Finite Range Random Fields and Energy Fields

WAYNE G. SULLIVAN*

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

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1. Introduction

In the lattice gas model of probability theory and statistical mechanics each point of the $\nu$-dimensional integer lattice $\mathbb{Z}^\nu$ can be vacant or occupied by any one of a finite set of distinct particles. A problem in probability theory is to determine which probability measures on the space of configurations of the lattice correspond to a given family of conditional probabilities for each point of the lattice (see Dobrushin [1]). In statistical mechanics a basic problem is to determine which probability measures correspond to a given potential on the lattice.

In the case of the nearest neighbor interaction, one-particle lattice gas, Spitzer [9] has shown that these two problems are essentially the same. We extend this analysis to the finite range interaction $\omega$-particle lattice gas.

One of the ideas underlying our approach comes from the fundamental energy-probability relationship of statistical mechanics. For a fixed temperature the probability of a state is proportional to the negative exponential of the energy of the state in appropriate energy units. We reverse this rule to get energies from probabilities.

The restriction to the finite range case simplifies proofs and allows such quantities as specific free energies to be expressed as finite sums. The set of interaction considered by Ruelle [7] is more general, but the finite range interactions are dense in this set.

2. Random Fields and Energy Fields

$W = \{0, 1, \ldots, \omega\}$ is the finite set of single particle states. The state space of the $\nu$-dimensional integer lattice $\mathbb{Z}^\nu$ is $\Omega = W^{\mathbb{Z}^\nu}$. $\mathcal{L}$ denotes the set of finite subsets of $\mathbb{Z}^\nu$. A finite cylinder set is a finite union of sets of the form

$$\{\omega \in \Omega: \omega = x \text{ on } A\}$$

(2.1)

* Present address: Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland.

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for $x \in \Omega$, $\Lambda \in \mathcal{L}$. $\mathcal{F}$ denotes the $\sigma$-field generated by the finite cylinder sets. The expression *measure* on $\Omega$ is used to mean an $\mathcal{F}$-defined, nonnegative, countably additive measure on $\Omega$. A measure on $\Omega$ is uniquely determined by its values on all sets of the form (2.1) or equivalently by its values on all finite cylinder sets. A measure on $\Omega$ is called *positive* if it takes on strictly positive values on all nonempty finite cylinder sets. We shall use without comment natural correspondences between $\Omega$ and $W^A$. A measure is not distinguished from the function which takes on the measure of each point.

Let $\mu$ be a positive probability measure on $\Omega$. Let $\Lambda, \Lambda' \in \mathcal{L}$, $\Lambda \cap \Lambda' = \phi$, $x, y \in \Omega$. The conditional probability of $x$ on $\Lambda$ given $y$ on $\Lambda'$ is expressed

$$
\mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \Lambda') = \mu(\omega = x \text{ on } \Lambda, \omega = y \text{ on } \Lambda')/\mu(\omega = y \text{ on } \Lambda').
$$

**Definition.** An *energy field* $E(\Lambda, x)$ is a real-valued function on $\mathcal{L} \times \Omega$ such that for a given $\Lambda$, $E(\Lambda, x)$ depends only on the values of $x$ on $\Lambda$. The *family* $\nu_\Lambda$ of probability measures on $W^A$ associated with $E(\Lambda, x)$ is defined

$$
\nu_\Lambda(\omega = x \text{ on } \Lambda) = \exp(-E(\Lambda, x))/z_\Lambda
$$

with the constant $z_\Lambda$ chosen so that $\nu_\Lambda$ is a probability measure on $W^A$.

**Definition.** A *positive random field* $\mu$ is a positive probability measure on $\Omega$. The energy field associated with $\mu$, $E_\mu(\Lambda, x)$, is defined

$$
E_\mu(\Lambda, x) = -\log(\mu(\omega = x \text{ on } \Lambda)).
$$

We now introduce some special notation. For $x \in \Omega$, $i, j \ldots k \in Z^r$

$$
ij\ldots kx
$$

denotes that element of $\Omega$ equal to zero on $\{i, j, \ldots, k\}$ and equal to $x$ elsewhere on $Z^r$. Let $E(\Lambda, x)$ be an energy field.

$$
\Delta^i E(\Lambda, x) = E(\Lambda, x) - E(\Lambda, j^i x)
$$
$$
\Delta^i E(\Lambda, x) = \Delta^k E(\Lambda, x) - \Delta^k E(\Lambda, j^i x).
$$

A simple computation shows that $\Delta^{jk} E(\Lambda, x) = \Delta^{kj} E(\Lambda, x)$.

In the following we shall be interested in recovering probability distributions from conditional probabilities.

**Lemma 1.** Let $\Lambda \in \mathcal{L}$. Let $q(j, x)$ be a real-valued function on $\Lambda \times W^A$ such that

1. $q(j, x) > 0$ for all $(j, x)$ in the domain.
2. For $j$ and $x$ fixed the sum over $W$ of terms $q(j, y)$ where $y$ differs from $x$ at most at $j$ is equal to 1.
For each $x \in W$, $j_k \in \Lambda$

\[(q(j, x)/q(k, x))(q(k, j_k x)/q(k, j x)) = (q(k, x)/q(k, j_k x))(q(j, x)/q(j, j_k x)).\]

Then there is a unique positive probability measure $\nu$ on $W$ such that

\[\nu(\omega_j = x_j | \omega = x \text{ on } \Lambda - \{j\}) = q(j, x). \quad (2.6)\]

Proof. We proceed to construct an energy function for $\nu$. Define $A^j E(x) = -\log(q(j, x)/q(j, j_k x))$. Then condition (3) becomes

\[A^j E(x) + A^k E(j_k x) = A^k E(x) + A^j E(k x). \quad (3^*)\]

Let $j_1, \ldots, j_m$ be an enumeration of $\Lambda$. We set

\[E(x) = A^j E(i \cdots i m x) + \cdots + A^{m-1} E(i m x) + A^m E(x). \quad (2.7)\]

Had the enumerations of $\Lambda$ differed from that given in two adjacent indices, then (3*) guarantees that the same $E(x)$ would be obtained. But any permutation of the enumeration can be effected by adjacent transpositions, so that $E(x)$ given by (2.7) is independent of the order of enumeration of $\Lambda$. Define $\nu(x) = (\exp - E(x))/x$ with $x$ determined so that $\nu$ is a probability measure on $W$. For a given $(j, x)$, we can take $j$ to be $j_m$ in (2.7); then

\[\nu(x)/\nu(j x) = \exp - (E(x) - E(i x)) = q(j, x)/q(j, i x). \quad (2.8)\]

Condition (2) of the lemma and (2.8) imply the conclusion (2.6). $\nu(x)/\nu(y)$ is determined by terms of the form (2.8), so uniqueness is assured for probability measures.

3. Finite Range Random Fields and Energy Fields

Definition. A finite range random field $\mu$ is a positive random field with the property that for each $j \in Z^*$ there exists a $\Lambda_j \in L$ not containing $j$ such that if $\Lambda$ is any element of $L$ not containing $j$ and $x \in \Omega$, $\mu(\omega_j = x_j | \omega = x$ on $\Lambda \cup \Lambda_j) = \mu(\omega_j = x_j | \omega = x$ on $\Lambda_j)$. Any such $\Lambda_j$ with this property is called a bounding set for $j$.

Lemma 2. Let $\mu$ be a finite range random field and let $E_\mu$ be the energy field associated with $\mu$ by (2.3). For fixed $j$ let $\Lambda$ be a bounding set for $j$. In the notation (2.5) let $\Lambda' \subset \{k \in \Lambda: A^k E_\mu(\Lambda \cup \{j\}, x) = 0$ for all $x \in \Omega\}$. Then $\Lambda - \Lambda'$ is also a bounding set for $j$. 

Proof. We must show $\mu(\omega_j = x_j | \omega = x \text{ on } \Lambda) = \mu(\omega_j = x_j | \omega = x \text{ on } \Lambda - \Lambda')$. Since both are measures on $W$, it is sufficient to show

$$\mu(\omega = x \text{ on } \Lambda \cup \{j\})/\mu(\omega = x \text{ on } \Lambda \cup \{j\})$$

$$= \mu(\omega = x \text{ on } \Lambda - \Lambda')/\mu(\omega = x \text{ on } \Lambda - \Lambda').$$

The right side of (3.1) equals

$$\sum_{y \in W'} \mu(\omega = x \text{ on } \Lambda \cup \{j\} - \Lambda', \omega = y \text{ on } \Lambda')/\sum_{y \in W'} \mu(\omega = x \text{ on } \Lambda \cup \{j\} - \Lambda', \omega = y \text{ on } \Lambda').$$

(3.2)

Consider the ratio of the expression (3.2) without the $\Sigma'$s. This ratio is independent of $y$ because $\Delta^k E_\mu(\Lambda \cup \{j\}, x) = 0$ for all $x \in \Omega$, $k \in \Lambda'$. The desired conclusion follows.

**Theorem 1.** Let $\mu$ be a finite range random field. Then each point $j \in Z^r$ has a unique minimal bounding set.

**Proof.** It is sufficient to show that the intersection of two bounding sets for $j$ is a bounding set for $j$. Let $\Lambda$ and $\Lambda'$ be bounding sets for $j$. Then $\Lambda \cup \Lambda'$ is a bounding set for $j$. We can express $\Delta^k E(\Lambda \cup \Lambda' \cup \{j\}, x)$ in terms of $\Delta^k E(\Lambda \cup \{j\}, x)$ or in terms of $\Delta^k E(\Lambda' \cup \{j\}, x)$, so that

$$\Delta^k E(\Lambda \cup \Lambda' \cup \{j\}, x) = 0$$

whenever $k \notin \Lambda \cap \Lambda'$. Lemma 2 implies that $\Lambda \cap \Lambda'$ is a bounding set for $j$.

**Definition.** Let $\mu$ be a finite range random field and let $j \in Z^r$. The boundary $\partial j$ of $j$ is the minimal bounding set of Theorem 1. The closure $\bar{j} = \{j\} \cup \partial j$. For $\Lambda \in \mathcal{L}$, the closure $\bar{\Lambda} = \{k: \text{there exists } j \in \Lambda \text{ with } k \in j\}$. The boundary $\partial \Lambda = \bar{\Lambda} - \Lambda$; the interior $\Lambda^0 = \{j \in \Lambda: j \subset \Lambda\}$.

**Definition.** A finite range energy field $E(\Lambda, x)$ is an energy field such that for each $j \in Z^r$ there exists a $\Lambda_j \in \mathcal{L}$ containing $j$ such that for each $\Lambda \in \mathcal{L}$, $\Lambda \supset \Lambda_j$ the following holds.

$$\Delta^k E(\Lambda, x) = \Delta^k E(\Lambda_j, x) \quad \text{for all } x \in \Omega.$$ 

(3.3)

Any such $\Lambda_j$ with this property is called a closing set for $j$.

Elementary considerations yield the following two lemmas.

**Lemma 3.** A finite linear combination of finite range energy fields is a finite range energy field.
4. Let \( g : \mathcal{L} \to \mathcal{L} \) be a function with the property that \( g(A) \cap A = \emptyset \) for all \( A \in \mathcal{L} \). Let \( \mu \) be a finite range random field and let \( y \in \Omega \). Then \( E(A, x) = -\log \mu(\omega = x \mid \omega = y \text{ on } g(A)) \) is a finite range energy field.

**Definition.** Let \( E \) and \( E' \) be finite range energy fields. \( E \) and \( E' \) are said to be **increment equivalent** provided that for each \( j \in \mathbb{Z}^r \) whenever \( A \) and \( A' \) are closing sets for \( j \) with respect to \( E \) and \( E' \) respectively

\[
\Delta^k E(A, x) = \Delta^k E'(A', x) \quad \text{for all } x \in \Omega. \tag{3.4}
\]

**Lemma 5.** Let \( E \) and \( E' \) be increment equivalent finite range energy fields, and let \( \nu_A \) and \( \nu_{A'} \) be the families of probability measures associated by (2.2) with \( E \) and \( E' \) respectively. Let \( \Sigma \in \mathcal{L} \) and let \( A \) and \( A' \) be closing sets for each point of \( \Sigma \) with respect to \( E \) and \( E' \). Then for all \( x \in \Omega \)

\[
\nu_A(\omega = x \mid \omega = x \text{ on } \Sigma - \Sigma) = \nu_{A'}(\omega = x \mid \omega = x \text{ on } \Sigma - \Sigma). \tag{3.5}
\]

**Proof.** Without loss of generality we may assume \( x \) to be fixed outside of \( \Sigma \). Define \( q(k, x) \) on \( \Sigma \times W^2 \) by \( q(k, x) = \nu_A(\omega_k = x_k \mid \omega = x \text{ on } A - \{k\}) \). Define \( q'(k, x) \) analogously with \( \nu_{A'} \). Then \( q \) and \( q' \) satisfy the hypotheses of Lemma 1. It is clear that the measure on \( W^2 \) determined by \( q \) is \( \nu_A(\omega = x \mid \omega = x \text{ on } A - \Sigma) \), so it is sufficient to prove that \( q = q' \). This follows from the normalization of \( q \) and \( q' \) and the fact that for each \( k \in \Sigma \), \( \Delta^k E(A, x) = \Delta^k E'(A', x) \).

**Corollary.** The family of probability measures \( \nu_A \) associated by (2.2) with the finite range energy field \( E \) is consistent in the sense of (3.5) with \( \nu = \nu' \).

The consistency of the family of probability measures allows the definition of interior, closure and boundary just as with finite range random fields. Increment equivalent finite range energy fields determine the same closure etc. in this sense. However, the finite range energy field closure of the point \( j \in \mathbb{Z}^r \) is not necessarily a closing set for \( j \), a difficulty not encountered with finite range random fields.

The following two lemmas are central to subsequent convergence and approximation theorems.

**Lemma 6.** Let \( x \in \Omega \) and let \( \mu \) be a finite range random field. Let \( A \) and \( A' \) be disjoint elements of \( \mathcal{L} \). Then

\[
\min_{y \in W^A} \mu(\omega = x \mid \omega = y \text{ on } \partial A) \\
\leq \mu(\omega = x \mid \omega = x \text{ on } A') \\
\leq \max_{y \in W^A} \mu(\omega = x \mid \omega = y \text{ on } \partial A) \tag{3.6}
\]
Proof.

\[ \mu(\omega = x \text{ on } A \mid \omega = x \text{ on } A') = \sum_{y \in \partial A - A'} \mu(\omega = y \text{ on } \partial A - A' \mid \omega = x \text{ on } A') \]

\[ \mu(\omega = x \text{ on } A \mid \omega = x \text{ on } A', \omega = y \text{ on } \partial A - A'). \]

**Lemma 7.** Let \( E \) be a finite range energy field with \( \nu_A \) the associated family of probability measures. Let \( x \in \Omega \), let \( \Sigma \in \mathcal{L} \) and let \( \Lambda \in \mathcal{L} \) be a closing set for each point of \( \Sigma \). Then there exists a \( \delta > 0 \) such that for each \( \Lambda' \in \mathcal{L}, \Lambda' \supset \Lambda, \nu_A'(\omega = x \text{ on } \Sigma) > \delta \). The proof is by an estimate similar to (3.6) using the closing set \( \Lambda \) instead of the boundary.

4. Convergence

There are natural topologies which endow both \( \Omega \) and the set of (\( \mathcal{F} \)-defined, countably additive, nonnegative) probability measures on \( \Omega \) with the important property of compactness. We shall not go into the topological details but shall state a property sufficient for our purposes.

**Sequential Compactness Property of Probability Measures on \( \Omega \)**

Let \( \{\mu_n\} \) be a sequence of probability measures on \( \Omega \). Then there exists a probability measure \( \mu \) on \( \Omega \) and a subsequence \( \{\mu_{n'}\} \) of \( \{\mu_n\} \) which converges to \( \mu \) on each finite cylinder set.

**Definition.** \( V_n = \{j \in \mathbb{Z}^r: |j_\alpha| \leq n, \alpha = 1,..., r\} \).

The following is a special case of a theorem of Dobrushin [1].

**Theorem 2.** Let \( E(\Lambda, x) \) be a finite range energy field. Then there exists a finite range random field \( \mu \) for which the energy field \( E_\mu(\Lambda, x) \) associated to \( \mu \) by (2.3) is increment equivalent to \( E(\Lambda, x) \).

**Proof.** Let \( \mu_n \) be the probability measure on \( W^{V_n} \) associated with \( E(\mathbb{V}_n, x) \) by (2.2). Let \( \mu_n^* \) denote the measure of \( \Omega \) which is the product of \( \mu_n \) on \( W^{V_n} \) with the probability measures of uniform density \( 1/(w + 1) \) on each of the remaining factor spaces \( W \) of \( \Omega \). Then the sequence \( \{\mu_n^*\} \) has a subsequence \( \{\mu_{n'}^*\} \) converging to a probability measure \( \mu \) on \( \Omega \). Lemma 7 implies that \( \mu \) is positive. Let \( E_\mu \) be the associated energy field. Take \( j \in \mathbb{Z}^r \) and let \( \Lambda \) be a closing set for \( j \). Then for all \( n' \) such that \( \Lambda \subset V_{n'} \), \( \mu_{n'}(\omega_j = x_j \mid \omega = x \text{ on } \Lambda - \{j\}) = \mu_n(\omega_j = x_j \mid \omega = x \text{ on } \Lambda - \{j\}) \) from Lemma 5 and the proof of Theorem 1, the subscripted measures being associated with the original energy field \( E \). By Lemma 7, \( \mu_{n'}^* \) is
bounded away from zero on any fixed finite cylinder set except \( \phi \), so conditional probabilities on finite cylinder sets can be obtained by taking limits. This implies that
\[
\mu_A(\omega_j = x_j \mid \omega = x) = \mu(\omega_j = x_j \mid \omega = x).
\]
Thus \( E_\mu \) is increment equivalent to \( E \), which in turn implies that \( \mu \) is finite range.

5. Translation Invariance

Let \( j \in \mathbb{Z}^r \). We shall use the symbol \( T_j \) to denote translation by \( j \) on several spaces. For \( A \in \mathcal{L} \), \( x \in \Omega \), \( A \in \mathcal{F} \), \( \mu \) a probability measure on \( \Omega \) and \( E(A, x) \) an energy field,

\[
T_j A = \{ j + k : k \in A \},
\]
\[
(T_j x)_k = x_{k-j},
\]
\[
T_j A = \{ T_j x : x \in A \},
\]
\[
(T_j \mu)(A) = \mu(T_j A),
\]
\[
T_j E(A, x) = E(T_j A, T_j x).
\]

**DEFINITION.** A probability measure \( \mu \) on \( \Omega \) or an energy field \( E \) is called translation invariant provided that it is left fixed by \( T_j \) for each \( j \in \mathbb{Z}^r \).

**Notation.** For \( A \in \mathcal{L} \), \( |A| \) denotes the number of elements in \( A \).

**Theorem 3.** Let \( E \) be a translation invariant finite range energy field. Then there exists a translation invariant finite range random field \( \mu \) such that \( E_\mu \) is increment equivalent to \( E \).

**Proof.** By Theorem 2 there is a random field \( \nu \) such that \( E_\nu \) is increment equivalent to \( E \). Let

\[
\mu_n = \sum_{j \in V_n} T_j \nu / | V_n |
\]

with \( V_n \) as defined preceding Theorem 2. The energy field associated with \( \mu_n \) is increment equivalent to \( E \). The sequence \( \{ \mu_n \} \) possesses a subsequence \( \{ \mu_n' \} \) which converges to a random field \( \mu \) on each finite cylinder set. Lemma 7 insures convergence of conditional probabilities on finite cylinder sets. Then \( E_\mu \) is increment equivalent to \( E \), which implies that \( \mu \) is finite range. Standard arguments show that the limit of terms of the form (5.2) is translation invariant.
Let $\mu$ be a finite range random field. In addition to $E_\mu$ of (2.3) we introduce some related energy fields. For $x, y \in \Omega, \Lambda \in \mathcal{L}$

$$E_\mu^y(\Lambda, x) = -\log \mu(\omega = x \text{ on } \Lambda \mid \omega = y \text{ on } \partial \Lambda).$$

$$E_\mu^{y*}(\Lambda, x) = E_\mu^y(\Lambda, x) - E_\mu^y(\Lambda, 0). \quad (6.1)$$

We use 0 to denote both the origin in $\mathbf{Z}^r$ and that element $x \in \Omega$ such that $x_j = 0$ for all $j \in \mathbf{Z}^r$.

**Lexicographical order for $\mathbf{Z}^r$**

If $j, k \in \mathbf{Z}^r$ we write $j < k$ to mean that $j \neq k$ and if $\alpha$ is the smallest index such that $j_\alpha \neq k_\alpha$, then $j_\alpha < k_\alpha$. We write $j < k$ to mean $j = k$ or $j < k$. Lexicographical order is translation invariant. Any translation invariant linear order will suffice for our purposes.

**Lemma 8.** Let $\mu$ be a finite range random field, $x, y \in \Omega, \Lambda \in \mathcal{L}$. Then

$$E_\mu^{y*}(\Lambda, x) = \sum_{j \in \Lambda} \Delta^j E_\mu(\Lambda, \{j\}_x) \quad (6.2)$$

with

\[
\{j_x\}_x = \begin{cases} 
  \gamma_k & \text{if } k \notin \Lambda, \\
  x_k & \text{if } k \in \Lambda \text{ and } k \leq j, \\
  0 & \text{if } k \in \Lambda \text{ and } j < k.
\end{cases} \quad (6.3)
\]

**Proof.** This summation formula arises in the proof of Lemma 1. By translation we get the following

**Corollary.** Let $\mu$ be translation invariant. Then

$$E_\mu^{y*}(\Lambda, x) = \sum_{j \in \Lambda} \Delta^0 E_\mu(\Lambda, \{j\}_x)) \quad (6.4)$$

where $\overline{0}$ denotes the closure of $0 \in \mathbf{Z}^r$.

**Lemma 9.** Let $\mu$ be a translation invariant finite range random field. Then there exists a real constant $c$ such that

$$| E_\mu(\Lambda, x) - E_\mu(\Lambda, 0) - E_\mu^{y*}(\Lambda, x) | \leq c | \Lambda - \Lambda^0 | \quad (6.5)$$

for all $x, y \in \Omega, \Lambda \in \mathcal{L}$. 
Proof. From Lemma 6 it follows that
\[ \max_{y \in \mathcal{W}^\mathcal{A}} E_{u^*}(A, y) \geq E_u(A, x) - E_u(A, 0) \geq \min_{y \in \mathcal{W}^\mathcal{A}} E_{u^*}(A, y). \quad (6.6) \]

Now by Lemma 8
\[ E_{u^*}(A, x) - E_{u^*}(A, y) = \sum_{j \in \mathcal{A} - z^0} \Delta^j E_{u}(\tilde{A}, j, x) - \Delta^j E_{u}(\tilde{A}, j, y). \quad (6.7) \]

So by (6.6), (6.7) and (6.4) we deduce that with \( c = 2 \max z \in \mathcal{W}^0 | \Delta^0 E_u(0, z)| \), (6.5) holds.

**Theorem 4.** Let \( \mu \) be a translation invariant finite range random field. Then with \( V_n \) as defined preceding Theorem 2
\[ \lim_{n \to \infty} E_{\mu}(V_n, 0)/|V_n| \text{ exists.} \quad (6.8) \]

*Proof.* Let \( n \) and \( n' \) be positive integers with \( n' > n \). We partition \( V_n \) into \( A_0, A_1, \ldots, A_m \) such that \( A_1, \ldots, A_m \) are isomorphic to \( V_n \) under translation. Then
\[ \mu(\omega = 0 \text{ on } V_{n'}) = \mu(\omega = 0 \text{ on } A_0) \mu(\omega = 0 \text{ on } A_1 | \omega = 0 \text{ on } A_0) \ldots \mu(\omega = 0 \text{ on } A_m | \omega = 0 \text{ on } A_0 \ldots A_{m-1}). \quad (6.9) \]

Using (6.9), Lemma 6 and the techniques of the proof of Lemma 9, we get an estimate of the form
\[ \left| E_{\mu}(V_{n'}, 0) - mE_{\mu}(V_n, 0) \right| \leq c(m|V_n - V_n^0| + |A_0|) \quad (6.10) \]
where \( c \) is a positive constant. By appropriate choice of \( n \) we can make \( c|V_n - V_n^0|/|V_n| \) as small as desired. For fixed \( n \) we can find \( n^* > n \) such that for all \( n' > n^* \) with the appropriate partition \( c|A_0|/|V_{n'}| \) is less than any specified positive value. Then by the Cauchy criterion we have the limit (6.8).

**Definition.** The \( \mu \)-specific energy of the zero state \( e_{\mu}(0) \) is the limit (6.8).

**Lemma 10.** Let \( \mu \) and \( \nu \) be translation invariant random fields. Let \( \mu \) be finite range. Then
\[ \lim_{n \to \infty} \sum_{x \in V_n} \nu(\omega = x \text{ on } V_n) E_{u^*}(V_n, x)/|V_n| \quad (6.11) \]
exists and equals
\[ \sum_{x \in W_1} \nu(\omega = x \text{ on } \Psi) \Delta^0 E_\mu(\bar{0}, \bar{0}, x), \quad \Psi = \{ j \in \bar{0}: j \leq 0 \} \] (6.12)

independent of \( y \in \Omega \).

**Proof.** Express \( E_\mu^y(V_n, x) \) as the sum in (6.4). By translation invariance the \( \nu \)-average of this sum has \( |V_n^0| \) terms equal to (6.12). The remaining \( |V_n - V_n^0| \) terms are each bounded by a single fixed constant. Since
\[ \lim_{n \to \infty} |V_n - V_n^0|/|V_n| = 0, \] (6.13)

the lemma follows.

**Definition.** The \( \mu \)-specific energy of \( \nu \), denoted \( e_\mu(\nu) \), is defined so that \( e_\mu(\nu) - e_\mu(0) \) is the sum (6.12) with \( e_\mu(0) \) as defined above.

**Theorem 5.** Let \( \nu \) be a probability measure on \( \Omega \). Let \( \mu \) be a finite range random field with associated energy field \( E_\mu \). Then, if \( \nu \) and \( \mu \) are translation invariant,
\[ \lim_{n \to \infty} \sum_{x \in W_n} \nu(\omega = x \text{ on } V_n) E_\mu(V_n, x)/|V_n| \] (6.14)

exists and equals \( e_\mu(\nu) \). Further, for \( y \in \Omega \) the use of \( E_\mu^y \) in place of \( E_\mu \) yields the same limit.

**Proof.** Theorem 5 follows from Lemmas 9 and 10 and Theorem 4.

**Lemma 11.** Let \( \mu \) and \( \nu \) be translation invariant random fields with \( \mu \) finite range. Then there exists a positive constant \( c \) such that
\[ |V_n| \left| e_\mu(\nu) - \sum_{x \in W_n} \nu(\omega = x \text{ on } V_n) E_\mu(V_n, x) \right| \leq c |V_n - V_n^0|. \] (6.15)

**Proof.** The proof of Lemma 10 shows that the inequality of the form (6.15) with \( e_\mu(\nu) \) replaced by \( e_\mu(\nu) - e_\mu(0) \) and \( E_\mu(V_n, x) \) replaced by \( E_\mu(V_n, x) - E_\mu(V_n, 0) \) is satisfied. Thus it is sufficient to show
\[ \left| |V_n| e_\mu(0) - E_\mu(V_n, 0) \right| \leq c' |V_n - V_n^0|. \] (6.16)

We obtain (6.16) by multiplying (6.10) by \( |V_n|/|V_n'| \) and taking the limit as \( n' \to \infty \).
DEFINITION. Let $\mu$ and $\nu$ be translation invariant finite range random fields. The specific entropy of $\nu$, $s(\nu)$, is $e_s(\nu)$. The $\mu$-specific free energy of $\nu$ is $e_s(\nu) - s(\nu)$.

**Theorem 6.** In the one-dimensional case, $\Omega = W^Z$, let $\mu$ and $\nu$ be translation invariant random fields with $\mu$ finite range. Then the $\mu$-specific free energy of $\nu$ is zero if and only if $\mu = \nu$.

**Proof.** With the inequality (6.16), $|V_n - V_n^0|$ being constant in the one dimensional case, the proof of Spitzer [10] for finite Markov chains works for finite range random fields as well.

The $\nu$-dimensional version of Theorem 6 has been proved for the one-particle lattice gas by Lanford-Ruelle [6] using partition functions and by Holley [5] using semigroup generators.

7. **Finite Range Potentials**

**Definition.** A finite range potential $U$ on the nonempty set $\Psi \in \mathcal{L}$ is a real-valued function on $W^\Psi$. The energy field $E_U$ of $U$, $\Psi$ is

$$E_U(A, x) = \sum_{i \in A'} U(T_i x)$$

for $x \in \Omega, A \in \mathcal{L}$ and

$$A' = \{ j \in Z^\nu: T_j \Psi \subseteq A \}.$$  

**Theorem 7.** The energy field $E_U$ of the finite range potential $U$ on $\Psi$ is a translation invariant finite range energy field, and $\Psi' = \{j - k: j, k \in \Psi\}$ is a closing set for $0 \in Z^\nu$.

**Proof.** A simple calculation shows that (7.1) is translation invariant. If $A \in \mathcal{L}, A \supset \Psi'$ then

$$\Delta^0 E_U(A, x) = \sum_{k \in \Psi} U(T_k x) - U(T_k^0 x).$$

Expression (7.3) depends only on the values of $x$ on $\Psi'$. The theorem follows from translation invariance.

**Theorem 8.** Let $\mu$ be a translation invariant finite range random field with associated energy field $E_\mu$. Then there exists $\Psi \in \mathcal{L}$ and a finite range potential $U$ on $\Psi$ such that $E_U$ is increment equivalent to $E_\mu$.

**Proof.** In the notation of Lemma 8 let

$$\Psi = \{j \in \delta: j \leq 0\}, \quad U(x) = \Delta^0 E_\mu(\delta, \delta^x).$$
Let $A \in \mathcal{L}$ be such that $A'$ given by (7.2) contains $\bar{0}$. In considering energy increments at 0, we may assume that $x_j = 0$ for $j \notin 0$. By Lemma 8, $E_U(A, x) = E_{\mu}(A', x)$. Hence $\Delta^{0}E_U(A, x) = \Delta^{0}E_{\mu}(A, x)$. Translation invariance implies the theorem.

By Theorem 3 we get the following

**Corollary.** Every finite range translation invariant energy field is increment equivalent to the energy field of a finite range potential.

In certain cases we can improve on Theorem 8 by giving a more precise form of potential.

**Notation.** Let $1, 2, \ldots, \nu$ denote the fundamental unit vectors in $Z^{\nu}$, i.e., $(1)_1 = 1$, $(1)_a = 0$ for all $a \neq 1$ etc.

**Definition.** A Markov random field $\mu$ is a translation invariant finite range random field such that $\partial 0$ is contained in the set of $2\nu$ points $\{\pm 1, \pm 2, \ldots, \pm \nu\}$.

The following is a generalization of a theorem of Spitzer [9].

**Theorem 9.** Let $\mu$ be a Markov random field. Then there exist finite range potentials $U_0, U_1, \ldots, U_\nu$ with $U_0$ on $\{0\}$, $U_\alpha$ on $\{0, \alpha\}$ for $\alpha = 1, \ldots, \nu$ such that if $E_0, E_1, \ldots, E_\nu$ are the corresponding energy fields, then $E_\alpha$ is increment equivalent to $E_0 + E_1 + \cdots + E_\nu$.

**Proof.** Let $A = \{0, \pm 1, \ldots, \pm \nu\}$. Define $U_\alpha(a, b)$ on $\{0, \alpha\}$ for $\alpha = 1, \ldots, \nu$, $a, b \in W$ by

$$U_\alpha(a, b) = E_{\mu}(A, y), \quad y_0 = a, \quad y_\alpha = b, \quad y_k = 0 \text{ otherwise.} \quad (7.6)$$

From (7.6) it follows that the corresponding energy field $E_\alpha(A, x)$ satisfies

$$\Delta^{0}E_\alpha(A, x) = \Delta^{0}E_{\mu}(A, x) \quad \text{if} \quad \beta = \alpha, \quad (7.7)$$

$$= 0 \quad \text{if} \quad \beta \neq \alpha,$$

for all $x \in \Omega$. Consider the energy field

$$E' = E_\mu - E_1 - \cdots - E_\nu. \quad (7.8)$$

Lemma 2 and (7.7) imply that $\{0\}$ is an $E'$ closing set for 0. Then Theorem 8 and its corollary yield a potential $U_0$ on $\{0\}$ whose energy field $E_0$ is increment equivalent to $E'$. 

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Theorem 10. Let $\mu$ be a translation invariant finite range random field on $\Omega = \mathbb{Z}^d$ which satisfies $\partial \Omega \subset \{(\pm 1, \pm 1), (0, \pm 1), (\pm 1, 0)\}$. Then there is a finite range potential on the set $\Psi = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ whose energy field is increment equivalent to $E_\mu$.

Proof. Let $A = \{(0, 0), (\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$. Define $U$ by

$$U \begin{pmatrix} ab \\ cd \end{pmatrix} = E_\mu^0(A, y), \quad y = \begin{pmatrix} ab \\ cd \end{pmatrix} \quad \text{on} \quad \begin{pmatrix} (0, 1) \\ (0, 0) \end{pmatrix}, \begin{pmatrix} (1, 1) \\ (1, 0) \end{pmatrix}$$

and

$$y_h = 0 \text{ otherwise.}$$

Then one computes that $E_\mu - E_U$ has the boundary property of a Markov random field. Theorem 9 gives potentials for $E_\mu - E_U$ which can be combined with $U$ in a notationally tedious, but essentially simple fashion to yield the required finite range potential on $\Psi$.

8. The Kirkwood–Salsburg Equations

Finally we show that the correlation function of a finite range random field satisfies the Kirkwood–Salsburg equations. Throughout this section $\mu$ is a given finite range random field which is not necessarily translation invariant. See Spitzer [11] for a discussion of the relation of the Kirkwood–Salsburg operations to the uniqueness problem for $\mu$.

Definition. The support of $x \in \Omega$, $\text{supp} \ x$, is defined

$$\text{supp} \ x = \{j \in \mathbb{Z}^d : x_j \neq 0\}. \quad (8.1)$$

The subspace of finite support $\Omega_0$ is defined

$$\Omega_0 = \{x \in \Omega: \text{supp} \ x \in \mathcal{L}\}. \quad (8.2)$$

Definition. The correlation function $\rho$ of $\mu$ is the real-valued function on $\Omega_0$ given by

$$\rho(x) = \mu(\omega = x \text{ on } \text{supp} \ x) \quad \text{for} \quad x \in \Omega_0. \quad (8.3)$$

Lemma 12. Let $x \in \Omega_0$, $A \in \mathcal{L}$, $A \supset \text{supp} \ x$. Then

$$\mu(\omega = x \text{ on } A) = \sum_{y \in A} (-1)^{|\text{supp} y|} \rho(x + y), \quad (8.4)$$

$$A - \{y \in \Omega_0: \text{supp} \ y \subset A - \text{supp} \ x\}. \quad (8.5)$$
Proof. Let \( h_j(y) = 1 \) if \( y_j \neq 0 \), \( h_j(y) = 0 \) if \( y_j = 0 \). Then

\[
\mu(\omega = x \text{ on } A) = \sum_{y \in A} \left[ \prod_{j \in \text{supp} z} (1 - h_j(y)) \right] \mu(\omega = x + y \text{ on } A). \tag{8.6}
\]

If we expand the product and perform the summation in (8.6), we get exactly the terms in (8.4).

**Theorem 11.** The correlation function \( \rho \) satisfies the equations

\[
\rho(x) = \sum_{y \in \Omega_0} K_j(x, i x + y) \rho(i x + y) \tag{8.7}
\]

for each \( x \in \Omega_0 \) and each \( j \in \text{supp } \omega \) with

\[
K_j(x, i x + y) = \sum_{\text{supp } z \subseteq \text{supp } y} (-1)^{|\text{supp } y - \text{supp } z|} \mu(\omega_j = x_j \mid \omega = x + z \text{ on } \partial j)
\]

if \( \text{supp } y \subseteq \partial j - \text{supp } x \), \( K_j(x, i x + y) = 0 \) otherwise. \tag{8.8}

Proof. Let \( V = \text{supp } x \setminus \{j\} \), \( V' = \partial j - \text{supp } x \), \( V'' = V' - \text{supp } y \). Then from (8.3)

\[
\rho(x) = \sum_{\text{supp } y \subset V'} \mu(\omega = x \text{ on } V, \omega = y \text{ on } V') \mu(\omega_j = x_j \mid \omega = x + y \text{ on } \partial j). \tag{8.9}
\]

With the substitution involving (8.4) equation (8.9) becomes

\[
\rho(x) = \sum_{y} \sum_{\text{supp } z \subset V'} (-1)^{|\text{supp } z|} \times \rho(i x + y + z) \mu(\omega_j = x_j \mid \omega = x + y \text{ on } \partial j). \tag{8.10}
\]

Collecting terms for a fixed value of \( y + z \) and making the change of variable \( y + z \rightarrow y \), we obtain (8.7) and (8.8).

**References**


