Techniques in matroid reconstruction

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Abstract

Brylawski first raised the questions of whether a matroid can be reconstructed from its multisets of hyperplanes, single-element deletions, or minors. This paper explores the relationships between these questions, and resolves them for certain classes of matroids. In particular, finite Dowling lattices are shown to be reconstructible both by deletions and by contractions.

0. Introduction

Matroid reconstruction questions were inspired by the Reconstruction Conjecture of Kelly [16] and Ulam [22]. This section begins with a brief review of those parts of graph reconstruction which are relevant to this paper. For more background on graph reconstruction, consult the surveys [3,4,20,21]. Also, [18] is suggested as a general reference for matroid theory, and as a guide to terminology used in this paper.

For a graph $G$ on vertex set $V = \{v_1, v_2, \ldots, v_k\}$, let $G_i = G - v_i$ be the vertex-induced subgraph on vertex set $V - v_i$. The multiset of (unlabeled) graphs $\{G_1, \ldots, G_k\}$ forms the deck of $G$. A reconstruction of $G$ is any graph with the same deck as $G$. A graph $G$ is reconstructible if every reconstruction of $G$ is isomorphic to $G$. In this setting, the original conjecture made by Kelly and Ulam is as follows:

The Reconstruction Conjecture. All finite simple graphs on at least three vertices are reconstructible.

Analogous problems have been posed for digraphs, hypergraphs, and related structures. Also several weakening of the notion of reconstruction have been examined. For instance, a graph property is reconstructible if whenever one reconstruction from a deck has that property, all reconstructions from that deck have it. Some reconstructible graph properties are the number of edges in a graph, the degree sequence, and the chromatic polynomial. Also, a class of graphs is recognizable if whenever a graph $G$ belongs to that class, every reconstruction of $G$ belongs to the
class. Examples of recognizable classes of graphs include regular graphs, trees, and 2-connected graphs. Finally, a class of graphs is reconstructible if every graph belonging to that class is reconstructible. For instance, regular graphs, trees, and disconnected graphs are all reconstructible classes.

Also of importance when examining matroid reconstruction questions is the Edge Reconstruction Conjecture [13], which states that any simple graph on more than three edges is reconstructible from its deck of single-edge deletions. In this setting the notions of deck and of reconstructibility are analogous to the notions defined above for vertex reconstruction.

The three main matroid analogs to the Graph Reconstruction Conjecture are:

(M1) Hyperplane Reconstruction Conjecture: Every binary matroid is reconstructible from its multiset (deck) of hyperplanes [6].

(M2) Deletion Reconstruction Conjecture: Every binary matroid other than a circuit or free matroid is reconstructible from its deck of single-point deletions [6].

(M3) Minor Reconstruction Conjecture: Every matroid is reconstructible from its deck of single-point deletions and single-point contractions.

The notions of property reconstructibility, and of class reconstructibility for matroids are also lifted directly from the graph reconstruction terminology.

Note that (M1) is closest in spirit to the Reconstruction Conjecture, since if a graph $G$ is 2-connected and has no loops, then the graph $G_i$ corresponds to a hyperplane of the cycle matroid of $G$. Typically, however, the cycle matroid of $G$ has some hyperplanes which do not arise in this fashion, so a complete list of hyperplanes may provide additional information. Also notice that (M2) is a direct analog of the Edge Reconstruction Conjecture. Finally, although Brylawski did not formulate (M3) as a conjecture, he was the first to raise the question of which matroids are reconstructible from their decks of minors [9]. Also it may be useful to view (M3) in terms of the following, more precise formulation. Let $M$ and $N$ be matroids on the same point set $S$. If $M - p \cong N - p$ and $M/p \cong N/p$ for every point $p$, then $M \cong N$.

There are counterexamples to (M1) and (M2) if the hypothesis that the matroid be binary is omitted. Brylawski [6] first observed that the two rank-3 simple matroids with six points and precisely two three-point lines (which intersect in one of the matroids, but not in the other) are nonbinary and indistinguishable from their decks of hyperplanes. For a general class of examples, note that any two nonisomorphic projective planes of the same order also have the same deck of hyperplanes, and hence are not hyperplane reconstructible. Brylawski [7] also discovered a pair of nonbinary matroids which are indistinguishable from their decks of deletions. Further, it is clear that a circuit on $n$ points will have the same deck of deletions as the free matroid on $n$ points. Hence, these two matroids are excluded from discussions of deletion reconstructibility.

This paper is divided into three sections. The first discusses some useful reconstructible matroid properties, and establishes the reconstructibility of certain classes of matroids. Section 2 focuses on the deletion reconstruction of several classes of matroids that have special properties which aid reconstruction. In particular,
Dowling lattices are described and finite Dowling lattices are shown to be reconstructible. The third section explores reconstruction of matroids from contractions, which is precisely the dual of deletion reconstruction. Finally, finite Dowling lattices are shown also to be contraction reconstructible.

1. General results

This section focuses on results analogous to results in graph reconstruction. Proposition 1.1 makes some observations which are useful in dismissing trivialities in later results. The straightforward proof of this proposition (which can be found in [17]) has been omitted.

**Proposition 1.1.** The following are deletion and minor reconstructible.

(a) The number of points of a matroid $M$.
(b) The rank of a matroid $M$.
(c) Matroids containing loops or isthmuses.

In approaching the minor reconstruction question, note that several interpretations are possible. Two particular interpretations are that either the deck of minors is given as single multiset, or that the deck of minors is divided into two distinguishable multisets: one of deletions and the other of contractions. However, except for the two rank-I matroids on two points, Proposition 1.1 holds without dividing the deck of minors into these two multisets. Therefore, because of Proposition 1.1, and the fact that one can distinguish between deletions and contractions in a deck of minors of a matroid with no isthmuses or loops, these two interpretations of the minor reconstruction question are equivalent, aside from the counterexamples mentioned above. Hence from here on, these counterexamples will be excluded from discussions of minor reconstructibility, and it will be assumed that the deck of minors is divided into a deck of deletions and a deck of contractions. Further, it follows that any result which can be obtained about deletion reconstruction can also be obtained for minor reconstruction. Hence, from here on, any matroid or property which is said to be deletion reconstructible will be understood to be minor reconstructible as well.

As a matroid's rank is one more than the rank of any of its hyperplanes, rank is hyperplane reconstructible. Hyperplane reconstruction of the number of points of $M$, however, is nontrivial, although Brylawski [8] was able to show two stronger results.

**Proposition 1.2 (Brylawski [8]).** For a matroid $M$, the multiset of flats of rank less than $rk(M)$ and the Tutte polynomial of $M$ are reconstructible from $M$'s deck of hyperplanes.

The following proposition is quite useful, as it gives access to the multiset of flats and the Tutte polynomial for deletion and minor reconstruction, via Proposition 1.2.
Proposition 1.3. The deck of hyperplanes of a matroid is deletion reconstructible.

Proof. Recall that circuits and free matroids are excluded from consideration. By Proposition 1.1, assume $M$ has no isthmuses. Let $\mathcal{H}$ be the multiset of all hyperplanes of all single-point deletions of $M$. Observe that $\mathcal{H}$ contains two types of sets: hyperplanes $H$ of $M$ (note that $H$ appears $|M - H|$ times in $\mathcal{H}$, namely as a hyperplane of $M - x$ for each $x \notin H$); and sets of the form $H - x$ where $H$ is a hyperplane of $M$, and $x$ is not an isthmus of $H$. In this latter case, $H - x$ appears once in $\mathcal{H}$, namely as a hyperplane of $M - x$.

Look for a member $H \in \mathcal{H}$ of maximum cardinality. Then $H$ is certainly a hyperplane of $M$, and occurs in $M$ with multiplicity.

\[
\text{multiplicity of } H \text{ in } \mathcal{H} = \frac{|M - H|}{\text{cardinality of } H}.
\]

Compute the multiplicities of all maximal cardinality hyperplanes of $M$ in this way.

Now, for each (including multiplicity) of these maximal cardinality hyperplanes, take each single-element deletion of it (excepting deletions of isthmuses, as these do not appear in $\mathcal{H}$), and remove it from $\mathcal{H}$. The maximal cardinality hyperplanes in what remains of $\mathcal{H}$ must now also be true hyperplanes of $M$, and their multiplicities can be determined as above. Continue this process of averaging and removing until $\mathcal{H}$ is empty. $\square$

Hence any reconstruction result which can be obtained from the deck of hyperplanes is also valid as a deletion or minor reconstruction result. For this reason, hereafter, any matroid or property which is proven to be hyperplane reconstructible will be understood to be deletion (and minor) reconstructible as well.

Recall from [18] that a parallel class of $M$ is a maximal loopless set of points such that any two are parallel. A parallel class is trivial if it contains only one point. In this paper, such points will be called simple; nonsimple points will refer to points from nontrivial parallel classes.

Proposition 1.4. The following statements hold.

(a) Simple matroids are hyperplane recognizable.

(b) The underlying simple matroid of a nonsimple matroid is deletion reconstructible.

(c) Let $M$ be a nonsimple matroid. Let $m_1 < m_2 < \cdots < m_k$ be the sizes of parallel classes of $M$. If there is a number $i$, $1 \leq i \leq k$, such that $m_i \neq i$, then $M$ is deletion reconstructible.

(d) The number of circuits of $M$ of any size less than $|M|$ is deletion reconstructible.

(e) Paving matroids are deletion recognizable.

Each statement in Proposition 1.4 is either straightforward or is an immediate consequence of Propositions 1.2 and 1.3. In fact, paving matroids are recognizable.
from hyperplanes alone, as every circuit of size $r$ or less must be a circuit of some hyperplane.

The next result about disconnected matroids makes use of the following notation. If a disconnected matroid $M$ is a direct product of $x$ copies of a connected matroid $N$, then the expression $M \cong N \oplus N \oplus \cdots \oplus N$ will be abbreviated as $M \cong N^x$.

**Proposition 1.5.** Disconnected matroids are hyperplane reconstructible.

**Proof.** Suppose $M \cong M_1^{x_1} \oplus M_2^{x_2} \oplus \cdots \oplus M_k^{x_k}$ for positive integers $x_1, \ldots, x_k$. By Proposition 1.2, the Tutte polynomial of $M$ is reconstructible; hence connectedness is recognizable, since it is reflected in the $\beta$-invariant (an evaluation of the Tutte polynomial).

Let $\mathcal{C}$ be the multiset of all components of all hyperplanes (multiplicities included). Note that a hyperplane of $M$ is isomorphic to

$$M_1^{x_1} \oplus \cdots \oplus M_q^{x_q-1} \oplus \text{(a hyperplane of } M_q) \oplus \cdots \oplus M_k^{x_k}.$$ 

Notice that a hyperplane of $M_q$ could have several connected components, each of which will appear separately in $\mathcal{C}$.

Let $M_i$ be a component in $\mathcal{C}$ of maximal rank. Then $M_i$ must be a component of $M$, and no occurrence of $M_i$ in $\mathcal{C}$ arises from a hyperplane of any component $M_j$.

Let $h$ be the number of hyperplanes of $M$. Let $h_j$ be the number of hyperplanes in $M_j$ for each $j$. Thus each of the $x_i$ copies of $M_i$ will appear $h - h_i$ times in $\mathcal{C}$. Therefore in $\mathcal{C}$, the number of times $M_i$ appears will be $\sum_i (h - h_i)$. Hence

$$\sum_i \frac{\text{multiplicity of } M_i \text{ in } \mathcal{C}}{h_i} = x_i(h - h_i).$$

Now for each copy of $M_i$ in $M$, remove each component of each of $M_i$'s hyperplanes from $\mathcal{C}$. At this point, components of highest rank remaining in $\mathcal{C}$ cannot have arisen by being in a hyperplane of another component. So they must be actual components of $M$. Compute the multiplicities of these components and continue this process until $\mathcal{C}$ is empty.

Thus each component and its multiplicity is reconstructed, which in turn uniquely reconstructs $M$. \[\square\]

2. Specialized techniques for deletion reconstruction

This section explores more techniques for reconstructing certain classes of matroids from their decks of deletions. The matroids examined here have special structure or properties which are exploited for reconstruction. In particular, hyperplane structure, line closure, and line cardinalities are used.
Proposition 2.1. If no hyperplane of $M$ is isomorphic to a deletion of another hyperplane of $M$, then $M$ is deletion reconstructible.

Proof. This condition is detected by reconstructing the deck hyperplanes. So pick any deletion $M - e$. Notice that $M - e$ has two kinds of hyperplanes:

1. Those of $M$ not containing $e$.
2. Those of $M$ having a circuit containing $e$, but which now lack $e$.

A hyperplane of this second type will not be isomorphic to a hyperplane of $M$. So viewing the hyperplanes as sets of points, add $e$ to each hyperplane of $M - e$ which is not isomorphic to a hyperplane of $M$.

Finally, there may still be some hyperplanes of $M$ which have not been labeled. These will be precisely the hyperplanes of which $e$ is an isthmus. Examine the colines of $M - e$. For a coline $X$ of $M - e$, the set $X \cup e$ will be a hyperplane of $M$ if and only if no hyperplane obtained above contains $X \cup e$. Hence, adding $e$ to all such colines gives all labeled hyperplanes of $M$ in which $e$ is an isthmus.

In this way all labeled hyperplanes of $M$ are reconstructed (although perhaps with new labels). This reconstructs $M$ up to relabeling. $\square$

One application of this proposition is that affine geometries, projective geometries, and (more generally) perfect matroid designs are deletion reconstructible. Also, cocycle matroids of bipartite graphs are reconstructible, since the cardinalities of all hyperplanes have the same parity.

The remainder of this section deals with a class of matroids called Dowling lattices. This class of geometric lattices was discovered by Dowling [11] and is based on finite groups. It is of particular interest to matroid theorists and has strong connections with linear coding theory and with arrangements of hyperplanes (see [2, 5, 10, 14, 15, 24, 25]). Dowling showed that these lattices are supersolvable. Hence they are also line-closed, as shown by Halsey [12]. Since viewing these matroids from the perspective of line-closure better suits the present purposes, a review of the notion of line-closure follows. Dowling lattices can then be defined via line closure.

Let $M$ be a geometry (simple matroid) on a point set $S$. A subset $T$ of $S$ is a line-flat (or line-closed set) in $M$ if and only if for every two points $x, y \in T$, the line $x \vee y$ is contained in $T$. Notice that every flat of $M$ is also a line flat of $M$. The geometry $M$ is called line-closed if every line flat in $M$ is also a flat. Hence line-closed geometries are completely determined by their (labeled) lines. Further, Halsey also showed that any geometry which has the rank and (labeled) lines of a line-closed geometry must be that line-closed geometry.

Thus one may define a line-closed geometry by specifying the set of points and the set of lines, and then verifying that these lines are the lines of a line-closed geometry.

Toward defining finite Dowling lattices, let $G$ be any finite group. The rank-$r$ Dowling lattice over $G$, denoted $Q_r(G)$, has the following points and lines. There are two kinds of points: coordinate points $\{b_1, b_2, \ldots, b_r\}$, which form a basis for $Q_r(G)$;
and noncoordinate points \( g_{ij} \), for every \( g \in G \) and every pair of indices \((1 \leq i < j \leq r)\) (so there is a copy of \( G \) for each pair of coordinate points). There are three types of lines: coordinate lines \( b_i \uplus b_j = b_i \uplus b_j \cup \{g_{ij} \mid g \in G\} \) (i.e., containing \( b_i, b_j \), and the \( ij \)-copy of \( G \)); transversal lines \( \{g_{ij}, h_{jk}, (gh)_k \} \) for each pair \( g, h \in G \) residing on intersecting coordinate lines; and trivial lines, connecting any two points not already on a common line. Note that the transversal lines in the coordinate planes \( b_i \uplus b_j \uplus b_k \) completely capture \( G \)'s group operation. It can be shown that line-closure indeed forms a matroid closure operator for a geometry with these points and lines.

Note that for \( r = 3 \), \( G \) need only be a quasi-group in order for line closure to induce a geometry on the points of \( Q_3(G) \). It has also been shown [11] that if \( G \) is trivial then \( Q_3(G) \) is isomorphic to the rank-\( r \) partition lattice \( \Pi_{r+1} \).

A useful result for reconstructing Dowling lattices is the axiom scheme developed by Bennett et al. [2]. This states that a geometric lattice \( L \) of rank \( r \geq 4 \) is a Dowling lattice if and only if \( L \) has a basis \( b_1, b_2, \ldots, b_r \) satisfying:

(Ax. 1) Each point of \( L \) lies on a coordinate line \( b_i \uplus b_j \).

(Ax. 2) No coordinate line \( b_i \uplus b_j \) is trivial.

(Ax. 3) For nonbasis points \( p \) and \( q \) on distinct coordinate lines \( b_i \uplus b_j \) and \( b_i \uplus b_k \) through \( b_i \), the line \( p \uplus q \) is nontrivial.

Note that the term coordinate line can be used in any matroid context to refer to a line spanned by two points in a specified basis. Also, as is justified by well-known cryptomorphisms, this paper will use the notions of geometric lattice and simple matroid interchangeably. Hence, the term Dowling lattice is used to refer to the geometry associated with that lattice, and mention of the partition lattice \( \Pi_{r+1} \) is actually referring to the matroid \( M(K_{r+1}) \), the cycle matroid of the complete graph \( K_{r+1} \).

Proposition 2.4 will show that Dowling lattices over finite groups are deletion reconstructible. To begin, the special case of partition lattices are shown to be reconstructible.

**Lemma 2.2.** Partition lattices \( \Pi_{r+1} \) are deletion reconstructible.

**Proof.** It will first be shown that any reconstruction of \( \Pi_{r+1} \) is graphic.

Let \( M \) be any reconstruction of \( \Pi_{r+1} \). One can see from the number of points and the rank that \( M \) is not among \( U_{2,4}, F_7, F_7^+, M^*(K_5) \), or \( M^*(K_{3, 3}) \) (i.e., the forbidden minors for graphic matroids). Since deletion and contraction commute, any forbidden minor in \( M \) involving both deletions and contractions would be a minor of some deletion of \( M \) (hence of any deletion of \( M \) since all deletions are isomorphic). Thus, no such minor can exist, since it would be a minor of the graphic matroid \( \Pi_{r+1} \). So if \( M \) contains a forbidden minor for graphic matroids, it is a contraction, say \( M/X \). Notice, however, that for any points \( x, y \in M \), the minor \( M/y - x \) has several nonsimple points (since \( \Pi_{r+1}/y - x \) does). Hence, any contraction of \( M \) must have nonsimple points, and some of these could be deleted before contracting to form
a minor forbidden to graphic matroids. Thus $M$ cannot contain any such minor solely as contraction of points. So $M$ must be graphic.

Finally, note that $M$ contains the maximum number of points possible for a graphic matroid of its rank. Therefore, it must be that $M \cong \Pi_{r+1}$. 

The following technical lemma is useful in proving that Dowling lattices over the 2-element group are deletion reconstructible. Its proof is a straightforward exercise in checking all possible cases of joining points to the various lines in a Dowling lattice.

**Lemma 2.3.** For $|G| = 2$, planes in $Q_r(G)$ can only have cardinalities 3, 4, 5, 6, or 9; the coordinate planes are the only planes with 9 points.

**Proposition 2.4.** Dowling lattices over finite groups are deletion reconstructible.

**Proof.** Let $M$ be any matroid with the same deck of deletions as $Q_r(G)$.

Case I: If $|G| \geq 3$, then coordinate lines have at least 5 points, and other lines have cardinality 2 and 3. Also, no coordinate point lies on any 3-point line. Reconstructing the line multiset shows this skip in line cardinalities, and one can spot in the deck of deletions that some deletions have the same number of 3-point lines as $M$ (corresponding precisely to deletions of coordinate points). Let $M - e$ be any of these (they are all isomorphic). Now $M - e$ will have only four cardinalities of lines (namely 2, 3, $|G| + 1$, and $|G| + 2$), so label $M - e$ and add $e$ to all $(|G| + 1)$-point lines. This gives the complete set of (nontrivial) labeled lines of $M$, which are also the labeled lines of $Q_r(G)$. So $M$ has the lines and rank of the line-closed matroid $Q_r(G)$ and is thus isomorphic to $Q_r(G)$.

Case II: If $|G| = 1$ then $Q_r(G) \cong r+1$, so by Lemma 2.2 the proof is complete.

Case III: For $|G| = 2$, the proof is divided into two parts. First assume $r > 3$. Notice from the reconstruction of the flat multiset that all 9-point planes are isomorphic. By Lemma 2.3 all other planes have fewer than eight points. But some deletions in the deck have 8-point planes in which not every pair of 3-point lines intersect (corresponding to the deletions of coordinate points). Let $M - e$ be any of these, label it, and examine the 8-point planes. In $M$ these must have been 9-point planes, and hence must have had all pairs of 3-point lines intersecting. There is only one way of adding $e$ which accomplishes this (namely extend the two nonintersecting 3-point lines to 4-point lines intersecting at $e$), so do so. These amended 8-point planes and the original 9-point planes of $M - e$ give all labeled 9-point planes of $M$. Now notice by comparing with the line multiset of $M$ that these labeled 9-point planes contain all (nontrivial) labeled lines of $M$ (also the labeled lines of $Q_r(G)$). So $M$ has the lines and rank of $Q_r(G)$ and hence is isomorphic to $Q_r(G)$.

For $|G| = 2$, $r = 3$, notice that exactly three deletions have two fewer 4-point lines than $M$. Let $M - e$ be one of these. So $e$ must have been the intersection point of two
4-point lines. All but one pair of 3-point lines intersect in \( M - e \), so the only way to add \( e \) is to add it to this nonintersecting pair. This makes \( M \cong Q_3(G) \), completing the proof. \( \square \)

3. Contraction reconstruction

Although the question of which matroids are reconstructible solely from their decks of contractions has never been asked explicitly, it is implicit as the dual to the deletion reconstruction question. In particular a matroid \( M \) is deletion reconstructible precisely when its dual \( M^* \) is contraction reconstructible. Hence, by dualizing the examples excluded in (M2), matroids consisting entirely of loops, and those consisting of a single point (simple or nonsimple) are not contraction reconstructible. Thus, all matroids referred to in this section are assumed to have rank at least two.

**Proposition 3.1** The following are reconstructible from the deck of contractions.

(a) The number of points and the rank of a matroid.
(b) The Tutte polynomial of a matroid.
(c) Disconnected matroids.
(d) Duals of affine and projective geometries.
(e) Duals of Dowling lattices.
(f) The multiset of flat cardinalities of a matroid \( M \).

**Proof.** Parts (a)–(e) follow from duality and results from previous section. For (f), note that it can be assumed that \( M \) has no loops. For any \( x \in M \) the rank-\( t \) (\( t \geq 0 \)) flats of \( M/x \) are precisely the rank-\( (t + 1) \) flats of \( M \) which contained \( x \) (with \( x \) now removed). So if there are \( m \) rank-\( t \) flats of cardinality \( k \) among all contractions of \( M \), then there will be \( m/(k + 1) \) rank-(\( t + 1 \)) flats of cardinality \( k + 1 \) in \( M \). \( \square \)

The final proposition proves contraction reconstructibility of Dowling lattices of ranks greater than three. Although it is straightforward to show that for any group \( G \), a matroid with the same deck of contractions as \( Q_3(G) \) must be a rank-3 Dowling lattice, it is also true that if \( G \) is not trivial, then \( Q_3(G) \) is not contraction reconstructible, as any two nonisomorphic (quasi) groups of the same order will yield rank-3 Dowling lattices with identical decks of contractions.

The proof of Proposition 3.2 requires use of the Scum Theorem of Higgs [18].

**The Scum Theorem.** Let \( M_1 \) be a simple minor of a simple matroid \( M(S) \). Then \( M \) has a flat \( F \) of rank \( \text{rk}(M) - \text{rk}(M_1) \) such that there is a one-to-one order-preserving map from the lattice \( \mathcal{L}(M_1) \) into the interval \( [F, S] \) of \( \mathcal{L}(M) \).

**Proposition 3.2.** The Dowling lattices \( Q_r(G) \) are contraction reconstructible for \( r > 3 \).
Proof. Let $M(S)$ be any reconstruction of $Q_r(G)$. Let $M/x$ be one of the $r$ contractions having $r - 1$ parallel classes of size $|G| + 1$. Since $M$ is simple, the nontrivial parallel classes of $M/x$ must have been lines through $x$, say $l_1, l_2, \ldots, l_{r-1}$.

Next, note by comparing the number of maximal-sized hyperplanes of $M$ with the number of maximal-sized hyperplanes of $M/x$ that there is at least one maximal-sized hyperplane $H$ of $M$ not containing $x$ (and unless $G$ is trivial, $H$ will be unique). The hyperplane $H$ may contain no more than one point from each line $l_i$ (else $H$ would contain $x$). But notice that in order for $H$ to have the correct cardinality, it must contain exactly one point from each of the $l_i$’s and all points outside of the $l_i$’s.

By the Scum Theorem, $M$ has a point $q$ such that $H$ embeds into $[q, S]$. However, the number of points in $H$ equals the number of atoms in $[q, S]$. Thus, $H \cong [q, S]$. But $[q, S] \cong Q_{r-1}(G)$ for every $q \in M$. Similarly, all maximal hyperplanes of $M$ are isomorphic to $Q_{r-1}(G)$.

Let $l_i \cap H = \{b_i\}$ for $i = 1, 2, \ldots, r - 1$, and let $b_r = x$. Since $H \cong Q_r(G)$ and $x \notin H$, note that the $b_i$’s form a basis for $M$. Note also that every point lies on a coordinate line $b_i \vee b_j$ and no coordinate line is trivial. Finally, since all maximal hyperplanes are isomorphic to $Q_{r-1}(G)$, they all satisfy Axiom 3 for coordinate bases; hence $M$ must also. So the $b_i$’s satisfy the axioms of a coordinate basis for a Dowling lattice. Further, the group of this Dowling lattice is known, since it is reflected in every coordinate plane of $M/x$. Thus $M \cong Q_r(G)$. □

It is straightforward to show that $\Pi_4$ is contraction reconstructible. This and Proposition 3.2 prove Corollary 3.3, which is in the spirit of results by Aigner [1], Stonesifer and Bogart [19], and Yoon [23], all of which concern reconstruction of partition lattices.

**Corollary 3.3.** The partition lattice $\Pi_{r+1}$ is contraction reconstructible.

**References**