Integral-type operators on continuous function spaces on the real line

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Abstract

In this paper we introduce some new sequences of positive linear operators, acting on a sufficiently large space of continuous functions on the real line, which generalize Gauss–Weierstrass operators.

We study their approximation properties and prove an asymptotic formula that relates such operators to a second order elliptic differential operator of the form \(Lu := \alpha u'' + \beta u' + \gamma u\).

Shape-preserving and regularity properties are also investigated.

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1. Introduction

In this paper we introduce and study a sequence of integral-type positive linear operators acting on a sufficiently large continuous function space which contains wide classes of weighted spaces of continuous functions on the real line. These operators depend on three given functions \(\alpha, \beta, \gamma \in C(\mathbb{R})\), \(\gamma\) bounded, and generalize the classical Gauss–Weierstrass convolution operators [4, Section 5.2.9] which are a particular case of them with \(\alpha = 1\) and \(\beta = \gamma = 0\).

The main motivation to introduce these operators rests on the aim to construct a sequence of positive linear operators which satisfies an asymptotic formula (with respect to a given weighted
norm) whose limit operator is a second order elliptic differential operator of the form
\[ Lu = xu'' + \beta u' + \gamma u. \]

The operators which we introduce in this paper satisfy, indeed, such an asymptotic formula opening the way to a possible investigation to find suitable domains on which the differential operator \( L \) generates a \( C_0 \)-semigroup of positive operators which can be represented in terms of iterates of these operators. However this aspect will be developed in a forthcoming paper.

Besides the connection with semigroup theory, our operators seem to have a possible interest in the weighted approximation of continuous functions on the real line for a wide class of weights.

We start our analysis by first introducing a sequence of integral-type positive linear operators depending only on the functions \( x \) and \( \beta \). We discuss their approximation properties in several spaces of continuous functions and we give some estimates of the rate of convergence. We also establish an asymptotic formula together with a saturation result. Subsequently we study some shape-preserving properties. Finally, in the last section, by modifying these operators, we obtain a further approximation process which involves the function \( \gamma \) and which verifies analogous qualitative properties, including the general above mentioned asymptotic formula.

2. The integral operators \( G_n \)

Throughout the paper we shall denote by \( C(\mathbb{R}) \) the space of all real valued continuous functions on \( \mathbb{R} \) and with \( C_b(\mathbb{R}) \) (resp. \( C_0(\mathbb{R}), UC_b(\mathbb{R}) \)) the Banach lattice of all bounded continuous functions (resp. continuous functions that vanish at infinity, uniformly continuous and bounded functions) endowed with the natural order and the uniform norm \( \| \cdot \|_\infty \).

We shall also consider the closed subspace \( C_*(\mathbb{R}) \) of functions \( f \in C_b(\mathbb{R}) \) such that \( \lim_{x \to \pm \infty} f(x) \in \mathbb{R} \).

If \( w \) is a weight on \( \mathbb{R} \), i.e. \( w \in C_b(\mathbb{R}) \) and it is strictly positive, we shall denote by \( C_w^b(\mathbb{R}) \) (resp. \( C_w^0(\mathbb{R}) \)) the space of all functions \( f \in C(\mathbb{R}) \) such that \( w f \in C_b(\mathbb{R}) \) (resp. \( C_0(\mathbb{R}) \)); these spaces with respect to the natural order and the weighted norm \( \| \cdot \|_w \) defined by \( \| f \|_w := \| w f \|_\infty \) (\( f \in C_w^b(\mathbb{R}) \)) are Banach lattices.

Observe that \( C_b(\mathbb{R}) \subset C_w^b(\mathbb{R}) \) and \( \| \cdot \|_w \leq \| w \|_\infty \| \cdot \|_\infty \) on \( C_b(\mathbb{R}) \); in particular, if \( w \in C_0(\mathbb{R}) \), then \( C_b(\mathbb{R}) \subset C_w^0(\mathbb{R}) \).

Let \( E(\mathbb{R}) \) be the space of all functions \( f \in C(\mathbb{R}) \) such that
\[ \int_{-\infty}^{+\infty} |f(ay + b)| e^{-y^2/2} dy < +\infty \quad \text{for every } a \geq 0 \text{ and } b \in \mathbb{R}. \] (2.1)

Clearly a continuous function \( f \) belongs to \( E(\mathbb{R}) \) if and only if, for every positive second degree polynomial \( P \), the integral \( \int_{-\infty}^{+\infty} |f(y)| e^{-P(y)} dy \) is finite.

All polynomials belong to \( E(\mathbb{R}) \).

Moreover, if \( w \) is a weight on \( \mathbb{R} \) such that, for every couple of compact sets \( I \subset \mathbb{R}_+ \) and \( J \subset \mathbb{R} \), there exists \( h \in L^1(\mathbb{R}) \) such that
\[ \frac{e^{-y^2/2}}{w(ay + b)} \leq h(y) \quad \text{for every } y \in \mathbb{R}, \ a \in I, \ b \in J, \] (2.2)

then
\[ C_w^b(\mathbb{R}) \subset E(\mathbb{R}). \] (2.3)
From now on we shall fix two functions $\alpha, \beta \in C(\mathbb{R})$ and we shall assume that

$$\alpha(x) \geq 0 \quad \text{for every } x \in \mathbb{R}.$$  \hfill (2.4)

Consider, for every $n \geq 1$, the linear positive operator $G_n$ defined by setting, for every $f \in E(\mathbb{R})$ and $x \in \mathbb{R}$

$$G_n(f)(x) := \frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{+\infty} f\left(\sqrt{\frac{2\alpha(x)}{n}} y + \frac{\beta(x)}{n} \right) e^{-y^2/2} dy. \hfill (2.5)$$

Note that, if $\alpha = 1$ and $\beta = 0$, the operator $G_n$ turns into the $n$th Gauss–Weierstrass convolution operator (see, e.g., [4, Section 5.2.9]).

If, for some $x \in \mathbb{R}$, $\alpha(x) > 0$, denoted by $Z_{n,x}$ a real random variable with normal distribution $N\left(x + \frac{\beta(x)}{n}, \frac{2\alpha(x)}{n}\right)$ (see, e.g., [7, p. 26]), then for every $f \in E(\mathbb{R})$ we can rewrite $G_n$ as

$$G_n(f)(x) = \frac{n}{\sqrt{4\pi \alpha(x)}} \int_{-\infty}^{+\infty} f(t) e^{-\frac{1}{4\alpha(x)} \left(t-x-\frac{\beta(x)}{n}\right)^2} dt = \int_{-\infty}^{+\infty} f dP_{Z_{n,x}} = \mathbb{E}\left(f \circ Z_{n,x}\right), \hfill (2.6)$$

where $\mathbb{E}\left(f \circ Z_{n,x}\right)$ and $P_{Z_{n,x}}$ denote the expected value of $f \circ Z_{n,x}$ and the distribution of $Z_{n,x}$.

Moreover, if $\beta(x) = 0$, by a Kolmogorov’s theorem (see [7, Corollary 9.5]), there exist a probability space $(\Omega, \mathcal{F}, P)$ and a sequence $(Y(k,x))_{k \geq 1}$ of independent real random variables with the same normal distribution $N\left(x, \frac{2\alpha(x)}{n}\right)$ so that $Z_{n,x} = \frac{Y(1,x) + \cdots + Y(n,x)}{n}$ and

$$G_n(f)(x) = \int_{\Omega} f\left(\frac{Y(1,x) + \cdots + Y(n,x)}{n}\right) dP. \hfill (2.7)$$

This formula will be useful to discuss the monotonicity of the sequence $(G_n)_{n \geq 1}$ on convex functions.

**Remark 2.1.** The operators $G_n$, $n \geq 1$, are well defined on the bigger linear subspace

$$\tilde{E}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is Borel-measurable and satisfies (2.1)} \} \hfill (2.8)$$

which contains the spaces $L^p(\mathbb{R})$, $1 \leq p \leq +\infty$.

However in this paper we limit ourselves to study the operators $G_n$ on continuous function spaces, while a similar analysis on Lebesgue spaces will be carried out in a subsequent paper.

Under suitable assumption the operators $G_n$ are continuous on some subspaces of $C_b^w(\mathbb{R})$.

**Theorem 2.2.** Under assumption (2.2), $G_n\left(C_b^w(\mathbb{R})\right) \subset C(\mathbb{R})$ for every $n \geq 1$. Moreover, if

$$G_n(w^{-1}) \in C_b^w(\mathbb{R}), \hfill (2.9)$$

then $G_n\left(C_b^w(\mathbb{R})\right) \subset C_b^w(\mathbb{R})$, $G_n$ is continuous on $C_b^w(\mathbb{R})$ and $\|G_n\|_{C_b^w(\mathbb{R})} = \|G_n(w^{-1})\|_w$.

In particular $G_n\left(C_b(\mathbb{R})\right) \subset C_b(\mathbb{R})$, $G_n$ is continuous on $C_b(\mathbb{R})$ and $\|G_n\|_{C_b(\mathbb{R})} = 1$. 

Proof. Let $n \geq 1$, $f \in C_w^b(\mathbb{R})$ and $x \in \mathbb{R}$. If $(x_p)_{p \geq 1}$ is a sequence of real numbers tending to $x$, then for every $y \in \mathbb{R}$

$$
\lim_{p \to \infty} f \left( \sqrt{\frac{2x(x_p)}{n}} y + x_p + \frac{\beta(x_p)}{n} \right) e^{-\frac{y^2}{2}} = f \left( \sqrt{\frac{2x(x)}{n}} y + x + \frac{\beta(x)}{n} \right) e^{-\frac{y^2}{2}}.
$$

On the other hand, if $I$ and $J$ are compact subsets of $\mathbb{R}_{+}$ and $\mathbb{R}$ such that $\left\{ \sqrt{\frac{2x(x_p)}{n}} | p \geq 1 \right\} \subset I$ and $\left\{ x_p + \frac{\beta(x_p)}{n} | p \geq 1 \right\} \subset J$, by (2.2) there exists $h \in L^1(\mathbb{R})$ such that, for every $y \in \mathbb{R}$ and $p \geq 1$,

$$
\left| f \left( \sqrt{\frac{2x(x_p)}{n}} y + x_p + \frac{\beta(x_p)}{n} \right) e^{-\frac{y^2}{2}} \right| \leq \frac{\| f \|_w e^{-\frac{y^2}{2}}}{w \left( \sqrt{\frac{2x(x_p)}{n}} y + x_p + \frac{\beta(x_p)}{n} \right)}
$$

$$
\leq \| f \|_w h(y),
$$

therefore $G_n(f)(x_p) \to G_n(f)(x)$ by the Lebesgue dominated convergence theorem.

Under assumption (2.9), if $f \in C_w^b(\mathbb{R})$, then for every $x \in \mathbb{R}$

$$
|w(x) G_n(f)(x)| \leq \frac{w(x)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| f \left( \sqrt{\frac{2x(x_p)}{n}} y + x + \frac{\beta(x_p)}{n} \right) \right| e^{-y^2/2} dy
$$

$$
\leq \frac{\| f \|_w}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{w(x)}{w \left( \sqrt{\frac{2x(x_p)}{n}} y + x + \frac{\beta(x_p)}{n} \right)} e^{-y^2/2} dy
$$

$$
= \| f \|_w w(x) G_n(w^{-1})(x) \leq \| G_n(w^{-1}) \|_w \| f \|_w,
$$

which implies that $G_n(f) \in C_w^b(\mathbb{R})$ and $\| G_n \|_{C_w^b(\mathbb{R})} \leq \| G_n(w^{-1}) \|_w$. The converse inequality is obvious.

The last statement follows from the previous one applied to the weight $w = 1$ which satisfies (2.2) and (2.9). □

Examples 2.3. 1. Conditions (2.2) and (2.9) are satisfied by every $w \in C_b(\mathbb{R})$ such that $\inf_{\mathbb{R}} w > 0$.

2. Consider $\phi \in C(\mathbb{R})$ and assume that $\phi$ is increasing on $[0, +\infty]$ and $\inf_{x \in \mathbb{R}} \phi(x) > 0$. Suppose that

$$
\phi(x) \leq \phi(|x|) \quad \text{for every } x \in \mathbb{R},
$$

and

$$
\int_{-\infty}^{+\infty} \phi(a |x| + b) e^{-x^2/2} dx < +\infty \quad \text{for every } a, b > 0.
$$

Then $w := \phi^{-1}$ satisfies (2.2).
Indeed, if $I$ and $J$ are compact subsets of $\mathbb{R}_+$ and $\mathbb{R}$, respectively, and if $M$ is a strictly positive constant such that $I \cup J \subseteq [-M, M]$ then, for every $a \in I$, $b \in J$ and $y \in \mathbb{R}$, we obtain
\[
\frac{e^{-y^2/2}}{w(a y + b)} = \varphi(a y + b) e^{-y^2/2} \leq \varphi(a |y| + |b|) e^{-y^2/2}
\]
and $h \in L^1(\mathbb{R})$.

For instance, we can consider $\varphi(x) := 1 + |x|^m$ or $\varphi(x) := e^{m|x|}$ ($x \in \mathbb{R}$, $m \geq 0$) and so $w_m(x) := (1 + |x|^m)^{-1}$ and $w_m(x) := e^{-m|x|}$ satisfy (2.2).

3. Assume that $x(x) = O\left(x^2\right)$ and $\beta(x) = O\left(|x|\right)$ as $x \to \pm \infty$. Then the polynomial weight $w_m(x) := (1 + |x|^m)^{-1}$ ($x \in \mathbb{R}$, $m \geq 0$) satisfies (2.9).

Moreover, if $m \in \mathbb{N}$ is even, there exists a constant $M_m > 0$, independent of $n$, such that
\[
\|G_n\|_{C^m_p(\mathbb{R})} \leq 1 + \frac{M_m}{n}, \tag{2.10}
\]
Actually there exists $C > 0$ such that $\sqrt{2\pi}(x) \leq C(1 + |x|)$ and $|\beta(x)| \leq C(1 + |x|)$. Note that, for every $x \in \mathbb{R}$,
\[
G_n\left(w_m^{-1}\right)(x) = 1 + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[ \frac{2\pi}{n} y + x + \frac{\beta(x)}{n} \right] e^{-y^2/2} \, dy
\]
\[
\leq 1 + \frac{(1 + |x|)^m}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \frac{C}{n} |y| + 1 + \frac{C}{n} \right) e^{-y^2/2} \, dy
\]
\[
\leq 1 + \frac{(1 + |x|)^m}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (C |y| + 1 + C)^m e^{-y^2/2} \, dy
\]
\[
= 1 + C_m (1 + |x|)^m,
\]
where $C_m := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (C |y| + 1 + C)^m e^{-y^2/2} \, dy$ and hence
\[
\sup_{x \in \mathbb{R}} w_m(x) G_n\left(w_m^{-1}\right)(x) \leq 1 + C_m.
\]

Assume now that $m = 2p$, $p \geq 1$, and evaluate
\[
\sup_{x \in \mathbb{R}} w_m(x) G_n\left(w_m^{-1}\right)(x) = \sup_{x \in \mathbb{R}} \frac{1 + G_n\left(e^{2p}\right)(x)}{1 + x^{2p}},
\]
where $e^{2p}(x) := x^{2p}$ ($x \in \mathbb{R}$).

Set $F := \{(i, j, k) \in \mathbb{N}^3 | i + j + k = 2p$ and two of the indices $i, j, k$ are not both null}. Then there exist some real constants $C_{ijk}$, $(i, j, k) \in F$, such that
\[
\left(\frac{2\pi}{n} y + x + \frac{\beta(x)}{n}\right)^{2p} = \left(\frac{2\pi}{n} \right)^p y^{2p} + x^{2p} + \left(\frac{\beta(x)}{n}\right)^{2p}
\]
\[
+ \sum_{(i,j,k) \in F} C_{ijk} \left(\frac{2\pi}{n}\right)^i y^i x^j \left(\frac{\beta(x)}{n}\right)^k;
\]
therefore
\[
G_n(e_{2p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{2\alpha(x)}{n}} y + x + \frac{\beta(x)}{n} \right)^{2p} e^{-y^2/2} \, dy
\]
\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{2\alpha(x)}{n} \right)^p \int_{-\infty}^{+\infty} y^{2p} e^{-y^2/2} \, dy + x^{2p} + \left( \frac{\beta(x)}{n} \right)^{2p}
\]
\[
+ \frac{1}{\sqrt{2\pi}} \sum_{(i,j,k) \in F} C_{ijk} \left( \sqrt{\frac{2\alpha(x)}{n}} \right)^i x^j \left( \frac{\beta(x)}{n} \right)^k \int_{-\infty}^{+\infty} y^i e^{-y^2/2} \, dy
\]
\[
= D \left( \frac{2\alpha(x)}{n} \right)^p + x^{2p} + \left( \frac{\beta(x)}{n} \right)^{2p} + \sum_{(i,j,k) \in F} D_{ijk} \left( \frac{2\alpha(x)}{n} \right)^{i/2} x^j \left( \frac{\beta(x)}{n} \right)^k
\]
where \( D := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^{2p} e^{-y^2/2} \, dy \) and \( D_{ijk} := \frac{C_{ijk}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} y^i e^{-y^2/2} \, dy \). Thus, setting
\[
\varphi(x) := D \left( 2\alpha(x) \right)^p + \left( \beta(x) \right)^{2p} + \sum_{(i,j,k) \in F} D_{ijk} \left( 2\alpha(x) \right)^{i/2} |x|^j |\beta(x)|^k
\]
and
\[
M_m := \sup_{\mathbb{R}} \frac{\varphi(x)}{(1 + x^m)},
\]
we obtain
\[
\sup_{x \in \mathbb{R}} \frac{1 + G_n(e_{m}) (x)}{1 + x^m} \leq 1 + \frac{M_m}{n},
\]
and hence the result follows.

**Theorem 2.4.** Let \( n \geq 1 \) and suppose that for almost every \( y \in \mathbb{R} \)
\[
\lim_{x \to \pm \infty} \left( \sqrt{\frac{2\alpha(x)}{n}} y + x + \frac{\beta(x)}{n} \right) \in \{-\infty, +\infty\}.
\]
Then for every \( f \in \mathcal{C}_*(\mathbb{R}) \)
\[
\lim_{x \to +\infty} G_n(f)(x) = \lim_{x \to +\infty} f(x) \quad \text{and} \quad \lim_{x \to -\infty} G_n(f)(x) = \lim_{x \to -\infty} f(x).
\]
In particular
(1) \( G_n \) maps \( \mathcal{C}_0(\mathbb{R}) \) into itself, it is continuous and \( \| G_n \|_{\mathcal{C}_0(\mathbb{R})} \leq 1 \),
(2) \( G_n \) maps \( \mathcal{C}_*(\mathbb{R}) \) into itself, it is continuous and \( \| G_n \|_{\mathcal{C}_*(\mathbb{R})} = 1 \).

**Proof.** Let \( f \in \mathcal{C}_*(\mathbb{R}) \); the continuity of \( G_n(f) \) follows from Theorem 2.2.
Suppose that
\[
\lim_{x \to +\infty} f(x) = 0.
\]
In order to prove that \( \lim_{x \to +\infty} G_n(f)(x) = 0 \), note that for every \( x, y \in \mathbb{R} \)

\[
\left| f\left( \sqrt{\frac{2\pi}{n}} y + x + \frac{\beta(x)}{n} \right) e^{-y^2/2} \right| \leq \| f \|_{\infty} e^{-y^2/2}
\]

and, by (2.12), for a.e. \( y \in \mathbb{R} \)

\[
\lim_{x \to +\infty} f\left( \sqrt{\frac{2\pi}{n}} y + x + \frac{\beta(x)}{n} \right) e^{-y^2/2} = 0
\]

so that, by Lebesgue’s dominated convergence theorem, \( \lim_{x \to +\infty} G_n(f)(x) = 0 \).

A similar reasoning shows that \( \lim_{x \to -\infty} G_n(f)(x) = 0 \) provided that \( \lim_{x \to -\infty} (f)(x) = 0 \), and hence part (1) follows.

Since \( G_n(1) = 1 \), part (2) easily follows by applying part (1) to \( f - f(+\infty) \) and to \( f - f(-\infty) \), where \( f(+\infty) := \lim_{x \to +\infty} f(x) \) and \( f(-\infty) := \lim_{x \to -\infty} f(x) \).

When \( \alpha \) is strictly positive “at infinity”, the operators \( G_n \) also map the space \( C^w_0(\mathbb{R}) \) into itself.

**Theorem 2.5.** Let \( w \in C_0(\mathbb{R}) \) a weight on \( \mathbb{R} \) satisfying (2.2) and (2.9). Assume that there exists \( \delta > 0 \) such that \( \alpha(x) > 0 \) for \( |x| \geq \delta \).

Then, for every \( n \geq 1 \), \( G_n \) is continuous from \( C^w_0(\mathbb{R}) \) into itself and \( \| G_n \|_{C^w_0(\mathbb{R})} \leq \| G_n(w^{-1}) \|_w \).

**Proof.** By virtue of Theorem 2.2 we just have to prove that \( \lim_{x \to \pm\infty} w(x)G_n(f)(x) = 0 \) for every \( f \in C^w_0(\mathbb{R}) \).

Let \( f \in C^w_0(\mathbb{R}) \) and \( \varepsilon > 0 \). Then there exists \( x_0 > 0 \) such that for \( |x| \geq x_0 \)

\[
|f(x)| \leq \frac{\varepsilon}{2w(x) \| G_n(w^{-1}) \|_w};
\]

hence, setting \( \delta_1 := \max (x_0, \delta) \), for \( |x| \geq \delta_1 \)

\[
|G_n(f)(x)| \leq \sqrt{\frac{n}{4\pi \alpha(x)}} \int_{|y| \leq \delta_1} |f(y)| e^{-\frac{n}{4\pi \alpha(x)}} (y - \frac{\beta(x)}{n} - x)^2 \, dy
\]

\[
+ \sqrt{\frac{n}{4\pi \alpha(x)}} \int_{|y| > \delta_1} |f(y)| e^{-\frac{n}{4\pi \alpha(x)}} (y - \frac{\beta(x)}{n} - x)^2 \, dy
\]

\[
\leq \sqrt{\frac{n}{4\pi \alpha(x)}} \max_{|y| \leq \delta_1} |f(y)| \int_{-\infty}^{+\infty} e^{-\frac{n}{4\pi \alpha(x)}} (y - \frac{\beta(x)}{n} - x)^2 \, dy
\]

\[
+ \frac{\varepsilon}{2 \| G_n(w^{-1}) \|_w} \sqrt{\frac{n}{4\pi \alpha(x)}} \int_{-\infty}^{+\infty} w(y)^{-1} e^{-\frac{n}{4\pi \alpha(x)}} (y - \frac{\beta(x)}{n} - x)^2 \, dy
\]

\[
\leq \max_{|y| \leq \delta_1} |f(y)| + \frac{\varepsilon}{2 \| G_n(w^{-1}) \|_w} G_n(w^{-1})(x),
\]
which implies that

\[ |w(x)G_n(f)(x)| \leq w(x) \max_{|y| \leq \delta_1} |f(y)| + \frac{\varepsilon}{2} \| G_n(w^{-1}) \|_w w(x)G_n(w^{-1})(x) \]

\[ \leq w(x) \max_{|y| \leq \delta_1} |f(y)| + \frac{\varepsilon}{2}. \]

Thus, since \( w \in C_0(\mathbb{R}) \), we obtain \( \lim_{x \to \pm \infty} w(x)G_n(f)(x) = 0 \).

The last inequality follows from the estimate

\[ |w(x)G_n(f)(x)| \leq \|f\|_w w(x)G_n(w^{-1})(x) \leq \|f\|_w \| G_n(w^{-1}) \|_w \]

which holds true for every \( x \in \mathbb{R} \). \( \square \)

3. Approximation properties

In this section we establish some approximation properties of the sequence \((G_n)_{n \geq 1}\) with respect to weighted norms and the uniform norm.

For every \( k \geq 0 \) and \( x \in \mathbb{R} \) we shall set

\[ e_k(t) := t^k \quad \text{and} \quad \psi_x(t) := t - x \quad (t \in \mathbb{R}). \quad (3.1) \]

Let us remark that, for every \( x \in \mathbb{R} \), the functions \( e_0, e_1, e_2, \psi_x, \psi_x^2, \psi_x^4 \in E(\mathbb{R}) \). Moreover, recalling that \( \int_{-\infty}^{+\infty} y^2 e^{-\frac{y^2}{2}} \, dy = \sqrt{2\pi} \) and \( \int_{-\infty}^{+\infty} y^4 e^{-\frac{y^2}{2}} \, dy = 3\sqrt{2\pi} \), for every \( n \geq 1 \) the following identities hold true:

\[ G_n(e_0)(x) = 1, \quad G_n(e_1)(x) = x + \frac{\beta(x)}{n}, \quad (3.2) \]

\[ G_n(e_2)(x) = \frac{2\sigma(x)}{n} + \left(x + \frac{\beta(x)}{n}\right)^2, \quad (3.3) \]

\[ G_n(\psi_x)(x) = \frac{\beta(x)}{n}, \quad G_n(\psi_x^2)(x) = \frac{2\sigma(x)}{n} + \left(\frac{\beta(x)}{n}\right)^2, \quad (3.4) \]

\[ G_n(\psi_x^4)(x) = 12 \left(\frac{\sigma(x)}{n}\right)^2 + 12 \frac{\sigma(x)}{n} \left(\frac{\beta(x)}{n}\right)^2 + \left(\frac{\beta(x)}{n}\right)^4. \quad (3.5) \]

We are now in the position to state the next result.

**Theorem 3.1.** Let \( w \in C_0(\mathbb{R}) \) a weight on \( \mathbb{R} \) satisfying (2.2). Assume that:

(i) there exists \( \delta > 0 \) such that \( \sigma(x) > 0 \) for \( |x| \geq \delta \);

(ii) \( e_2 \in C_0^w(\mathbb{R}) \);

(iii) \( \sigma, \beta^2 \in C_b^w(\mathbb{R}) \);

(iv) there exist \( M > 0 \) and \( v \geq 1 \) such that, for every \( n \geq v \),

\[ \sup_{x \in \mathbb{R}} w(x)G_n(w^{-1})(x) \leq M. \quad (3.6) \]
Then for every \( f \in C_{0}^{w}(\mathbb{R}) \)
\[
\lim_{n \to \infty} G_{n}(f) = f \quad \text{in} \quad (C_{0}^{w}(\mathbb{R}), \|\cdot\|_{w})
\]
and the convergence is uniform on compact subsets of \( \mathbb{R} \). In particular, for every \( f \in C_{b}(\mathbb{R}) \)
\[
\lim_{n \to \infty} \lim_{n \to \infty} G_{n}(f) = f \quad \text{uniformly on compact subsets of} \ \mathbb{R}.
\]

**Proof.** According to Theorem 2.5, for \( n \geq v \) we have \( G_{n}(C_{0}^{w}(\mathbb{R})) \subset C_{0}^{w}(\mathbb{R}) \) and, by (3.6), the sequence \((G_{n})_{n \geq v}\) is equicontinuous on \( C_{0}^{w}(\mathbb{R}) \).

From condition (ii) it follows that \( \{e_{0}, e_{1}, e_{2}\} \subset C_{0}^{w}(\mathbb{R}) \). This set is a Korovkin subset in \( C_{0}^{w}(\mathbb{R}) \) (see [5, Example 2.3.3], [6, Example 4.9]) so, in order to obtain (3.7), it suffices to prove that \( G_{n}(e_{i}) \to e_{i} \) with respect to the weighted norm \( \|\cdot\|_{w} \), for \( i = 0, 1, 2 \). These limit relations easily follow from (3.2) and (3.3) and the proof is complete. \( \square \)

In some cases the convergence is uniform on the whole real line.

**Theorem 3.2.** Let \( \alpha, \beta \in C_{b}(\mathbb{R}) \). Then, for every \( f \in C_{0}(\mathbb{R}) \)
\[
\lim_{n \to \infty} G_{n}(f) = f \quad \text{uniformly on} \ \mathbb{R}.
\]
Moreover, if \( \alpha \) is strictly positive, the above limit relation holds for every \( f \in UC_{b}(\mathbb{R}) \).

**Proof.** Given \( f \in C_{0}(\mathbb{R}) \subset UC_{b}(\mathbb{R}) \), since \( \beta \) is bounded, we get
\[
\lim_{n \to \infty} f \left( x + \frac{\beta(x)}{n} \right) = f(x)
\]
uniformly with respect to \( x \in \mathbb{R} \).

Accordingly, it is enough to prove that
\[
\lim_{n \to \infty} G_{n}(f)(x) - f \left( x + \frac{\beta(x)}{n} \right) = 0,
\]
uniformly with respect to \( x \in \mathbb{R} \). To this purpose, consider the space
\[
\mathcal{H} := \left\{ C_{0}(\mathbb{R}) \cap C^{1}(\mathbb{R}) \mid f' \in C_{b}(\mathbb{R}) \right\}
\]
and let \( f \in \mathcal{H} \); then
\[
\left| G_{n}(f)(x) - f \left( x + \frac{\beta(x)}{n} \right) \right| \\
\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| f \left( \sqrt{2\pi} x \right) t + x + \frac{\beta(x)}{n} - f \left( x + \frac{\beta(x)}{n} \right) \right| e^{-t^{2}/2} \, dt \\
\leq \frac{\|f'\|_{\infty}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{2\pi x} \frac{2\pi x}{n} |t| e^{-t^{2}/2} \, dt \leq 2 \sqrt{\frac{x}{n\pi} \|f'\|_{\infty}}
\]
and hence (2) holds true. On the other hand, by virtue of Stone–Weierstrass’s Theorem (see, e.g., [4, Theorem 4.4.4]), the space \( \mathcal{H} \) is dense in \( (C_{0}(\mathbb{R}), \|\cdot\|_{\infty}) \), so the limit (2) holds true for every \( f \in C_{0}(\mathbb{R}) \).
If \( \alpha \) is strictly positive the statement follows from Theorem 5.2.2 in [4], taking (2.6) into account. \( \square \)

**Remark 3.3.** By using Theorem 5.1.2 of [4] and Theorem 2.2.1 of [9] it is possible to state some quantitative pointwise estimates of the above convergence results.

The following uniform estimate can also be deduced from Theorem 2.2.1 of [9] and we leave the details of the proof to the reader.

As usual \( (f, \Delta) \) and \( (f, \Delta^2) \) denote the ordinary first and second moduli of continuity (see, e.g., [4, p. 266]). Then, if \( \alpha, \beta \in C_b(\mathbb{R}) \), for every \( f \in C_b(\mathbb{R}) \)

\[
\|G_n(f) - f\|_{\infty} \leq \frac{\|\beta\|_{\infty}}{\sqrt{\|\alpha\|_{\infty} + \|\beta\|_{\infty}^2}} \omega \left(f, \sqrt{\frac{2\|\alpha\|_{\infty} + \|\beta\|_{\infty}^2}{n}}\right)
\]

\[
+ \frac{3}{2} \omega \left(f, \sqrt{\frac{2\|\alpha\|_{\infty} + \|\beta\|_{\infty}^2}{n}}\right).
\]

(3.9)

Let \( M \geq 0 \) and \( k > 0 \) and consider the class \( \text{Lip}(k, M) \) of all functions \( f \in C(\mathbb{R}) \) such that

\[
|f(x) - f(y)| \leq M|x - y|^k \quad \text{for every } x, y \in \mathbb{R}.
\]

Observe that, by (2.2) and Example 2.3-2, \( \text{Lip}(k, M) \subset E(\mathbb{R}) \). As regards this class of functions, we establish the following estimates.

**Proposition 3.4.** For \( n \geq 1, f \in \text{Lip}(k, M) \) and \( x \in \mathbb{R} \)

\[
|G_n(f)(x) - f(x)| \leq MG_n\left(\left|\psi_x\right|^k\right)(x).
\]

(3.10)

In particular, if \( f \in E(\mathbb{R}) \) is differentiable and \( f' \in C_b(\mathbb{R}) \)

\[
|G_n(f)(x) - f(x)| \leq \|f'\|_{\infty} \sqrt{\frac{2\alpha(x)}{n} + \left(\frac{\beta(x)}{n}\right)^2}.
\]

(3.11)

**Proof.** Let \( f \in \text{Lip}(k, M) \) and \( n \geq 1 \); then, for every \( x \in \mathbb{R} \)

\[-M|x - e_1|^k \leq f - f(x) \cdot 1 \leq M|x - e_1|^k.
\]

Since \( G_n \) is linear and positive, we have that

\[-MG_n\left(\left|\psi_x\right|^k\right) \leq G_n\left(f - f(x) \cdot 1\right) \leq MG_n\left(\left|\psi_x\right|^k\right),
\]

and hence (3.10) holds true. If \( f \) is differentiable and \( f' \in C_b(\mathbb{R}) \), clearly \( f \in \text{Lip}(1, \|f'\|_{\infty}) \), and therefore, by virtue of Cauchy–Schwarz’s inequality, (see, e.g., [4, p. 21])

\[
|G_n(f)(x) - f(x)| \leq \|f'\|_{\infty} G_n\left(\left|\psi_x\right|\right)(x) \leq \|f'\|_{\infty} \sqrt{G_n\left(\left|\psi_x\right|^2\right)(x)} \sqrt{G_n(1)(x)}
\]

\[
= \|f'\|_{\infty} \sqrt{\frac{2\alpha(x)}{n} + \left(\frac{\beta(x)}{n}\right)^2}. \quad \square
\]
In the sequel the symbol $UC^2_b(\mathbb{R})$ will stand for the space of all twice differentiable functions on $\mathbb{R}$ with uniformly continuous and bounded second derivative. Note that $UC^2_b(\mathbb{R}) \subset E(\mathbb{R})$ because all the functions $f \in C(\mathbb{R})$ such that $\sup_{x \in \mathbb{R}} \frac{|f(x)|}{1+x^2} < +\infty$ belong to $E(\mathbb{R})$.

**Theorem 3.5.** Let $w$ be a weight on $\mathbb{R}$ and assume that $\alpha^2, \beta^4 \in C^b_w(\mathbb{R})$. Then for every $f \in UC^2_b(\mathbb{R})$

$$\lim_{n \to \infty} w \left[ n(G_n(f) - f) - (\alpha f'' + \beta f') \right] = 0$$

(3.12)

uniformly on $\mathbb{R}$. In particular

$$\lim_{n \to \infty} n(G_n(f) - f) = \alpha f'' + \beta f'$$

uniformly on compact subsets of $\mathbb{R}$.

**Proof.** To get the result we shall use Theorem 1 in [1]. First, note that for every $x \in \mathbb{R}$

$$\left| w(x) \left( nG_n \left( \psi^2_x \right)(x) - 2\alpha(x) \right) \right| = \left| w(x) \left( n \left( \frac{2\alpha(x)}{n} + \left( \frac{\beta(x)}{n} \right)^2 \right) - 2\alpha(x) \right) \right|$$

$$= \frac{w(x) \beta(x)^2}{n} \leq \frac{w(x) (1 + \beta(x)^4)}{n}$$

$$\leq \frac{1}{n} \left( \|w\|_\infty + \|\beta^4\|_w \right),$$

and hence $\lim_{n \to \infty} w(x) \left( nG_n \left( \psi^2_x \right)(x) - 2\alpha(x) \right) = 0$ uniformly on $\mathbb{R}$.

On the other hand, for every $x \in \mathbb{R}$ and $k \in \mathbb{N}$,

$$\left| w(x) x^k \left( nG_n \left( \psi^2_x \right)(x) - \beta(x) \right) \right| = \left| w(x) x^k \left( n \frac{\beta(x)}{n} - \beta(x) \right) \right| = 0$$

and hence $\lim_{n \to \infty} w(x) x^k \left[ nG_n \left( \psi^2_x \right)(x) - \beta(x) \right] = 0$ uniformly on $\mathbb{R}$.

Furthermore, if $x \in \mathbb{R}$,

$$w(x) nG_n \left( \psi^4_x \right)(x) = w(x) n \left[ 12 \frac{\alpha(x)^2}{n} + 12 \frac{\alpha(x) \beta(x)^2}{n} + \frac{\beta(x)^4}{n} \right]$$

$$\leq \frac{12 \|\alpha^2\|_w}{n} + \frac{12 \sqrt{\|\alpha\|_w} \sqrt{\|\beta^4\|_w}}{n^2} + \frac{\|\beta^4\|_w}{n^3}$$

$$\leq \frac{12 \|\alpha^2\|_w}{n} + \frac{12 \sqrt{\|\alpha^2\|_w} \sqrt{\|\beta^4\|_w}}{n^2} + \frac{\|\beta^4\|_w}{n^3}$$

so that $\lim_{n \to \infty} w(x) nG_n \left( \psi^4_x \right)(x) = 0$ uniformly on $\mathbb{R}$.

Finally

$$\sup_{x \in \mathbb{R}} \sup_{n \geq 1} w(x) nG_n \left( \psi^2_x \right)(x) = \sup_{x \in \mathbb{R}} \sup_{n \geq 1} w(x) n \left( \frac{2\alpha(x)}{n} + \left( \frac{\beta(x)}{n} \right)^2 \right)$$

$$= \sup_{x \in \mathbb{R}} \sup_{n \geq 1} w(x) \left( 2\alpha(x) + \frac{\beta(x)^2}{n} \right)$$
\[
\begin{align*}
\leq & \sup_{x \in \mathbb{R}} w(x) \left( 2\alpha(x) + (\beta(x))^2 \right) \\
\leq & \sup_{x \in \mathbb{R}} w(x) \left( 2 + 2\alpha(x)^2 + 1 + (\beta(x))^4 \right) \\
= & 3 \|w\|_{\infty} + \left\| 2\alpha^2 + \beta^4 \right\|_w < +\infty
\end{align*}
\]

and hence the result follows. \(\square\)

An immediate consequence of the previous result is indicated below.

**Corollary 3.6.** Under the same assumptions of Theorem 3.5, further suppose that \(\alpha(x) > 0\) for every \(x \in \mathbb{R}\). Then, for every \(f \in UC_b^2(\mathbb{R})\), the following statements are equivalent:

(a) \(\|G_n(f) - f\|_w = o \left( \frac{1}{n} \right)\) as \(n \to +\infty\);
(b) there exist \(c_1, c_2 \in \mathbb{R}\) such that

\[
f(x) = c_1 \int_0^x \exp \left( -\int_0^t \frac{\beta(s)}{\alpha(s)} ds \right) dt + c_2 \quad (x \in \mathbb{R}).
\]

(3.13)

### 4. Qualitative properties

Let \(J\) be a real interval. We say that a function \(f : \mathbb{R} \to \mathbb{R}\) is affine (resp. convex, increasing) on \(J\) if the restriction of \(f\) to \(J\) is affine (resp. convex, increasing).

The proof of the next result is straightforward and so we omit it.

**Theorem 4.1.** The following statements hold true:

1. If \(\beta\) is affine on \(J\), then \(G_n\) maps affine functions on \(\mathbb{R}\) into affine functions on \(J\). In particular, if \(\beta = 0\), then \(G_n(f) = f\) for every affine function \(f\) on \(\mathbb{R}\).
2. If \(\sqrt{\alpha}\) and \(\beta\) are affine on \(J\), then \(G_n\) maps convex functions on \(\mathbb{R}\) into convex functions on \(J\).
3. If \(\sqrt{\alpha}\) is affine on \(J\) and \(\beta\) is convex on \(J\), then \(G_n\) maps increasing convex functions on \(\mathbb{R}\) into convex functions on \(J\).
4. If \(\alpha\) is constant on \(J\) and \(e_1 + \frac{\beta}{n}\) is increasing on \(J\), then \(G_n\) maps increasing functions on \(\mathbb{R}\) into increasing functions on \(J\).

On the class of convex functions we can obtain further information about the convergence of \((G_n)_{n \geq 1}\).

**Theorem 4.2.** For every \(n \geq 1\), \(x \in \mathbb{R}\) and for every convex function \(f \in E(\mathbb{R})\)

\[
G_n(f)(x) \geq f \left( x + \frac{\beta(x)}{n} \right) \quad (4.1)
\]

In particular, if \(\beta(x) = 0\)

\[
G_n(f)(x) \geq G_{n+1}(f)(x) \geq f(x) \quad (4.2)
\]
Proof. Let $n \geq 1$, $x \in \mathbb{R}$ and consider a convex function $f \in E(\mathbb{R})$. The inequality (4.1) follows from Jensen’s inequality [7, Theorem 3.9], because

$$G_n(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f \left( \sqrt{\frac{2z(x)}{n}} y + x + \frac{\beta(x)}{n} \right) e^{-y^2/2} dy \geq f \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{2z(x)}{n}} y + x + \frac{\beta(x)}{n} \right) e^{-y^2/2} dy \right) = f \left( x + \frac{\beta(x)}{n} \right).$$

Concerning (4.2), if $\alpha(x) = \beta(x) = 0$ then

$$G_n(f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-y^2/2} dy = f(x).$$

While, if $\alpha(x) > 0$ and $\beta(x) = 0$, we can prove (4.2) by using a technique introduced by Khan in [8]. Indeed, by (2.7),

$$G_n(f)(x) = \int_{\Omega} f \left( \frac{Y(1,x) + \cdots + Y(n,x)}{n} \right) dP = \mathbb{E} \left( f \circ \frac{Z(n,x)}{n} \right),$$

where $Z(n,x) := Y(1,x) + \cdots + Y(n,x)$ and, since

$$\mathbb{E} \left( \frac{Z(n,x)}{n} \bigg| Z(n+1,x) \right) = \frac{Z(n+1,x)}{n+1} \quad P-a.e.$$

(see [8, Lemma 6]), where $\mathbb{E} \left( \frac{Z(n,x)}{n} \bigg| Z(n+1,x) \right)$ denotes the conditional expectation of $\frac{Z(n,x)}{n}$ given $Z(n+1,x)$ (see [7, Definition 15.2]), by applying once more Jensen’s inequality, we have

$$G_n(f)(x) = \mathbb{E} \left( f \circ \frac{Z(n,x)}{n} \right) = \mathbb{E} \left( \mathbb{E} \left( f \circ \frac{Z(n,x)}{n} \bigg| Z(n+1,x) \right) \right) \geq \mathbb{E} \left( f \left( \mathbb{E} \left( \frac{Z(n,x)}{n} \bigg| Z(n+1,x) \right) \right) \right) = \mathbb{E} \left( f \circ \frac{Z(n+1,x)}{n+1} \right) = G_{n+1}(f)(x).$$

From (4.1) it also follows that $G_n(f)(x) \geq f(x)$ and the proof is complete. □

The regularity of $\alpha$ and $\beta$ as well as the shapes of $\alpha$ and $\beta$ affect the ones of $G_n(f)$. We have, indeed, the following result.

Theorem 4.3. If $\alpha$ and $\beta$ are polynomials, then, for every $n \geq 1$, $G_n$ maps polynomials into polynomials.

Proof. In order to prove the statement, it is enough to show that, for every $p \geq 1$, the function $G_n(e_p)$ is a polynomial. Actually, for every $x \in \mathbb{R}$,

$$G_n(e_p)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{2\alpha(x)}{n}} y + x + \frac{\beta(x)}{n} \right)^p e^{-y^2/2} dy.$$
Theorem 4.4. If $\sqrt{x} \in \text{Lip}(1, M_{\xi})$ and $\beta \in \text{Lip}(1, M_{\beta})$, for some $M_{\xi} \geq 0$ and $M_{\beta} \geq 0$, then, for every $n \geq 1$, $k > 0$ and $M \geq 0$,

$$G_n (\text{Lip}(k, M)) \subset \text{Lip} \left( k, M \left( 1 + \frac{2}{\sqrt{n}} \int_{-\infty}^{0} e^{-s^2} ds \right) \exp \left( \frac{kM_{\beta} + k^2 M_{\xi}^2}{n} \right) \right).$$

Proof. Let $n \geq 1$ and $f \in \text{Lip}(k, M) \subset E(\mathbb{R})$. For every $x, y \in \mathbb{R}$

$$|G_n (f)(x) - G_n (f)(y)|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| f \left( \sqrt{\frac{2\alpha (x)}{n}} t + x + \frac{\beta (x)}{n} \right) - f \left( \sqrt{\frac{2\alpha (y)}{n}} t + y + \frac{\beta (y)}{n} \right) \right| e^{-\frac{t^2}{2}} dt$$

$$\leq \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left| \sqrt{\frac{2\alpha (x)}{n}} t + x + \frac{\beta (x)}{n} - \left( \sqrt{\frac{2\alpha (y)}{n}} t + y + \frac{\beta (y)}{n} \right) \right|^k e^{-\frac{t^2}{2}} dt$$

$$\leq \frac{M}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left( \sqrt{\frac{2\alpha (x)}{n}} - \sqrt{\frac{2\alpha (y)}{n}} \right) |t| + |x - y| + \left| \frac{\beta (x)}{n} - \frac{\beta (y)}{n} \right|^k e^{-\frac{t^2}{2}} dt.$$

Then, since $\sqrt{x} \in \text{Lip}(1, M_{\xi})$ and $\beta \in \text{Lip}(1, M_{\beta})$, we have

$$|G_n(f)(x) - G_n(f)(y)|$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} M \left[ \sqrt{\frac{2}{n} M_{\xi} |x - y| |t| + |x - y| + \frac{M_{\beta}}{n} |x - y|} \right]^k e^{-\frac{t^2}{2}} dt$$

$$\leq \frac{M}{\sqrt{2\pi}} |x - y|^k \int_{-\infty}^{+\infty} \left[ \sqrt{\frac{2}{n} M_{\xi} |t| + 1 + \frac{M_{\beta}}{n} |x - y|} \right]^k e^{-\frac{t^2}{2}} dt$$

$$\leq \frac{M}{\sqrt{2\pi}} |x - y|^k \int_{-\infty}^{+\infty} e \left( \sqrt{\frac{2}{n} M_{\xi} |t| + \frac{M_{\beta}}{n}} \right) e^{-\frac{t^2}{2}} dt$$

$$= \frac{M}{\sqrt{2\pi}} |x - y|^k \sqrt{\frac{2}{n}} \sqrt{2} e e^{\frac{k^2 M_{\xi}^2 + k M_{\beta}}{n}} \left( \frac{\sqrt{\pi}}{2} + \int_{-\frac{k M_{\xi}}{\sqrt{n}}}^{0} e^{-s^2} ds \right)$$

$$\leq M \left( 1 + \frac{2}{\sqrt{\pi}} \int_{-\frac{k M_{\xi}}{\sqrt{n}}}^{0} e^{-s^2} ds \right) \exp \left( \frac{k M_{\beta} + k^2 M_{\xi}^2}{n} \right) |x - y|^k. \quad \square$$
5. Modifying the integral operators $G_n$

In the spirit of [2] we shall modify the operators $G_n$, in order to obtain a further positive approximation process on $C_0^w(\mathbb{R})$, which satisfies an asymptotic formula generating a complete second order differential operator on the real line (see Theorem 5.2).

Let $\gamma \in C_b(\mathbb{R})$. Then, by the same considerations made in Section 2, we define, for every $n \geq 1$, the linear operator $G_n^*$ by setting

$$G_n^*(f) := G_n \left( \left( 1 + \frac{\gamma}{n} \right) f \right) \quad (f \in E(\mathbb{R})).$$  \hfill (5.1)

For $n \geq \|\gamma\|_{\infty}$ the operators $G_n^*$ are positive. Moreover, under the same assumptions of Theorems 2.2, 2.4 and 2.5, we have that

1. $G_n^*$ maps continuously $C_b^w(\mathbb{R})$ into itself;
2. $G_n^*$ maps continuously $C_0^w(\mathbb{R})$ into itself;
3. $G_n^*$ maps continuously $C_0(\mathbb{R})$ into itself and, if $\gamma \in C_0(\mathbb{R})$, then it maps continuously $C_*(\mathbb{R})$ into itself.

In any case

$$\|G_n^*\| \leq \|G_n\| \left( 1 + \frac{\|\gamma\|_{\infty}}{n} \right).$$  \hfill (5.2)

We also obtain some approximation results.

**Theorem 5.1.** (1) Under the same assumptions of Theorem 3.1, for every $f \in C_0^w(\mathbb{R})$

$$\lim_{n \to \infty} G_n^*(f) = f \quad \text{in} \quad \left( C_0^w(\mathbb{R}), \|\cdot\|_w \right)$$

and the convergence is uniform on compact subsets of $\mathbb{R}$.

(2) Let $\alpha, \beta \in C_b(\mathbb{R})$. Then, for every $f \in C_0(\mathbb{R})$

$$\lim_{n \to \infty} G_n^*(f) = f \quad \text{uniformly on} \quad \mathbb{R}.$$  

Moreover, if $\alpha$ is strictly positive, the above limit relation holds for every $f \in UC_b(\mathbb{R})$.

**Proof.** Statement (1) follows from Theorem 3.1 by simply observing that, for $f \in C_0^w(\mathbb{R})$ and $n \geq \nu$,

$$\|G_n^*(f) - f\|_w = \left\| G_n \left( \left( 1 + \frac{\gamma}{n} \right) f \right) - f \right\|_w = \left\| G_n(f) + \frac{1}{n} G_n(\gamma f) - f \right\|_w$$

$$\leq \|G_n(f) - f\|_w + \frac{M}{n} \|\gamma\|_{\infty} \|f\|_w.$$

Statement (2) follows from Theorem 3.2. \hfill $\Box$

The next result could be related to the approximation of strongly continuous semigroups, generated by second order differential operators on the real line, by means of iterates of the operators $G_n^*$. This subject will be treated in forthcoming papers.
Theorem 5.2. Let \( w \) be a weight on \( \mathbb{R} \) and assume that:

(i) \( e_1 \sqrt{\alpha} e_1 \beta, \alpha^2, \beta^4 \in C^w_b(\mathbb{R}) \);

(ii) \( \lim_{n \to \infty} w(x) x^k (G_n(\gamma)(x) - \gamma(x)) = 0 \) uniformly with respect to \( x \in \mathbb{R} \), for \( k = 0, 2 \).

Then for every \( f \in UC^2_b(\mathbb{R}) \)

\[
\lim_{n \to \infty} w \left[ n \left( G_n^* (f) - f \right) - \left( \alpha f'' + \beta f' + \gamma f \right) \right] = 0
\quad \text{(5.3)}
\]

uniformly on \( \mathbb{R} \). In particular

\[
\lim_{n \to \infty} n \left( G_n^* (f) - f \right) = \alpha f'' + \beta f' + \gamma f
\]

uniformly on compact subsets of \( \mathbb{R} \).

**Proof.** On account of Theorem 1 of [1], we need to show that

(1) \( \lim_{n \to \infty} w(x) \left[ nG_n^* \left( \psi^2_x \right)(x) - 2\alpha(x) \right] = 0; \)

(2) \( \lim_{n \to \infty} w(x) x^k \left[ nG_n^* \left( \psi^2_x \right)(x) - \beta(x) \right] = 0, \) for \( k = 0, 1; \)

(3) \( \lim_{n \to \infty} w(x) x^k \left[ n \left( G_n^* (1)(x) - 1 \right) - \gamma(x) \right] = 0, \) for \( k = 0, 2; \)

(4) \( \lim_{n \to \infty} w(x) nG_n^* \left( \psi^4_x \right)(x) = 0, \) uniformly with respect to \( x \in \mathbb{R}; \)

(5) \( \sup_{x \in \mathbb{R}} \sup_{n \geq 1} w(x) nG_n^* \left( \psi^2_x \right)(x) < +\infty. \)

To this aim, notice that, as we showed in the proof of Theorem 3.5, the operators \( G_n \) satisfy the analogous conditions (1), (2), (4) and (5). Moreover, by (3.4), (3.5) and (i), it follows that

\[
\lim_{n \to \infty} w(x) nG_n^* \left( \gamma \psi^k_x \right)(x) = 0 \quad \text{for } k = 2, 4
\]

and

\[
\lim_{n \to \infty} w(x) x^k nG_n^* \left( \gamma \psi_x \right)(x) = 0 \quad \text{for } k = 0, 1.
\]

Therefore, for every \( x \in \mathbb{R} \)

\[
w(x) \left[ nG_n^* \left( \psi^2_x \right)(x) - 2\alpha(x) \right] = w(x) \left[ nG_n \left( \left( 1 + \frac{\gamma}{n} \right) \psi_x^2 \right)(x) - 2\alpha(x) \right]
\]

\[
= w(x) \left[ nG_n \left( \psi_x^2 \right)(x) - 2\alpha(x) + G_n \left( \gamma \psi_x^2 \right)(x) \right],
\]

which implies (1). For \( k \in \mathbb{N} \)

\[
w(x) x^k \left[ nG_n^* \left( \psi_x \right)(x) - \beta(x) \right] = w(x) x^k \left[ nG_n \left( \left( 1 + \frac{\gamma}{n} \right) \psi_x \right)(x) - \beta(x) \right]
\]

\[
v = w(x) x^k \left[ nG_n \left( \psi_x \right)(x) - \beta(x) + G_n \left( \gamma \psi_x \right)(x) \right],
\]

and

\[
w(x) x^k \left[ n \left( G_n^* (1)(x) - 1 \right) - \gamma(x) \right] = w(x) x^k \left[ n \left( G_n \left( \left( 1 + \frac{\gamma}{n} \right) \right)(x) - 1 \right) - \gamma(x) \right]
\]

\[
= w(x) x^k \left[ G_n \left( \gamma \right)(x) - \gamma(x) \right].
\]
whence (2) and (3) are fulfilled. Finally, since

\[ w(x) nG_n^* \left( \psi^2 \right) (x) = w(x) nG_n \left( \left( 1 + \frac{\gamma}{n} \right) \psi^4 \right) (x) = w(x) nG_n \left( \psi^2 \right) (x) + w(x) G_n \left( \gamma \psi^2 \right) (x), \]

(4) and (5) are also satisfied. □

**Remark 5.3.** Let \( z(x) = O \left( x^2 \right) \) and \( \beta(x) = O \left( |x| \right) \) as \( x \to \pm \infty \) and consider the polynomial weight \( w_m(x) := \left( 1 + |x|^m \right)^{-1} \), \( x \in \mathbb{R}, m \in \mathbb{N} \).

In Examples 2.3 we proved that \( w_m \) satisfies conditions (2.2) and (3.6). Then, all the assumptions of Theorems 2.2, 3.1 and 5.2 are satisfied if \( m \geq 1 \), \( m > 2 \) and \( m > 4 \) \( (m \geq 4 \), if \( \gamma \) is constant), respectively.

Finally, by making use of the results in Section 4, we can obtain some qualitative properties of the operators \( G_n^* \). For the sake of brevity we omit the proof.

**Theorem 5.4.** Let \( n \geq 1 \) and \( J \) be a real interval. The following statements hold true:

1. Let \( \beta \) be affine on \( J \) and \( \gamma \) be constant. Then \( G_n^* \) maps affine functions into affine functions on \( J \). In particular, if \( \beta = 0 \), \( G_n^* (f) = \left( 1 + \frac{\gamma}{n} \right) f \), for every affine function \( f \) on \( \mathbb{R} \).
2. Let \( \sqrt{\alpha} \) and \( \beta \) be affine on \( J \) and assume that, for every convex function \( f \in E(\mathbb{R}) \), \( \left( 1 + \frac{\gamma}{n} \right) f \) is convex as well. Then \( G_n^* \) maps convex functions into convex functions on \( J \).
3. Let \( \sqrt{\alpha} \) be affine on \( J \), let \( \beta \) be convex on \( J \) and assume that, for every increasing convex function \( f \in E(\mathbb{R}) \), \( \left( 1 + \frac{\gamma}{n} \right) f \) is increasing and convex as well. Then \( G_n^* \) maps increasing convex functions into convex functions on \( J \).
4. Let \( \alpha \) be constant on \( J \), let \( e_1 + \frac{\beta}{n} \) be increasing on \( J \) and assume that, for every increasing \( f \in E(\mathbb{R}) \), \( \left( 1 + \frac{\gamma}{n} \right) f \) is increasing as well. Then \( G_n^* \) maps increasing functions into increasing functions on \( J \).
5. Let \( k > 0 \) and \( M_\alpha, M_\beta, M_\gamma \geq 0 \). If \( \sqrt{\alpha} \in \text{Lip} \left( 1, M_\alpha \right) \), \( \beta \in \text{Lip} \left( 1, M_\beta \right) \) and \( \gamma \in \text{Lip} \left( k, M_\gamma \right) \) (resp. \( \gamma \) is constant), then for every \( M \geq 0 \)

\[
G_n^* \left( \text{Lip}(k, M) \cap C_b(\mathbb{R}) \right) \subset \text{Lip} \left( k, \left( M + \frac{M_\alpha \| \gamma \|_\infty}{n} + \frac{M_\gamma \| f \|_\infty}{n} \right) \right)
\]

\[
\times \left( 1 + \frac{2}{\sqrt{\pi}} \int_{-\frac{kM_\beta + k^2 M_\alpha^2}{\sqrt{n}}}^{0} e^{-s^2} ds \exp \left( \frac{kM_\beta + k^2 M_\alpha^2}{n} \right) \right)
\]

\[
\times \left( \text{resp. } G_n^* \left( \text{Lip} \left( k, M \right) \right) \subset \text{Lip} \left( k, M \left| 1 + \frac{\gamma}{n} \right| \left( 1 + \frac{2}{\sqrt{\pi}} \int_{-\frac{kM_\beta + k^2 M_\gamma^2}{\sqrt{n}}}^{0} e^{-s^2} ds \right) \right) \right)
\]

\[
\times \exp \left( \frac{kM_\beta + k^2 M_\gamma^2}{n} \right). \)

6. If \( \alpha, \beta \) and \( \gamma \) are polynomials then \( G_n^* \) maps polynomials into polynomials.
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References