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A condition for a hamiltonian bipartite graph to be bipancyclic

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Abstract

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Let G be a hamiltonian bipartite graph of order $2n$ and let $C=(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1)$ be a hamiltonian cycle of G. G is said to be bipancyclic if it contains a cycle of length $2l$, for every l, $2 \leqslant l \leqslant n$. Suppose the vertices x_1 and x_2 are such that $d(x_1)+d(x_2) \geqslant n+1$. Then G is either:
(1) bipancyclic,
(2) missing a 4-cycle (then n is odd and the structure of G is known),
(3) missing a $(n+1)$-cycle (then n is odd and the structure of G is known).

1. Introduction

Let G be a finite graph of order n. Various sufficient conditions for a graph to be hamiltonian have been given in terms of vertex-degree and size of the graph [5–8]. Almost all these conditions imply the graph to be pancyclic, [2, 4, 12], following the meta-conjecture of Bondy. But if the graph is supposed to be hamiltonian, we need weaker conditions in order for the graph to be pancyclic [1, 3]. When G is a balanced bipartite graph, we have analogous results [9–11]. In [13] Schmeichel and Hakimi proved the following theorem.

Theorem. *If there exist two vertices x_1 and x_2, consecutive on a hamiltonian cycle of a graph G of order n, such that $d(x_1)+d(x_2) \geqslant n$, then G is either*:
(1) *pancyclic*,
(2) *bipartite*,
(3) *missing only an $(n-1)$-cycle*.

We establish the analogous result for hamiltonian bipartite graphs.

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Theorem. *If there exist two vertices x_1 and x_2, the distance between which is two, on a hamiltonian cycle of a bipartite graph G of order $2n$, such that:*

$$d(x_1) + d(x_2) \geqslant n + 1$$

then G is bipancyclic except in two cases:

(1) *n is odd,*

$$N(x_1) = \{y_j : j \text{ odd}, 1 \leqslant j \leqslant n\},$$

$$N(x_2) = \{y_j : j \text{ even}, 2 \leqslant j \leqslant n-1\} \cup \{y_1\},$$

so that G contains no 4-cycle.

(2) *n is odd, $n = 2l - 1$,*

$$N(x_1) = N(x_2) = \{y_1, y_2, \ldots, y_d\} \cup \{y_{l+d}, \ldots, y_{2l-1}\}.$$

with $d \leqslant l - 1$.

Then G contains no $2l$-cycle (i.e. *an $(n+1)$-cycle*).

Notations. Let $G = (V(G), E(G))$ with: $V(G) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. Let $C = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1)$ be a hamiltonian cycle of G. For x_i and y_j in G, let:

$$\delta(x_i, y_j) = 1 \text{ if the edge } (x_i, y_j) \text{ is in } G,$$

$$\delta(x_i, y_j) = 0 \text{ if the edge } (x_i, y_j) \text{ is not in } G.$$

For $i = 1, 2$, $N(x_i)$ denotes the set of the neighbours of x_i and $d(x_i)$ the degree of x_i.

2. Proof of the theorem

We suppose that G does not contain a $2l$-cycle for some l, $2 \leqslant l \leqslant n-1$. Then, G does not contain $2l$-cycle with two edges incident with x_1 or x_2, and the other edges on the hamiltonian cycle of G.

Lemma 1. *If G does not contain a $2l$-cycle:*

$$\text{for } j,\ 1 \leqslant j \leqslant n - l + 1,\ \delta(x_1, y_j) \cdot \delta(x_1, y_{j+l-1}) = 0,$$

$$\text{for } j,\ 2 \leqslant j \leqslant n - l + 2,\ \delta(x_2, y_j) \cdot \delta(x_2, y_{j+l-1}) = 0$$

(*the indices are taken modulo n*).

2.1. Case $l = 2$

If G does not contain a 4-cycle, then x_1 and x_2 have no common neighbour but y_1 and for $2 \leqslant i \leqslant n-1$, x_1 or x_2 are not neighbours of both y_i and y_{i+1}. The

only possible case is: n is odd,

$$N(x_1) = \{y_j, j \text{ odd}\},$$

$$N(x_2) = \{y_j, j = 1 \text{ or } j \text{ even}\}.$$

If there are no other edges in G, it is easy to see that G contains no 4-cycle, but contains a $2l$-cycle, for $3 \leq l \leq n$.

2.2. *First conditions when $3 \leq l \leq n-1$*

For $3 \leq l \leq n-1$, G does not contain a $2l$-cycle with two edges incident with x_1 or x_2, and the other edges on the hamiltonian cycle. If for some j, $2 \leq j \leq n-l+2$, $\delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l-2})$, $(x_1, y_1, x_2, y_j, \ldots, y_{j+l-2}, x_1)$ is a $2l$-cycle in G. If for some j, $n-l+3 \leq j \leq n-1$, $\delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l-n})$, $(x_1, y_{j+l-n}, \ldots, y_2, x_2, y_j, \ldots, y_n, x_1)$ is a $2l$-cycle in G.

Lemma 2. *If G does not contain a $2l$-cycle for some l, $3 \leq l \leq n-1$,*

$$\delta(x_2, y_j) + \delta(x_1, y_{j+l-2}) \leq 1 \quad \textit{for } 2 \leq j \leq n-l+2,$$

$$\delta(x_2, y_j) + \delta(x_1, y_{j+l-n}) \leq 1 \quad \textit{for } n-l+3 \leq j \leq n-1,$$

$$j \in [2, n-l+2] \Leftrightarrow j+l-2 \in [l, n],$$

$$j \in [n-l+3, n-1] \Leftrightarrow j+l-n \in [3, l-1].$$

We have then:

$$\begin{aligned} d(x_1) + d(x_2) = & \sum_{j=2}^{n-l+2} \delta(x_2, y_j) + \delta(x_1, y_{j+l-2}) \\ & + \sum_{j=n-l+3}^{n-1} \delta(x_2, y_j) + \delta(x_1, y_{j+l-n}) \\ & + \delta(x_2, y_1) + \delta(x_2, y_n) + \delta(x_1, y_1) + \delta(x_1, y_2) \end{aligned}$$

By Lemma 2: $d(x_1) + d(x_2) \leq n + \delta(x_2, y_n) + \delta(x_1, y_2)$. The condition $d(x_1) + d(x_2) \geq n+1$ implies: $1 \leq \delta(x_2, y_n) + \delta(x_1, y_2) \leq 2$.

First case: $\delta(x_2, y_n) = \delta(x_1, y_2) = 1$.

By Lemma 1: $\delta(x_1, y_{l+1}) = 0 = \delta(x_2, y_{n-l+1})$.

By Lemma 2 and the condition $d(x_1) + d(x_2) \geq n+1$:

$$\delta(x_1, y_{l+1}) + \delta(x_2, y_3) + \delta(x_2, y_{n-l+1}) + \delta(x_1, y_{n-1}) \geq 1.$$

Then

$$\delta(x_2, y_3) + \delta(x_1, y_{n-1}) \geq 1.$$

By an argument of symmetry, we can suppose $\delta(x_2, y_3) = 1$.

Second case: $\delta(x_1, y_2) = 1$, $\delta(x_2, y_n) = 0$.

By Lemma 1: $\delta(x_1, y_{l+1}) = 0$. The condition $d(x_1) + d(x_2) \geq n + 1$ implies: $\delta(x_1, y_{l+1}) + \delta(x_2, y_3) = 1$. Then $\delta(x_2, y_3) = 1$.

Third case: $\delta(x_1, y_2) = 0$, $\delta(x_2, y_n) = 1$.

This case is symmetric to the second case. For the following, we can suppose: $\delta(x_1, y_2) = 1 = \delta(x_2, y_3)$.

2.3. *New conditions when $\delta(x_1, y_2) = \delta(x_2, y_3) = 1$*

G contains cycles of lengths 4, 6 and $2n-2$. We suppose G does not contain a $2l$-cycle, for some l, $4 \leq l \leq n-2$. If for one j, $4 \leq j \leq n-l+2$, $\delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l-3})$, $(x_1, y_2, x_3, y_3, x_2, y_j, \ldots, y_{j+l-3}, x_1)$ is a $2l$-cycle of G.

If for one j, $n-l+3 \leq j \leq n-1$, $\delta(x_2, y_j) = 1 = \delta(x_1, y_{j+l-n+1})$, $(x_1, y_n, \ldots, y_j, x_2, y_3, \ldots, y_{j+l-n+1}, x_1)$ is a $2l$-cycle of G. Moreover, if $\delta(x_2, y_l) = 1$, $(x_1, y_2, \ldots, y_l, x_2, y_1, x_1)$ is a $2l$-cycle of G. Then, with Lemma 2, we can obtain the following.

Lemma 3. *If G does not contain a $2l$-cycle, for some l, $4 \leq l \leq n-2$, and if $\delta(x_2, y_3) = 1 = \delta(x_1, y_2)$:*

$$\text{For } 3 \leq j \leq n-l+2, \quad \delta(x_2, y_j) = 1 \Rightarrow \begin{cases} \delta(x_1, y_{j+l-3}) = 0, \\ \delta(x_1, y_{j+l-2}) = 0. \end{cases}$$

$$\text{For } n-l+3 \leq j \leq n-1, \quad \delta(x_2, y_j) = 1 \Rightarrow \begin{cases} \delta(x_1, y_{j+l-n}) = 0, \\ \delta(x_1, y_{j+l-n+1}) = 0. \end{cases}$$

2.4. *Proof of the theorem*

We define 'intervals of neighbours' for x_2, and then, applying Lemma 3, we determine 'forbidden intervals' for the neighbours of x_1.

There exist integers $r \geq 1$, $s \geq 0$, intervals of $\mathbb{N}$, I_k, $1 \leq k \leq r$, and J_h, $1 \leq h \leq s$, such that:

For $1 \leq k \leq r$, $I_k \subset [1, n-l+2]$ (resp. for $1 \leq h \leq s$, $J_h \subset [n-l+3, n]$).

For $1 \leq k \leq r-1$, $\max(I_k) \leq \min(I_{k+1}) - 2$ (resp. for $1 \leq h \leq s-1$, $\max(J_h) \leq \min(J_{h+1}) - 2$) with the property:

$$N(x_2) = \left\{ y_j, j \in \left(\bigcup_{k=1}^{r} I_k \right) \cup \left(\bigcup_{h=1}^{s} J_h \right) \right\}.$$

Then:

$$d(x_2) = \sum_{k=1}^{r} |I_k| + \sum_{h=1}^{s} |J_h|.$$

Let $I_1 = [1, d]$, with $d \geq 3$, then for $2 \leq k \leq r$, $I_k \subset [d+2, n-l+2]$. We define:

$$T(I_1) = [l, d+l-2] \quad \text{and for} \quad 2 \leq k \leq r, \tag{i}$$
$$T(I_k) = ((l-2) + I_k) \cup ((l-3) + I_k),$$

where $(l-2)+I_k$ and $(l-3)+I_k$ mean translations of the set I_k, then for $2 \leqslant k \leqslant r$:

$$\begin{aligned} &T(I_k) \subset [d+l-1, n], \\ &|T(I_k)| = |I_k| + 1; \qquad |T(I_1)| = |I_1| - 1. \end{aligned} \tag{ii}$$

If $s > 0$, let α and β such that: $0 \leqslant \beta \leqslant \alpha$, and $J_s = [n-\alpha, n-\beta]$, then for $1 \leqslant h \leqslant s-1$

$$J_h \subset [n-l+3, n-\alpha-2].$$

We define

$$\begin{aligned} &T(J_s) = [l-\alpha, l-\beta+1], \quad \text{if } \beta > 1, \\ &T(J_s) = [l-\alpha, l-1], \quad \text{if } \beta \leqslant 1 \leqslant \alpha, \\ &T(J_s) = \emptyset \quad \text{if } \beta = \alpha = 0, \end{aligned} \tag{iii}$$

and for $1 \leqslant h \leqslant s-1$,

$$T(J_h) = ((l-n)+J_h) \cup ((l-n+1)+J_h),$$

then for $1 \leqslant h \leqslant s-1$:

$$\begin{aligned} &T(J_h) \subset [3, l-\alpha-1], \\ &|T(J_h)| = |J_h| + 1. \end{aligned} \tag{iv}$$

The relations (i), (ii), (iii), (iv) and the property of the non-adjacency of the intervals I_k (resp. J_h) allow us to conclude that, for $1 \leqslant k \leqslant r$ and $1 \leqslant h \leqslant s$, the intervals $T(I_k)$ and $T(J_h)$ are disjoint. Lemma 3 implies that $N(x_1)$ is independent of any vertex in the set

$$\left\{y_j, j \in \left(\bigcup_{k=1}^{r} T(I_k)\right) \cup \left(\bigcup_{h=1}^{s} T(J_h)\right)\right\}.$$

Then

$$n - d(x_1) \geqslant \sum_{k=1}^{r} |T(I_k)| + \sum_{h=1}^{s} |T(J_h)|.$$

For $s=0$ or $s=1$ and $\beta > 1$: $d(x_1)+d(x_2) \leqslant n-(r-1)-(s-1)$. The condition $d(x_1)+d(x_2) \geqslant n+1$ implies: $r=1$, $s=0$.

For $s \geqslant 1$, $\beta = 1$: $d(x_1)+d(x_2) \leqslant n+3-r-s$ then $r=1=s$.

For $s \geqslant 1$, $\beta = 0$: $d(x_1)+d(x_2) \leqslant n+4-r-s$ then $2 \leqslant r+s \leqslant 3$, with $r \geqslant 1$, $s \geqslant 1$.

We will study these three cases.

First case: $r=1$, $s=0$.

$$N(x_2) = \{y_1, y_2, \ldots, y_d\}, \quad \text{with } d \leqslant n-l+2.$$

By Lemma 3: $N(x_1)=\{y_k, k\notin[l, d+l-2]\}$. As $d\leqslant l-1$, then $\delta(x_1, y_d)=1$ and: $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ is a $2l$-cycle.

Second case: $\beta=1$, $r=s=1$.

$$N(x_2)=\{y_1, \ldots, y_d\}\cup\{y_{n-\alpha}, \ldots, y_{n-1}\}, \quad \text{with } 3\leqslant d\leqslant l-1,\ 1\leqslant\alpha\leqslant l-3.$$

By Lemma 3: $N(x_1)=\{y_k, k\notin[l-\alpha, d+l-2)\}$.

If $d+l-1\leqslant n-l+1$: $(x_1, y_{n-l+1}, \ldots, y_n, x_1)$ is a $2l$-cycle.

If $d+l-1>n-l+1$:

(i) For $n\geqslant 2l$, $n-l+d-1\geqslant d+l-1$, then $\delta(x_1, y_{n-l+d-1})=1$, $(x_1, y_{n-l+d-1}, \ldots, y_{n-1}, x_2, y_d, \ldots, y_2, x_1)$ is a $2l$-cycle.

(ii) For $n\leqslant 2l-1$, by Lemma 1, $d+l-1\leqslant n-\alpha-1$, then $d\leqslant n-l-\alpha\leqslant l-\alpha-1$ and $\delta(x_1, y_d)=1=\delta(x_1, y_{d+l-1})$, $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ is a $2l$-cycle.

Third case: $\beta=0$, $2\leqslant r+s\leqslant 3$, $r\geqslant 1$, $s\geqslant 1$.

(a) $r=s=1$.

$$N(x_2)=\{y_1, \ldots, y_d\}\cup\{y_{n-\alpha}, \ldots, y_n\}, \quad \text{with } 3\leqslant d\leqslant l-1 \text{ and } \alpha\leqslant l-3.$$

By Lemma 3: $N(x_1)\subset\{y_k, k\notin[l-\alpha, d+l-2]\}$, and at most one y_k, $k\notin[l-\alpha, d+l-2]$ is independent of x_1.

(i) For $n\geqslant 2l$, $n-l+d-1\geqslant d+l-1$, then $\delta(x_1, y_{n-l+d-1})=1$ or $\delta(x_1, y_{n-l+d})=1$. $(x_1, y_{n-l+d-1}, \ldots, y_n, x_2, y_{d-1}, \ldots, y_2, x_1)$ or $(x_1, y_{n-l+d}, \ldots, y_n, x_2, y_d, \ldots, y_2, x_1)$ is a $2l$-cycle.

(ii) For $n\leqslant 2l-2$, by Lemma 1, $d+l-1\leqslant n-\alpha-1$, then $d\leqslant n-l-\alpha\leqslant l-\alpha-2$. $\delta(x_1, y_d)\cdot\delta(x_1, y_{d+l-1})=1$ or $\delta(x_1, y_{d+1})\cdot\delta(x_1, y_{d+1})=1$. Then $(x_1, y_d, \ldots, y_{d+l-1}, x_1)$ or $(x_1, y_{d+1}, \ldots, y_{d+l}, x_1)$ is a $2l$-cycle.

(iii) For $n=2l-1$, $d-l-1=n-l+d$, if $\delta(x_1, y_{d+l-1})=1$: $(x_1, y_{n-l+d}, \ldots, y_n, x_2, y_d, \ldots, y_2, x_1)$ is a $2l$-cycle. We suppose $\delta(x_1, y_{d+l-1})=0$. If $(x_1, y_{d+1}, \ldots, y_{d+l}, x_1)$ is not a $2l$-cycle in G, then $d+1\geqslant l-\alpha$. By Lemma 1: $d\leqslant n-l-\alpha=l-\alpha-1$. Then $d=l-\alpha-1$ and

$$N(x_1)=N(x_2)=\{y_1, y_2, \ldots, y_d\}\cup\{y_{l+d}, \ldots, y_{2l-1}\}, \quad \text{with } 3\leqslant d\leqslant l-1.$$

If there is no other edge in G than the edges of the hamiltonian cycle and the edges incident with x_1 or x_2, G contains cycles of every even length but $2l$ for $l=(n+1)/2$.

(b) $r=2$, $s=1$.

$$N(x_2)=\{y_j: j\in[1, d]\cup[p, q]\cup[n-\alpha, n]\}, \quad \text{with}$$
$$3\leqslant d\leqslant l-1,\ d+2\leqslant p\leqslant q\leqslant n-l+2,\ 0\leqslant\alpha\leqslant l-3.$$

By Lemma 3:

$$N(x_1)=\{y_k: k\notin[l-\alpha, d+l-2]\cup[p+l-3, q+l-2]\}.$$

By Lemma 1, $q+l-1 \leqslant n-\alpha-1$, then $\delta(x_1, y_{n-\alpha-1})=1$. $(x_1, y_{n-\alpha-1}, \ldots, y_n, x_2, y_2, \ldots, y_{l-\alpha-1}, x_1)$ is a $2l$-cycle.

(c) $r=1$, $s=2$.

$$N(x_2)=\{y_j: j \in [1, d] \cup [p, q] \cup [n-\alpha, n]\}, \quad \text{with}$$
$$3 \leqslant d \leqslant l-1,\ n-1+3 \leqslant p \leqslant q \leqslant n-\alpha-2,\ 0 \leqslant \alpha \leqslant l-3.$$

$$N(x_1)=\{y_k: k \notin [l-\alpha, d+l-2] \cup [p+l-n, q+l-n+1]\}.$$

(i) For $n \geqslant 2l-1$, $d+l-1 \leqslant n-l+d$, then $\delta(x_1, y_{n-l+d})=1$, $(x_1, y_{n-l+d}, \ldots, y_n, x_2, y_d, \ldots, y_2, x_1)$ is a $2l$-cycle.

(ii) For $n \leqslant 2l-2$: If $q<n-\alpha-2$, $\delta(x_1, y_{l-\alpha-1})=1$, and by Lemma 1: $d+l-1 \leqslant n-\alpha-1$, then $\delta(x_1, y_{n-\alpha-1})=1$, $(x_1, y_{n-\alpha-1}, \ldots, y_n, x_2, y_2, \ldots, y_{l-\alpha-1}, x_1)$ is a $2l$-cycle.

If $q=n-\alpha-2$, by Lemma 1: $d+l-1 \leqslant n-\alpha-1$, then $q \geqslant d+l-2$, $\delta(x_2, y_{d+l+1})=\delta(x_2, y_{d+l-2})=0$, then: $q \geqslant p \geqslant d+l$. For $1 \leqslant j \leqslant p+l-n-1$, and $d+l-1 \leqslant k \leqslant n$: $\delta(x_1, y_j)=1$ and $\delta(x_1, y_k)=1$;

$$(d+l-1)-(p+l-n-1)=(d-p)+n \leqslant -l+2l-1=l-1.$$

Then there exist j, $1 \leqslant j \leqslant p+l-n-1$, and there exist $k=j+l-1 \leqslant n$, such that: $\delta(x_1, y_j) \cdot \delta(x_1, y_k)=1$.

We obtain a contradiction with Lemma 1.

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