $\delta(x_1, y_{n-l+d-1}) = 1$ or $\delta(x_1, y_{n-l+d-1}, \ldots, y_n, x_1)$ is a 2l-cycle.

(ii) For $n \leq 2l - 1$, by Lemma 1, $d + l - 1 \leq n - \alpha - 1$, then $d \leq n - l - \alpha \leq l - \alpha - 1$ and $\delta(x_1, y_d) = 1$. $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ is a 2l-cycle.

\textbf{Third case: } $\beta = 0$, $2 \leq r + s \leq 3, r \geq 1, s \geq 1$.

(a) $r = s = 1$.

$N(x_2) = \{y_1, \ldots, y_d\} \cup \{y_{n-\alpha}, \ldots, y_n\}$, with $3 \leq d \leq l - 1$ and $\alpha \leq l - 3$.

By Lemma 3: $N(x_1) \subset \{y_k, k \notin [l - \alpha, d + l - 2]\}$, and at most one $y_k, k \notin [l - \alpha, d + l - 2]$ is independent of $x_1$.

(i) For $n \geq 2l$, $n - l + d - 1 \geq d + l - 1$, then $\delta(x_1, y_{n-l+d-1}) = 1$ or $\delta(x_1, y_{n-l+d-1}, \ldots, y_n, x_1)$ is a 2l-cycle.

(ii) For $n \leq 2l - 2$, by Lemma 1, $d + l - 1 \leq n - \alpha - 1$, then $d \leq n - l - \alpha \leq l - \alpha - 2$. $\delta(x_1, y_d) \cdot \delta(x_1, y_{d+l-1}) = 1$ or $\delta(x_1, y_{d+l-1}) = 1$. Then $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ is a 2l-cycle.

(iii) For $n = 2l - 1$, $d + l - 1 \leq n - l + d$, if $\delta(x_1, y_{d+l-1}) = 1$: $(x_1, y_{n-l+d}, \ldots, y_n, x_2, y_d, \ldots, y_2, x_1)$ is a 2l-cycle. We suppose $\delta(x_1, y_{d+l-1}) = 0$. If $(x_1, y_{d+l-1})$ is not a 2l-cycle in $G$, then $d + 1 \geq l - \alpha$. By Lemma 1: $d \leq n - l - \alpha = l - \alpha - 1$. Then $d = l - \alpha - 1$ and

$N(x_1) = N(x_2) = \{y_1, y_2, \ldots, y_d\} \cup \{y_{r+d}, \ldots, y_{2l-1}\}$, with $3 \leq d \leq l - 1$.

If there is no other edge in $G$ than the edges of the hamiltonian cycle and the edges incident with $x_1$ or $x_2$, $G$ contains cycles of every even length but 2l for $l = (n + 1)/2$.

(b) $r = 2, s = 1$.

$N(x_2) = \{y_j: j \in [1, d] \cup [p, q] \cup [n - \alpha, n]\}, \text{ with } 3 \leq d \leq l - 1, d + 2 \leq p \leq q \leq n - l + 2, 0 \leq \alpha \leq l - 3$.

By Lemma 3:

$N(x_1) = \{y_k: k \notin [l - \alpha, d + l - 2] \cup [p + l - 3, q + l - 2]\}$. 
By Lemma 1, \( q + l - 1 \leq n - \alpha - 1 \), then \( \delta(x_1, y_{n-\alpha-1}) = 1 \). (\( x_1, y_{n-\alpha-1}, \ldots, y_n, x_2, y_2, \ldots, y_{l-\alpha-1}, x_1 \)) is a 2l-cycle.

(c) \( r = 1, s = 2 \).

\[ N(x_2) = \{ y_j : j \in [1, d] \cup [p, q] \cup [n - \alpha, n] \}, \quad \text{with} \]
\[ 0 \leq d \leq l - 1, n - 1 + 3 \leq p \leq q \leq n - \alpha - 2, 0 \leq \alpha \leq l - 3. \]

\[ N(x_1) = \{ y_k : k \notin [l - \alpha, d + l - 2] \cup [p + l - n, q + l - n + 1] \}. \]

(i) For \( n \geq 2l - 1, d + l - 1 \leq n - l + d \), then \( \delta(x_1, y_{n-l+d}) = 1 \), (\( x_1, y_{n-l+d}, \ldots, y_n, x_2, y_d, \ldots, y_2, x_1 \)) is a 2l-cycle.

(ii) For \( n \leq 2l - 2 \): If \( q < n - \alpha - 2 \), \( \delta(x_1, y_{l-\alpha-1}) = 1 \), and by Lemma 1: \( d + l - 1 \leq n - \alpha - 1 \), then \( \delta(x_1, y_{n-\alpha-1}) = 1 \), (\( x_1, y_{n-\alpha-1}, \ldots, y_n, x_2, y_2, \ldots, y_{l-\alpha-1}, x_1 \)) is a 2l-cycle.

If \( q = n - \alpha - 2 \), by Lemma 1: \( d + l - 1 \leq n - \alpha - 1 \), then \( q \geq d + l - 2 \), \( \delta(x_2, y_{d+l-2}) = 0 \), then: \( q \geq p \geq d + l \). For \( 1 \leq j \leq p + l - n - 1 \), and \( d + l - 1 \leq k \leq n \): \( \delta(x_1, y_j) = 1 \) and \( \delta(x_1, y_k) = 1 \);

\[ (d + l - 1) - (p + l - n - 1) = (d - p) + n \leq -l + 2l - 1 = l - 1. \]

Then there exist \( j, 1 \leq j \leq p + l - n - 1 \), and there exist \( k - j + l - 1 \leq n \), such that: \( \delta(x_1, y_j) \cdot \delta(x_1, y_k) = 1 \).

We obtain a contradiction with Lemma 1.

References